

CBSE Class 12 Mathematics Question Paper 2017 with Solution PDF-Set 65-2-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) **All** questions are compulsory.
- (ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 3 questions of four marks each and 3 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Questions number 1 to 4 carry 1 mark each.

1. If $|A| = 3$ and $A^{-1} = \begin{pmatrix} 3 & -1 \\ -\frac{5}{3} & \frac{2}{3} \end{pmatrix}$, then write the $\text{adj } A$.

[1 mark]

Solution:

Concept:

- For an invertible square matrix A , the adjoint satisfies $A^{-1} = \frac{\text{adj } A}{|A|}$.
- Thus $\text{adj } A = |A|A^{-1}$; the determinant multiplies every entry of the inverse.

Step 1: Use the inverse-adjoint identity The given determinant is nonzero, so the inverse-adjoint relation applies. Multiplying both sides of $A^{-1} = \frac{\text{adj } A}{|A|}$ by the scalar $|A| = 3$ gives $\text{adj } A = 3A^{-1}$.

Step 2: Multiply each entry Using the printed inverse,

$$\text{adj } A = 3 \begin{pmatrix} 3 & -1 \\ -\frac{5}{3} & \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 9 & -3 \\ -5 & 2 \end{pmatrix}.$$

The negative signs in the second column of the first row and the first column of the second row are preserved.

Step 3: Conclusion $\text{adj } A = \begin{pmatrix} 9 & -3 \\ -5 & 2 \end{pmatrix}$.

Quick Tip to Solve: Multiply the inverse by the determinant, entry by entry; do not transpose it again.

2. For what value of k is the function $f(x) = \begin{cases} \frac{\sin 5x}{3x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ continuous at $x = 0$? [1 mark]

Solution:

Concept:

- Continuity at $x = 0$ requires $\lim_{x \rightarrow 0} f(x) = f(0) = k$.
- The standard radian limit is $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$, and cosine is continuous.

Step 1: Evaluate the limiting expression For $x \neq 0$, rewrite the sine term to match the standard limit:

$$\frac{\sin 5x}{3x} = \frac{5}{3} \frac{\sin 5x}{5x}.$$

As $x \rightarrow 0$, also $5x \rightarrow 0$, and this quotient tends to $5/3$ from both sides.

Quick Tip to Solve: First make the sine argument and denominator identical, then add the cosine limit.

Step 2: Apply the continuity condition Adding the cosine limit gives

$$\lim_{x \rightarrow 0} f(x) = \frac{5}{3} + \cos 0 = \frac{5}{3} + 1 = \frac{8}{3}.$$

The separately assigned value at zero must equal this two-sided limit.

Step 3: Conclusion The required value is $k = \frac{8}{3}$.

Quick Tip to Solve: First make the sine argument and denominator identical, then add the cosine limit.

3. Evaluate: $\int_0^{2\pi} \cos^5 x \, dx$.

[1 mark]

Solution:

Concept:

- Cosine satisfies $\cos(x + \pi) = -\cos x$. An odd power preserves this change of sign.
- Split the definite integral into two intervals of length π , and translate the second interval.

Step 1: Split one complete period Write the integral as

$$I = \int_0^{\pi} \cos^5 x \, dx + \int_{\pi}^{2\pi} \cos^5 x \, dx.$$

In the second integral put $x = u + \pi$, so $dx = du$, with new limits 0 and π .

Step 2: Cancel the two contributions Since $\cos^5(u + \pi) = -\cos^5 u$,

$$I = \int_0^{\pi} \cos^5 x \, dx - \int_0^{\pi} \cos^5 u \, du = 0.$$

The integration variable is a dummy symbol, so the two displayed integrals have identical magnitudes.

Step 3: Conclusion $\int_0^{2\pi} \cos^5 x \, dx = 0$.

Quick Tip to Solve: On a full cosine period, an odd power changes sign after a shift of π , so the halves cancel.

4. Write the distance of the point $(3, -5, 12)$ from x -axis.

[1 mark]

Solution:

Concept:

- Every point on the x -axis has the form $(a, 0, 0)$, where a is real.
- The perpendicular foot of (x, y, z) on this axis is $(x, 0, 0)$; its distance is $\sqrt{y^2 + z^2}$.

Step 1: Locate the perpendicular foot For $P = (3, -5, 12)$, choose $H = (3, 0, 0)$ on the x -axis. The segment PH has zero change in its x -coordinate and is perpendicular to the axis. Thus PH is the

Quick Tip to Solve: For distance from the x -axis, square only the y - and z -coordinates and add them.

shortest distance sought.

Step 2: Calculate the distance By the three-dimensional distance formula,

$$PH = \sqrt{(3-3)^2 + (-5-0)^2 + (12-0)^2} = \sqrt{25 + 144} = \sqrt{169}.$$

The coordinate along the axis does not contribute to this perpendicular distance.

Step 3: Conclusion The distance from the x -axis is 13 units.

Quick Tip to Solve: For distance from the x -axis, square only the y - and z -coordinates and add them.

Section B (16 marks)

Questions number 5 to 12 carry 2 marks each.

5. If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$, then find the value of $|2AB|$. [2 marks]

Solution:

Concept:

- For square matrices of the same order, $|AB| = |A| |B|$.
- For a matrix M of order n , multiplying all entries by c gives $|cM| = c^n |M|$, because each of its n rows contributes one factor c .

Step 1: Evaluate the product determinant Both matrices have order 3, so their product also has order 3. Apply the determinant product property directly:

$$|AB| = |A| |B| = (-1)(3) = -3.$$

No individual entries of either matrix are needed: the supplied determinants completely determine the determinant of their product.

Step 2: Account for the scalar factor In $2AB$, the scalar 2 multiplies each of the three rows of AB . Taking one factor 2 out of each row yields

$$|2AB| = 2^3 |AB| = 8(-3) = -24.$$

Using merely $2|AB|$ would count the scalar for only one row and therefore miss the other two factors. The positive scalar factor leaves the negative sign of $|AB|$ unchanged.

Step 3: Conclusion $|2AB| = -24$.

Quick Tip to Solve: Before extracting a scalar from a determinant, raise it to the order of the matrix.

