

CBSE Class 12 Mathematics Question Paper 2017 with Solution PDF-Set 65-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 3 questions of four marks each and 3 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. Find: $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx.$

[1 mark]

Solution:

Concept:

- Divide the numerator termwise by $\sin x \cos x$ to obtain familiar trigonometric functions.
- The tangent antiderivative is $\int \tan x dx = -\log |\cos x|.$
- The cotangent antiderivative is $\int \cot x dx = \log |\sin x|.$

Step 1: Simplify the integrand On any interval where $\sin x \cos x \neq 0$, the integrand becomes

$$\frac{\sin^2 x - \cos^2 x}{\sin x \cos x} = \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} = \tan x - \cot x.$$

Step 2: Integrate the two terms Thus

$$\begin{aligned} I &= \int (\tan x - \cot x) dx \\ &= -\log |\cos x| - \log |\sin x| + C. \end{aligned}$$

The absolute values allow the logarithms on every interval of the original domain, and C is an arbitrary constant.

Step 3: Conclusion The required integral is $\boxed{-\log |\sin x \cos x| + C}.$

Quick Tip to Solve: Split the fraction before integrating; retain the minus sign before the cotangent term.

2. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$.

[1 mark]

Solution:

Concept:

- For parallel planes $ax + by + cz = d_1$ and $ax + by + cz = d_2$, the perpendicular distance is $\frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}.$
- The coefficients of the normal vector must be identical before using this formula.

Step 1: Put the planes in the same normal form Dividing the second plane by 2.5 gives

$$2x - y + 2z = 8.$$

The first plane is $2x - y + 2z = 5$. Both now have normal vector $(2, -1, 2)$, whose length is $\sqrt{4 + 1 + 4} = 3$.

Quick Tip to Solve: Normalize the second equation first: subtracting 20 and 5 directly would give the wrong distance.

Step 2: Apply the distance formula The separation along their common normal is therefore

$$d = \frac{|8 - 5|}{\sqrt{2^2 + (-1)^2 + 2^2}} = \frac{3}{3}.$$

Taking the absolute value makes the distance independent of the order in which the planes are named.

Step 3: Conclusion The distance between the planes is 1 unit.

Quick Tip to Solve: Normalize the second equation first: subtracting 20 and 5 directly would give the wrong distance.

3. If for any 2×2 square matrix A , $A(\text{adj } A) = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$, then write the value of $|A|$. [1 mark]

Solution:

Concept:

- For every square matrix A , the adjugate identity is $A(\text{adj } A) = |A|I$, where $|A|$ is its determinant.
- For order two, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; multiplying it by a scalar places that scalar in both diagonal entries.

Step 1: Use the adjugate identity Since A is a 2×2 matrix,

$$A(\text{adj } A) = \begin{pmatrix} |A| & 0 \\ 0 & |A| \end{pmatrix}.$$

This identity holds without needing the individual entries of A .

Step 2: Compare diagonal entries The given product is $\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = 8I$. Comparing either diagonal entry with $|A|I$ gives the required determinant directly. Taking the determinant of the product would instead give $|A|^2 = 64$, which unnecessarily loses the sign information already present in the matrix equation.

Step 3: Conclusion The value of the determinant is $|A| = 8$.

Quick Tip to Solve: Read the scalar multiplying the identity matrix; do not take a square root of the product determinant.

4. Determine the value of k for which the following function is continuous at $x = 3$: $f(x) = \begin{cases} \frac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$. [1 mark]

Solution:

Concept:

- Continuity at $x = 3$ requires $\lim_{x \rightarrow 3} f(x) = f(3) = k$.
- A removable factor can be cancelled for $x \neq 3$ when evaluating the limit.

Step 1: Factor the numerator Using the difference of squares,

$$(x+3)^2 - 36 = (x+3-6)(x+3+6) = (x-3)(x+9).$$

Hence for $x \neq 3$,

$$f(x) = \frac{(x-3)(x+9)}{x-3} = x+9.$$

This cancellation determines the nearby values, rather than redefining the stated value at $x = 3$.

Step 2: Evaluate the limit and impose continuity The simplified polynomial has the same limit from both sides:

$$\lim_{x \rightarrow 3^-} f(x) = \lim_{x \rightarrow 3^+} f(x) = 3+9 = 12.$$

Therefore the value assigned at the point must equal this common limit.

Step 3: Conclusion The function is continuous at $x = 3$ when $k = 12$.

Quick Tip to Solve: Factor the difference of squares before substituting; direct substitution produces an indeterminate ratio.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event “number obtained is even” and B be the event “number obtained is red”. Find if A and B are independent events. [2 marks]

Solution:

Concept:

- For a fair die, each outcome in $S = \{1, 2, 3, 4, 5, 6\}$ has probability $1/6$.
- Two events are independent exactly when $P(A \cap B) = P(A)P(B)$. The intersection contains outcomes satisfying both conditions.

Step 1: List the events and their intersection The even outcomes give $A = \{2, 4, 6\}$. The red faces

Quick Tip to Solve: Count red and even outcomes jointly; the intersection is only face 2, not all three even faces.

give $B = \{1, 2, 3\}$. The only red even outcome is 2, so $A \cap B = \{2\}$. Consequently,

$$P(A) = \frac{3}{6} = \frac{1}{2}, \quad P(B) = \frac{3}{6} = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{6}.$$

Green even faces belong to A , but do not belong to the intersection.

Step 2: Test the independence condition The product of the individual probabilities is

$$P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

This differs from $P(A \cap B) = 1/6$. Equivalently, knowing that the result is red leaves three equally likely outcomes, only one of which is even; thus $P(A | B) = 1/3 \neq 1/2 = P(A)$. The information about colour changes the chance of obtaining an even number.

Step 3: Conclusion As $\frac{1}{6} \neq \frac{1}{4}$, the events A and B are not independent.

Quick Tip to Solve: Count red and even outcomes jointly; the intersection is only face 2, not all three even faces.

6. Two tailors, A and B , earn ₹ 300 and ₹ 400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work and if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP. [2 marks]

Solution:

Concept:

- Let x and y denote the numbers of days worked by tailors A and B , respectively.
- An “at least production target creates a lower-bound inequality; the labour cost is the sum of daily wages times days worked.

Step 1: Express the production requirements Tailor A produces $6x$ shirts and $4x$ pairs of trousers; tailor B produces $10y$ shirts and $4y$ pairs of trousers. The stated minimum targets therefore require

$$6x + 10y \geq 60, \quad 4x + 4y \geq 32.$$

The latter inequality is equivalently $x + y \geq 8$. Since days worked cannot be negative, $x \geq 0$ and $y \geq 0$. The directions are lower bounds because production may exceed the required amounts.

Step 2: Form the cost objective The wage cost of A is $300x$ rupees and that of B is $400y$ rupees. Hence the objective function is

$$Z = 300x + 400y.$$

It must be minimized subject to the two production inequalities and non-negativity. These statements constitute the requested linear programming formulation; a graphical optimization is not needed for a formulation question.

Quick Tip to Solve: Translate “at least as greater than or equal to, and attach each daily wage to its own tailor.

Step 3: Conclusion The LPP is to minimize $Z = 300x + 400y$, subject to

$$\begin{aligned} 6x + 10y &\geq 60, \\ 4x + 4y &\geq 32, \\ x &\geq 0, \quad y \geq 0. \end{aligned}$$

Here x and y are the days of work for tailors A and B , respectively.

Quick Tip to Solve: Translate “at least as greater than or equal to, and attach each daily wage to its own tailor.

7. The x -coordinate of a point on the line joining the points $P(2, 2, 1)$ and $Q(5, 1, -2)$ is 4. Find its z -coordinate. [2 marks]

Solution:

Concept:

- The line through points with position vectors $\mathbf{p}$ and $\mathbf{q}$ has equation $\mathbf{r} = \mathbf{p} + t(\mathbf{q} - \mathbf{p})$.
- The same parameter t must be used in all three coordinates of a point on the line.

Step 1: Write the coordinates parametrically For $P(2, 2, 1)$ and $Q(5, 1, -2)$, the direction vector is

$$Q - P = (5 - 2, 1 - 2, -2 - 1) = (3, -1, -3).$$

Thus a general point on the line has coordinates

$$(x, y, z) = (2 + 3t, 2 - t, 1 - 3t).$$

Here $t = 0$ gives P and $t = 1$ gives Q , confirming both endpoints.

Step 2: Use the given x -coordinate Substituting $x = 4$ in the first coordinate gives

$$2 + 3t = 4 \implies t = \frac{2}{3}.$$

The third coordinate is then

$$z = 1 - 3\left(\frac{2}{3}\right) = 1 - 2 = -1.$$

As a check, the same parameter gives $y = 2 - 2/3 = 4/3$, so the full point is $(4, 4/3, -1)$. It lies between the endpoints since $0 < t < 1$.

Step 3: Conclusion The required z -coordinate is $\boxed{-1}$.

Quick Tip to Solve: Find the line parameter from the x -coordinate, then substitute it into the z -coordinate with its negative direction.

8. Find: $\int \frac{dx}{5 - 8x - x^2}$.

[2 marks]

Solution:

Concept:

- Completing the square changes this denominator into a difference of squares.
- Split a difference-of-squares denominator into two linear fractions and integrate each using logarithms.

Step 1: Complete the square Write $5 - 8x - x^2 = -(x^2 + 8x - 5) = 21 - (x + 4)^2$. Set $u = x + 4$, so that $du = dx$, and let $a = \sqrt{21}$. Thus the required integral becomes

$$I = \int \frac{du}{a^2 - u^2}.$$

The substitution is a translation, so no extra numerical factor arises in the differential.

Step 2: Integrate using partial fractions The identity

$$\frac{1}{a^2 - u^2} = \frac{1}{2a} \left(\frac{1}{a + u} + \frac{1}{a - u} \right)$$

gives

$$I = \frac{1}{2a} (\log |a + u| - \log |a - u|) + C.$$

The second logarithm has a minus sign because the derivative of $a - u$ is -1 . Substituting $a = \sqrt{21}$ and $u = x + 4$ now expresses the antiderivative in the original variable. This formula applies on any interval avoiding the two zeros of the original denominator.

Step 3: Conclusion

$$\int \frac{dx}{5 - 8x - x^2} = \frac{1}{2\sqrt{21}} \log \left| \frac{\sqrt{21} + x + 4}{\sqrt{21} - x - 4} \right| + C.$$

Quick Tip to Solve: Complete the square before integrating; retain the minus sign from integrating $1/(a - u)$.

