

CBSE Class 12 Mathematics Question Paper 2017 with Solution PDF-Set 65-1-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) This question paper contains **29** questions.
- (iii) Questions **1-4** in Section A are very short-answer type questions carrying **1** mark each.
- (iv) Questions **5-12** in Section B are short-answer type questions carrying **2** marks each.
- (v) Questions **13-23** in Section C are long-answer I type questions carrying **4** marks each.
- (vi) Questions **24-29** in Section D are long-answer II type questions carrying **6** marks each.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. If a line makes angles 90° and 60° respectively with the positive directions of x and y axes, find the angle which it makes with the positive direction of z -axis. [1 mark]

Solution:

Concept:

- The direction cosines of a line are $l = \cos \alpha$, $m = \cos \beta$, $n = \cos \gamma$, where the angles are measured from the positive coordinate axes.
- They satisfy $l^2 + m^2 + n^2 = 1$, with each direction angle in $[0^\circ, 180^\circ]$.

Step 1: Substitute the two given angles Here $l = \cos 90^\circ = 0$ and $m = \cos 60^\circ = \frac{1}{2}$. Thus

$$0^2 + \left(\frac{1}{2}\right)^2 + \cos^2 \gamma = 1.$$

Step 2: Determine the possible direction angles It follows that $\cos^2 \gamma = \frac{3}{4}$, so $\cos \gamma = \pm \frac{\sqrt{3}}{2}$. Both signs are permitted because the given angles do not specify the sign of the component along the positive z -axis. In the allowed range of direction angles these give $\gamma = 30^\circ$ or 150° .

Step 3: Conclusion The possible angle with the positive z -axis is 30° or 150° .

Quick Tip to Solve: Keep both signs when taking the square root of a squared direction cosine.

2. Evaluate: $\int_2^3 3^x dx$. [1 mark]

Solution:

Concept:

- For a positive base $a \neq 1$, $\frac{d}{dx} a^x = a^x \ln a$, where $\ln$ denotes the natural logarithm.
- A definite integral is evaluated by subtracting the antiderivative at the lower limit from its value at the upper limit.

Step 1: Find the antiderivative Since the derivative of 3^x contains the factor $\ln 3$, divide by this constant:

$$\int 3^x dx = \frac{3^x}{\ln 3} + C.$$

Differentiating this expression recovers 3^x .

Step 2: Apply the limits in the correct order Therefore

$$\int_2^3 3^x dx = \left[\frac{3^x}{\ln 3} \right]_2^3 = \frac{3^3 - 3^2}{\ln 3} = \frac{27 - 9}{\ln 3}.$$

The value is positive, as expected because the integrand is positive throughout the interval.

Quick Tip to Solve: For an exponential with base other than e , retain the divisor $\ln$ of the base.

Step 3: Conclusion The integral equals $\boxed{\frac{18}{\ln 3}}$.

Quick Tip to Solve: For an exponential with base other than e, retain the divisor ln of the base.

3. Determine the value of the constant k so that the function $f(x) = \begin{cases} \frac{kx}{|x|}, & \text{if } x < 0 \\ 3, & \text{if } x \geq 0 \end{cases}$ is continuous at $x = 0$. [1 mark]

Solution:

Concept:

- Continuity at $x = 0$ requires the left-hand limit, the right-hand limit and $f(0)$ to be equal.
- For $x < 0$, the absolute value is $|x| = -x$.

Step 1: Evaluate the left-hand limit On the negative side of zero,

$$f(x) = \frac{kx}{|x|} = \frac{kx}{-x} = -k.$$

Thus $\lim_{x \rightarrow 0^-} f(x) = -k$. We simplify only for $x < 0$, where division by x is valid.

Step 2: Match the limit to the value at zero For $x \geq 0$, $f(x) = 3$, so $\lim_{x \rightarrow 0^+} f(x) = 3 = f(0)$. The required equality is therefore $-k = 3$. With $k = -3$, both branches have value 3 near zero, verifying continuity.

Step 3: Conclusion The required constant is $\boxed{k = -3}$.

Quick Tip to Solve: On the left of zero replace absolute x by negative x before taking the limit.

4. If A is a 3×3 invertible matrix, then what will be the value of k if $\det(A^{-1}) = (\det A)^k$. [1 mark]

Solution:

Concept:

- For an invertible square matrix A , $AA^{-1} = I$ and $\det A \neq 0$.
- Determinants multiply: $\det(BC) = \det B \det C$, while $\det I = 1$.

Step 1: Take determinants of the inverse identity Using $AA^{-1} = I_3$, we obtain

$$\det A \det(A^{-1}) = \det I_3 = 1.$$

Since the determinant of A is nonzero, division by it is legitimate.

Step 2: Express the reciprocal as a power Hence

$$\det(A^{-1}) = \frac{1}{\det A} = (\det A)^{-1}.$$

Quick Tip to Solve: Use the determinant of AA inverse equals one; the inverse makes a reciprocal.

Comparing this universal inverse-determinant formula with the stated expression gives the required exponent. The matrix order does not change this reciprocal rule.

Step 3: Conclusion The required exponent in the determinant identity is $k = -1$.

Quick Tip to Solve: Use the determinant of AA^{-1} equals one; the inverse makes a reciprocal.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5. Prove that if E and F are independent events, then the events E and F' are also independent.

[2 marks]

Solution:

Concept:

- Independence of E and F means $P(E \cap F) = P(E)P(F)$.
- The complementary event F' has probability $P(F') = 1 - P(F)$. Events $E \cap F$ and $E \cap F'$ are disjoint and together form E .

Step 1: Partition E using F and its complement Every outcome belonging to E either belongs to F or does not. These possibilities cannot occur together, so

$$E = (E \cap F) \cup (E \cap F').$$

By addition of probabilities for disjoint events,

$$P(E) = P(E \cap F) + P(E \cap F').$$

Rearranging gives $P(E \cap F') = P(E) - P(E \cap F)$.

Step 2: Use the given independence Substitute the product formula for the intersection with F :

$$\begin{aligned} P(E \cap F') &= P(E) - P(E)P(F) \\ &= P(E)[1 - P(F)] \\ &= P(E)P(F'). \end{aligned}$$

This is exactly the defining equality for independence of E and F' . No division by a probability is needed, so the reasoning also covers zero probabilities.

Step 3: Conclusion Therefore E and F' are independent.

Quick Tip to Solve: Split E into its intersections with F and the complement, then subtract.

6. A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes one hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹ 300. Formulate an L.P.P. for finding how many of each should be produced daily to maximize the profit? It is being given that at least one of each must be produced. [2 marks]

Solution:

Concept:

- Let x be the number of necklaces and y the number of bracelets produced per day.
- The profit to maximize is $Z = 100x + 300y$, measured in rupees; each resource limit becomes a linear inequality.

Step 1: Translate the production and time limits The firm handles at most 24 articles in total, giving

$$x + y \leq 24.$$

A necklace takes half an hour and a bracelet takes one hour. Thus the total daily manufacturing time is $\frac{1}{2}x + y$ hours. With at most 16 hours available,

$$\frac{x}{2} + y \leq 16, \quad \text{equivalently } x + 2y \leq 32.$$

Both inequalities use an upper bound because unused capacity is permitted.

Step 2: Include the minimum production and objective At least one of each must be produced, hence $x \geq 1$ and $y \geq 1$; these conditions also ensure nonnegativity. The profits add because each necklace contributes 100 rupees and each bracelet contributes 300 rupees. Consequently the objective function is $Z = 100x + 300y$. The question asks for formulation, so the feasible constraints and objective complete the requested model. Since the variables count articles, practical production quantities are integers.

Step 3: Conclusion The L.P.P. is

$$\text{Maximize } Z = 100x + 300y$$

subject to

$$x + y \leq 24, \quad x + 2y \leq 32, \quad x \geq 1, \quad y \geq 1.$$

Here x and y count daily necklaces and bracelets respectively.

Quick Tip to Solve: Assign variables first, then translate article capacity and hours separately.

7. Find: $\int \frac{dx}{x^2 + 4x + 8}.$

[2 marks]

Solution:

Concept:

- Complete the square to express a quadratic denominator as $(x + h)^2 + a^2$.

Quick Tip to Solve: Complete the square first; the square root of the remaining constant is a.

- The standard formula is $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1}(u/a) + C$, for $a > 0$.

Step 1: Complete the square and substitute The denominator becomes

$$x^2 + 4x + 8 = (x + 2)^2 + 4 = (x + 2)^2 + 2^2.$$

Put $u = x + 2$, so $du = dx$. The integral is now

$$\int \frac{du}{u^2 + 2^2}.$$

This denominator is positive for every real x , so the antiderivative can be used throughout the real line.

Step 2: Apply the inverse-tangent formula and check Using $a = 2$, the antiderivative is $\frac{1}{2} \tan^{-1}(u/2) + C$. Replace u with $x + 2$. To check the coefficient by differentiation,

$$\frac{d}{dx} \left[\frac{1}{2} \tan^{-1} \left(\frac{x+2}{2} \right) \right] = \frac{1/4}{1 + (x+2)^2/4} = \frac{1}{(x+2)^2 + 4}.$$

This is precisely the original integrand.

Step 3: Conclusion

$$\int \frac{dx}{x^2 + 4x + 8} = \frac{1}{2} \tan^{-1} \left(\frac{x+2}{2} \right) + C.$$

Quick Tip to Solve: Complete the square first; the square root of the remaining constant is a.

8. Find the vector equation of the line passing through the point $A(1, 2, -1)$ and parallel to the line

$$5x - 25 = 14 - 7y = 35z.$$

[2 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$, parallel to a nonzero direction vector $\vec{b}$, has equation $\vec{r} = \vec{a} + \lambda \vec{b}$, where $\lambda \in \mathbb{R}$.
- In the symmetric form $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$, the direction ratios are l, m, n . Multiplying all three by the same nonzero number preserves the direction.

Step 1: Extract the direction from the given line Let the common value of $5x - 25 = 14 - 7y = 35z$ be t . Then

$$x = 5 + \frac{t}{5}, \quad y = 2 - \frac{t}{7}, \quad z = \frac{t}{35}.$$

Thus its direction ratios are $\frac{1}{5}, -\frac{1}{7}, \frac{1}{35}$. Multiplying all three by 35 gives the simpler ratios 7, -5, 1. In particular, the negative sign in the second ratio follows from the term $-7y$.

Step 2: Use the specified point and parallel direction The position vector of $A(1, 2, -1)$ is $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$. A vector parallel to the given line is $\vec{b} = 7\hat{i} - 5\hat{j} + \hat{k}$. Substitute these into $\vec{r} = \vec{a} + \lambda \vec{b}$. When $\lambda = 0$,

Quick Tip to Solve: Set the three expressions equal to one parameter; their parameter coefficients give direction ratios with the correct signs.

the point obtained is exactly A ; changing λ moves only along the required parallel direction.

Step 3: Conclusion The required vector equation is

$$\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k}), \quad \lambda \in \mathbb{R}.$$

Quick Tip to Solve: Set the three expressions equal to one parameter; their parameter coefficients give direction ratios with the correct signs.

9. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on $\mathbb{R}$.

[2 marks]

Solution:

Concept:

- To test monotonicity, differentiate the function and determine the sign of $f'(x)$ on its domain.
- A nonnegative derivative implies a nondecreasing function. To establish strict increase when the derivative vanishes at one point, compare the function values at any two distinct inputs.

Step 1: Find the derivative and its sign Differentiating the polynomial term by term gives

$$f'(x) = 12x^2 - 36x + 27 = 3(2x - 3)^2.$$

A square is nonnegative for every real number. Hence $f'(x) \geq 0$ on $\mathbb{R}$, with equality only at $x = \frac{3}{2}$. The derivative is positive everywhere else; there is no interval on which it is negative.

Step 2: Verify increase even across the stationary point For a direct comparison, rewrite the original polynomial as

$$f(x) = 4\left(x - \frac{3}{2}\right)^3 + \frac{13}{2}.$$

The real cube function is strictly increasing: if $u < v$, then $u^3 < v^3$. Thus for any $a < b$, we have $a - \frac{3}{2} < b - \frac{3}{2}$, and multiplication by the positive number 4 and addition of $\frac{13}{2}$ preserve this strict inequality. Consequently $f(a) < f(b)$, including when the two inputs straddle $\frac{3}{2}$.

Step 3: Conclusion Therefore f is strictly increasing on $\mathbb{R}$. The isolated zero of its derivative at $x = \frac{3}{2}$ does not prevent strict increase.

Quick Tip to Solve: Factor the derivative as a perfect square; a single stationary point does not create a decreasing interval.

10. The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm. [2 marks]

Solution:

Concept:

- For a sphere of radius r , volume and surface area are $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. Here r, V, S vary with time t .
- The chain rule relates the rates: $\frac{dV}{dt} = \frac{dV}{dr} \frac{dr}{dt}$ and $\frac{dS}{dt} = \frac{dS}{dr} \frac{dr}{dt}$.

Step 1: Determine the instantaneous radius growth Differentiate the volume formula with respect to time, keeping in mind that the radius depends on time:

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

The supplied rate is $\frac{dV}{dt} = 3 \text{ cm}^3 \text{ s}^{-1}$. At the instant when $r = 2 \text{ cm}$,

$$3 = 16\pi \frac{dr}{dt}, \quad \frac{dr}{dt} = \frac{3}{16\pi} \text{ cm s}^{-1}.$$

This rate is positive, consistent with an expanding sphere.

Step 2: Relate the surface area rate to the radius rate Differentiating $S = 4\pi r^2$ gives

$$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}.$$

At the same instant, substitute $r = 2$ and the radius rate already calculated:

$$\frac{dS}{dt} = 8\pi(2) \left(\frac{3}{16\pi} \right) = 3 \text{ cm}^2 \text{ s}^{-1}.$$

The area rate has square centimetres per second as its units, whereas the supplied volume rate has cubic centimetres per second.

Step 3: Conclusion The surface area is increasing at $3 \text{ cm}^2 \text{ s}^{-1}$ when the radius is 2 cm.

Quick Tip to Solve: Divide the two differentiated formulas to get $dS/dt = (2/r)(dV/dt)$, then insert the radius at the specified instant.

11. Show that all the diagonal elements of a skew symmetric matrix are zero. [2 marks]

Solution:

Concept:

- A square real matrix A is skew symmetric when $A^T = -A$, where A^T denotes its transpose.
- Transposition interchanges row and column indices: if $A = (a_{ij})$, then the entry of A^T in position (i, j) is a_{ji} . In particular, each diagonal entry remains in its own position.

Step 1: Compare corresponding matrix entries Let $A = (a_{ij})$ be an arbitrary skew symmetric matrix of

Quick Tip to Solve: Use $A \text{ transpose} = -A$ at a diagonal position: the entry is unchanged by transpose but changes sign on negation.

order n . The defining equality $A^T = -A$ is an equality of matrices of the same order, so corresponding entries must be equal. Its (i, j) entry therefore gives

$$a_{ji} = -a_{ij} \quad (1 \leq i, j \leq n).$$

Off the diagonal this states that opposite entries have equal magnitude and opposite signs.

Step 2: Apply the relation to every diagonal position On the diagonal, choose $j = i$ in the entry relation. Transposition does not move a_{ii} , so

$$a_{ii} = -a_{ii} \implies 2a_{ii} = 0 \implies a_{ii} = 0.$$

This argument applies to every index $i = 1, 2, \dots, n$, not just to the first diagonal position. Since the entries are real numbers, division by the nonzero scalar 2 is valid.

Step 3: Conclusion Hence $a_{11} = a_{22} = \dots = a_{nn} = 0$: every diagonal element of a skew symmetric matrix is zero.

Quick Tip to Solve: Use $A^T = -A$ at a diagonal position: the entry is unchanged by transpose but changes sign on negation.

12. If $y = \sin^{-1}(6x\sqrt{1-9x^2})$, $-\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$, then find $\frac{dy}{dx}$. [2 marks]

Solution:

Concept:

- For the principal inverse sine, $\sin^{-1}(\sin u) = u$ whenever $-\frac{\pi}{2} \leq u \leq \frac{\pi}{2}$. Checking this range is essential before cancelling sine and inverse sine.
- The double-angle identity is $\sin 2\theta = 2 \sin \theta \cos \theta$, and $\frac{d}{dx} \sin^{-1}(3x) = \frac{3}{\sqrt{1-9x^2}}$ wherever $|3x| < 1$.

Step 1: Use a substitution consistent with the specified interval Set $\theta = \sin^{-1}(3x)$, so $3x = \sin \theta$. The supplied bounds on x give $-\frac{1}{\sqrt{2}} < 3x < \frac{1}{\sqrt{2}}$, hence $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. On this interval $\cos \theta > 0$, so $\sqrt{1-9x^2} = \cos \theta$. Therefore

$$6x\sqrt{1-9x^2} = 2 \sin \theta \cos \theta = \sin 2\theta.$$

Step 2: Select the principal inverse-sine branch and differentiate Because $-\frac{\pi}{2} < 2\theta < \frac{\pi}{2}$, the principal inverse sine returns 2θ itself. Thus

$$y = \sin^{-1}(\sin 2\theta) = 2\theta = 2 \sin^{-1}(3x).$$

Differentiating this expression by the chain rule yields

$$\frac{dy}{dx} = 2 \frac{1}{\sqrt{1-(3x)^2}} \frac{d(3x)}{dx} = \frac{6}{\sqrt{1-9x^2}}.$$

Quick Tip to Solve: Put $3x = \sin \theta$ and check that 2θ lies in the principal inverse-sine range before simplifying.

The denominator is positive throughout the stated open interval, so the derivative exists at every allowed input.

Step 3: Conclusion For $-\frac{1}{3\sqrt{2}} < x < \frac{1}{3\sqrt{2}}$,

$$\frac{dy}{dx} = \frac{6}{\sqrt{1-9x^2}}.$$

Quick Tip to Solve: Put $3x = \sin \theta$ and check that 2θ lies in the principal inverse-sine range before simplifying.

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, then

(a) Let $c_1 = 1$ and $c_2 = 2$, find c_3 which makes $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

(b) If $c_2 = -1$ and $c_3 = 1$, show that no value of c_1 can make $\vec{a}$, $\vec{b}$ and $\vec{c}$ coplanar.

[4 marks]

Solution:

Concept:

- Three vectors are coplanar if and only if their scalar triple product is zero: $[\vec{a}, \vec{b}, \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c}) = 0$. This product measures the signed volume of the parallelepiped determined by the vectors.
- For component vectors, the scalar triple product is the determinant whose rows are their components. Here $\vec{a} = (1, 1, 1)$, $\vec{b} = (1, 0, 0)$ and $\vec{c} = (c_1, c_2, c_3)$.

Step 1: Obtain the general coplanarity condition Substituting the given components into the determinant gives

$$\begin{aligned} [\vec{a}, \vec{b}, \vec{c}] &= \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= 1(0) - 1(c_3) + 1(c_2) \\ &= c_2 - c_3. \end{aligned}$$

Hence coplanarity requires $c_2 - c_3 = 0$, or $c_2 = c_3$. The coefficient c_1 does not occur in this condition because changing the component along $\vec{b} = \hat{i}$ does not move a vector outside the plane generated by $\vec{a}$ and $\vec{b}$.

Step 2: Apply the condition to part (a) With $c_1 = 1$ and $c_2 = 2$, the determinant becomes $2 - c_3$. Set it equal to zero:

$$2 - c_3 = 0 \implies c_3 = 2.$$

As an independent check, the resulting vector is $\vec{c} = (1, 2, 2) = 2\vec{a} - \vec{b}$. It is therefore a linear combination of the two given vectors and lies in their plane.

Step 3: Test part (b) for every possible value of the first component With $c_2 = -1$ and $c_3 = 1$, the

Quick Tip to Solve: Since the second vector is along the x-axis, coplanarity here reduces immediately to equality of the y- and z-components of the third vector.

scalar triple product is

$$[\vec{a}, \vec{b}, \vec{c}] = -1 - 1 = -2.$$

This is nonzero and is independent of c_1 . Thus changing c_1 cannot make the determinant vanish. Equivalently, every vector in the plane spanned by $\vec{a}$ and $\vec{b}$ has the form $s\vec{a} + t\vec{b} = (s+t, s, s)$, so its second and third components must be equal; the prescribed components -1 and 1 cannot satisfy that requirement.

Step 4: Conclusion (a) $c_3 = 2$. (b) No value of c_1 makes the vectors coplanar, since their scalar triple product remains $-2 \neq 0$.

Quick Tip to Solve: Since the second vector is along the x-axis, coplanarity here reduces immediately to equality of the y- and z-components of the third vector.

14. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors of equal magnitudes, show that the vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to $\vec{a}, \vec{b}$ and $\vec{c}$. Also, find the angle which $\vec{a} + \vec{b} + \vec{c}$ makes with $\vec{a}$ or $\vec{b}$ or $\vec{c}$. [4 marks]

Solution:

Concept:

- Perpendicular vectors have zero dot product. Thus the given mutual perpendicularity implies $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$.
- For nonzero vectors $\vec{u}$ and $\vec{v}$, the angle $\theta \in [0, \pi]$ between them satisfies $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$. The angle question presupposes nonzero magnitudes.

Step 1: Calculate the magnitude of the sum Let the common magnitude be $m > 0$, so that $|\vec{a}| = |\vec{b}| = |\vec{c}| = m$, and write $\vec{s} = \vec{a} + \vec{b} + \vec{c}$. Expanding its squared magnitude using the dot product gives

$$\begin{aligned} |\vec{s}|^2 &= (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) \\ &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\vec{a} \cdot \vec{b} + 2\vec{b} \cdot \vec{c} + 2\vec{c} \cdot \vec{a} \\ &= m^2 + m^2 + m^2 = 3m^2. \end{aligned}$$

Therefore $|\vec{s}| = \sqrt{3}m$. In particular, the sum is nonzero and all three angles mentioned in the question are well defined.