6. The radius r of a right circular cylinder is decreasing at the rate of 3 cm/min and its height h is increasing at the rate of 2 cm/min. When $r = 7$ cm and $h = 2$ cm, find the rate of change of the volume of cylinder. [Use $\pi = \frac{22}{7}$]. [2 marks]

Solution:

Concept:

- The cylinder volume is $V = \pi r^2 h$, where r and h both depend on time t .
- The product and chain rules give $\frac{dV}{dt} = \pi \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right)$. A decreasing radius has a negative derivative.

Step 1: Assign the signed rates and differentiate Take t in minutes. The given rates are $dr/dt = -3$ cm/min and $dh/dt = 2$ cm/min. Since both factors vary, differentiating $V = \pi r^2 h$ requires two terms:

$$\frac{dV}{dt} = \pi \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right).$$

Holding the radius fixed would omit the loss of volume caused by its decrease.

Step 2: Substitute the instantaneous dimensions At $r = 7$ cm and $h = 2$ cm,

$$\begin{aligned} \frac{dV}{dt} &= \frac{22}{7} [2(7)(2)(-3) + 7^2(2)] \\ &= \frac{22}{7} (-84 + 98) = \frac{22}{7} (14) = 44 \text{ cm}^3/\text{min}. \end{aligned}$$

The height contribution exceeds the radius contribution in magnitude, so the net rate is positive.

Step 3: Conclusion The cylinder volume is increasing at $44 \text{ cm}^3/\text{min}$.

Quick Tip to Solve: Write a negative rate for the shrinking radius before substituting into the product rule.

7. Find $\frac{dy}{dx}$ at $t = \frac{2\pi}{3}$ when $x = 10(t - \sin t)$ and $y = 12(1 - \cos t)$. [2 marks]

Solution:

Concept:

- For parametric equations $x = x(t)$ and $y = y(t)$, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ when $dx/dt \neq 0$.
- The derivatives of $\sin t$ and $\cos t$ are $\cos t$ and $-\sin t$, respectively, with t measured in radians.

Step 1: Differentiate both parametric equations From $x = 10(t - \sin t)$ and $y = 12(1 - \cos t)$,

$$\frac{dx}{dt} = 10(1 - \cos t), \quad \frac{dy}{dt} = 12 \sin t.$$

Thus the slope is the ratio $\frac{12 \sin t}{10(1 - \cos t)}$. The two coefficients are different and must both be retained; differentiating the constant term in y contributes zero.

Step 2: Evaluate the ratio at the given parameter At $t = 2\pi/3$, $\sin t = \sqrt{3}/2$ and $\cos t = -1/2$.

Quick Tip to Solve: Differentiate x and y with respect to t first, then substitute the angle into their ratio.

Therefore $dx/dt = 10(1 + 1/2) = 15 \neq 0$, so the derivative exists and

$$\left. \frac{dy}{dx} \right|_{t=2\pi/3} = \frac{12(\sqrt{3}/2)}{10(1 + 1/2)} = \frac{6\sqrt{3}}{15} = \frac{2\sqrt{3}}{5}.$$

Both numerator and denominator are positive at this parameter.

Step 3: Conclusion $\boxed{\left. \frac{dy}{dx} \right|_{t=2\pi/3} = \frac{2\sqrt{3}}{5}}.$

Quick Tip to Solve: Differentiate x and y with respect to t first, then substitute the angle into their ratio.

8. Show that the function f given by $f(x) = \tan^{-1}(\sin x + \cos x)$ is decreasing for all $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. [2 marks]

Solution:

Concept:

- The chain rule gives $\frac{d}{dx} \tan^{-1} u = \frac{u'}{1+u^2}$ for a differentiable real function u .
- A differentiable function is strictly decreasing on an interval if its derivative is negative at every point of that interval.

Step 1: Differentiate the composite function Put $u = \sin x + \cos x$. Since $u' = \cos x - \sin x$, the chain rule gives

$$f'(x) = \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}.$$

The denominator is strictly positive for every real x , so only the sign of the numerator must be checked.

Step 2: Establish the sign on the given interval For $\pi/4 < x < \pi/2$, we have $0 < x - \pi/4 < \pi/4$. Thus

$$\cos x - \sin x = -\sqrt{2} \sin(x - \pi/4) < 0.$$

Consequently $f'(x) < 0$ throughout the entire open interval. The derivative test therefore applies at every interior point; the zero derivative at the excluded endpoint $\pi/4$ does not affect the result.

Step 3: Conclusion Hence $f(x) = \tan^{-1}(\sin x + \cos x)$ is strictly decreasing on $(\pi/4, \pi/2)$.

Quick Tip to Solve: Check the numerator of the derivative only: the inverse-tangent denominator is always positive.

9. A line passes through the point with position vector $2\hat{i} - 3\hat{j} + 4\hat{k}$ and is perpendicular to the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 5\hat{k}) = 7$. Find the equation of the line in cartesian and vector forms. [2 marks]

Solution:

Concept:

- A line perpendicular to a plane has a direction vector parallel to the normal vector of that plane.
- A line through position vector $\vec{a}$ with direction vector $\vec{b}$ has equation $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\lambda \in \mathbb{R}$.

Step 1: Identify the point and the line direction The given point is $P(2, -3, 4)$, with position vector $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$. The plane is $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 5\hat{k}) = 7$; hence its normal is $\vec{n} = 3\hat{i} + 4\hat{j} - 5\hat{k}$. Because the required line is perpendicular to the plane, choose its direction ratios as 3, 4, -5. The constant 7 fixes the plane position, not its normal direction.

Step 2: Write the vector and cartesian equations Using the given point and normal direction,

$$\vec{r} = (2\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda(3\hat{i} + 4\hat{j} - 5\hat{k}), \quad \lambda \in \mathbb{R}.$$

Its coordinate equations are $x = 2 + 3\lambda$, $y = -3 + 4\lambda$, and $z = 4 - 5\lambda$. Eliminating the common parameter gives the symmetric cartesian form. At $\lambda = 0$, the line passes through the required point.

Step 3: Conclusion The vector form is $\vec{r} = 2\hat{i} - 3\hat{j} + 4\hat{k} + \lambda(3\hat{i} + 4\hat{j} - 5\hat{k})$. The cartesian form is $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-4}{-5}$.