9. If A is a skew-symmetric matrix of order 3, then prove that $\det A = 0$.

[2 marks]

Solution:

Concept:

- A skew-symmetric matrix satisfies $A^T = -A$, where A^T denotes its transpose.
- Transposition leaves the determinant unchanged. Multiplying each of the n rows by -1 multiplies the determinant by $(-1)^n$.

Step 1: Apply the skew-symmetry condition Since A is skew-symmetric, its transpose is the negative

Quick Tip to Solve: Count all three row factors in $\det(-A)$; an odd number gives the sign change needed for the proof.

of the original matrix: $A^T = -A$. Take determinants on both sides to obtain

$$\det(A^T) = \det(-A).$$

Here the determinant exists because the matrix is square, and its order is three. We will evaluate each side using standard determinant properties, without needing the individual entries of A .

Step 2: Use the odd order of the matrix The determinant of a transpose equals the original determinant, so $\det(A^T) = \det A$. In $-A$, each of the three rows of A is multiplied by -1 ; consequently,

$$\det(-A) = (-1)^3 \det A = -\det A.$$

Combining these two properties with the preceding equality yields $\det A = -\det A$, and hence $2 \det A = 0$. Since the entries are real and $2 \neq 0$, division by two is valid.

Step 3: Conclusion Therefore $\boxed{\det A = 0}$, as required. In particular, the matrix is singular.

Quick Tip to Solve: Count all three row factors in $\det(-A)$; an odd number gives the sign change needed for the proof.

10. Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $[-\sqrt{3}, 0]$. [2 marks]

Solution:

Concept:

- Rolle's theorem requires continuity on $[a, b]$, differentiability on (a, b) , and equal endpoint values $f(a) = f(b)$.
- Under these conditions, there exists $c \in (a, b)$ such that $f'(c) = 0$. The required point is strictly inside the interval.

Step 1: Verify the hypotheses The given function $f(x) = x^3 - 3x$ is a polynomial. It is therefore continuous on $[-\sqrt{3}, 0]$ and differentiable on $(-\sqrt{3}, 0)$. At the endpoints,

$$f(-\sqrt{3}) = (-\sqrt{3})^3 - 3(-\sqrt{3}) = -3\sqrt{3} + 3\sqrt{3} = 0, \quad f(0) = 0.$$

Thus the endpoint values are equal and all the hypotheses of Rolle's theorem hold on the specified interval.

Step 2: Find and select the stationary point Differentiating gives $f'(x) = 3x^2 - 3$. The point furnished by the theorem must satisfy

$$f'(c) = 0 \implies 3c^2 - 3 = 0 \implies c^2 = 1.$$

The algebraic candidates are $c = -1$ and $c = 1$. Since $-\sqrt{3} < -1 < 0$, the negative candidate lies in the required open interval. The positive candidate is outside this interval and cannot serve as the Rolle point for the given endpoints.

Step 3: Conclusion The required value is $\boxed{c = -1}$.

Quick Tip to Solve: After solving $f'(c) = 0$, always discard roots outside the specified open interval.

11. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on $\mathbb{R}$.

[2 marks]

Solution:

Concept:

- If a differentiable function has $f'(x) > 0$ throughout an interval, it is strictly increasing on that interval.
- Completing the square can establish the sign of a quadratic derivative for every real value of x .

Step 1: Differentiate the function The function $f(x) = x^3 - 3x^2 + 6x - 100$ is a polynomial and is differentiable for every real x . Differentiate each term:

$$f'(x) = 3x^2 - 6x + 6.$$

The constant -100 contributes zero, while the linear term contributes 6. Hence it is the sign of this quadratic expression, rather than the sign of the original cubic, that determines whether the function is increasing.

Step 2: Establish positivity on the whole real line Complete the square in the derivative:

$$f'(x) = 3(x^2 - 2x + 2) = 3((x - 1)^2 + 1).$$

For every real number x , the square $(x - 1)^2$ is nonnegative. Therefore $(x - 1)^2 + 1 \geq 1$, giving $f'(x) \geq 3 > 0$ everywhere on $\mathbb{R}$. There are no points at which the derivative vanishes or becomes negative. The standard increasing-function criterion thus applies throughout the domain.

Step 3: Conclusion Therefore $f(x) = x^3 - 3x^2 + 6x - 100$ is strictly increasing on $\mathbb{R}$.

Quick Tip to Solve: Write the derivative as $3((x - 1)^2 + 1)$; the added positive constant proves positivity immediately.

12. The length x , of a rectangle is decreasing at the rate of 5 cm/minute and the width y , is increasing at the rate of 4 cm/minute. When $x = 8$ cm and $y = 6$ cm, find the rate of change of the area of the rectangle.

[2 marks]

Solution:

Concept:

- The area of a rectangle with time-dependent length x and width y is $A = xy$.
- The product rule gives $dA/dt = x dy/dt + y dx/dt$. A decreasing dimension has a negative derivative.

Step 1: Translate the stated rates into signed derivatives Let t be measured in minutes and let the dimensions x and y be measured in centimetres. Since the length is decreasing at 5 cm/minute,

$$\frac{dx}{dt} = -5 \text{ cm/minute.}$$

Since the width is increasing at 4 cm/minute, $dy/dt = 4$ cm/minute. At the specified instant, the dimensions are $x = 8$ cm and $y = 6$ cm. Both dimensions vary, so both terms of the product rule must be retained.

Quick Tip to Solve: Assign a negative sign to the decreasing length before substituting into the product rule.

Step 2: Differentiate the area and substitute From $A = xy$, differentiation with respect to time gives

$$\frac{dA}{dt} = x \frac{dy}{dt} + y \frac{dx}{dt}.$$

Substituting the instantaneous dimensions and signed rates,

$$\frac{dA}{dt} = 8(4) + 6(-5) = 32 - 30 = 2 \text{ cm}^2/\text{minute}.$$

The gain due to the growing width exceeds the loss due to the shrinking length. Because the resulting derivative is positive, the rectangle's area is increasing at this instant.

Step 3: Conclusion The area is increasing at $2 \text{ cm}^2/\text{minute}$.

Quick Tip to Solve: Assign a negative sign to the decreasing length before substituting into the product rule.

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13(a). Evaluate: $\int_0^{\pi} \frac{x \tan x}{\sec x + \tan x} dx.$

[4 marks]

Solution:

Concept:

- The reflection property of definite integrals is $\int_0^a f(x) dx = \int_0^a f(a-x) dx$. If a factor is unchanged by reflection, adding the two forms removes the variable multiplier.
- Dividing numerator and denominator by $\sec x$ gives $\tan x / (\sec x + \tan x) = \sin x / (1 + \sin x)$. Also, $1 + \cos t = 2 \cos^2(t/2)$.

Step 1: Simplify the trigonometric quotient Let the required integral be I . Multiplying the numerator and denominator by $\cos x$, wherever the original quotient is defined, gives

$$I = \int_0^{\pi} \frac{x \sin x}{1 + \sin x} dx.$$

At $x = \pi/2$, the original tangent and secant are undefined, but this simplified expression has a finite limit. Thus the original integral is interpreted with this removable discontinuity; the continuous extension gives the same integral. On $[0, \pi]$, the denominator $1 + \sin x$ is positive.

Step 2: Apply the reflection property Replace x by $\pi - x$. Since $\sin(\pi - x) = \sin x$,

$$I = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \sin x} dx.$$

Adding this to the preceding expression yields

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \sin x} dx = \pi [\pi - K], \quad K = \int_0^{\pi} \frac{dx}{1 + \sin x}.$$

Quick Tip to Solve: First simplify the quotient to $\sin x / (1 + \sin x)$; reflecting about $\pi/2$ then replaces the multiplier x by $\pi/2$.

Here the decomposition used is $\sin x/(1 + \sin x) = 1 - 1/(1 + \sin x)$.

Step 3: Evaluate the remaining integral Put $t = x - \pi/2$, so that $dx = dt$, $\sin x = \cos t$, and the limits become $-\pi/2$ and $\pi/2$. Therefore

$$\begin{aligned} K &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sec^2(t/2) dt \\ &= [\tan(t/2)]_{-\pi/2}^{\pi/2} \\ &= \tan(\pi/4) - \tan(-\pi/4) = 2. \end{aligned}$$

The half-angle tangent is finite throughout these limits, so the endpoint evaluation is valid. Substitution in $2I = \pi(\pi - K)$ now evaluates the required integral.

Step 4: Conclusion The required value is $\boxed{\frac{\pi}{2}(\pi - 2)}$.

Quick Tip to Solve: First simplify the quotient to $\sin x/(1 + \sin x)$; reflecting about $\pi/2$ then replaces the multiplier x by $\pi/2$.

OR

13(b). Evaluate: $\int_1^4 \{|x - 1| + |x - 2| + |x - 4|\} dx$.

[4 marks]

Solution:

Concept:

- For a real number u , $|u| = u$ when $u \geq 0$, and $|u| = -u$ when $u < 0$. A modulus integral is split at the zeros of its arguments inside the interval.
- Definite integrals are additive: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, for $a \leq c \leq b$.

Step 1: Determine the signs on the interval Write the required integral as I . On the whole interval $1 \leq x \leq 4$, $x - 1 \geq 0$ and $x - 4 \leq 0$. Hence $|x - 1| = x - 1$ and $|x - 4| = 4 - x$. Their sum is the constant 3. The remaining expression $x - 2$ changes sign at the interior point $x = 2$: it is non-positive on $[1, 2]$ and non-negative on $[2, 4]$. Equality at an endpoint causes no problem because both descriptions give zero there.

Step 2: Remove the moduli in each interval For $1 \leq x \leq 2$, the integrand is

$$(x - 1) + (2 - x) + (4 - x) = 5 - x.$$

For $2 \leq x \leq 4$, it is

$$(x - 1) + (x - 2) + (4 - x) = x + 1.$$

Both expressions equal 3 at $x = 2$, confirming that the pieces join correctly. Thus the integral becomes

$$I = \int_1^2 (5 - x) dx + \int_2^4 (x + 1) dx.$$

Quick Tip to Solve: Combine $|x - 1| + |x - 4| = 3$ on $[1, 4]$, and split only the remaining modulus $|x - 2|$ at $x = 2$.