Step 2: Evaluate the dot product with each original vector Distributivity of the dot product and mutual perpendicularity yield

$$\begin{aligned} \vec{s} \cdot \vec{a} &= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{c} \cdot \vec{a} = m^2, \\ \vec{s} \cdot \vec{b} &= \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b} + \vec{c} \cdot \vec{b} = m^2, \\ \vec{s} \cdot \vec{c} &= \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{c} = m^2. \end{aligned}$$

In each calculation, the product with that same vector contributes its squared magnitude; the other two products vanish.

Step 3: Compare the three angle cosines Let α, β, γ be the angles of $\vec{s}$ with $\vec{a}, \vec{b}, \vec{c}$, respectively. The

Quick Tip to Solve: For three mutually perpendicular equal vectors, their sum has magnitude $\sqrt{3}$ times the common magnitude; each angle cosine therefore reduces to $1/\sqrt{3}$.

angle formula now gives

$$\cos \alpha = \cos \beta = \cos \gamma = \frac{m^2}{(\sqrt{3}m)m} = \frac{1}{\sqrt{3}}.$$

Since cosine is one-to-one on the interval $[0, \pi]$ used for angles between vectors, these equal cosines imply equal angles. Their positive cosine also shows that each angle is acute, so no supplementary angle is admissible.

Step 4: Conclusion The vector $\vec{a} + \vec{b} + \vec{c}$ is equally inclined to all three vectors, and the common angle is $\cos^{-1}(1/\sqrt{3})$, approximately 54.74° .

Quick Tip to Solve: For three mutually perpendicular equal vectors, their sum has magnitude $\sqrt{3}$ times the common magnitude; each angle cosine therefore reduces to $1/\sqrt{3}$.

15. The random variable X can take only the values 0, 1, 2, 3. Given that $P(X = 0) = P(X = 1) = p$ and $P(X = 2) = P(X = 3)$ such that $\sum p_i x_i^2 = 2 \sum p_i x_i$, find the value of p . [4 marks]

Solution:

Concept:

- The probabilities of all possible values of a discrete random variable add to one: $\sum_i p_i = 1$, where $p_i = P(X = x_i)$.
- The moments are $E(X) = \sum_i p_i x_i$ and $E(X^2) = \sum_i p_i x_i^2$. The given condition states that $E(X^2) = 2E(X)$.

Step 1: Express the probability distribution Let $q = P(X = 2) = P(X = 3)$. The distribution is

x_i	0	1	2	3
p_i	p	p	q	q

These are the only possible values, so normalization gives

$$2p + 2q = 1, \quad p + q = \frac{1}{2}. \quad (1)$$

Both unknowns must be nonnegative because they represent probabilities.

Step 2: Evaluate the two moments Multiplying each value by its probability gives

$$E(X) = 0p + 1p + 2q + 3q = p + 5q.$$

For the second moment, first square each possible value:

$$E(X^2) = 0^2p + 1^2p + 2^2q + 3^2q = p + 13q.$$

The printed condition consequently becomes

$$p + 13q = 2(p + 5q), \quad 3q = p. \quad (2)$$

Quick Tip to Solve: Use one symbol for the equal probabilities at 2 and 3, and combine normalization with the moment equation to solve two linear equations.

Here the second moment is a weighted sum of squares, not the square of the first moment. The term for $X = 0$ vanishes in both sums, although its probability still contributes to normalization.

Step 3: Solve and check the probabilities Substituting $p = 3q$ from (2) into (1), we obtain

$$3q + q = \frac{1}{2}, \quad q = \frac{1}{8}, \quad p = \frac{3}{8}.$$

The distribution is therefore $3/8, 3/8, 1/8, 1/8$. All four probabilities lie between zero and one, and their sum is one. Check the supplied relation directly:

$$E(X) = \frac{3}{8} + \frac{5}{8} = 1, \quad E(X^2) = \frac{3}{8} + \frac{13}{8} = 2.$$

Thus $E(X^2) = 2E(X)$, so the computed distribution satisfies both independent conditions in the question.

Step 4: Conclusion The required value is $p = \frac{3}{8}$.

Quick Tip to Solve: Use one symbol for the equal probabilities at 2 and 3, and combine normalization with the moment equation to solve two linear equations.

16. Often it is taken that a truthful person commands more respect in the society. A man is known to speak the truth 4 out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually a six. Do you also agree that the value of truthfulness leads to more respect in the society? [4 marks]

Solution:

Concept:

- Bayes' theorem finds the probability of a cause after observing a report. For complementary events A and A' , $P(A | B) = \frac{P(A)P(B | A)}{P(A)P(B | A) + P(A')P(B | A')}$.
- A fair die has six equally likely outcomes. The man's truthfulness gives probability $4/5$ of a truthful report and $1/5$ of a false report.

Step 1: Define the actual outcome and the reported outcome Let A be the event that the die actually shows six, and let B be the event that the man reports a six. Its complement A' means the die shows one of the other five faces. Thus

$$P(A) = \frac{1}{6}, \quad P(A') = \frac{5}{6}.$$

We require $P(A | B)$, since the report has already been heard. It is a conditional probability, not simply the prior probability $1/6$.

Step 2: Find the likelihood of the report in each case If a six was thrown, reporting six is a truthful statement, so

$$P(B | A) = \frac{4}{5}.$$

Quick Tip to Solve: Separate the actual six from the reported six, and include both truthful and false routes to the report in the Bayes denominator.

If a six was not thrown, the assertion that it was six is false; using the truth-or-false-report model of this question,

$$P(B | A') = \frac{1}{5}.$$

The total probability of hearing the report is consequently

$$\begin{aligned} P(B) &= P(A)P(B | A) + P(A')P(B | A') \\ &= \frac{1}{6} \cdot \frac{4}{5} + \frac{5}{6} \cdot \frac{1}{5} = \frac{4}{30} + \frac{5}{30} = \frac{9}{30}. \end{aligned}$$

Both routes must be included: a true report after a six and a false report after a non-six.

Step 3: Apply Bayes' theorem The probability that a six actually occurred together with the report is $P(A \cap B) = 4/30$. Therefore

$$P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{4/30}{9/30} = \frac{4}{9}.$$

The other possibility has conditional probability $5/9$, and these conditional probabilities sum to one. The answer differs from $4/5$, since truthfulness alone does not account for the five possible non-six outcomes.

Step 4: Answer the question about truthfulness Yes, I agree. Truthfulness makes a person's statements reliable and builds trust. People can depend on an honest person in their dealings, which encourages confidence and respect in society.

Step 5: Conclusion The probability that the die actually shows six is $\boxed{\frac{4}{9}}$. Yes, truthfulness promotes trust and earns respect in society.

Quick Tip to Solve: Separate the actual six from the reported six, and include both truthful and false routes to the report in the Bayes denominator.

17(a). Using properties of determinants, prove that
$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y). \quad [4 \text{ marks}]$$

Solution:

Concept:

- Adding other columns to one column preserves a determinant: $C_1 \rightarrow C_1 + C_2 + C_3$ does not change its value.
- A common factor in a column can be taken outside the determinant. Subtracting one row from another also preserves the determinant.
- A column with two zero entries permits direct cofactor expansion; the second-order minor is evaluated by $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Quick Tip to Solve: Equal row sums suggest replacing one column by the sum of all columns, then subtracting the first row from each remaining row.

Step 1: Use the equal row sums Write the determinant as Δ . Each row contains x , $x + y$ and $x + 2y$, whose sum is $3(x + y)$. Apply $C_1 \rightarrow C_1 + C_2 + C_3$, then take the common factor out of the first column:

$$\Delta = \begin{vmatrix} 3(x+y) & x+y & x+2y \\ 3(x+y) & x & x+y \\ 3(x+y) & x+2y & x \end{vmatrix} = 3(x+y) \begin{vmatrix} 1 & x+y & x+2y \\ 1 & x & x+y \\ 1 & x+2y & x \end{vmatrix}.$$

The column replacement is valid because adding a multiple of another column leaves a determinant unchanged. Taking a common factor outside does not involve division by $x + y$, so the calculation also covers $x + y = 0$.

Step 2: Make two zeros in the first column Apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$. The first row stays unchanged throughout. In the second row, the last two entries become $x - (x + y) = -y$ and $(x + y) - (x + 2y) = -y$. In the third row they become $(x + 2y) - (x + y) = y$ and $x - (x + 2y) = -2y$. Thus

$$\Delta = 3(x+y) \begin{vmatrix} 1 & x+y & x+2y \\ 0 & -y & -y \\ 0 & y & -2y \end{vmatrix}.$$

Neither row replacement changes the value of this determinant.

Step 3: Evaluate the minor Expand along the first column. Only its first entry contributes, and its cofactor sign is positive:

$$\begin{aligned} \Delta &= 3(x+y) \begin{vmatrix} -y & -y \\ y & -2y \end{vmatrix} \\ &= 3(x+y) [(-y)(-2y) - (-y)(y)] \\ &= 3(x+y)(2y^2 + y^2). \end{aligned}$$

The off-diagonal product is $-y^2$, so subtracting it contributes $+y^2$. This also makes the squared factor in y explicit without cancelling y ; the argument remains valid for $y = 0$.

Step 4: Conclusion Hence $\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y)$, as required.

Quick Tip to Solve: Equal row sums suggest replacing one column by the sum of all columns, then subtracting the first row from each remaining row.

17(b). Let $A = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 5 \\ 3 & 8 \end{pmatrix}$; find a matrix D such that $CD - AB = O$.

[4 marks]

Solution:**Concept:**

- The matrix equation $CD - AB = O$, where O is the zero matrix, is equivalent to $CD = AB$.
- If $\det C \neq 0$, then C is invertible. For $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its inverse is $\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- Matrix multiplication is order-dependent. To remove C from CD , multiply both sides on the left by C^{-1} , obtaining $D = C^{-1}AB$.

Step 1: Calculate the product on the right The required equation gives $CD = AB$. Using row-by-column multiplication,

$$\begin{aligned} AB &= \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 2 \\ 7 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 2(5) - 1(7) & 2(2) - 1(4) \\ 3(5) + 4(7) & 3(2) + 4(4) \end{pmatrix} \\ &= \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix}. \end{aligned}$$

In particular, the upper-right entry is zero because $4 - 4 = 0$; this is a product entry, not an assumption about D .

Step 2: Check invertibility and find the inverse of C For the given matrix,

$$\det C = \begin{vmatrix} 2 & 5 \\ 3 & 8 \end{vmatrix} = 2(8) - 5(3) = 16 - 15 = 1.$$

Since this determinant is nonzero, the equation has a unique solution. Interchanging the diagonal entries and changing the signs of the other two entries gives

$$C^{-1} = \frac{1}{1} \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix}.$$

Left-multiplication by this inverse gives $C^{-1}CD = C^{-1}AB$, hence $D = C^{-1}(AB)$. Right-multiplication would not cancel the factor C in the correct position.