Quick Tip to Solve: Use the plane normal as the line direction, and subtract each coordinate of the given point.

10. If $P(A) = 0.4$, $P(B) = p$, $P(A \cup B) = 0.6$ and A and B are given to be independent events, find the value of p . [2 marks]

Solution:

Concept:

- For any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Independence gives $P(A \cap B) = P(A)P(B)$; it does not require the intersection to be empty.

Step 1: Use independence to find the intersection probability The given probabilities are $P(A) = 0.4$, $P(B) = p$, and $P(A \cup B) = 0.6$. Since A and B are independent,

$$P(A \cap B) = 0.4p.$$

This intersection must be subtracted once when adding the individual probabilities, because outcomes belonging to both events would otherwise be counted twice.

Quick Tip to Solve: For independent events, first replace the intersection by the product before solving the union equation.

Step 2: Substitute in the addition rule and solve The addition rule now yields

$$0.6 = 0.4 + p - 0.4p,$$

$$0.2 = 0.6p,$$

$$p = \frac{0.2}{0.6} = \frac{1}{3}.$$

The resulting number lies between 0 and 1, as required for a probability. To check it, the intersection probability is $2/15$, and $2/5 + 1/3 - 2/15 = 3/5 = 0.6$, the given union probability.

Step 3: Conclusion Hence the required value is $p = \frac{1}{3}$.

Quick Tip to Solve: For independent events, first replace the intersection by the product before solving the union equation.

11. A company produces two types of goods A and B, that require gold and silver. Each unit of type A requires 3 g of silver and 1 g of gold while that of type B requires 1 g of silver and 2 g of gold. The company can procure a maximum of 9 g of silver and 8 g of gold. If each unit of type A brings a profit of ₹ 40 and that of type B ₹ 50, formulate LPP to maximize profit. [2 marks]

Solution:

Concept:

- Choose non-negative decision variables to represent the quantities of the two goods produced.
- Total resource consumption cannot exceed the available supply; the objective is total profit from both goods.

Step 1: Define the variables and objective function Let x and y denote the number of units of goods A and B, respectively. Production quantities satisfy $x \geq 0$ and $y \geq 0$. Each unit of A earns Rs. 40 and each unit of B earns Rs. 50, so the total profit, in rupees, is

$$Z = 40x + 50y.$$

The objective is to maximize this linear expression.

Step 2: Translate the material limits into inequalities Type A consumes 3 g of silver and type B consumes 1 g of silver per unit. Thus total silver consumption is $3x + y$ g. The supply is at most 9 g, giving $3x + y \leq 9$. Similarly, the gold consumption is $x + 2y$ g, because the two per-unit requirements are 1 g and 2 g. Its limit of 8 g gives $x + 2y \leq 8$. These inequalities allow unused material, which is consistent with maximum available supplies.

Step 3: Conclusion The required LPP is: maximize $Z = 40x + 50y$, subject to $3x + y \leq 9$, $x + 2y \leq 8$, $x \geq 0$, and $y \geq 0$.

Quick Tip to Solve: Build one inequality for each material; read its coefficients from the per-unit requirements of A and B.

12. Find: $\int \frac{dx}{\sqrt{3-2x-x^2}}.$

[2 marks]

Solution:

Concept:

- Complete the square to convert a quadratic radicand into the standard form $a^2 - u^2$.
- The standard formula is $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}(u/a) + C$, for $a > 0$ and $|u| < a$.

Step 1: Complete the square in the denominator Write

$$3 - 2x - x^2 = 4 - (x^2 + 2x + 1) = 4 - (x + 1)^2.$$

Thus the given integral is

$$I = \int \frac{dx}{\sqrt{4 - (x + 1)^2}}.$$

For a real-valued integrand, the radicand must be strictly positive, so $|x + 1| < 2$, or $-3 < x < 1$. This is also the interval on which the standard inverse-sine derivative is valid.

Step 2: Apply the inverse-sine formula Put $u = x + 1$, so $du = dx$. Since $4 = 2^2$,

$$I = \int \frac{du}{\sqrt{2^2 - u^2}} = \sin^{-1}\left(\frac{u}{2}\right) + C.$$

Substitute $u = x + 1$ back. A derivative check gives

$$\frac{d}{dx} \sin^{-1}\left(\frac{x+1}{2}\right) = \frac{1/2}{\sqrt{1 - (x+1)^2/4}} = \frac{1}{\sqrt{3-2x-x^2}},$$

which recovers the original integrand on its real domain.

Step 3: Conclusion $\int \frac{dx}{\sqrt{3-2x-x^2}} = \boxed{\sin^{-1}\left(\frac{x+1}{2}\right) + C}$, for $-3 < x < 1$.

Quick Tip to Solve: Complete the square before choosing a substitution; the constant becomes 4, so the inverse-sine scale is 2.

Section C (44 marks)

Questions number 13 to 23 carry 4 marks each.

13. Find the value of $\cot \left[\frac{1}{2} \left(\cos^{-1} \frac{2x}{1+x^2} + \sin^{-1} \frac{1-y^2}{1+y^2} \right) \right]$, $|x| < 1$, $y > 0$ and $xy < 1$. [4 marks]

Solution:

Concept:

- Principal values satisfy $\cos^{-1} t \in [0, \pi]$, $\sin^{-1} t \in [-\pi/2, \pi/2]$ and $\tan^{-1} t \in (-\pi/2, \pi/2)$. These ranges must be checked before removing an inverse function.

Quick Tip to Solve: Set x and y equal to tangents first, but check the principal ranges separately before simplifying either inverse function.

- If $x = \tan u$, then $2x/(1+x^2) = \sin 2u$. If $y = \tan v$, then $(1-y^2)/(1+y^2) = \cos 2v$.
- The addition formula is $\tan(u+v) = (\tan u + \tan v)/(1 - \tan u \tan v)$, provided the denominator is nonzero.