Step 3: Integrate and substitute the limits Using the elementary antiderivatives of a constant and of x , we obtain

$$\begin{aligned} I &= \left[5x - \frac{x^2}{2} \right]_1^2 + \left[\frac{x^2}{2} + x \right]_2^4 \\ &= \left(10 - 2 - 5 + \frac{1}{2} \right) + (8 + 4 - 2 - 2) \\ &= \frac{7}{2} + 8. \end{aligned}$$

As a check, the constant part 3 contributes $3(4-1) = 9$. The integral of $|x-2|$ contributes $1/2 + 2 = 5/2$, giving the same total as the two interval evaluations.

Step 4: Conclusion The required value is $\boxed{\frac{23}{2}}$.

Quick Tip to Solve: Combine $|x-1| + |x-4| = 3$ on $[1, 4]$, and split only the remaining modulus $|x-2|$ at $x = 2$.

14. Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence find the area of the triangle. [4 marks]

Solution:

Concept:

- Subtract position vectors to find a joining vector: $\overrightarrow{PQ} = \vec{q} - \vec{p}$. A vector (x, y, z) has magnitude $\sqrt{x^2 + y^2 + z^2}$.
- Two non-zero vectors are perpendicular precisely when their scalar product is zero. The area of a right-angled triangle is half the product of its perpendicular side lengths.

Step 1: Obtain the vectors along the sides The coordinates are $A = (2, -1, 1)$, $B = (1, -3, -5)$ and $C = (3, -4, -4)$. Subtracting the coordinates of the initial vertex from those of the final vertex gives

$$\begin{aligned} \overrightarrow{AB} &= -\hat{i} - 2\hat{j} - 6\hat{k}, \\ \overrightarrow{AC} &= \hat{i} - 3\hat{j} - 5\hat{k}, \\ \overrightarrow{BC} &= 2\hat{i} - \hat{j} + \hat{k}. \end{aligned}$$

The three vectors are non-zero, so the vertices are distinct. To check the angle at C , use vectors directed from C towards A and B .

Step 2: Prove that the angle at C is a right angle Reversing the corresponding side vectors gives

$$\overrightarrow{CA} = -\hat{i} + 3\hat{j} + 5\hat{k}, \quad \overrightarrow{CB} = -2\hat{i} + \hat{j} - \hat{k}.$$

Their scalar product is

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-1)(-2) + (3)(1) + (5)(-1) = 2 + 3 - 5 = 0.$$

Quick Tip to Solve: After a pair of side vectors has zero dot product, use those two side lengths directly as the base and height for the area.

Both vectors are non-zero, and their scalar product is zero. Hence CA and CB are perpendicular and $\angle ACB = 90^\circ$. This also establishes that the three points are not collinear and form a triangle. As an independent check, $AB^2 = 1 + 4 + 36 = 41$, while $AC^2 + BC^2 = 35 + 6 = 41$, as required by Pythagoras' theorem.

Step 3: Use the perpendicular sides to find the area The lengths of the sides meeting at the right angle are

$$CA = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{35}, \quad CB = \sqrt{(-2)^2 + 1^2 + (-1)^2} = \sqrt{6}.$$

Taking CA as the base, CB is the corresponding perpendicular height. Thus

$$\text{Area}(\triangle ABC) = \frac{1}{2}(CA)(CB) = \frac{1}{2}\sqrt{35}\sqrt{6}.$$

The hypotenuse AB is not used as the height, since it is not perpendicular to either of the other sides.

Step 4: Conclusion The points form a triangle right-angled at C , with area $\frac{\sqrt{210}}{2}$ square units.

Quick Tip to Solve: After a pair of side vectors has zero dot product, use those two side lengths directly as the base and height for the area.

15. There are 4 cards numbered 1, 3, 5 and 7, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X . [4 marks]

Solution:

Concept:

- For two cards drawn uniformly without replacement, the $\binom{4}{2} = 6$ unordered pairs are equally likely. Each pair therefore has probability $1/6$.
- For a discrete random variable, $E(X) = \sum_x xP(X = x)$ and $E(X^2) = \sum_x x^2P(X = x)$. Its variance is $\text{Var}(X) = E(X^2) - [E(X)]^2$.

Step 1: List the equally likely pairs and their sums Since the cards are drawn without replacement, the same card cannot occur twice. Since X is the sum, the order in which the two cards are drawn does not affect its value. There are $\binom{4}{2} = 6$ unordered pairs, and they give the following sums:

Pair	(1, 3)	(1, 5)	(1, 7)	(3, 5)	(3, 7)	(5, 7)
X	4	6	8	8	10	12

Each pair is equally likely. The sum 8 occurs for two different pairs, while each other listed sum occurs for one pair. Thus the five distinct values of X are not equally likely.

Step 2: Form the probability distribution Collecting together the pairs with the same sum gives

x	4	6	8	10	12
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Quick Tip to Solve: List the six unordered card pairs first; count repeated sums before assigning probabilities, since the five distinct sums are not equally likely.

All probabilities are non-negative, and their sum is $(1 + 1 + 2 + 1 + 1)/6 = 1$, so this is a valid distribution. In particular, $P(X = 8) = 1/3$, because either $(1, 7)$ or $(3, 5)$ produces that sum.

Step 3: Calculate the mean Multiply each possible sum by its probability and add:

$$E(X) = \frac{4 + 6 + 2(8) + 10 + 12}{6} = \frac{48}{6} = 8.$$

Step 4: Calculate the second moment and the variance First square each possible value and weight it by its probability:

$$\begin{aligned} E(X^2) &= \frac{4^2 + 6^2 + 2(8^2) + 10^2 + 12^2}{6} \\ &= \frac{16 + 36 + 128 + 100 + 144}{6} = \frac{424}{6} = \frac{212}{3}. \end{aligned}$$

Consequently,

$$\text{Var}(X) = \frac{212}{3} - 8^2 = \frac{212 - 192}{3} = \frac{20}{3}.$$

Step 5: Conclusion The mean of X is $\boxed{8}$, and its variance is $\boxed{\frac{20}{3}}$.

Quick Tip to Solve: List the six unordered card pairs first; count repeated sums before assigning probabilities, since the five distinct sums are not equally likely.

16. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance? Is regularity required only in school? Justify your answer. [4 marks]

Solution:

Concept:

- Let R denote a student with 100% attendance, I an irregular student, and A a student attaining A grade. The events R and I partition the school population.
- Total probability gives $P(A) = P(R)P(A | R) + P(I)P(A | I)$. Bayes' theorem gives $P(R | A) = P(R \cap A)/P(A)$.

Step 1: Identify the given probabilities Selection at random makes the attendance proportions the corresponding probabilities:

$$P(R) = \frac{3}{10}, \quad P(I) = \frac{7}{10}, \quad P(A | R) = \frac{7}{10}, \quad P(A | I) = \frac{1}{10}.$$

The last two probabilities describe grade attainment within each attendance group. The required probability reverses the conditioning: attendance must be determined among students already known to have an A grade.

Quick Tip to Solve: Among A-grade students, use the ratio of regular contributors to total contributors: 21 divided by 21+7.

Step 2: Apply total probability and Bayes' theorem Both groups contribute to the A-grade students. Thus

$$\begin{aligned} P(A) &= \frac{3}{10} \cdot \frac{7}{10} + \frac{7}{10} \cdot \frac{1}{10} \\ &= \frac{21}{100} + \frac{7}{100} = \frac{28}{100}. \end{aligned}$$

The joint probability of regular attendance and an A grade is $P(R \cap A) = 21/100$. Therefore

$$P(R | A) = \frac{P(R \cap A)}{P(A)} = \frac{21/100}{28/100} = \frac{21}{28} = \frac{3}{4}.$$

A representative group of 100 students would contain 21 regular A-grade students and 7 irregular A-grade students. Of the 28 A-grade students, 21 are regular, which checks the conditional probability.

Step 3: Justify the value of regularity in life Regularity is required beyond school too. Regular study, exercise and work develop discipline, maintain health and help a person complete responsibilities on time. Consistent practice builds skills and prevents unfinished tasks from accumulating. Regularity is therefore useful at home, in a profession and in fulfilling social responsibilities, as well as in attending school.

Step 4: Conclusion The required probability is $\frac{3}{4}$, or 75%. No, regularity is not required only in school; disciplined and consistent effort is valuable throughout life.

Quick Tip to Solve: Among A-grade students, use the ratio of regular contributors to total contributors: 21 divided by 21+7.

17. If $\tan^{-1}\left(\frac{x-3}{x-4}\right) + \tan^{-1}\left(\frac{x+3}{x+4}\right) = \frac{\pi}{4}$, then find the value of x .

[4 marks]

Solution:

Concept:

- The principal value of $\tan^{-1} t$ lies in $(-\pi/2, \pi/2)$. Taking tangents yields candidate roots, which must be checked against these principal values.
- For angles α, β , $\tan(\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta)$, provided the denominator is nonzero.

Step 1: Set up the tangent addition formula The original expressions require $x \neq 4, -4$. Put

$$a = \frac{x-3}{x-4}, \quad b = \frac{x+3}{x+4}, \quad \alpha = \tan^{-1} a, \quad \beta = \tan^{-1} b.$$

Since the given sum is $\pi/4$, its tangent is 1. Simplifying the numerator and denominator separately gives

$$\begin{aligned} a + b &= \frac{(x-3)(x+4) + (x+3)(x-4)}{x^2 - 16} = \frac{2x^2 - 24}{x^2 - 16}, \\ 1 - ab &= 1 - \frac{x^2 - 9}{x^2 - 16} = \frac{-7}{x^2 - 16}. \end{aligned}$$

Quick Tip to Solve: After taking tangents, check the signs of both arguments; positivity puts the sum in $(0, \pi)$, fixing its branch.

The denominator is nonzero everywhere in the original domain, so the tangent formula is applicable.

Step 2: Obtain the candidate values Consequently, the equation implies

$$\frac{a+b}{1-ab} = \frac{24-2x^2}{7} = 1.$$

Hence $24 - 2x^2 = 7$, so $2x^2 = 17$. The candidates are

$$x = \pm \sqrt{\frac{17}{2}} = \pm \frac{\sqrt{34}}{2}.$$

Neither candidate equals an excluded value 4 or -4 . However, taking tangents loses information about multiples of π , so both candidates still require a principal-value check.