Quick Tip to Solve: In an equation $CD = AB$, cancel C by multiplying on the left; compute AB first and verify the final matrix by multiplying CD .

Step 3: Multiply to obtain D and verify the original equation Now calculate

$$\begin{aligned} D &= \begin{pmatrix} 8 & -5 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix} \\ &= \begin{pmatrix} 24 - 215 & 0 - 110 \\ -9 + 86 & 0 + 44 \end{pmatrix} \\ &= \begin{pmatrix} -191 & -110 \\ 77 & 44 \end{pmatrix}. \end{aligned}$$

As a check, substitute these entries directly:

$$CD = \begin{pmatrix} 2(-191) + 5(77) & 2(-110) + 5(44) \\ 3(-191) + 8(77) & 3(-110) + 8(44) \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 43 & 22 \end{pmatrix} = AB.$$

Consequently every entry of $CD - AB$ is zero, which verifies the required matrix equation.

Step 4: Conclusion The required matrix is $D = \begin{pmatrix} -191 & -110 \\ 77 & 44 \end{pmatrix}$.

Quick Tip to Solve: In an equation $CD = AB$, cancel C by multiplying on the left; compute AB first and verify the final matrix by multiplying CD.

18(a). Differentiate the function $(\sin x)^x + \sin^{-1} \sqrt{x}$ with respect to x .

[4 marks]

Solution:

Concept:

- For a positive function raised to a variable power, logarithmic differentiation converts the power into a product: if $u = (\sin x)^x$, then $\log u = x \log(\sin x)$.
- The product rule is $(fg)' = f'g + fg'$. The inverse sine derivative, combined with the chain rule, is $\frac{d}{dx} \sin^{-1} v = \frac{v'}{\sqrt{1-v^2}}$.

Step 1: Separate the two terms Write the given function as $y = u + v$, where $u = (\sin x)^x$ and $v = \sin^{-1} \sqrt{x}$. Differentiation is linear, so the derivative of the sum is the sum of the two derivatives. The first term has both a variable base and a variable exponent; the ordinary power rule alone cannot differentiate it. For a real-valued logarithmic calculation we require $\sin x > 0$, and the derivative of the inverse sine term is considered for $0 < x < 1$. These conditions hold throughout the common open interval $(0, 1)$.

Step 2: Apply logarithmic differentiation Since $u > 0$, take the natural logarithm of both sides, and then differentiate with respect to x :

$$\begin{aligned} \log u &= x \log(\sin x), \\ \frac{1}{u} \frac{du}{dx} &= \log(\sin x) + x \frac{\cos x}{\sin x}. \end{aligned}$$

Quick Tip to Solve: For $(\sin x)^x$, differentiate $x \log(\sin x)$ first and then multiply by the original power; for the inverse sine term, keep the derivative of $\sqrt{x}$ as a separate chain-rule factor.

The first term on the right comes from differentiating x ; the second comes from differentiating $\log(\sin x)$ using the chain rule. Multiplying by u gives

$$\frac{du}{dx} = (\sin x)^x [\log(\sin x) + x \cot x].$$

Step 3: Differentiate the inverse sine term Set the inner function equal to $w = \sqrt{x}$. Then $dw/dx = 1/(2\sqrt{x})$, while the derivative of $\sin^{-1} w$ with respect to w is $1/\sqrt{1-w^2}$. Thus

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

Both square-root factors are positive on $0 < x < 1$, so this simplification preserves the sign of the derivative.

Step 4: Conclusion Adding the two derivatives, we obtain

$$\boxed{\frac{dy}{dx} = (\sin x)^x [\log(\sin x) + x \cot x] + \frac{1}{2\sqrt{x(1-x)}}}, \quad 0 < x < 1.$$

Quick Tip to Solve: For $(\sin x)^x$, differentiate $x \log(\sin x)$ first and then multiply by the original power; for the inverse sine term, keep the derivative of $\sqrt{x}$ as a separate chain-rule factor.

OR

18(b). If $x^m y^n = (x + y)^{m+n}$, prove that $\frac{d^2 y}{dx^2} = 0$.

[4 marks]

Solution:

Concept:

- Logarithmic differentiation of an implicit relation uses $\frac{d}{dx} \log y = \frac{y'}{y}$, where $y' = dy/dx$, and m, n are constants.
- Work on a differentiable branch where the given powers are defined, with constants m, n not both zero. Logarithms of absolute values give the same derivatives on nonzero real branches.

Step 1: Take logarithms and differentiate Take logarithms of $x^m y^n = (x + y)^{m+n}$ on a positive branch:

$$m \log x + n \log y = (m + n) \log(x + y).$$

Differentiating, remembering that y depends on x , gives

$$\frac{m}{x} + \frac{n}{y} y' = \frac{m + n}{x + y} (1 + y').$$

Quick Tip to Solve: After logarithmic differentiation, factor the equation as $(nx - my)(xy' - y) = 0$. Check the straight-line branch before cancelling, then use $y' = y/x$ to make the second derivative vanish.

Step 2: Simplify the first derivative Multiply by $xy(x+y)$ and collect the terms containing y' :

$$\begin{aligned}my(x+y) + nx(x+y)y' &= (m+n)xy(1+y'), \\x(nx-my)y' &= y(nx-my), \\(nx-my)(xy' - y) &= 0.\end{aligned}$$

On an interval where $nx - my \neq 0$, cancellation therefore gives

$$y' = \frac{y}{x}.$$

If $nx - my$ vanishes identically, then $my = nx$, so the branch is already a straight line. For $m = 0, n \neq 0$, the factor is nonzero when $x \neq 0$. Any permitted horizontal axis branch likewise has constant slope.

Step 3: Differentiate once more For the regular branch, apply the quotient rule to $y' = y/x$ and substitute this first derivative:

$$y'' = \frac{xy' - y}{x^2} = \frac{x(y/x) - y}{x^2} = \frac{y - y}{x^2}.$$

The exceptional straight-line branch also has zero second derivative directly. If $m = n = 0$, the relation is an identity; the usual nondegeneracy of the constants is necessary for the stated result.

Step 4: Conclusion Hence, on every nondegenerate twice-differentiable branch of the given relation,

$$\boxed{\frac{d^2y}{dx^2} = 0}.$$

Quick Tip to Solve: After logarithmic differentiation, factor the equation as $(nx - my)(xy' - y) = 0$. Check the straight-line branch before cancelling, then use $y' = y/x$ to make the second derivative vanish.

19(a). Evaluate: $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx.$

[4 marks]

Solution:

Concept:

- Reflection property: $\int_0^a f(x) dx = \int_0^a f(a-x) dx$. Reflecting about the midpoint helps when part of the integrand is unchanged.
- If $g(\pi - x) = g(x)$, then adding the reflected expressions for $\int_0^\pi xg(x) dx$ replaces x by $\pi/2$.
- With $t = \cos x$, $dt = -\sin x dx$, and $\int dt/(1+t^2) = \tan^{-1} t + C$.

Step 1: Reflect about the midpoint Write $I = \int_0^\pi x \sin x / (1 + \cos^2 x) dx$. Its integrand is continuous because the denominator is positive throughout. Using the reflection property together with $\sin(\pi -$

Quick Tip to Solve: Replace x by $\pi - x$ before integration by parts: the trigonometric factor stays unchanged, and adding removes x .

$x) = \sin x$ and $\cos(\pi - x) = -\cos x$, we obtain

$$I = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx.$$

The squared cosine is unchanged by reflection. Thus the two expressions differ only in their factors x and $\pi - x$.

Step 2: Add the two expressions Adding the original and reflected expressions combines their numerators using $x + (\pi - x) = \pi$. Hence

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx.$$

This removes the factor x without integration by parts. Let the remaining integral be J , so $I = (\pi/2)J$. The factor one-half is necessary because we have added two copies of the original integral.

Step 3: Evaluate the remaining integral Set $t = \cos x$. At the lower limit $x = 0$, $t = 1$; at the upper limit $x = \pi$, $t = -1$. The differential contributes a negative sign, which reverses the bounds:

$$\begin{aligned} J &= - \int_1^{-1} \frac{dt}{1 + t^2} = \int_{-1}^1 \frac{dt}{1 + t^2} \\ &= [\tan^{-1} t]_{-1}^1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}. \end{aligned}$$

The inverse tangent values used here are the principal values in $(-\pi/2, \pi/2)$. Substituting this value of J into the relation from the previous step gives the required evaluation. Its positive sign agrees with the nonnegative original integrand on the whole interval.

Step 4: Conclusion Therefore, $I = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$.

Quick Tip to Solve: Replace x by $\pi - x$ before integration by parts: the trigonometric factor stays unchanged, and adding removes x .

OR

19(b). Evaluate: $\int_0^{3/2} |x \sin \pi x| dx$.

[4 marks]

Solution:

Concept:

- A modulus integral must be split where its inner expression changes sign: $|f(x)| = f(x)$ when $f(x) \geq 0$, and $|f(x)| = -f(x)$ when $f(x) < 0$.
- Integration by parts gives $\int u dv = uv - \int v du$. For $x \sin(\pi x)$, choose $u = x$, whose derivative is constant.

Step 1: Determine the sign and split the interval Let $I = \int_0^{3/2} |x \sin \pi x| dx$. Since $x \geq 0$ on the entire interval, its sign is determined by $\sin \pi x$. This sine is nonnegative for $0 \leq x \leq 1$ and negative for

Quick Tip to Solve: Split at $x = 1$ before integrating: sine changes sign there, so subtract the signed integral from 1 to $3/2$.

$1 < x \leq 3/2$. It vanishes at the splitting point $x = 1$. Therefore

$$I = \int_0^1 x \sin \pi x \, dx - \int_1^{3/2} x \sin \pi x \, dx.$$

The second minus sign changes the negative signed integral into its positive contribution to the modulus integral. The endpoint $x = 0$ also gives zero but creates no further interval.

Step 2: Find an antiderivative by integration by parts Take $u = x$, $dv = \sin \pi x \, dx$, so $du = dx$ and $v = -\cos \pi x / \pi$. Then

$$\begin{aligned} F(x) &= \int x \sin \pi x \, dx \\ &= -\frac{x \cos \pi x}{\pi} + \frac{1}{\pi} \int \cos \pi x \, dx \\ &= -\frac{x \cos \pi x}{\pi} + \frac{\sin \pi x}{\pi^2}. \end{aligned}$$

An additive constant is unnecessary when evaluating definite differences. As a check, differentiation cancels the two cosine terms and leaves $F'(x) = x \sin \pi x$, as required.

Step 3: Evaluate both definite contributions At the relevant endpoints, $\cos 0 = 1$, $\cos \pi = -1$, $\cos(3\pi/2) = 0$, while the corresponding sine values are $0, 0, -1$. Consequently

$$F(0) = 0, \quad F(1) = \frac{1}{\pi}, \quad F(3/2) = -\frac{1}{\pi^2}.$$

Hence the first signed integral is $F(1) - F(0) = 1/\pi$. The second signed integral is $F(3/2) - F(1) = -1/\pi^2 - 1/\pi$, which is negative, consistently with the sign analysis. In the modulus integral this second contribution must be subtracted:

$$I = \frac{1}{\pi} - \left(-\frac{1}{\pi^2} - \frac{1}{\pi} \right).$$

Step 4: Conclusion The required value is $\boxed{\frac{2}{\pi} + \frac{1}{\pi^2} = \frac{2\pi + 1}{\pi^2}}.$

Quick Tip to Solve: Split at $x = 1$ before integrating: sine changes sign there, so subtract the signed integral from 1 to $3/2$.