Step 1: Choose angles with the correct principal ranges Put $u = \tan^{-1} x$ and $v = \tan^{-1} y$. The restrictions $|x| < 1$ and $y > 0$ give

$$-\frac{\pi}{4} < u < \frac{\pi}{4}, \quad 0 < v < \frac{\pi}{2}.$$

Hence $\pi/2 - 2u \in (0, \pi)$ and $\pi/2 - 2v \in (-\pi/2, \pi/2)$. These are precisely the permitted principal-value ranges for the cosine and sine inverses respectively.

Step 2: Simplify both inverse trigonometric terms Using the double-angle identities and the ranges established above,

$$\begin{aligned} \cos^{-1} \frac{2x}{1+x^2} &= \cos^{-1} \left[\cos \left(\frac{\pi}{2} - 2u \right) \right] = \frac{\pi}{2} - 2u, \\ \sin^{-1} \frac{1-y^2}{1+y^2} &= \sin^{-1} \left[\sin \left(\frac{\pi}{2} - 2v \right) \right] = \frac{\pi}{2} - 2v. \end{aligned}$$

Therefore half the sum in the question is $\pi/2 - u - v$. This step also covers negative values of x ; replacing an inverse by an angle without checking its range would not be justified.

Step 3: Apply the complementary-angle and addition identities The condition $xy < 1$ gives $1 - xy > 0$. Moreover,

$$\cos(u+v) = \cos u \cos v (1 - xy) > 0,$$

because both u and v lie between $-\pi/2$ and $\pi/2$. Since $-\pi/4 < u+v < 3\pi/4$, this implies $-\pi/4 < u+v < \pi/2$. Thus the cotangent in the question is defined, and

$$\cot \left(\frac{\pi}{2} - u - v \right) = \tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}.$$

Step 4: Conclusion Substituting $\tan u = x$ and $\tan v = y$, the required value is $\boxed{\frac{x+y}{1-xy}}$.

Quick Tip to Solve: Set x and y equal to tangents first, but check the principal ranges separately before simplifying either inverse function.

14(a). Using properties of determinants show that $\begin{vmatrix} 1 & 1 & 1+x \\ 1 & 1+y & 1 \\ 1+z & 1 & 1 \end{vmatrix} = xyz + yz + zx + xy.$ [4 marks]

Solution:

Concept:

- Adding a multiple of one row to another leaves a determinant unchanged. In particular, $R_i \rightarrow R_i - R_j$ is allowed without changing its value.

Quick Tip to Solve: Subtract the first row from the other two, then retain the alternating cofactor signs; $x=y=z=1$ quickly checks the overall sign.

- Cofactor expansion along the first row uses the alternating signs $+, -, +$. A 2×2 determinant is $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.
- An identity asserted for arbitrary variables must hold for every choice of those variables. A single unequal evaluation disproves such an identity.

Step 1: Simplify the determinant using row operations Denote the determinant printed in the question by D . Apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$; in both operations R_1 is unchanged. The resulting determinant is

$$D = \begin{vmatrix} 1 & 1 & 1+x \\ 1 & 1+y & 1 \\ 1+z & 1 & 1 \end{vmatrix} \\ = \begin{vmatrix} 1 & 1 & 1+x \\ 0 & y & -x \\ z & 0 & -x \end{vmatrix}.$$

The entries $-x$ arise from $1 - (1 + x)$. Neither operation changes the determinant sign.

Step 2: Expand with the correct cofactor signs Expanding along the first row gives

$$D = 1 \begin{vmatrix} y & -x \\ 0 & -x \end{vmatrix} - 1 \begin{vmatrix} 0 & -x \\ z & -x \end{vmatrix} + (1+x) \begin{vmatrix} 0 & y \\ z & 0 \end{vmatrix} \\ = -xy - xz - (1+x)yz \\ = -xy - xz - yz - xyz.$$

The middle minor is xz , but its cofactor contribution is $-xz$. The final minor is $-yz$. Thus every term has the opposite sign to the asserted right-hand side.

Step 3: Check the identity as printed As an independent check, set $x = y = z = 1$. Direct expansion of the printed determinant gives $1(2 - 1) - 1(1 - 2) + 2(1 - 4) = -4$. The asserted right-hand side gives $1 + 1 + 1 + 1 = 4$. Therefore the displayed equality fails even for this simple choice. It holds only when $xyz + yz + zx + xy = 0$, which the question does not impose.

Step 4: Conclusion For the matrix printed in the question, $D = -(xyz + yz + zx + xy)$. The asserted equality has a sign error and cannot be proved as an identity; its right-hand side must carry a minus sign.

Quick Tip to Solve: Subtract the first row from the other two, then retain the alternating cofactor signs; $x=y=z=1$ quickly checks the overall sign.

14(b). Find matrix X so that $X \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$.

[4 marks]

Solution:

Concept:

- In a product, the number of columns of the first matrix must equal the number of rows of the second. If XA has order 2×3 and A has order 2×3 , then X has order 2×2 .
- Matrix multiplication uses the row-column rule. Equality of two matrices means equality of every pair of corresponding entries, giving simultaneous linear equations for the entries of X .

Step 1: Introduce the unknown entries Write $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Using the row-column rule, the equation becomes

$$\begin{pmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}.$$

Thus the first row requires $a+4b = -7$, $2a+5b = -8$ and $3a+6b = -9$. The second row requires $c+4d = 2$, $2c+5d = 4$ and $3c+6d = 6$. We determine each row independently and then check the remaining equations.

Step 2: Determine the first row of the unknown matrix Twice the equation $a+4b = -7$ is $2a+8b = -14$. Subtract $2a+5b = -8$ to obtain $3b = -6$, so $b = -2$. Substitution in the first equation gives $a-8 = -7$, hence $a = 1$. The unused third entry is consistent because $3a+6b = 3-12 = -9$.

Step 3: Determine the second row and verify the whole product Twice $c+4d = 2$ gives $2c+8d = 4$. Subtracting $2c+5d = 4$ yields $3d = 0$, so $d = 0$. It follows that $c = 2$; the third equation also holds, since $3c+6d = 6$. As a direct check,

$$\begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1-8 & 2-10 & 3-12 \\ 2 & 4 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}.$$

The coefficient determinant of the first two equations in each row is $1 \cdot 5 - 2 \cdot 4 = -3 \neq 0$, so the entries found are unique.

Step 4: Conclusion The required matrix is $X = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$.