Step 3: Verify the principal-value branch for both candidates For either candidate, $x^2 = 17/2 < 9$, so $-3 < x < 3$. Then $x - 3$ and $x - 4$ are both negative, while $x + 3$ and $x + 4$ are both positive. Thus $a > 0$ and $b > 0$, which implies

$$0 < \alpha < \frac{\pi}{2}, \quad 0 < \beta < \frac{\pi}{2}, \quad 0 < \alpha + \beta < \pi.$$

The only angle in this last interval with tangent 1 is $\pi/4$. Therefore the tangent equation gives the required sum, rather than another branch, at both candidates. This checks each solution in the original inverse-tangent equation.

Step 4: Conclusion Both values satisfy the printed equation: $x = \pm \frac{\sqrt{34}}{2}$.

Quick Tip to Solve: After taking tangents, check the signs of both arguments; positivity puts the sum in $(0, \pi)$, fixing its branch.

18(a). Using properties of determinants, prove that $\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a - 1)^3$. [4 marks]

Solution:

Concept:

- Adding a multiple of one row to another leaves a determinant unchanged. Equal entries in a column can therefore be used to create zeros.
- A common factor in a row can be taken outside the determinant. Cofactor expansion along a column with two zeros reduces the calculation to a second-order determinant.

Step 1: Create zeros in the third column Let the given determinant be D . Apply $R_1 \rightarrow R_1 - R_3$ and $R_2 \rightarrow R_2 - R_3$, keeping the third row unchanged. These operations preserve the determinant and

Quick Tip to Solve: For a column of ones, subtract one row from the others before expanding; here that also exposes two factors of $a - 1$.

give

$$D = \begin{vmatrix} a^2 + 2a - 3 & 2a - 2 & 0 \\ 2a - 2 & a - 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}.$$

The third column now has only one nonzero entry. This avoids the longer expansion of the original determinant, while keeping all dependence on a explicit.

Step 2: Factor the first two rows The entries can be written using

$$a^2 + 2a - 3 = (a - 1)(a + 3), \quad 2a - 2 = 2(a - 1).$$

Hence $a - 1$ is common to the first row and also to the second row. Extracting one factor from each gives

$$D = (a - 1)^2 \begin{vmatrix} a + 3 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}.$$

We have factored polynomial expressions, without dividing by $a - 1$. Consequently this equality remains valid at $a = 1$, where both of the original first two transformed rows vanish.

Step 3: Expand along the third column The entry in position $(3, 3)$ is 1, and its cofactor sign is positive because $(-1)^{3+3} = 1$. All other terms in the column expansion vanish. Therefore

$$\begin{aligned} D &= (a - 1)^2 \begin{vmatrix} a + 3 & 2 \\ 2 & 1 \end{vmatrix} \\ &= (a - 1)^2 [(a + 3) \cdot 1 - 2 \cdot 2] \\ &= (a - 1)^2 (a + 3 - 4). \end{aligned}$$

The second-order determinant contributes the remaining linear factor.

Step 4: Conclusion Thus $D = (a - 1)^2(a - 1) = (a - 1)^3$, which proves the stated identity for every value of a .

Quick Tip to Solve: For a column of ones, subtract one row from the others before expanding; here that also exposes two factors of $a - 1$.

OR

18(b). Find matrix A such that $\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} A = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}.$

[4 marks]

Solution:

Concept:

- For matrix multiplication, each entry of the product is the scalar product of a row of the first matrix with a column of the second matrix. Equality of matrices means equality of corresponding entries.

Quick Tip to Solve: Start with the row $(1, 0)$: it gives the first row of A directly. Then use $(2, -1)$ and check the final row.

- A product of a 3×2 matrix with A has two columns here, so A must be a 2×2 matrix. Use independent rows to determine its entries and the remaining row to verify consistency.

Step 1: Introduce the unknown entries Write

$$A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Multiplying the given matrix by A row by column yields

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} A = \begin{pmatrix} 2x - z & 2y - w \\ x & y \\ -3x + 4z & -3y + 4w \end{pmatrix}.$$

Step 2: Compare the second and first rows Comparing the second row with the second row $(1, -2)$ of the prescribed product gives

$$x = 1, \quad y = -2.$$

Now compare the first row, whose entries must be -1 and -8 :

$$\begin{aligned} 2x - z = -1 &\implies 2 - z = -1 \implies z = 3, \\ 2y - w = -8 &\implies -4 - w = -8 \implies w = 4. \end{aligned}$$

The first two rows determine the entries uniquely; the remaining row must also be checked.

Step 3: Verify the third row and the full product Substituting the calculated entries in the third-row expressions gives

$$-3(1) + 4(3) = 9, \quad -3(-2) + 4(4) = 22.$$

Both values agree with the third row of the required matrix. Indeed the complete multiplication is

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}.$$

All six corresponding entries agree, verifying the matrix equation.

Step 4: Conclusion The required matrix is $A = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$.

Quick Tip to Solve: Start with the row $(1, 0)$: it gives the first row of A directly. Then use $(2, -1)$ and check the final row.

19(a). If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.

[4 marks]

Solution:

Concept:

- Variable powers can be written as $x^y = e^{y \ln x}$ and $y^x = e^{x \ln y}$, on the positive-base domain $x > 0, y > 0$.
- Use implicit differentiation, the product rule and the chain rule, taking y as a function of x . The quantities a, b are constants.

Step 1: Differentiate the first term Write $x^y = e^{y \ln x}$. The derivative of its exponent is found by the product rule, because y is variable:

$$\frac{d}{dx}(y \ln x) = \frac{dy}{dx} \ln x + \frac{y}{x}.$$

Applying the chain rule to the exponential therefore gives

$$\frac{d}{dx}(x^y) = x^y \left(\ln x \frac{dy}{dx} + \frac{y}{x} \right).$$

Thus differentiating a variable power requires accounting for both its changing base and its changing exponent.

Step 2: Differentiate the second term and the given relation Likewise, $y^x = e^{x \ln y}$. Here the derivative of $\ln y$ is $\frac{1}{y} \frac{dy}{dx}$, so

$$\frac{d}{dx}(y^x) = y^x \left(\ln y + \frac{x}{y} \frac{dy}{dx} \right).$$

Since a^b is constant, differentiation of the entire equation gives

$$x^y \left(\ln x \frac{dy}{dx} + \frac{y}{x} \right) + y^x \left(\ln y + \frac{x}{y} \frac{dy}{dx} \right) = 0.$$

No logarithm of the sum is needed; differentiating its two terms separately preserves their simple exponential forms.

Step 3: Isolate the derivative Collecting the derivative terms on the left and the remaining terms on the right gives

$$(x^y \ln x + xy^{x-1}) \frac{dy}{dx} = -yx^{y-1} - y^x \ln y.$$

Division by the coefficient of dy/dx , when it is nonzero, determines the derivative. If this coefficient vanishes, the formula cannot be used to assign a finite slope at that point.

Step 4: Conclusion

$$\boxed{\frac{dy}{dx} = -\frac{yx^{y-1} + y^x \ln y}{x^y \ln x + xy^{x-1}}}$$

The expression applies for $x > 0, y > 0$ wherever its denominator is nonzero.

Quick Tip to Solve: Rewrite both variable powers as exponentials, then collect all terms containing dy/dx before dividing.

19(b). If $e^y(x+1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

[4 marks]

Solution:

Concept:

- Implicit differentiation uses the product rule on $e^y(x+1)$, with $\frac{d}{dx}(e^y) = e^y \frac{dy}{dx}$.
- The second derivative is obtained by differentiating the first derivative once more. Comparing it with the square of the first derivative proves the required identity.

Step 1: Identify the domain and differentiate the equation The given equation is $e^y(x+1) = 1$. Since $e^y > 0$ for every real y , this equation requires $x+1 > 0$, hence $x > -1$. In particular, division by $x+1$ is valid throughout the real domain of the relation. Differentiate both sides with respect to x . Applying the product rule on the left gives

$$e^y \frac{dy}{dx}(x+1) + e^y = 0.$$

The right-hand derivative is zero because 1 is constant, while the second product-rule term is e^y because the derivative of $x+1$ is 1.

Step 2: Determine the first derivative Factor the nonzero quantity e^y from the differentiated equation:

$$e^y \left((x+1) \frac{dy}{dx} + 1 \right) = 0.$$

Divide by e^y and rearrange to obtain

$$(x+1) \frac{dy}{dx} = -1, \quad \frac{dy}{dx} = -\frac{1}{x+1}.$$

This also agrees with taking logarithms of the original equation, which yields $y = -\ln(x+1)$ on its real domain.

Step 3: Differentiate once more and compare the two expressions Write the first derivative as $-(x+1)^{-1}$. By the power rule and chain rule,

$$\frac{d^2y}{dx^2} = -(-1)(x+1)^{-2} \cdot 1 = \frac{1}{(x+1)^2}.$$

Independently, squaring the first derivative gives

$$\left(\frac{dy}{dx}\right)^2 = \left(-\frac{1}{x+1}\right)^2 = \frac{1}{(x+1)^2}.$$

The minus sign in the first derivative disappears upon squaring, just as the two negative factors produce a positive second derivative.

Quick Tip to Solve: Cancel the positive factor e^y immediately after the first differentiation; then the second differentiation is just a power-rule calculation.

Step 4: Conclusion Both expressions equal $\frac{1}{(x+1)^2}$. Therefore,

$$\boxed{\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2}, \quad x > -1.$$

Quick Tip to Solve: Cancel the positive factor e^y immediately after the first differentiation; then the second differentiation is just a power-rule calculation.

20. Find: $\int \frac{\sin \theta d\theta}{(4 + \cos^2 \theta)(2 - \sin^2 \theta)}.$

[4 marks]

Solution:

Concept:

- Use $\sin^2 \theta + \cos^2 \theta = 1$ to express both denominator factors in terms of $\cos \theta$.
- With $u = \cos \theta$, the differential is $du = -\sin \theta d\theta$. The minus sign must be retained.
- Partial fractions reduce the rational integrand to standard forms, with $\int du/(u^2 + a^2) = a^{-1} \tan^{-1}(u/a) + C$, where $a > 0$.

Step 1: Rewrite the denominator and substitute Denote the required integral by I . Since

$$2 - \sin^2 \theta = 2 - (1 - \cos^2 \theta) = 1 + \cos^2 \theta,$$

putting $u = \cos \theta$ gives

$$I = - \int \frac{du}{(u^2 + 4)(u^2 + 1)}.$$

Both factors are positive for every real u , so the substitution introduces no singularity.