20. Solve the following L.P.P. graphically: Maximise $Z = 20x + 10y$ subject to constraints $x + 2y \leq 28$, $3x + y \leq 24$, $x \geq 2$, $x, y \geq 0$. [4 marks]

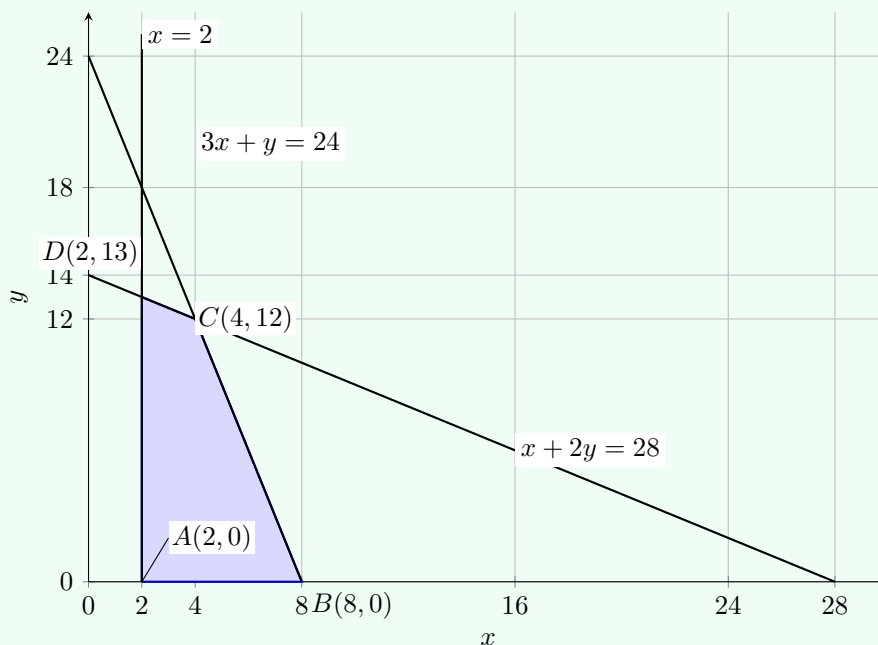
Solution:

Concept:

- The feasible region is the common region satisfying all the inequalities. A nonempty bounded feasible polygon has an optimum of a linear objective function at a vertex.
- Draw each boundary as a solid line, since equality is allowed. For $Z = 20x + 10y$, evaluate Z at every vertex of the feasible polygon.

Quick Tip to Solve: At the boundary $x=2$, use the smaller of the two permitted y -values; this gives $(2,13)$, not $(2,18)$.

Step 1: Plot the boundary lines and select the common region The line $x + 2y = 28$ passes through $(0, 14)$ and $(28, 0)$, while $3x + y = 24$ passes through $(0, 24)$ and $(8, 0)$. For both inequalities, the required half-plane is below the corresponding line, since the origin satisfies each inequality. The condition $x \geq 2$ selects the region to the right of the vertical line $x = 2$. Finally, $x, y \geq 0$ restricts the graph to the first quadrant. The shaded polygon below is the intersection of these regions; its boundary is included.



The shaded feasible region $ABCD$ and its four vertices.

Step 2: Determine the vertices of the feasible polygon On $y = 0$, the feasible interval is $2 \leq x \leq 8$, giving $A(2, 0)$ and $B(8, 0)$. The intersection of the two sloping boundaries satisfies

$$\begin{aligned}x + 2y &= 28, \\3x + y &= 24.\end{aligned}$$

Substituting $y = 24 - 3x$ into the first equation gives $x + 48 - 6x = 28$, hence $x = 4$ and $y = 12$. Thus $C = (4, 12)$. At $x = 2$, the first inequality gives $y \leq 13$ and the second gives $y \leq 18$; hence the upper vertex is $D(2, 13)$. All four vertices satisfy every constraint.

Step 3: Compare the objective function at all vertices The feasible region $ABCD$ is nonempty and bounded. Therefore apply the corner-point method:

Vertex	$Z = 20x + 10y$
$A(2, 0)$	40
$B(8, 0)$	160
$C(4, 12)$	200
$D(2, 13)$	170

The largest entry is obtained at C . Adjacent vertices have strictly smaller values, so the objective is not constant along an edge of the feasible polygon.

Quick Tip to Solve: At the boundary $x=2$, use the smaller of the two permitted y -values; this gives $(2, 13)$, not $(2, 18)$.

Step 4: Conclusion The maximum value is $Z_{\max} = 200$, attained uniquely at $x = 4, y = 12$.

Quick Tip to Solve: At the boundary $x=2$, use the smaller of the two permitted y -values; this gives $(2,13)$, not $(2,18)$.

21. Show that the family of curves for which $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$, is given by $x^2 - y^2 = cx$. [4 marks]

Solution:

Concept:

- A homogeneous differential equation has a right-hand side expressible as a function of y/x . Put $y = vx$, where v is a function of x ; then $dy/dx = v + x dv/dx$.
- Separate the variables and integrate. The given equation is defined only where $xy \neq 0$, so its solution curves are considered on intervals avoiding these singular points.

Step 1: Reduce the equation using the homogeneous substitution Since the numerator and denominator are both homogeneous expressions of degree two, write

$$\frac{dy}{dx} = \frac{1 + (y/x)^2}{2(y/x)}.$$

On an interval where $x \neq 0$, set $y = vx$. Differentiating by the product rule and substituting gives

$$\begin{aligned} v + x \frac{dv}{dx} &= \frac{1 + v^2}{2v}, \\ x \frac{dv}{dx} &= \frac{1 - v^2}{2v}. \end{aligned}$$

Here $v \neq 0$, since the original equation requires $y \neq 0$.

Step 2: Separate and integrate the variables For $v^2 \neq 1$, separate the variables to obtain

$$\frac{2v}{1 - v^2} dv = \frac{dx}{x}.$$

Since the derivative of $1 - v^2$ is $-2v$, integration yields

$$-\ln |1 - v^2| = \ln |x| + C.$$

Equivalently, $\ln |x(1 - v^2)|$ is constant. Absorbing the magnitude and sign into an arbitrary real constant c , we get $x(1 - v^2) = c$. Substituting back $v = y/x$ therefore gives

$$x \left(1 - \frac{y^2}{x^2} \right) = c, \quad x^2 - y^2 = cx.$$

During separation we excluded $v = 1$ and $v = -1$. These constant values give $y = x$ and $y = -x$, which both satisfy the original equation away from the origin and are already included by setting $c = 0$.

Quick Tip to Solve: When separating, check $v=1$ and $v=-1$ separately; both are included in the final family with $c=0$.

Step 3: Verify the family directly Differentiate $x^2 - y^2 = cx$, keeping c constant:

$$2x - 2y \frac{dy}{dx} = c.$$

From the family, $c = (x^2 - y^2)/x$. Therefore, where $xy \neq 0$,

$$\begin{aligned} 2y \frac{dy}{dx} &= 2x - \frac{x^2 - y^2}{x} \\ &= \frac{x^2 + y^2}{x}, \end{aligned}$$

which recovers precisely the stated differential equation. Thus both integration and direct differentiation establish the required family.

Step 4: Conclusion Hence the family is $x^2 - y^2 = cx$, where c is an arbitrary real constant, on portions of the curves where $xy \neq 0$.

Quick Tip to Solve: When separating, check $v=1$ and $v=-1$ separately; both are included in the final family with $c=0$.

22. Find: $\int \frac{(3 \sin x - 2) \cos x}{13 - \cos^2 x - 7 \sin x} dx.$

[4 marks]

Solution:

Concept:

- The substitution $t = \sin x$, with $dt = \cos x dx$, converts an integrand containing $\sin x$ and a factor $\cos x dx$ into a rational function of t .
- Use $\cos^2 x = 1 - \sin^2 x$, factor the denominator, and decompose the resulting proper rational function into partial fractions.
- For a constant a , $\int \frac{dt}{t - a} = \ln |t - a| + C$, where C is an arbitrary constant.

Step 1: Express the denominator in terms of sine Let the required integral be I . Applying the identity for $\cos^2 x$, we obtain

$$13 - \cos^2 x - 7 \sin x = 12 + \sin^2 x - 7 \sin x = (\sin x - 3)(\sin x - 4).$$

Thus the denominator is a quadratic polynomial in $\sin x$. The numerator already contains $\cos x dx$, which is the differential of $\sin x$, so this substitution removes the trigonometric functions from the integration step.

Step 2: Substitute and find the partial fractions Set $t = \sin x$, so that $dt = \cos x dx$. Then

$$I = \int \frac{3t - 2}{(t - 3)(t - 4)} dt.$$

Quick Tip to Solve: First replace $\cos^2 x$ by $1 - \sin^2 x$; then use $t = \sin x$ and evaluate the partial-fraction identity at $t = 3, 4$ to find both coefficients quickly.

Write

$$\frac{3t-2}{(t-3)(t-4)} = \frac{A}{t-3} + \frac{B}{t-4}.$$

Multiplying by the denominator gives $3t-2 = A(t-4) + B(t-3)$. At $t=3$, this gives $7 = -A$, and at $t=4$, it gives $10 = B$. Hence $A = -7$ and $B = 10$. As a check, their sum is 3 and the constant term $-4A - 3B$ is -2 , as required.

Step 3: Integrate and restore the original variable Integrating the two simple fractions separately,

$$\begin{aligned} I &= -7 \int \frac{dt}{t-3} + 10 \int \frac{dt}{t-4} \\ &= -7 \ln |t-3| + 10 \ln |t-4| + C. \end{aligned}$$

Now replace t by $\sin x$. Since $-1 \leq \sin x \leq 1$, neither logarithm has a zero argument. Differentiating the resulting expression gives

$$\cos x \left(\frac{-7}{\sin x - 3} + \frac{10}{\sin x - 4} \right) = \frac{(3 \sin x - 2) \cos x}{(\sin x - 3)(\sin x - 4)},$$

which is the original integrand.

Step 4: Conclusion $I = 10 \ln |\sin x - 4| - 7 \ln |\sin x - 3| + C$.

Quick Tip to Solve: First replace $\cos^2 x$ by $1 - \sin^2 x$; then use $t = \sin x$ and evaluate the partial-fraction identity at $t=3, 4$ to find both coefficients quickly.