Quick Tip to Solve: Solve each row of X using the first two columns, then check the third column to confirm consistency.

15(a). If $xy = e^{(x-y)}$, then show that $\frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$.

[4 marks]

Solution:

Concept:

- For an implicit relation, regard y as a differentiable function of x and write $y' = \frac{dy}{dx}$. The product rule gives $\frac{d}{dx}(xy) = y + xy'$.
- The chain rule gives $\frac{d}{dx}e^{u(x)} = e^{u(x)}u'(x)$. Here $u = x - y(x)$, so $u' = 1 - y'$. The original relation can then replace the exponential by xy .

Step 1: Differentiate both sides of the implicit relation We are given $xy = e^{x-y}$. Differentiating with respect to x , and applying the product rule on the left and chain rule on the right, gives

$$y + x\frac{dy}{dx} = e^{x-y} \left(1 - \frac{dy}{dx}\right).$$

Both factors in xy depend on x , and the exponent also contains $y(x)$. Therefore the derivative terms on both sides must be retained.

Step 2: Use the given equation and collect the derivative terms Substitute $e^{x-y} = xy$ into the differentiated relation:

$$\begin{aligned}y + xy' &= xy(1 - y'), \\y + xy' &= xy - xy y', \\xy' + xy y' &= xy - y, \\x(1 + y)y' &= y(x - 1).\end{aligned}$$

In the last line we factor xy' on the left and y on the right. This isolates the derivative with a single coefficient, avoiding the need to solve explicitly for y .

Step 3: Isolate the derivative and check the denominator The exponential is strictly positive, so $xy = e^{x-y} > 0$. In particular, $x \neq 0$ and $y \neq 0$. The case $y = -1$ requires $-x = e^{x+1}$, with $x < 0$. Writing $t = -x > 0$, this becomes $\log t + t = 1$. The left side is strictly increasing and equals 1 only at $t = 1$, so the only such point is $(x, y) = (-1, -1)$. At this point the equation $x(1 + y)y' = y(x - 1)$ would read $0 = 2$, so the curve cannot be a differentiable function of x there. Thus, wherever the requested finite derivative exists, $x(y + 1) \neq 0$, and division gives the required expression.

Step 4: Conclusion Hence, at points where y is differentiable as a function of x , $\frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$, as required.

Quick Tip to Solve: After differentiating, replace the exponential by xy before collecting the terms containing dy/dx .

15(b). If $\log y = \tan^{-1} x$, then show that $(1 + x^2) \frac{d^2 y}{dx^2} + (2x - 1) \frac{dy}{dx} = 0$.

[4 marks]

Solution:

Concept:

- For a positive differentiable function $y = y(x)$, the chain rule gives $\frac{d}{dx}(\log y) = \frac{1}{y} \frac{dy}{dx}$, where $\log$ denotes the natural logarithm.
- The derivative of inverse tangent is $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1 + x^2}$. Differentiate a rearranged first derivative relation using the product rule to obtain the second derivative identity.

Step 1: Differentiate the given relation once Since $\log y = \tan^{-1} x$, we have $y > 0$. Differentiating both sides with respect to x , remembering that y depends on x , gives

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{1 + x^2}.$$

Multiply by $y(1 + x^2)$ to obtain

$$(1 + x^2) \frac{dy}{dx} = y.$$

This rearrangement avoids differentiating a quotient in the next step. It is valid for every real x , because $1 + x^2$ is positive, and the given logarithmic relation ensures that y is never zero.

Step 2: Differentiate the rearranged identity Apply the product rule to $(1 + x^2) \frac{dy}{dx}$. The derivative of the first factor is $2x$, and that of the second factor is $\frac{d^2 y}{dx^2}$. Hence

$$2x \frac{dy}{dx} + (1 + x^2) \frac{d^2 y}{dx^2} = \frac{dy}{dx}.$$

Move the first derivative on the right to the left:

$$(1 + x^2) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} - \frac{dy}{dx} = 0.$$

Combining the last two terms produces the coefficient $2x - 1$. This coefficient is obtained by subtracting the derivative of the right-hand side, while the factor $1 + x^2$ remains attached to the second derivative.

Step 3: Conclusion Therefore, $(1 + x^2) \frac{d^2 y}{dx^2} + (2x - 1) \frac{dy}{dx} = 0$, as required.

Quick Tip to Solve: Rearrange the first derivative as $(1 + x^2)y' = y$ before differentiating again; the product rule gives the required identity directly.

16. Find: $\int \frac{e^x dx}{(2 + e^x)(4 + e^{2x})}$.

[4 marks]

Solution:

Concept:

- When an integral contains $e^x dx$ and an algebraic expression in e^x , substitute $t = e^x$; then $dt = e^x dx$ and $t > 0$.
- For these linear and quadratic factors, use $\frac{A}{t+2} + \frac{Bt+C}{t^2+4}$; the quadratic factor requires a linear numerator.
- Use $\int \frac{dt}{t+a} = \log|t+a|$, $\int \frac{t dt}{t^2+a^2} = \frac{1}{2} \log(t^2+a^2)$, and $\int \frac{dt}{t^2+a^2} = \frac{1}{a} \tan^{-1}(t/a)$.

Step 1: Replace the exponential by a single variable Let I denote the required integral. Put $t = e^x$, so that $dt = e^x dx$ and $e^{2x} = t^2$. Thus

$$I = \int \frac{dt}{(t+2)(t^2+4)}.$$

The numerator is exactly the differential of the substituted variable.

Step 2: Find the partial fractions Write

$$\frac{1}{(t+2)(t^2+4)} = \frac{A}{t+2} + \frac{Bt+C}{t^2+4}.$$

Multiplying through by the denominator gives

$$\begin{aligned} 1 &= A(t^2+4) + (Bt+C)(t+2) \\ &= (A+B)t^2 + (2B+C)t + (4A+2C). \end{aligned}$$

Equating coefficients yields $A+B=0$, $2B+C=0$, and $4A+2C=1$. Therefore $B=-A$, $C=2A$, and $8A=1$. Hence $A=\frac{1}{8}$, $B=-\frac{1}{8}$, and $C=\frac{1}{4}$.