Step 2: Resolve the rational function into partial fractions Write

$$\frac{1}{(u^2 + 4)(u^2 + 1)} = \frac{A}{u^2 + 1} + \frac{B}{u^2 + 4}.$$

Multiplication by the denominator yields

$$1 = A(u^2 + 4) + B(u^2 + 1) = (A + B)u^2 + 4A + B.$$

Comparing coefficients gives $A + B = 0$ and $4A + B = 1$. Hence $A = 1/3$, $B = -1/3$, and

$$I = -\frac{1}{3} \int \frac{du}{u^2 + 1} + \frac{1}{3} \int \frac{du}{u^2 + 4}.$$

Step 3: Integrate and verify the antiderivative Applying the standard inverse-tangent formula gives

$$I = -\frac{1}{3} \tan^{-1} u + \frac{1}{6} \tan^{-1} \left(\frac{u}{2} \right) + C.$$

Quick Tip to Solve: Rewrite the sine-squared factor first, then set u equal to cosine theta; retain the negative differential before taking partial fractions.

Substitute $u = \cos \theta$. To check the result, its derivative is

$$\begin{aligned}\frac{dI}{d\theta} &= \frac{\sin \theta}{3(1 + \cos^2 \theta)} - \frac{\sin \theta}{3(4 + \cos^2 \theta)} \\ &= \frac{\sin \theta}{(1 + \cos^2 \theta)(4 + \cos^2 \theta)},\end{aligned}$$

which is the original integrand after the trigonometric identity is applied.

Step 4: Conclusion

$$I = \frac{1}{6} \tan^{-1} \left(\frac{\cos \theta}{2} \right) - \frac{1}{3} \tan^{-1}(\cos \theta) + C.$$

Quick Tip to Solve: Rewrite the sine-squared factor first, then set u equal to cosine theta; retain the negative differential before taking partial fractions.

21. Solve the following linear programming problem graphically: Maximise $Z = 34x + 45y$ under the following constraints $x + y \leq 300$, $2x + 3y \leq 70$, $x \geq 0$, $y \geq 0$. [4 marks]

Solution:

Concept:

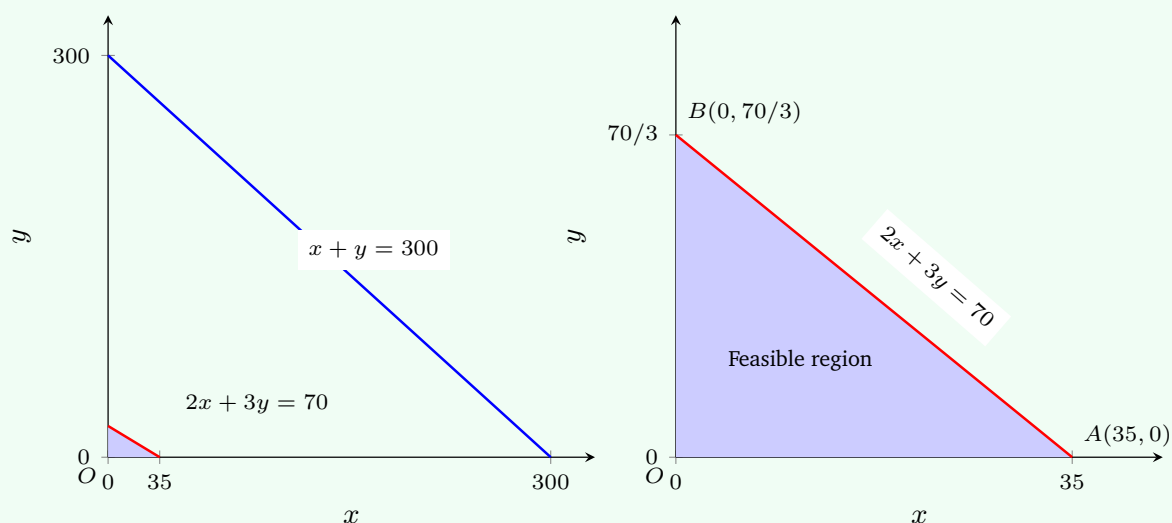
- A feasible region is the common region satisfying every constraint, including $x \geq 0$ and $y \geq 0$.
- Plot each boundary line from its intercepts and use the origin to identify the required half-plane for an inequality of the form $ax + by \leq c$, with $c > 0$.
- A linear objective function attains its maximum on a nonempty bounded feasible polygon at a vertex. Evaluate $Z = 34x + 45y$ at each vertex.

Step 1: Plot the boundary lines and select the half-planes The line $x + y = 300$ has intercepts $(300, 0)$ and $(0, 300)$. The line $2x + 3y = 70$ has intercepts

$$A = (35, 0), \quad B = \left(0, \frac{70}{3}\right).$$

For both inequalities the origin satisfies the strict inequality, so select the side containing the origin. The nonnegativity conditions restrict the graph to the first quadrant. The left graph shows both boundary lines; the right graph enlarges the region near the origin so its vertices can be read clearly.

Quick Tip to Solve: Before comparing vertices, check whether one constraint is redundant: here $2x + 3y$ at most 70 forces $x + y$ at most 35 in the first quadrant.



Both boundary lines and an enlarged view of the shaded feasible triangle OAB .

Step 2: Identify the feasible polygon In the first quadrant,

$$2(x + y) \leq 2x + 3y \leq 70 \implies x + y \leq 35 < 300.$$

Therefore every point satisfying the second inequality and nonnegativity also satisfies $x + y \leq 300$. The first inequality imposes no additional restriction on the feasible region. Thus the common region is the shaded triangle OAB , including its boundary, with vertices

$$O = (0, 0), \quad A = (35, 0), \quad B = \left(0, \frac{70}{3}\right).$$

It is nonempty and bounded, so the corner-point method applies. The intersection of the two oblique boundary lines lies outside the first quadrant and contributes no feasible vertex.

Step 3: Evaluate the objective function at every vertex The objective values are

Vertex (x, y)	$Z = 34x + 45y$
$(0, 0)$	0
$(35, 0)$	$34(35) = 1190$
$(0, 70/3)$	$45(70/3) = 1050$

The largest value is at A . This point satisfies both inequalities: $35 + 0 \leq 300$ and $2(35) + 3(0) = 70$, as well as both nonnegativity conditions. Since the objective values at the three vertices are distinct, the maximum is attained uniquely at A .

Step 4: Conclusion

$$Z_{\max} = 1190 \quad \text{at} \quad x = 35, y = 0.$$

Quick Tip to Solve: Before comparing vertices, check whether one constraint is redundant: here $2x + 3y$ at most 70 forces $x + y$ at most 35 in the first quadrant.

22. Find the value of x such that the points $A(3, 2, 1)$, $B(4, x, 5)$, $C(4, 2, -2)$ and $D(6, 5, -1)$ are coplanar.

[4 marks]

Solution:

Concept:

- Four points are coplanar if the three displacement vectors drawn from one of them have zero scalar triple product. This product is the signed volume of the parallelepiped formed by the vectors.
- For vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, the scalar triple product is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$, evaluated as the determinant with their components as rows.

Step 1: Form three vectors from the common point Take $A(3, 2, 1)$ as the common initial point. Subtract its coordinates from those of the other points, keeping the same order of subtraction throughout:

$$\overrightarrow{AB} = (4 - 3, x - 2, 5 - 1) = (1, x - 2, 4),$$

$$\overrightarrow{AC} = (4 - 3, 2 - 2, -2 - 1) = (1, 0, -3),$$

$$\overrightarrow{AD} = (6 - 3, 5 - 2, -1 - 1) = (3, 3, -2).$$

These vectors lie in a common plane precisely when their scalar triple product vanishes.

Step 2: Apply and simplify the coplanarity condition Thus the required condition is

$$\begin{vmatrix} 1 & x - 2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0.$$

Expanding along the first row, with alternating signs, gives

$$\begin{aligned} 1\{0(-2) - (-3)(3)\} - (x - 2)\{1(-2) - (-3)(3)\} \\ + 4\{1(3) - 0(3)\} &= 0, \\ 9 - 7(x - 2) + 12 &= 0, \\ 35 - 7x &= 0. \end{aligned}$$

Solving this linear equation gives $x = 5$.

Step 3: Check the value directly At $x = 5$, the vector $\overrightarrow{AB}$ is $(1, 3, 4)$. Also,

$$\overrightarrow{AC} \times \overrightarrow{AD} = (9, -7, 3).$$

Its scalar product with $\overrightarrow{AB}$ is $9 - 21 + 12 = 0$, as required. This cross product is nonzero, so A, C, D determine a definite plane and the condition really places B in that plane.

Step 4: Conclusion The four given points are coplanar when $x = 5$.

Quick Tip to Solve: Subtract one common point from the other three before forming the determinant; this reduces the coplanarity test to a three-by-three determinant.

23. Find the general solution of the differential equation $y dx - (x + 2y^2) dy = 0$.

[4 marks]

Solution:

Concept:

- A differential equation linear in x , with y as the independent variable, has the form $\frac{dx}{dy} + P(y)x = Q(y)$.
- Its integrating factor is $I(y) = e^{\int P(y) dy}$. Multiplication by this factor makes the left side the derivative $\frac{d}{dy}(Ix)$, which can then be integrated.

Step 1: Identify the dependent variable for the linear form The given equation is

$$y dx - (x + 2y^2) dy = 0.$$

Treat x as a function of y . On an interval where $y \neq 0$, rearrangement and division by y give

$$y \frac{dx}{dy} = x + 2y^2, \quad \frac{dx}{dy} - \frac{x}{y} = 2y.$$

This is linear in x , even though it would not have the usual linear form if we treated y as the dependent variable. Here $P(y) = -1/y$ and $Q(y) = 2y$.

Step 2: Obtain the integrating factor and integrate The integrating factor is

$$e^{\int -\frac{1}{y} dy} = e^{-\log |y|} = \frac{1}{|y|}.$$

On either interval $y > 0$ or $y < 0$, multiplying an integrating factor by a nonzero constant does not affect the method, so we may use $I(y) = 1/y$. Multiplying the linear equation by $1/y$ yields

$$\frac{1}{y} \frac{dx}{dy} - \frac{x}{y^2} = 2 \implies \frac{d}{dy} \left(\frac{x}{y} \right) = 2.$$

Integrating with respect to y ,

$$\frac{x}{y} = 2y + C, \quad x = 2y^2 + Cy,$$

where C is an arbitrary real constant.