23. Solve the following equation for x : $\cos(\tan^{-1} x) = \sin\left(\cot^{-1} \frac{3}{4}\right)$.

[4 marks]

Solution:

Concept:

- The principal value $\theta = \tan^{-1} x$ lies in $(-\pi/2, \pi/2)$. Consequently $\cos \theta > 0$ and $\cos(\tan^{-1} x) = 1/\sqrt{1+x^2}$.
- The principal value of $\cot^{-1} y$ lies in $(0, \pi)$. For $y > 0$, this angle is in the first quadrant, so its sine and cosine are both positive.
- Squaring an equation is reversible when both sides are nonnegative, but all obtained values should still be checked in the original equation.

Step 1: Evaluate the fixed angle on the right Put $\alpha = \cot^{-1}(3/4)$. By the principal-value convention, $0 < \alpha < \pi$. Since $\cot \alpha = 3/4 > 0$, the angle lies in the first quadrant. A right triangle with adjacent side 3 and opposite side 4 has hypotenuse $\sqrt{3^2 + 4^2} = 5$. Therefore

$$\sin\left(\cot^{-1} \frac{3}{4}\right) = \sin \alpha = \frac{4}{5}.$$

Here the numerator of the cotangent ratio represents the adjacent side.

Quick Tip to Solve: Use the principal range of $\tan^{-1} x$ to select the positive cosine, and retain both signs when the final equation determines only x^2 .

Step 2: Express the left side algebraically Let $\theta = \tan^{-1} x$. Then $\tan \theta = x$, and $-\pi/2 < \theta < \pi/2$. Using $\sec^2 \theta = 1 + \tan^2 \theta$, together with $\cos \theta > 0$ throughout this interval, gives

$$\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+x^2}}.$$

Hence the equation becomes $1/\sqrt{1+x^2} = 4/5$, with both sides positive.

Step 3: Solve the algebraic equation Both sides are positive, so taking reciprocals and squaring preserves equivalence:

$$\begin{aligned}\sqrt{1+x^2} &= \frac{5}{4}, \\ 1+x^2 &= \frac{25}{16}, \\ x^2 &= \frac{9}{16}, \\ x &= \pm \frac{3}{4}.\end{aligned}$$

The domain of $\tan^{-1} x$ is all real numbers, so both signs are admissible.

Step 4: Check both values in the original equation At either $x = 3/4$ or $x = -3/4$,

$$\cos(\tan^{-1} x) = \frac{1}{\sqrt{1+9/16}} = \frac{1}{5/4} = \frac{4}{5}.$$

Both values therefore satisfy the original equation. All transformations were equivalent, so there are no further solutions.

Step 5: Conclusion The complete solution set is $x \in \left\{ -\frac{3}{4}, \frac{3}{4} \right\}$.

Quick Tip to Solve: Use the principal range of $\tan^{-1} x$ to select the positive cosine, and retain both signs when the final equation determines only x^2 .

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24(a). Using integration, find the area of region bounded by the triangle whose vertices are $(-2, 1)$, $(0, 4)$ and $(2, 3)$. [6 marks]

Solution:

Concept:

- The area between two curves is $\int_a^b (y_{\text{upper}} - y_{\text{lower}}) dx$, provided their order is fixed on the interval.
- A line through (x_1, y_1) and (x_2, y_2) has slope $m = (y_2 - y_1)/(x_2 - x_1)$. Split an integral where the upper boundary changes.

Step 1: Determine the three sides of the triangle Write $A = (-2, 1)$, $B = (0, 4)$ and $C = (2, 3)$. The slope of AB is $(4 - 1)/(0 + 2) = 3/2$, so its equation is $y = \frac{3}{2}x + 4$. For BC , the slope is

Quick Tip to Solve: Split at the middle vertex's x-coordinate, here zero; subtract the same lower side from each of the two upper sides.

$(3 - 4)/(2 - 0) = -1/2$, giving $y = 4 - \frac{1}{2}x$. Finally, the slope of AC is $(3 - 1)/(2 + 2) = 1/2$, giving $y = \frac{1}{2}x + 2$. Substitution of the endpoints in these equations checks that each line passes through the required vertices.

Step 2: Identify the upper and lower boundaries The triangle extends from $x = -2$ to $x = 2$. Its lower side throughout this interval is AC . For $-2 \leq x \leq 0$, its upper side is AB , while for $0 \leq x \leq 2$, its upper side is BC . The height of a vertical strip is therefore $x + 2$ on the first interval and $2 - x$ on the second. Both heights are nonnegative, vanish at the outer vertices, and equal 2 at $x = 0$; thus the two integrals cover the whole triangle without overlap.

Step 3: Integrate the vertical strip heights The required area is

$$\begin{aligned} S &= \int_{-2}^0 \left[\left(\frac{3}{2}x + 4 \right) - \left(\frac{1}{2}x + 2 \right) \right] dx \\ &\quad + \int_0^2 \left[\left(4 - \frac{1}{2}x \right) - \left(\frac{1}{2}x + 2 \right) \right] dx \\ &= \int_{-2}^0 (x + 2) dx + \int_0^2 (2 - x) dx \\ &= \left[\frac{x^2}{2} + 2x \right]_{-2}^0 + \left[2x - \frac{x^2}{2} \right]_0^2 \\ &= (0 - (-2)) + (2 - 0) = 4. \end{aligned}$$

Each portion contributes 2 square units. Since this calculation integrates the actual distances between the sides, it yields the enclosed area directly rather than a signed area beneath a single line.

Step 4: Conclusion The area of the triangular region is 4 square units.

Quick Tip to Solve: Split at the middle vertex's x-coordinate, here zero; subtract the same lower side from each of the two upper sides.

OR

24(b). Find the area bounded by the circle $x^2 + y^2 = 16$ and the line $\sqrt{3}y = x$ in the first quadrant, using integration. [6 marks]

Solution:

Concept:

- In the first quadrant the circle $x^2 + y^2 = 16$ has upper boundary $y = \sqrt{16 - x^2}$, with radius 4.
- The enclosed first-quadrant region between the positive x-axis, the line and the circular arc can be integrated using vertical strips, splitting where the line meets the circle.
- For $|x| < a$, $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}(x/a) + C$, with inverse trigonometric angles in radians.

Step 1: Find the intersection in the first quadrant The given line is $y = x/\sqrt{3}$. Substitute this into the

Quick Tip to Solve: Split at $x = 2\sqrt{3}$, where the upper boundary changes from the line to the circle.

circle:

$$x^2 + \frac{x^2}{3} = 16 \implies x^2 = 12.$$

Since both coordinates are positive at the required intersection, it is $P = (2\sqrt{3}, 2)$. The circle meets the positive x-axis at $A = (4, 0)$, while the line meets it at the origin $O = (0, 0)$. These points bound the required sector.

Step 2: Set up the area with vertical strips From $x = 0$ to $x = 2\sqrt{3}$, a strip in the region runs from the x-axis up to the line, so its height is $x/\sqrt{3}$. From $x = 2\sqrt{3}$ to $x = 4$, its top lies on the circular arc, so its height is $\sqrt{16 - x^2}$. Consequently,

$$S = \int_0^{2\sqrt{3}} \frac{x}{\sqrt{3}} dx + \int_{2\sqrt{3}}^4 \sqrt{16 - x^2} dx.$$

The upper boundary changes at the intersection, so the integral must be split there.

Step 3: Evaluate the two integrals The first contribution is

$$\left[\frac{x^2}{2\sqrt{3}} \right]_0^{2\sqrt{3}} = 2\sqrt{3}.$$

For the second, use the circular-segment antiderivative:

$$\begin{aligned} \int_{2\sqrt{3}}^4 \sqrt{16 - x^2} dx &= \left[\frac{x}{2} \sqrt{16 - x^2} + 8 \sin^{-1} \frac{x}{4} \right]_{2\sqrt{3}}^4 \\ &= 4\pi - \left(2\sqrt{3} + \frac{8\pi}{3} \right) \\ &= \frac{4\pi}{3} - 2\sqrt{3}. \end{aligned}$$

Here $\sin^{-1}(1) = \pi/2$ and $\sin^{-1}(\sqrt{3}/2) = \pi/3$. Adding the contributions cancels the $2\sqrt{3}$ terms.

Step 4: Conclusion The required area is $\boxed{\frac{4\pi}{3} \text{ square units}}$.

Quick Tip to Solve: Split at $x = 2\sqrt{3}$, where the upper boundary changes from the line to the circle.

25(a). Find the equation of the plane through the line of intersection of $\vec{r} \cdot (2\hat{i} - 3\hat{j} + 4\hat{k}) = 1$ and $\vec{r} \cdot (\hat{i} - \hat{j}) + 4 = 0$ and perpendicular to the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 8 = 0$. Hence find whether the plane thus obtained contains the line $x - 1 = 2y - 4 = 3z - 12$. [6 marks]

Solution:

Concept:

- If two planes have equations $P_1 = 0$ and $P_2 = 0$, then $P_1 + \lambda P_2 = 0$ represents a plane through their line of intersection. Every point common to the two planes satisfies this equation.
- Two planes are perpendicular when their normal vectors have zero scalar product. A line lies in

Quick Tip to Solve: To test line containment, substitute its parameter form into the plane: both the constant term and the coefficient of the parameter must vanish.

a plane when a point on the line lies in the plane and its direction vector is perpendicular to the plane's normal.

Step 1: Form the family of planes through the intersection Writing $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, the two given planes are

$$2x - 3y + 4z - 1 = 0, \quad x - y + 4 = 0.$$

Therefore a plane through their intersection is

$$(2x - 3y + 4z - 1) + \lambda(x - y + 4) = 0,$$

or

$$(2 + \lambda)x + (-3 - \lambda)y + 4z - 1 + 4\lambda = 0.$$

Its normal vector is $\vec{n} = (2 + \lambda)\hat{i} + (-3 - \lambda)\hat{j} + 4\hat{k}$. The second given plane by itself is not perpendicular to the prescribed plane, since $(1, -1, 0) \cdot (2, -1, 1) = 3 \neq 0$, so the required member is included in this family.

Step 2: Impose the perpendicularity condition The plane $2x - y + z + 8 = 0$ has normal $\vec{m} = 2\hat{i} - \hat{j} + \hat{k}$. Thus the required condition is

$$\begin{aligned} \vec{n} \cdot \vec{m} &= 0, \\ 2(2 + \lambda) - (-3 - \lambda) + 4 &= 0, \\ 11 + 3\lambda &= 0, \\ \lambda &= -\frac{11}{3}. \end{aligned}$$

Substitution in the family and multiplication by 3 give

$$3(2x - 3y + 4z - 1) - 11(x - y + 4) = 0,$$

which simplifies to $-5x + 2y + 12z - 47 = 0$. Its normal $(-5, 2, 12)$ satisfies $(-5, 2, 12) \cdot (2, -1, 1) = -10 - 2 + 12 = 0$, verifying perpendicularity.

Step 3: Test every point of the given line Let the common value in $x - 1 = 2y - 4 = 3z - 12$ be t . Then

$$x = 1 + t, \quad y = 2 + \frac{t}{2}, \quad z = 4 + \frac{t}{3}.$$

Substituting these coordinates into the plane's left-hand side yields

$$\begin{aligned} -5x + 2y + 12z - 47 &= -5(1 + t) + 2\left(2 + \frac{t}{2}\right) + 12\left(4 + \frac{t}{3}\right) - 47 \\ &= (-5 + 4 + 48 - 47) + (-5 + 1 + 4)t \\ &= 0. \end{aligned}$$

This holds for every real t , establishing containment of the whole line, rather than merely an intersection at one point.