Step 3: Integrate and substitute back Splitting the quadratic numerator into two terms, we obtain

$$I = \frac{1}{8} \int \frac{dt}{t+2} - \frac{1}{8} \int \frac{t dt}{t^2+4} + \frac{1}{4} \int \frac{dt}{t^2+4}.$$

In the second integral, the derivative of the denominator is $2t$; in the third, the quadratic has the form t^2+2^2 . Consequently

$$I = \frac{1}{8} \log|t+2| - \frac{1}{16} \log(t^2+4) + \frac{1}{8} \tan^{-1}\left(\frac{t}{2}\right) + C.$$

Substitute $t = e^x$. As $e^x + 2 > 0$, remove the first logarithm's absolute value.

Step 4: Conclusion

$$I = \frac{1}{8} \log(e^x + 2) - \frac{1}{16} \log(e^{2x} + 4) + \frac{1}{8} \tan^{-1}\left(\frac{e^x}{2}\right) + C.$$

Quick Tip to Solve: First use $t = e^x$; then keep a linear numerator over $t^2 + 4$, so its two terms integrate to a logarithm and an inverse tangent.

17(a). Evaluate: $\int_{-2}^1 |x^3 - x| dx$.

[4 marks]

Solution:

Concept:

- For an absolute-value integral, determine the sign of the expression between its successive roots. Use $|f(x)| = f(x)$ where $f(x) \geq 0$, and $|f(x)| = -f(x)$ where $f(x) < 0$.
- The polynomial factors as $x^3 - x = x(x-1)(x+1)$, and its antiderivative is $F(x) = x^4/4 - x^2/2$.

Step 1: Determine the signs between the roots The roots are $-1, 0, 1$. Of these, -1 and 0 lie inside the integration interval; 1 is an endpoint. On $(-2, -1)$, all three factors are negative, so $x^3 - x < 0$. On $(-1, 0)$, two factors are negative and one is positive, so $x^3 - x > 0$. On $(0, 1)$, only $x - 1$ is negative, so $x^3 - x < 0$. The integrand vanishes at each root, so assigning a root to either adjacent interval makes no difference.

Step 2: Split the integral and remove the modulus Using the signs just obtained gives

$$I = - \int_{-2}^{-1} (x^3 - x) dx + \int_{-1}^0 (x^3 - x) dx - \int_0^1 (x^3 - x) dx.$$

Each term represents a nonnegative contribution. We must split at both interior roots, since the sign changes at each of them. Taking the absolute value only after integrating over the entire interval would wrongly allow positive and negative contributions to cancel.

Step 3: Evaluate the three contributions Substituting the endpoints into F ,

$$F(-2) = 2, \quad F(-1) = -\frac{1}{4}, \quad F(0) = 0, \quad F(1) = -\frac{1}{4}.$$

Therefore

$$\begin{aligned} I &= - \left(-\frac{1}{4} - 2 \right) + \left(0 + \frac{1}{4} \right) - \left(-\frac{1}{4} - 0 \right) \\ &= \frac{9}{4} + \frac{1}{4} + \frac{1}{4}. \end{aligned}$$

In particular, the equal contributions on $[-1, 0]$ and $[0, 1]$ agree with the symmetry of $|x^3 - x|$ about the vertical axis. The remaining contribution on $[-2, -1]$ is larger and must also be included.

Step 4: Conclusion Thus $\int_{-2}^1 |x^3 - x| dx = \boxed{\frac{11}{4}}$.

Quick Tip to Solve: Factor the polynomial inside the modulus, then split at every interior root where its sign changes before using an antiderivative.

17(b). Find: $\int e^{2x} \sin(3x + 1) dx$.

[4 marks]

Solution:

Concept:

- Integration by parts gives $\int u dv = uv - \int v du$. An exponential multiplied by a sine returns to the original integral after applying this rule twice.
- The chain rule gives $\frac{d}{dx} \sin(3x + 1) = 3 \cos(3x + 1)$ and $\frac{d}{dx} \cos(3x + 1) = -3 \sin(3x + 1)$; also $\int e^{2x} dx = e^{2x}/2$.

Step 1: Apply integration by parts to the sine integral Write $I = \int e^{2x} \sin(3x + 1) dx$, and introduce $J = \int e^{2x} \cos(3x + 1) dx$. Choose $u = \sin(3x + 1)$ and $dv = e^{2x} dx$, so that $du = 3 \cos(3x + 1) dx$ and $v = e^{2x}/2$. Integration by parts then gives

$$I = \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3}{2} J.$$

Step 2: Apply integration by parts to the cosine integral Now choose $u = \cos(3x + 1)$, $dv = e^{2x} dx$. Here $du = -3 \sin(3x + 1) dx$, so the subtraction in the integration-by-parts formula changes this negative derivative into a positive contribution:

$$J = \frac{e^{2x}}{2} \cos(3x + 1) + \frac{3}{2} I.$$

Substitute this expression for J into the first equation:

$$I = \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3e^{2x}}{4} \cos(3x + 1) - \frac{9}{4} I.$$

Constants of integration are collected into one arbitrary constant after solving for I .

Step 3: Collect the original integral and check the result Bringing $-9I/4$ to the left gives

$$\frac{13}{4} I = \frac{e^{2x}}{4} (2 \sin(3x + 1) - 3 \cos(3x + 1)).$$

For verification, differentiate the resulting expression. By the product and chain rules, its derivative is

$$\begin{aligned} \frac{e^{2x}}{13} [4 \sin(3x + 1) - 6 \cos(3x + 1) \\ + 6 \cos(3x + 1) + 9 \sin(3x + 1)] \\ = e^{2x} \sin(3x + 1). \end{aligned}$$

Step 4: Conclusion The required antiderivative is

$$\boxed{\frac{e^{2x}}{13} (2 \sin(3x + 1) - 3 \cos(3x + 1)) + C}.$$

Quick Tip to Solve: Apply integration by parts twice, keeping the derivative factor 3; then move the returning integral to the left and solve algebraically.