Step 3: Verify in the original differential equation Differentiating $x = 2y^2 + Cy$ gives $dx = (4y + C) dy$. Therefore,

$$\begin{aligned} y dx - (x + 2y^2) dy &= \{y(4y + C) - (4y^2 + Cy)\} dy \\ &= 0. \end{aligned}$$

Thus the family satisfies the original equation. The division by y used above excludes the separate constant solution $y = 0$; substituting $y = 0$ and $dy = 0$ directly in the original equation confirms this additional solution.

Step 4: Conclusion The general solution is $x = 2y^2 + Cy$, with arbitrary constant C . The original equation also admits the separate solution $y = 0$.

Quick Tip to Solve: When the equation is nonlinear in y , check whether swapping the dependent and independent variables makes it linear in x .

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24(a). Find the coordinates of the point where the line through the points $(3, -4, -5)$ and $(2, -3, 1)$, crosses the plane determined by the points $(1, 2, 3)$, $(4, 2, -3)$ and $(0, 4, 3)$. [6 marks]

Solution:

Concept:

- A plane through a point (x_0, y_0, z_0) with nonzero normal vector (a, b, c) has equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$. The normal is perpendicular to both independent directions lying in the plane.
- The line through position vectors $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{r} = \mathbf{u} + t(\mathbf{v} - \mathbf{u})$, where t is a real parameter. Substitution in the plane equation determines the parameter at the intersection.

Step 1: Find two directions in the required plane Denote the three given points by $P = (1, 2, 3)$, $Q = (4, 2, -3)$, and $R = (0, 4, 3)$. The vectors joining P to the other two points are

$$\overrightarrow{PQ} = (3, 0, -6), \quad \overrightarrow{PR} = (-1, 2, 0).$$

These vectors are not parallel, so the three points determine a unique plane. Let its normal vector be (a, b, c) . Perpendicularity to these two vectors gives

$$3a - 6c = 0, \quad -a + 2b = 0.$$

Thus $a = 2c$ and $b = c$. Choosing $c = 1$, we may take the normal vector as $(2, 1, 1)$.

Step 2: Obtain and check the plane equation Using the point P , the equation is

$$2(x - 1) + (y - 2) + (z - 3) = 0,$$

or $2x + y + z = 7$. As a check, substitution of Q gives $8 + 2 - 3 = 7$, and substitution of R gives $0 + 4 + 3 = 7$, so all three specified points lie on this plane.

Step 3: Express the given line in parametric form Let $U = (3, -4, -5)$ and $V = (2, -3, 1)$. A direction vector along their line is $V - U = (-1, 1, 6)$. Hence every point on the line can be written as

$$x = 3 - t, \quad y = -4 + t, \quad z = -5 + 6t.$$

The choices $t = 0$ and $t = 1$ recover U and V , respectively, verifying the signs in the parametrisation.

Step 4: Determine the intersection parameter and verify it Substituting the line coordinates in $2x + y + z = 7$,

$$2(3 - t) + (-4 + t) + (-5 + 6t) = 7.$$

Collecting terms gives $-3 + 5t = 7$, so $t = 2$. The coordinates at this parameter are $x = 3 - 2 = 1$, $y = -4 + 2 = -2$, and $z = -5 + 12 = 7$. Their plane expression is $2(1) - 2 + 7 = 7$, as required. Moreover, the scalar product of the line direction and plane normal is $-2 + 1 + 6 = 5 \neq 0$, so the line crosses the plane at a unique point.

Quick Tip to Solve: Use the two differences from the same plane point to find its normal, then substitute a parametric line to avoid solving three simultaneous coordinate equations.

Step 5: Conclusion The given line crosses the plane at $(1, -2, 7)$.

Quick Tip to Solve: Use the two differences from the same plane point to find its normal, then substitute a parametric line to avoid solving three simultaneous coordinate equations.

OR

24(b). A variable plane which remains at a constant distance $3p$ from the origin cuts the coordinate axes at A, B, C . Show that the locus of the centroid of triangle ABC is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$. [6 marks]

Solution:

Concept:

- A plane cutting the coordinate axes at $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, with finite nonzero intercepts, has intercept equation $X/a + Y/b + Z/c = 1$.
- The centroid of a triangle is obtained by averaging the three vertex coordinates. The distance from the origin to a plane $lX + mY + nZ + d = 0$ is $|d|/\sqrt{l^2 + m^2 + n^2}$.

Step 1: Represent the variable plane by its intercepts Let the variable plane cut the coordinate axes at $A = (a, 0, 0)$, $B = (0, b, 0)$, and $C = (0, 0, c)$. Here a, b, c are finite and nonzero, since the plane cuts all three axes and has a positive distance from the origin. Write its equation as

$$\frac{X}{a} + \frac{Y}{b} + \frac{Z}{c} - 1 = 0.$$

Capital letters X, Y, Z denote an arbitrary point on the plane; lower-case x, y, z will denote the centroid coordinates. This distinction lets us eliminate the intercepts without confusing the plane point with the centroid.

Step 2: Use the specified distance from the origin Applying the point-to-plane distance formula at the origin gives

$$\text{dist}(O, \text{plane}) = \frac{1}{\sqrt{1/a^2 + 1/b^2 + 1/c^2}}.$$

The given constant distance is $3p$, so $p > 0$. Therefore

$$\frac{1}{\sqrt{1/a^2 + 1/b^2 + 1/c^2}} = 3p.$$

Squaring both positive sides and taking reciprocals yields

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{9p^2}.$$

Signed intercepts cause no difficulty because the distance depends on their squares.

Step 3: Express the intercepts through the centroid If the centroid of triangle ABC is $G = (x, y, z)$,

Quick Tip to Solve: Replace each axis intercept by three times the corresponding centroid coordinate before simplifying the plane-distance relation.

then the coordinate-average formula gives

$$x = \frac{a + 0 + 0}{3} = \frac{a}{3}, \quad y = \frac{0 + b + 0}{3} = \frac{b}{3}, \quad z = \frac{0 + 0 + c}{3} = \frac{c}{3}.$$

Consequently $a = 3x$, $b = 3y$, and $c = 3z$. Substituting these expressions in the distance relation gives

$$\frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2} = \frac{1}{9p^2}.$$

Multiplication by 9 eliminates the intercepts and gives the required equation in the centroid coordinates.

Step 4: Check that the equation describes the full locus Conversely, choose any point (x, y, z) with nonzero coordinates satisfying the resulting equation. Construct the plane with intercepts $a = 3x$, $b = 3y$, and $c = 3z$. Its distance from the origin is

$$\frac{1}{\sqrt{\frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2}}} = \frac{1}{\sqrt{1/(9p^2)}} = 3p.$$

Its intercept triangle has centroid (x, y, z) . Thus every point satisfying the equation is realised by an admissible plane, and the relation is sufficient as well as necessary.

Step 5: Conclusion Hence the locus of the centroid is $\boxed{\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}}$, with $xyz \neq 0$.

Quick Tip to Solve: Replace each axis intercept by three times the corresponding centroid coordinate before simplifying the plane-distance relation.

25. Find the particular solution of the differential equation $(x - y)\frac{dy}{dx} = (x + 2y)$, given that $y = 0$ when $x = 1$. [6 marks]

Solution:

Concept:

- A homogeneous first-order differential equation has its right-hand side expressible as a function of y/x . Put $y = vx$, where v is a function of x , so that $dy/dx = v + x dv/dx$.
- Separate the variables and integrate. For a quadratic denominator, complete the square to obtain an inverse tangent integral; use the initial condition to determine the integration constant.
- The solution through $(1, 0)$ is considered on an interval containing $x = 1$. On this interval we may take $x > 0$; the original equation also requires $x - y \neq 0$ wherever dy/dx is finite.

Step 1: Recognise homogeneity and substitute From the printed equation,

$$\frac{dy}{dx} = \frac{x + 2y}{x - y} = \frac{1 + 2(y/x)}{1 - y/x}.$$

Quick Tip to Solve: After putting $y = vx$, split $1 - v$ as $-\frac{1}{2}(2v + 1) + \frac{3}{2}$; this immediately separates the logarithm and inverse tangent integrals.

Numerator and denominator have the same degree. Thus set $y = vx$. Substituting the derivative gives

$$(1 - v) \left(v + x \frac{dv}{dx} \right) = 1 + 2v.$$

Expanding and collecting terms produces

$$x(1 - v) \frac{dv}{dx} = 1 + v + v^2.$$

Since $v^2 + v + 1 = (v + 1/2)^2 + 3/4 > 0$, division by this quadratic loses no real constant-ratio solution.

Step 2: Separate variables and integrate the quadratic expression Separation gives

$$\frac{1 - v}{v^2 + v + 1} dv = \frac{dx}{x}.$$

Write the numerator in terms of the derivative of the denominator:

$$1 - v = -\frac{1}{2}(2v + 1) + \frac{3}{2}.$$

Hence, using completion of the square,

$$\begin{aligned} \int \frac{1 - v}{v^2 + v + 1} dv &= -\frac{1}{2} \ln(v^2 + v + 1) + \frac{3}{2} \int \frac{dv}{(v + 1/2)^2 + 3/4} \\ &= -\frac{1}{2} \ln(v^2 + v + 1) + \sqrt{3} \tan^{-1} \left(\frac{2v + 1}{\sqrt{3}} \right). \end{aligned}$$

Therefore this expression equals $\ln x + C$ on the interval through the specified initial point.

Step 3: Apply the initial condition and restore the original variables At $x = 1, y = 0$, we have $v = 0$. Since $\tan^{-1}(1/\sqrt{3}) = \pi/6$, substitution yields

$$C = \sqrt{3} \frac{\pi}{6} = \frac{\pi}{2\sqrt{3}}.$$

Now $v = y/x$ and, for $x > 0$,

$$\ln(v^2 + v + 1) = \ln(x^2 + xy + y^2) - 2 \ln x.$$

Substitution cancels the $\ln x$ terms. Multiplying by -2 gives an implicit relation in x, y . The inverse tangent is its continuous principal branch near $(1, 0)$.