Step 4: Conclusion The required plane is $\boxed{-5x + 2y + 12z - 47 = 0}$, or $\boxed{\vec{r} \cdot (-5\hat{i} + 2\hat{j} + 12\hat{k}) = 47}$.

Quick Tip to Solve: To test line containment, substitute its parameter form into the plane: both the constant term and the coefficient of the parameter must vanish.

It contains the line $x - 1 = 2y - 4 = 3z - 12$.

Quick Tip to Solve: To test line containment, substitute its parameter form into the plane: both the constant term and the coefficient of the parameter must vanish.

OR

25(b). Find the vector and Cartesian equations of a line passing through $(1, 2, -4)$ and perpendicular to the two lines $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$. [6 marks]

Solution:

Concept:

- In the symmetric equation $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$, the denominators (a, b, c) are direction ratios of the line; the numerators specify a point on it.
- A vector perpendicular to two nonparallel direction vectors $\vec{u}$ and $\vec{v}$ is their cross product $\vec{u} \times \vec{v}$. Its scalar product with each of the two vectors is zero.
- The line through a point with position vector $\vec{a}$, in a nonzero direction $\vec{d}$, has vector equation $\vec{r} = \vec{a} + t\vec{d}$, where $t \in \mathbb{R}$. Any nonzero multiple of $\vec{d}$ gives the same line.

Step 1: Extract the two direction vectors From the denominators of the given line equations, their direction vectors are

$$\vec{u} = 3\hat{i} - 16\hat{j} + 7\hat{k}, \quad \vec{v} = 3\hat{i} + 8\hat{j} - 5\hat{k}.$$

The offsets 8, -19, 10 and 15, 29, 5 locate points on the given lines. They do not affect the perpendicular direction required here. The required line must instead pass through the specified point $A(1, 2, -4)$.

Step 2: Calculate a direction perpendicular to both lines Taking the cross product,

$$\begin{aligned}\vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} \\ &= [80 - 56]\hat{i} - [(-15) - 21]\hat{j} + [24 + 48]\hat{k} \\ &= 24\hat{i} + 36\hat{j} + 72\hat{k}.\end{aligned}$$

Dividing by 12, choose the simpler direction vector $\vec{d} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. Since the cross product is nonzero, the original directions are nonparallel and determine a unique perpendicular direction, up to multiplication by a nonzero scalar.

Step 3: Write the line through the prescribed point The position vector of A is $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$. Thus the vector equation is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + t(2\hat{i} + 3\hat{j} + 6\hat{k}), \quad t \in \mathbb{R}.$$

Equating components gives $x = 1 + 2t$, $y = 2 + 3t$, $z = -4 + 6t$. Eliminating t gives the Cartesian

Quick Tip to Solve: Use the denominators of the given symmetric equations in the cross product, simplify the result, and then insert the prescribed point in the numerators.

equation

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}.$$

At $t = 0$ the point is exactly $(1, 2, -4)$, as required.

Step 4: Verify the perpendicular directions Direct scalar-product checks give

$$\vec{d} \cdot \vec{u} = 2(3) + 3(-16) + 6(7) = 6 - 48 + 42 = 0,$$

$$\vec{d} \cdot \vec{v} = 2(3) + 3(8) + 6(-5) = 6 + 24 - 30 = 0.$$

Hence the constructed line has a direction perpendicular to each of the two given lines, and satisfies both direction conditions independently.

Step 5: Conclusion The required vector equation is $\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + t(2\hat{i} + 3\hat{j} + 6\hat{k}), t \in \mathbb{R}$. Its

Cartesian equation is $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}$.

Quick Tip to Solve: Use the denominators of the given symmetric equations in the cross product, simplify the result, and then insert the prescribed point in the numerators.

26(a). Consider $f : \mathbb{R}_+ \rightarrow [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with $f^{-1}(y) = \frac{\sqrt{y+6}-1}{3}$. Hence find (i) $f^{-1}(10)$ (ii) y if $f^{-1}(y) = \frac{4}{3}$, where $\mathbb{R}_+$ is the set of all non-negative real numbers.

[6 marks]

Solution:

Concept:

- A function is invertible precisely when it is bijective: distinct inputs have distinct outputs, and every member of the codomain has a preimage.
- Complete the square: $9x^2 + 6x - 5 = (3x+1)^2 - 6$. Since $x \geq 0$, the factor $3x+1$ is positive, fixing the sign when a square root is taken.

Step 1: Prove that the function is one-one Let $u, v \in \mathbb{R}_+$, and suppose $f(u) = f(v)$. Subtracting the two expressions gives

$$\begin{aligned} 9u^2 + 6u - 5 &= 9v^2 + 6v - 5, \\ (u-v)\{9(u+v) + 6\} &= 0. \end{aligned}$$

Because $u, v \geq 0$, the second factor is at least 6, so it cannot vanish. Therefore $u = v$. This proves injectivity on the stated domain; the restriction to non-negative inputs is essential to this argument.

Step 2: Prove that the function is onto its codomain Take any $y \in [-5, \infty)$. Solving $y = f(x)$ by completing the square,

$$y + 6 = (3x + 1)^2.$$

Quick Tip to Solve: Write the quadratic as $(3x+1)^2 - 6$ first; the non-negative domain selects the positive square root and immediately gives the inverse.

On $x \geq 0$, we have $3x + 1 \geq 1$, so the admissible root is

$$x = \frac{\sqrt{y+6} - 1}{3}.$$

Since $y \geq -5$, we have $\sqrt{y+6} \geq 1$, and this x is non-negative. Substitution yields $f(x) = y$. Thus every value in the specified codomain is attained, proving surjectivity. Together with injectivity, this proves that f is invertible.

Step 3: Identify and verify the inverse The unique preimage obtained above defines

$$f^{-1} : [-5, \infty) \longrightarrow \mathbb{R}_+, \quad f^{-1}(y) = \frac{\sqrt{y+6} - 1}{3}.$$

The composition $f(f^{-1}(y)) = y$ follows directly from the completed square. Conversely, for $x \geq 0$,

$$f^{-1}(f(x)) = \frac{\sqrt{(3x+1)^2} - 1}{3} = \frac{3x+1-1}{3} = x,$$

since $3x + 1 > 0$. Both composition identities hold on their respective domains.

Step 4: Evaluate the two requested values For part (i), insert $y = 10$:

$$f^{-1}(10) = \frac{\sqrt{16} - 1}{3} = \frac{4 - 1}{3} = 1.$$

For part (ii), set the inverse equal to $4/3$:

$$\frac{\sqrt{y+6} - 1}{3} = \frac{4}{3} \implies \sqrt{y+6} = 5 \implies y + 6 = 25.$$

The root is non-negative, so squaring introduces no extraneous value. As a check, $f(4/3) = 16 + 8 - 5 = 19$, which lies in the codomain.

Step 5: Conclusion The function is bijective and hence invertible, with $f^{-1}(y) = (\sqrt{y+6} - 1)/3$. The requested results are (i) $\boxed{1}$ and (ii) $\boxed{y = 19}$.

Quick Tip to Solve: Write the quadratic as $(3x + 1)^2 - 6$ first; the non-negative domain selects the positive square root and immediately gives the inverse.

26(b). Discuss the commutativity and associativity of binary operation $*$ defined on $A = \mathbb{Q} - \{1\}$ by the rule $a * b = a - b + ab$ for all $a, b \in A$. Also find the identity element of $*$ in A and hence find the invertible elements of A . [6 marks]

Solution:

Concept:

- A binary operation on A must map every pair from $A \times A$ back into A . Commutativity requires $a * b = b * a$; associativity requires $(a * b) * c = a * (b * c)$.
- An identity must be two-sided: $a * e = e * a = a$ for every $a \in A$. An invertible element has a two-sided inverse relative to such an identity.

Step 1: Check closure for the printed rule Here $A = \mathbb{Q} \setminus \{1\}$ and the printed rule is $a * b = a - b + ab$. Although the expression is rational for rational inputs, the excluded value can occur. In fact $0, -1 \in A$, but

$$0 * (-1) = 0 - (-1) + 0(-1) = 1 \notin A.$$

Thus the stated rule is not closed on A , and strictly it does not define a binary operation on this set. The following calculations discuss the literal rule without changing its signs, and show that the requested identity cannot exist.

Step 2: Test commutativity For the two orders of the inputs,

$$a * b = a - b + ab, \quad b * a = b - a + ab.$$

Their difference is $2(a - b)$, which is not always zero. For example, $0, 2 \in A$, and both resulting values are in A , but

$$0 * 2 = -2, \quad 2 * 0 = 2.$$

Therefore the printed rule is not commutative, even for these pairs whose outputs remain in the stated set.

Step 3: Test associativity Expanding the two bracketings algebraically gives

$$\begin{aligned} (a * b) * c &= a - b + ab - c + ac - bc + abc, \\ a * (b * c) &= a - b + c - bc + ab - ac + abc. \end{aligned}$$

Hence their difference is $2c(a - 1)$, which need not vanish. A counterexample with all intermediate results allowed in A is $a = 0, b = 0, c = 2$:

$$(0 * 0) * 2 = 0 * 2 = -2, \quad 0 * (0 * 2) = 0 * (-2) = 2.$$

The two values differ, so the literal rule is not associative. This failure is independent of the earlier failure of closure.

Step 4: Determine whether a two-sided identity exists If e were an identity, the requirement $a * e = a$ would give

$$a - e + ae = a \implies e(a - 1) = 0.$$

Quick Tip to Solve: Check closure and both identity equations before finding inverses; for this printed subtraction rule, $a * 0 = a$ but $0 * a = -a$.

As every $a \in A$ differs from 1, this forces $e = 0$. Indeed $a * 0 = a$, so 0 is a right identity for the formula. However,

$$0 * a = -a,$$

which is not equal to a for every member of A ; for example $0 * 2 = -2 \neq 2$. Thus 0 is not a left identity, and no two-sided identity exists. Consequently invertible elements in the usual binary-operation sense cannot be defined here: that definition presupposes a two-sided identity. Solving an equation against the right identity alone would not produce the inverses asked for.

Step 5: Conclusion For the rule as printed, closure fails, so it is not a binary operation on A . The formula is neither commutative nor associative. It has right identity 0, but no two-sided identity; hence there are no invertible elements in the usual identity-based sense.

Quick Tip to Solve: Check closure and both identity equations before finding inverses; for this printed subtraction rule, $a * 0 = a$ but $0 * a = -a$.