18. Find the particular solution of the differential equation $2ye^{x/y} dx + (y - 2xe^{x/y}) dy = 0$, given that $x = 0$ when $y = 1$. [4 marks]

Solution:

Concept:

- A homogeneous equation depends on the ratio of the variables. Here use $v = x/y$ and regard x as a function of y .
- For $x = vy$, the product rule gives $dx/dy = v + y dv/dy$. Integrate using $\int e^v dv = e^v$ and $\int dy/y = \ln |y|$.

Step 1: Rewrite in homogeneous form The original equation requires $y \neq 0$. Dividing by dy and rearranging gives

$$2ye^{x/y} \frac{dx}{dy} + y - 2xe^{x/y} = 0.$$

Divide by $2ye^{x/y}$, which is nonzero, to obtain

$$\frac{dx}{dy} = \frac{x}{y} - \frac{1}{2}e^{-x/y}.$$

The right side depends only on x/y , so the equation is homogeneous in the ratio required for substitution.

Step 2: Substitute and integrate Put $v = x/y$, so $x = vy$. Substitution yields

$$v + y \frac{dv}{dy} = v - \frac{1}{2}e^{-v}, \quad 2e^v dv = -\frac{dy}{y}.$$

The two ratio terms cancel, leaving separated variables. Integrate both sides:

$$2e^v = -\ln |y| + C, \quad 2e^{x/y} + \ln |y| = C.$$

The integration constant belongs to the entire implicit relation.

Step 3: Use the initial condition and check At $x = 0$, $y = 1$, we have $v = 0$ and $\ln 1 = 0$, giving $C = 2e^0 + 0 = 2$. The branch containing this point has $y > 0$, hence $\ln |y| = \ln y$. To check the result, differentiate its left side:

$$2e^{x/y} \left(\frac{dx}{y} - \frac{x dy}{y^2} \right) + \frac{dy}{y} = 0.$$

Multiplying through by y^2 recovers $2ye^{x/y} dx + (y - 2xe^{x/y}) dy = 0$, exactly the printed equation. The initial point also satisfies the relation.

Step 4: Conclusion The particular solution is $\boxed{2e^{x/y} + \ln y = 2}$, on the branch $y > 0$ containing $(0, 1)$.

Quick Tip to Solve: Treat x as a function of y and put $x=vy$; the ratio terms cancel immediately, leaving a separable equation.

19. Find the area of a parallelogram $ABCD$ whose side AB and the diagonal AC are given by the vectors $3\hat{i} + \hat{j} + 4\hat{k}$ and $4\hat{i} + 5\hat{k}$ respectively. [4 marks]

Solution:

Concept:

- The area of a parallelogram is $|\mathbf{a} \times \mathbf{b}|$, where $\mathbf{a}$ and $\mathbf{b}$ are its adjacent side vectors.
- The diagonal satisfies $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD}$. Bilinearity also gives $\overrightarrow{AB} \times \overrightarrow{AC} = \overrightarrow{AB} \times \overrightarrow{AD}$.

Step 1: Find the adjacent side Write $\mathbf{a} = \overrightarrow{AB} = 3\hat{i} + \hat{j} + 4\hat{k}$ and $\mathbf{c} = \overrightarrow{AC} = 4\hat{i} + 5\hat{k}$. By the parallelogram law,

$$\overrightarrow{AD} = \mathbf{c} - \mathbf{a} = (4 - 3)\hat{i} + (0 - 1)\hat{j} + (5 - 4)\hat{k} = \hat{i} - \hat{j} + \hat{k}.$$

This is the side adjacent to AB , so its cross product with AB has magnitude equal to the required parallelogram area.

Step 2: Compute the cross product Expand the determinant along its first row, retaining the negative sign on the middle cofactor:

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AD} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix} \\ &= \hat{i}(1 + 4) - \hat{j}(3 - 4) + \hat{k}(-3 - 1) \\ &= 5\hat{i} + \hat{j} - 4\hat{k}.\end{aligned}$$

As an independent arithmetic check, this vector is perpendicular to each side: its dot products are $15 + 1 - 16 = 0$ and $5 - 1 - 4 = 0$, respectively.

Step 3: Take the magnitude The magnitude of the cross product is

$$\sqrt{5^2 + 1^2 + (-4)^2} = \sqrt{25 + 1 + 16} = \sqrt{42}.$$

This magnitude is the area of the entire parallelogram. No factor of one-half is used, since that factor would give the area of a triangle with the same adjacent sides. One could equivalently cross AB directly with AC because $\mathbf{a} \times (\mathbf{a} + \mathbf{b}) = \mathbf{a} \times \mathbf{b}$.

Step 4: Conclusion The area of parallelogram $ABCD$ is $\boxed{\sqrt{42} \text{ square units}}$.

Quick Tip to Solve: Cross the given side directly with the given diagonal: their cross product equals that of the two adjacent sides.

20. If $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b} = 4\hat{i} - 7\hat{j} + \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 6$.

[4 marks]

Solution:

Concept:

- Write the unknown vector as $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$. Equality of vectors means equality of their corresponding components.
- Evaluate the cross product using a determinant and the dot product by multiplying corresponding components. The cross product fixes the perpendicular component of $\vec{c}$, and the dot product fixes its component parallel to $\vec{a}$.

Step 1: Form the cross-product equations Taking $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$, we have

$$\vec{a} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ x & y & z \end{vmatrix} = (y + z)\hat{i} - (x + 2z)\hat{j} + (2y - x)\hat{k}.$$

Comparing the coefficients with $\vec{b} = 4\hat{i} - 7\hat{j} + \hat{k}$ gives

$$y + z = 4, \quad x + 2z = 7, \quad 2y - x = 1.$$

The first two equations give $y = 4 - z$ and $x = 7 - 2z$. The third is consistent with these expressions, since $2(4 - z) - (7 - 2z) = 1$. Thus one parameter remains undetermined by the cross product alone. This occurs because adding a multiple of $\vec{a}$ to $\vec{c}$ does not change $\vec{a} \times \vec{c}$.

Step 2: Determine the remaining component using the dot product The dot-product condition gives $2x + y - z = 6$. Substituting the expressions obtained above,

$$\begin{aligned} 2(7 - 2z) + (4 - z) - z &= 6, \\ 18 - 6z &= 6, \\ z &= 2. \end{aligned}$$

Therefore $y = 4 - 2 = 2$ and $x = 7 - 2 \cdot 2 = 3$. The additional scalar equation fixes a unique vector because $\vec{a}$ is nonzero and its squared magnitude is $2^2 + 1^2 + (-1)^2 = 6$.