Step 4: Verify the implicit relation Let $D = x^2 + xy + y^2$. Differentiating the resulting relation yields

$$\frac{2x + y + (x + 2y)y'}{D} = \frac{3(xy' - y)}{D}.$$

Cancelling D and rearranging gives $2(x + 2y) = 2(x - y)y'$, which is the required differential equation. At $(1, 0)$, the logarithm is zero and the inverse tangent term is $\pi/\sqrt{3}$, so the initial condition is also satisfied.

Quick Tip to Solve: After putting $y = vx$, split $1 - v$ as $-\frac{1}{2}(2v + 1) + \frac{3}{2}$; this immediately separates the logarithm and inverse tangent integrals.

Step 5: Conclusion The required particular solution, on the interval through $(1, 0)$, is

$$\ln(x^2 + xy + y^2) = 2\sqrt{3} \tan^{-1} \left(\frac{x + 2y}{\sqrt{3}x} \right) - \frac{\pi}{\sqrt{3}}.$$

Quick Tip to Solve: After putting $y = vx$, split $1 - v$ as $-\frac{1}{2}(2v + 1) + \frac{3}{2}$; this immediately separates the logarithm and inverse tangent integrals.

26(a). Using the method of integration, find the area of the triangle ABC , coordinates of whose vertices are $A(4, 1)$, $B(6, 6)$ and $C(8, 4)$. [6 marks]

Solution:

Concept:

- A non-vertical line through (x_1, y_1) and (x_2, y_2) has equation $y - y_1 = m(x - x_1)$, where $m = (y_2 - y_1)/(x_2 - x_1)$.
- The area between two curves over $[a, b]$ is $\int_a^b (y_{\text{upper}} - y_{\text{lower}}) dx$. When the upper boundary changes at a vertex, split the integral at that vertex's x -coordinate.
- For this triangle, the side AC is the lower boundary throughout $4 \leq x \leq 8$. The upper boundary is AB up to $x = 6$, and BC thereafter.

Step 1: Find the equations of all three sides Using $A(4, 1)$, $B(6, 6)$ and $C(8, 4)$, the slope of AB is

$$m_{AB} = \frac{6 - 1}{6 - 4} = \frac{5}{2}.$$

Thus $y - 1 = \frac{5}{2}(x - 4)$, or $y = \frac{5}{2}x - 9$. Similarly,

$$m_{BC} = \frac{4 - 6}{8 - 6} = -1, \quad BC : y = 12 - x,$$
$$m_{AC} = \frac{4 - 1}{8 - 4} = \frac{3}{4}, \quad AC : y = \frac{3}{4}x - 2.$$

Each expression gives the stated endpoint coordinates, so the three boundary equations are consistent with the vertices.

Step 2: Choose the vertical-strip boundaries The vertices occur in increasing order of x at 4, 6, 8. Between 4 and 6, a vertical strip extends from AC to AB . Its height is

$$\left(\frac{5}{2}x - 9 \right) - \left(\frac{3}{4}x - 2 \right) = \frac{7}{4}x - 7 = \frac{7}{4}(x - 4) \geq 0.$$

Between 6 and 8, the upper side is BC , so the height becomes

$$(12 - x) - \left(\frac{3}{4}x - 2 \right) = 14 - \frac{7}{4}x = \frac{7}{4}(8 - x) \geq 0.$$

Consequently the area must be the sum of two integrals. Using AB as the upper boundary all the way

Quick Tip to Solve: Order the vertices by their x -coordinates and split at $x = 6$; the lower side stays AC , while the upper side changes from AB to BC .

to $x = 8$ would extend outside the triangle.

Step 3: Integrate over the interval from the first vertex to the middle vertex The left portion has area

$$\begin{aligned} S_1 &= \int_4^6 \left(\frac{7}{4}x - 7 \right) dx \\ &= \left[\frac{7}{8}x^2 - 7x \right]_4^6 \\ &= \left(\frac{63}{2} - 42 \right) - (14 - 28) \\ &= -\frac{21}{2} + 14 = \frac{7}{2}. \end{aligned}$$

The positive result agrees with the nonnegative strip height on this interval.

Step 4: Integrate over the remaining interval and combine the areas For the right portion,

$$\begin{aligned} S_2 &= \int_6^8 \left(14 - \frac{7}{4}x \right) dx \\ &= \left[14x - \frac{7}{8}x^2 \right]_6^8 \\ &= (112 - 56) - \left(84 - \frac{63}{2} \right) \\ &= 56 - \frac{105}{2} = \frac{7}{2}. \end{aligned}$$

These two portions have disjoint interiors and together cover the triangle. Thus their areas add. As a check on the limits and signs, the strip height is zero at $x = 4$ and $x = 8$, and both expressions give height $7/2$ at $x = 6$; the two integrations join continuously at the middle vertex.

Step 5: Conclusion By integration, the area of triangle ABC is

$$S = S_1 + S_2 = \frac{7}{2} + \frac{7}{2} = 7 \text{ square units}.$$

Quick Tip to Solve: Order the vertices by their x -coordinates and split at $x = 6$; the lower side stays AC , while the upper side changes from AB to BC .

26(b). Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

[6 marks]

Solution:

Concept:

- The area between two curves is the integral of the upper ordinate minus the lower ordinate over their interval of intersection.
- For the given parabola, $y_P = \frac{3}{4}x^2$, and for the line, $y_L = \frac{3}{2}x + 6$. The limits must be obtained by equating these ordinates.
- An area is non-negative; the sign of $y_L - y_P$ determines which curve bounds the region above.

Step 1: Find the intersection points Substituting $y = \frac{3}{4}x^2$ into the equation of the line gives

$$3x - \frac{3}{2}x^2 + 12 = 0 \implies x^2 - 2x - 8 = 0.$$

Factoring, $(x - 4)(x + 2) = 0$, so the two abscissae are $x = -2$ and $x = 4$. Their ordinates, found from the parabola, are 3 and 12, respectively. Thus the boundaries meet at $(-2, 3)$ and $(4, 12)$; the enclosed region extends across the interval $[-2, 4]$.

Step 2: Identify the upper boundary throughout the interval The difference of the ordinates is

$$y_L - y_P = \frac{3}{2}x + 6 - \frac{3}{4}x^2 = \frac{3}{4}(4 - x)(x + 2).$$

Both factors $4 - x$ and $x + 2$ are non-negative for $-2 \leq x \leq 4$. Therefore the straight line is above the parabola throughout the interval, with equality only at the endpoints. Each vertical strip has height $y_L - y_P$ and width dx , so a single integral suffices; there is no reversal of upper and lower boundaries inside the region.

Step 3: Integrate the difference of the ordinates The required integral is

$$\begin{aligned} A &= \int_{-2}^4 \left(\frac{3}{2}x + 6 - \frac{3}{4}x^2 \right) dx \\ &= \left[\frac{3}{4}x^2 + 6x - \frac{1}{4}x^3 \right]_{-2}^4 \\ &= (12 + 24 - 16) - (3 - 12 + 2) \\ &= 20 - (-7). \end{aligned}$$

The antiderivative differentiates to the strip height, and the lower-limit value must be subtracted as a whole. In particular, its negative value contributes positively to the area; integrating the line or parabola separately without subtracting their ordinates would not give the enclosed area.

Step 4: Conclusion The area enclosed between the parabola and the straight line is 27 square units.

Quick Tip to Solve: Factor the ordinate difference to check that the line stays above the parabola before integrating from -2 to 4.

27(a). Consider $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \rightarrow \mathbb{R} - \left\{\frac{4}{3}\right\}$ given by $f(x) = \frac{4x+3}{3x+4}$. Show that f is bijective. Find the inverse of f and hence find $f^{-1}(0)$ and x such that $f^{-1}(x) = 2$. [6 marks]

Solution:

Concept:

- A function is bijective when it is both injective (equal images imply equal inputs) and surjective (each element of the codomain has a preimage).
- The inverse of a bijective function is obtained by solving $y = f(x)$ for x ; its domain is the codomain of f .
- Here the domain is $D = \mathbb{R} \setminus \{-4/3\}$ and the codomain is $C = \mathbb{R} \setminus \{4/3\}$. All algebraic divisions must respect these exclusions.

Step 1: Check that the function takes its values in the stated codomain For $x \in D$, the denominator $3x + 4$ is nonzero. Moreover,

$$f(x) - \frac{4}{3} = \frac{3(4x+3) - 4(3x+4)}{3(3x+4)} = \frac{-7}{3(3x+4)} \neq 0.$$

Hence no allowed input has image $4/3$, and $f : D \rightarrow C$ is well-defined.

Step 2: Prove that the function is injective Let $a, b \in D$ and suppose $f(a) = f(b)$. Since the denominators are nonzero, cross-multiplication gives

$$\begin{aligned}(4a+3)(3b+4) &= (4b+3)(3a+4), \\ 12ab + 16a + 9b + 12 &= 12ab + 16b + 9a + 12, \\ 7a &= 7b.\end{aligned}$$

Therefore $a = b$. Equal images can only arise from equal inputs, proving that f is one-one.

Step 3: Prove that the function is surjective and obtain its inverse Take any $y \in C$, so $3y - 4 \neq 0$. Solving the equation $y = f(x)$ gives

$$y(3x+4) = 4x+3 \implies x = \frac{3-4y}{3y-4}.$$

This candidate belongs to D , because

$$x + \frac{4}{3} = \frac{3(3-4y) + 4(3y-4)}{3(3y-4)} = \frac{-7}{3(3y-4)} \neq 0.$$

To verify the preimage explicitly, substitution gives

$$4x+3 = \frac{-7y}{3y-4}, \quad 3x+4 = \frac{-7}{3y-4}, \quad f(x) = y.$$

Thus every element of C is attained, proving surjectivity. Together with injectivity this establishes

Quick Tip to Solve: After solving $y=f(x)$, check the proposed preimage avoids $-4/3$; for f inverse of x equal to 2 , compute $f(2)$ directly.

bijectivity. Consequently the inverse is

$$f^{-1} : C \rightarrow D, \quad f^{-1}(y) = \frac{3-4y}{3y-4}.$$

Step 4: Evaluate the two requested inverse-function quantities Since $0 \in C$, substitution into the inverse formula gives

$$f^{-1}(0) = \frac{3-4(0)}{3(0)-4} = -\frac{3}{4}.$$

For the second requirement, $f^{-1}(x) = 2$ with $x \in C$ yields

$$\frac{3-4x}{3x-4} = 2 \implies 3-4x = 6x-8 \implies 10x = 11.$$

The value $11/10$ is allowed in C , and it agrees with the direct check $f(2) = (8+3)/(6+4) = 11/10$.