27. If the sum of lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$. [6 marks]

Solution:

Concept:

- For a right angled triangle with perpendicular sides x, y and hypotenuse h , Pythagoras' theorem gives $y^2 = h^2 - x^2$, and its area is $\mathcal{A} = \frac{1}{2}xy$.
- With the prescribed sum fixed, express the area in one variable. An interior maximum must satisfy $\mathcal{A}'(x) = 0$; a change of derivative from positive to negative establishes a maximum.
- The cosine of the acute angle between a side and the hypotenuse equals that side divided by the hypotenuse.

Step 1: Express the lengths using the fixed sum Let the given positive sum be s , let the side included in that sum have length x , and let the hypotenuse have length h . Then

$$h + x = s, \quad h = s - x.$$

Denote the other perpendicular side by y . By Pythagoras' theorem,

$$y^2 = (s - x)^2 - x^2 = s^2 - 2sx, \quad y = \sqrt{s^2 - 2sx}.$$

For a non-degenerate triangle both x and y are positive, so the admissible interval is $0 < x < s/2$. In particular, the hypotenuse is longer than the chosen side throughout this interval.

Step 2: Differentiate the area and locate its stationary point The area as a function of the chosen side is

$$\mathcal{A}(x) = \frac{x}{2} \sqrt{s^2 - 2sx}.$$

Quick Tip to Solve: Put the chosen side equal to x and the fixed sum equal to s ; Pythagoras immediately reduces the other side to $\sqrt{s^2 - 2sx}$.

Since s is constant, the product and chain rules give

$$\begin{aligned} A'(x) &= \frac{1}{2} \sqrt{s^2 - 2sx} - \frac{sx}{2\sqrt{s^2 - 2sx}} \\ &= \frac{s^2 - 3sx}{2\sqrt{s^2 - 2sx}} = \frac{s(s - 3x)}{2\sqrt{s^2 - 2sx}}. \end{aligned}$$

The denominator is positive on the admissible interval. Thus the only stationary point satisfies $s - 3x = 0$, giving $x = s/3$, which lies strictly between 0 and $s/2$.

Step 3: Establish the maximum and identify the angle For $0 < x < s/3$, the factor $s - 3x$ is positive, so the area increases. For $s/3 < x < s/2$, it is negative, so the area decreases. Therefore the stationary point is the unique absolute maximum on the interval. Also, the area tends to zero at both limiting degenerate triangles, which is consistent with this conclusion. At this maximum,

$$x = \frac{s}{3}, \quad h = s - x = \frac{2s}{3}, \quad y = \frac{s}{\sqrt{3}}.$$

Let θ be the angle between the chosen side and the hypotenuse. This is an acute angle, and

$$\cos \theta = \frac{x}{h} = \frac{s/3}{2s/3} = \frac{1}{2}.$$

The unique acute angle with this cosine is $\pi/3$. The corresponding maximum area is $s^2/(6\sqrt{3})$.

Step 4: Conclusion Hence the area of the triangle is maximum when the angle between the hypotenuse and the side whose sum with it is fixed is $\theta = \pi/3$.

Quick Tip to Solve: Put the chosen side equal to x and the fixed sum equal to s ; Pythagoras immediately reduces the other side to $\sqrt{s^2 - 2sx}$.

28. If $A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 2 \\ -3 & 1 & -1 \end{pmatrix}$, find A^{-1} and hence solve the system of equations $2x + y - 3z = 13$, $3x + 2y + z = 4$, $x + 2y - z = 8$. [6 marks]

Solution:

Concept:

- A square matrix A is invertible when $|A| \neq 0$, and $A^{-1} = \text{adj } A / |A|$. The adjugate is the transpose of the cofactor matrix.
- The cofactor of entry a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant obtained by deleting row i and column j .
- The coefficient matrix must be read directly from the equations. Here it is A^T , and $(A^T)^{-1} = (A^{-1})^T$.

Quick Tip to Solve: Read coefficients across each equation before multiplying: they form the transpose of A , so transpose the computed inverse to solve this system.

Step 1: Evaluate the determinant Expanding along the first row of the given matrix,

$$\begin{aligned}|A| &= 2\{2(-1) - 2(1)\} - 3\{1(-1) - 2(-3)\} \\ &\quad + 1\{1(1) - 2(-3)\} \\ &= 2(-4) - 3(5) + 7 = -16.\end{aligned}$$

Thus $|A| \neq 0$, so the inverse exists and can be obtained by the adjugate method.

Step 2: Calculate all cofactors and the inverse Including the alternating signs in the cofactor definition, we obtain

$$\begin{aligned}C_{11} &= 2(-1) - 2(1) = -4, \\ C_{12} &= -\{1(-1) - 2(-3)\} = -5, \\ C_{13} &= 1(1) - 2(-3) = 7, \\ C_{21} &= -\{3(-1) - 1(1)\} = 4, \\ C_{22} &= 2(-1) - 1(-3) = 1, \\ C_{23} &= -\{2(1) - 3(-3)\} = -11, \\ C_{31} &= 3(2) - 1(2) = 4, \\ C_{32} &= -\{2(2) - 1(1)\} = -3, \\ C_{33} &= 2(2) - 3(1) = 1.\end{aligned}$$

Consequently,

$$C = \begin{pmatrix} -4 & -5 & 7 \\ 4 & 1 & -11 \\ 4 & -3 & 1 \end{pmatrix}, \quad \text{adj } A = C^T = \begin{pmatrix} -4 & 4 & 4 \\ -5 & 1 & -3 \\ 7 & -11 & 1 \end{pmatrix}.$$

Hence

$$A^{-1} = \frac{1}{16} \begin{pmatrix} 4 & -4 & -4 \\ 5 & -1 & 3 \\ -7 & 11 & -1 \end{pmatrix}.$$

Step 3: Use the inverse to solve the equations Write $X = (x, y, z)^T$ and $B = (13, 4, 8)^T$. The printed equations have coefficient matrix

$$\begin{pmatrix} 2 & 1 & -3 \\ 3 & 2 & 1 \\ 1 & 2 & -1 \end{pmatrix} = A^T,$$

so they give $A^T X = B$. Multiplying by the inverse of A^T ,

$$\begin{aligned}X &= (A^{-1})^T B \\ &= \frac{1}{16} \begin{pmatrix} 4 & 5 & -7 \\ -4 & -1 & 11 \\ -4 & 3 & -1 \end{pmatrix} \begin{pmatrix} 13 \\ 4 \\ 8 \end{pmatrix} \\ &= \frac{1}{16} \begin{pmatrix} 52 + 20 - 56 \\ -52 - 4 + 88 \\ -52 + 12 - 8 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}.\end{aligned}$$

Quick Tip to Solve: Read coefficients across each equation before multiplying: they form the transpose of A, so transpose the computed inverse to solve this system.

The non-zero determinant also ensures that this solution is unique.

Step 4: Check the solution in each equation Substitution in the three original left-hand sides gives

$$\begin{aligned}2(1) + 2 - 3(-3) &= 13, \\3(1) + 2(2) + (-3) &= 4, \\1 + 2(2) - (-3) &= 8.\end{aligned}$$

Each equals its prescribed right-hand side, verifying the signs and the use of the transposed coefficient matrix.

Step 5: Conclusion

$$A^{-1} = \frac{1}{16} \begin{pmatrix} 4 & -4 & -4 \\ 5 & -1 & 3 \\ -7 & 11 & -1 \end{pmatrix}, \quad \boxed{x = 1, y = 2, z = -3}.$$

Quick Tip to Solve: Read coefficients across each equation before multiplying: they form the transpose of A , so transpose the computed inverse to solve this system.

29. Find the particular solution of the differential equation $\tan x \frac{dy}{dx} = 2x \tan x + x^2 - y$; ($\tan x \neq 0$)
given that $y = 0$ **when** $x = \frac{\pi}{2}$. [6 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$. Its integrating factor is $IF = e^{\int P(x) dx}$; multiplying by it makes the left-hand side the derivative of y IF.
- Here division by $\tan x$ gives $P(x) = \cot x$. On an interval with $\sin x \neq 0$, a convenient integrating factor is $\sin x$, since $\frac{d}{dx}(\sin x) = \cos x$.
- The product rule $\frac{d}{dx}(x^2 \sin x) = 2x \sin x + x^2 \cos x$ directly evaluates the integral that occurs after multiplication by the integrating factor.

Step 1: Put the differential equation in linear form From the given equation,

$$\tan x \frac{dy}{dx} + y = 2x \tan x + x^2.$$

Dividing by $\tan x$, as allowed by the printed condition, yields

$$\frac{dy}{dx} + y \cot x = 2x + x^2 \cot x.$$

Thus $P(x) = \cot x$ and $Q(x) = 2x + x^2 \cot x$. The condition at $x = \pi/2$ is applied to the continuous

Quick Tip to Solve: After multiplying by $\sin x$, recognize $2x \sin x + x^2 \cos x$ as the derivative of $x^2 \sin x$; this avoids a separate integration by parts.

solution of this linear form, which is regular there.

Step 2: Calculate and apply the integrating factor Since $\int \cot x \, dx = \log |\sin x|$, the integrating factor can be taken as $\sin x$ on an interval containing $\pi/2$, where $\sin x > 0$. Multiplying the linear equation by this factor gives

$$\sin x \frac{dy}{dx} + y \cos x = 2x \sin x + x^2 \cos x.$$

The left-hand side is $\frac{d}{dx}(y \sin x)$, while the right-hand side is $\frac{d}{dx}(x^2 \sin x)$. Therefore

$$\frac{d}{dx}(y \sin x) = \frac{d}{dx}(x^2 \sin x).$$

Step 3: Integrate and determine the arbitrary constant Integrating both sides with respect to x , we obtain

$$y \sin x = x^2 \sin x + C,$$

where C is a constant. Now substitute the given data $y = 0$ at $x = \pi/2$. Because $\sin(\pi/2) = 1$,

$$0 = \left(\frac{\pi}{2}\right)^2 + C, \quad C = -\frac{\pi^2}{4}.$$

The initial condition thus fixes the constant uniquely on the interval under consideration.

Step 4: Verify the particular solution Dividing the relation by $\sin x$, we have $y = x^2 - \frac{\pi^2}{4} \csc x$. Its derivative is

$$\frac{dy}{dx} = 2x + \frac{\pi^2}{4} \csc x \cot x.$$

Substitution into the linear equation gives

$$\begin{aligned} \frac{dy}{dx} + y \cot x &= 2x + \frac{\pi^2}{4} \csc x \cot x + x^2 \cot x - \frac{\pi^2}{4} \csc x \cot x \\ &= 2x + x^2 \cot x. \end{aligned}$$

It also gives $y(\pi/2) = \pi^2/4 - \pi^2/4 = 0$. Hence both the differential relation and the initial condition are satisfied, with the original tangent form used wherever it is defined.

Step 5: Conclusion The required particular solution is

$$\boxed{y = x^2 - \frac{\pi^2}{4 \sin x}}.$$

Equivalently, $y \sin x = x^2 \sin x - \pi^2/4$; near the given initial point the solution is defined on $0 < x < \pi$.

Quick Tip to Solve: After multiplying by $\sin x$, recognize $2x \sin x + x^2 \cos x$ as the derivative of $x^2 \sin x$; this avoids a separate integration by parts.