Step 5: Conclusion The function is bijective, with $f^{-1}(x) = \frac{3-4x}{3x-4}$ for $x \neq 4/3$. Hence $f^{-1}(0) = -3/4$, and $f^{-1}(x) = 2$ gives $x = 11/10$.

Quick Tip to Solve: After solving $y=f(x)$, check the proposed preimage avoids $-4/3$; for f inverse of x equal to 2, compute $f(2)$ directly.

OR

27(b). Let $A = \mathbb{Q} \times \mathbb{Q}$ and let $*$ be a binary operation on A defined by $(a, b) * (c, d) = (ac, b + ad)$ for $(a, b), (c, d) \in A$. Determine, whether $*$ is commutative and associative. Then, with respect to $*$ on A : (i) Find the identity element in A . (ii) Find the invertible elements of A . [6 marks]

Solution:

Concept:

- Commutativity means $u * v = v * u$ for every pair of elements; one counterexample disproves it.
- Associativity means $(u * v) * w = u * (v * w)$ for every triple. An identity must act on both sides, and an inverse must give the identity in both orders.

Step 1: Check closure and commutativity Since products and sums of rational numbers are rational, $(ac, b + ad) \in \mathbb{Q} \times \mathbb{Q}$, so the operation is closed on A . Reversing the order gives

$$(c, d) * (a, b) = (ca, d + cb).$$

Although the first coordinates agree, the second need not. For example,

$$(2, 0) * (1, 1) = (2, 2), \quad (1, 1) * (2, 0) = (2, 1).$$

These are distinct elements of A , so commutativity fails.

Step 2: Check associativity for arbitrary elements Let $(a, b), (c, d), (e, f) \in A$. Applying the given rule

Quick Tip to Solve: For this operation, solve the first coordinate of the inverse equation first: it immediately gives both the condition a nonzero and the reciprocal.

twice on the left gives

$$\begin{aligned}((a, b) * (c, d)) * (e, f) &= (ac, b + ad) * (e, f) \\ &= (ace, b + ad + acf).\end{aligned}$$

On the right, first $(c, d) * (e, f) = (ce, d + cf)$, hence

$$\begin{aligned}(a, b) * ((c, d) * (e, f)) &= (a, b) * (ce, d + cf) \\ &= (ace, b + a(d + cf)) \\ &= (ace, b + ad + acf).\end{aligned}$$

Both coordinates agree for every triple, which proves associativity.

Step 3: (i) Determine the two-sided identity Suppose the identity is (u, v) . The condition

$$(u, v) * (a, b) = (ua, v + ub) = (a, b)$$

for every rational a, b forces $u = 1$ and $v = 0$. Conversely,

$$(1, 0) * (a, b) = (a, b), \quad (a, b) * (1, 0) = (a, b).$$

Thus the candidate works in both orders.

Step 4: (ii) Determine all invertible elements If (c, d) is an inverse of (a, b) , then

$$(a, b) * (c, d) = (ac, b + ad) = (1, 0).$$

Therefore $ac = 1$, so necessarily $a \neq 0$, and then

$$c = \frac{1}{a}, \quad d = -\frac{b}{a}.$$

Both numbers are rational whenever a, b are rational and $a \neq 0$. To verify the other order as well,

$$\left(\frac{1}{a}, -\frac{b}{a}\right) * (a, b) = \left(1, -\frac{b}{a} + \frac{b}{a}\right) = (1, 0).$$

The original order also yields $(1, b + a(-b/a)) = (1, 0)$. If $a = 0$, the first coordinate of its product with any element is zero, so it cannot yield the identity.

Step 5: Conclusion The operation is not commutative but is associative. Its identity is $(1, 0)$. The invertible elements are precisely $\{(a, b) \in \mathbb{Q} \times \mathbb{Q} : a \neq 0\}$, with $(a, b)^{-1} = (1/a, -b/a)$.

Quick Tip to Solve: For this operation, solve the first coordinate of the inverse equation first: it immediately gives both the condition a nonzero and the reciprocal.

28. AB is the diameter of a circle and C is any point on the circle. Show that the area of triangle ABC is maximum, when it is an isosceles triangle. [6 marks]

Solution:

Concept:

- The angle subtended by a diameter at a point on the circle is a right angle, so $\angle ACB = 90^\circ$.
- For a right triangle with perpendicular sides x, y , area is $S = \frac{1}{2}xy$, and Pythagoras gives $x^2 + y^2 = AB^2$.
- An interior stationary point of a differentiable area function is found by setting its derivative equal to zero. The sign of the derivative and the endpoint areas establish the global maximum.

Step 1: Express the area using one variable Let the radius of the given circle be $r > 0$, so its fixed diameter is $AB = 2r$. For a nondegenerate triangle, let $AC = x$ and $BC = y$, where $0 < x < 2r$. By the angle-in-a-semicircle theorem, the triangle is right angled at C . Thus

$$x^2 + y^2 = 4r^2, \quad y = \sqrt{4r^2 - x^2}.$$

Its area as a function of x is consequently

$$S(x) = \frac{1}{2}x\sqrt{4r^2 - x^2}, \quad 0 < x < 2r.$$

As C approaches an endpoint of the diameter, one of these perpendicular sides approaches zero and the area approaches zero. We can therefore include the endpoints by continuity with $S(0) = S(2r) = 0$.

Step 2: Differentiate and find the stationary point Using the product rule and chain rule,

$$\begin{aligned} S'(x) &= \frac{1}{2} \left(\sqrt{4r^2 - x^2} - \frac{x^2}{\sqrt{4r^2 - x^2}} \right) \\ &= \frac{2r^2 - x^2}{\sqrt{4r^2 - x^2}}. \end{aligned}$$

The denominator is positive throughout $0 < x < 2r$. Hence the stationary-point equation is

$$2r^2 - x^2 = 0, \quad x = \sqrt{2}r,$$

where the positive root is taken because x is a length. This value lies strictly inside the allowed interval.

Step 3: Establish a maximum and identify the triangle For $0 < x < \sqrt{2}r$, the numerator $2r^2 - x^2$ is positive, so $S'(x) > 0$. For $\sqrt{2}r < x < 2r$, it is negative, so $S'(x) < 0$. The area therefore increases up to the stationary point and decreases thereafter. Together with the zero endpoint areas, this proves that the stationary point gives the global maximum. At that point,

$$y = \sqrt{4r^2 - 2r^2} = \sqrt{2}r = x.$$

Thus $AC = BC$, so the maximizing triangle is isosceles (and right angled at C). Conversely, if $AC = BC$, Pythagoras gives $2x^2 = 4r^2$, hence precisely the stationary-point value. The corresponding

Quick Tip to Solve: Use the right angle at C first: the fixed diameter becomes the hypotenuse, leaving just one variable in the area function.

area is

$$S_{\max} = \frac{1}{2}(\sqrt{2}r)(\sqrt{2}r) = r^2.$$

Points on either semicircle giving equal sides have this same area, since reflection across the diameter preserves all lengths.

Step 4: Conclusion The area of $\triangle ABC$ is maximum exactly when $AC = BC$, so the triangle is isosceles. Its maximum area is $r^2 = AB^2/4$.

Quick Tip to Solve: Use the right angle at C first: the fixed diameter becomes the hypotenuse, leaving just one variable in the area function.

29. If $A = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, find A^{-1} . Hence using A^{-1} solve the system of equations $2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$. [6 marks]

Solution:

Concept:

- A square matrix A is invertible if and only if its determinant $|A|$ is nonzero. Its inverse is $A^{-1} = \frac{\text{adj } A}{|A|}$.
- The cofactor of entry a_{ij} is $C_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the determinant obtained by deleting row i and column j . The adjoint is the transpose of the cofactor matrix.
- A system written as $AX = B$, with $X = (x, y, z)^T$, has the unique solution $X = A^{-1}B$ when A is invertible.

Step 1: Calculate the determinant and establish invertibility Expand along the first row, keeping the alternating signs:

$$\begin{aligned}|A| &= 2\{2(-2) - (-4)(1)\} - (-3)\{3(-2) - (-4)(1)\} \\ &\quad + 5\{3(1) - 2(1)\} \\ &= 2(0) + 3(-2) + 5(1) = -1.\end{aligned}$$

Since $|A| \neq 0$, the inverse exists. We can therefore use the adjoint formula and subsequently multiply the equations by A^{-1} .

Step 2: Find all cofactors and transpose their matrix The first row of cofactors is

$$C_{11} = 0, \quad C_{12} = -(-6 + 4) = 2, \quad C_{13} = 3 - 2 = 1.$$

For the second row, deleting each indicated row and column gives

$$\begin{aligned}C_{21} &= -\{(-3)(-2) - 5(1)\} = -1, \\ C_{22} &= 2(-2) - 5(1) = -9, \\ C_{23} &= -\{2(1) - (-3)(1)\} = -5.\end{aligned}$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; then multiply each inverse row by the constants column in its original order.

The third row is

$$C_{31} = (-3)(-4) - 5(2) = 2,$$

$$C_{32} = -\{2(-4) - 5(3)\} = 23,$$

$$C_{33} = 2(2) - (-3)(3) = 13.$$

Thus the cofactor matrix and its transpose are

$$C = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{pmatrix}, \quad \text{adj } A = \begin{pmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{pmatrix}.$$

Dividing every entry of the adjoint by the determinant -1 , we obtain

$$A^{-1} = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix}.$$

Step 3: Use the inverse to solve the matrix equation With the coefficients in the same order as the given equations, write

$$AX = B, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix}.$$

Multiplying on the left by the inverse gives $A^{-1}AX = A^{-1}B$, so

$$\begin{aligned} X &= \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix} \begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix} \\ &= \begin{pmatrix} -5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \end{aligned}$$

Check by substitution: $2(1) - 3(2) + 5(3) = 11$, $3(1) + 2(2) - 4(3) = -5$, and $1 + 2 - 2(3) = -3$. All three original equations are satisfied, confirming the multiplication and signs.

Step 4: Conclusion

$$A^{-1} = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix}, \quad x = 1, y = 2, z = 3.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; then multiply each inverse row by the constants column in its original order.