Step 3: Check both requirements Using the computed components, directly evaluate

$$\begin{aligned} \vec{a} \times \vec{c} &= (2 + 2)\hat{i} - (3 + 2 \cdot 2)\hat{j} + (2 \cdot 2 - 3)\hat{k} \\ &= 4\hat{i} - 7\hat{j} + \hat{k} = \vec{b}, \\ \vec{a} \cdot \vec{c} &= 2 \cdot 3 + 1 \cdot 2 - 1 \cdot 2 = 6. \end{aligned}$$

Both prescribed conditions are satisfied, including the order of the cross product. Reversing this order would produce $-\vec{b}$, so the sign of the middle component must be handled carefully.

Step 4: Conclusion The required vector is $\boxed{\vec{c} = 3\hat{i} + 2\hat{j} + 2\hat{k}}$.

Quick Tip to Solve: Use two cross-product component equations to express two unknowns in terms of the third, then apply the dot product and check all three components.

21. There are 4 cards numbered 1 to 4, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X . [4 marks]

Solution:

Concept:

- For a discrete random variable, $E(X) = \sum_x xP(X = x)$ and $E(X^2) = \sum_x x^2P(X = x)$, where the sum runs over the possible values of X .
- The variance is $\text{Var}(X) = E(X^2) - [E(X)]^2$. When two cards are selected uniformly without replacement, the $\binom{4}{2}$ unordered pairs are equally likely.

Step 1: List the equally likely selections and their sums Because the cards are drawn without replacement, the two numbers must be distinct. Since their sum does not depend on the order of drawing, use unordered pairs as the elementary outcomes. There are

$$\binom{4}{2} = \frac{4 \cdot 3}{2} = 6$$

such pairs, each with probability $1/6$. Their sums are

Pair	{1, 2}	{1, 3}	{1, 4}	{2, 3}	{2, 4}	{3, 4}
X	3	4	5	5	6	7

The sum 5 occurs for two different pairs; all the other sums occur once.

Step 2: Form the probability distribution and calculate its mean Combining outcomes with the same sum gives

x	3	4	5	6	7
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

These probabilities add to 1, as required. Hence the mean is

$$E(X) = \frac{3 + 4 + 2 \cdot 5 + 6 + 7}{6} = \frac{30}{6} = 5.$$

Step 3: Calculate the second moment and variance Square the possible sums before weighting them by their probabilities:

$$\begin{aligned} E(X^2) &= \frac{3^2 + 4^2 + 2 \cdot 5^2 + 6^2 + 7^2}{6} \\ &= \frac{9 + 16 + 50 + 36 + 49}{6} = \frac{160}{6} = \frac{80}{3}. \end{aligned}$$

Therefore

$$\text{Var}(X) = \frac{80}{3} - 5^2 = \frac{80 - 75}{3} = \frac{5}{3}.$$

Step 4: Conclusion The mean of X is $\boxed{5}$, and its variance is $\boxed{\frac{5}{3}}$.

Quick Tip to Solve: List the six unordered pairs first: the sum 5 appears twice, so its probability must be twice that of any other possible sum.

22. In a shop X , 30 tins of pure ghee and 40 tins of adulterated ghee which look alike, are kept for sale while in shop Y , similar 50 tins of pure ghee and 60 tins of adulterated ghee are there. One tin of ghee is purchased from one of the randomly selected shops and is found to be adulterated. Find the probability that it is purchased from shop Y . What measures should be taken to stop adulteration? [4 marks]

Solution:

Concept:

- Bayes' theorem reverses a conditional probability. If A denotes adulterated ghee and X, Y denote the selected shops, then $P(Y | A) = P(Y)P(A | Y)/P(A)$.
- The shops form mutually exclusive and exhaustive alternatives, so $P(A) = P(X)P(A | X) + P(Y)P(A | Y)$. Random selection of a shop gives each shop probability $1/2$.

Step 1: Determine the prior and conditional probabilities Let X and Y be the events that the tin was bought from the corresponding shop, and let A be the event that it is adulterated. Because one of the two shops is selected randomly,

$$P(X) = P(Y) = \frac{1}{2}.$$

Shop X contains $30 + 40 = 70$ tins, of which 40 are adulterated. Shop Y contains $50 + 60 = 110$ tins, of which 60 are adulterated. Hence

$$P(A | X) = \frac{40}{70} = \frac{4}{7}, \quad P(A | Y) = \frac{60}{110} = \frac{6}{11}.$$

These are probabilities within the selected shop; the different total numbers of tins do not change the equal probabilities of selecting the shops.

Step 2: Apply total probability and Bayes' theorem The probability of buying an adulterated tin is

$$P(A) = \frac{1}{2} \cdot \frac{4}{7} + \frac{1}{2} \cdot \frac{6}{11} = \frac{2}{7} + \frac{3}{11} = \frac{43}{77}.$$

The probability that the tin is both from shop Y and adulterated is $P(Y \cap A) = \frac{1}{2} \cdot \frac{6}{11} = \frac{3}{11}$. Conditioning on the observed adulteration gives

$$P(Y | A) = \frac{P(Y \cap A)}{P(A)} = \frac{3/11}{43/77} = \frac{21}{43}.$$

Step 3: State measures to prevent adulteration Regular testing and inspection of food products should be carried out, and strict action should be taken against sellers and manufacturers who adulterate them. Consumers should be made aware of quality standards and encouraged to buy certified products and report suspected adulteration. Such measures protect public health and discourage dishonest practices.

Step 4: Conclusion The required probability is $\frac{21}{43}$. Regular quality checks, strict enforcement against adulteration and consumer awareness should be used to prevent it.

Quick Tip to Solve: Weight each shop's adulterated fraction by its selection probability; do not pool all tins when the shop is selected first.

23. Solve the following LPP graphically: Maximise $Z = 1000x + 600y$ subject to the constraints $x + y \leq 200$, $x \geq 20$, $y - 4x \geq 0$, $x, y \geq 0$. [4 marks]

Solution:

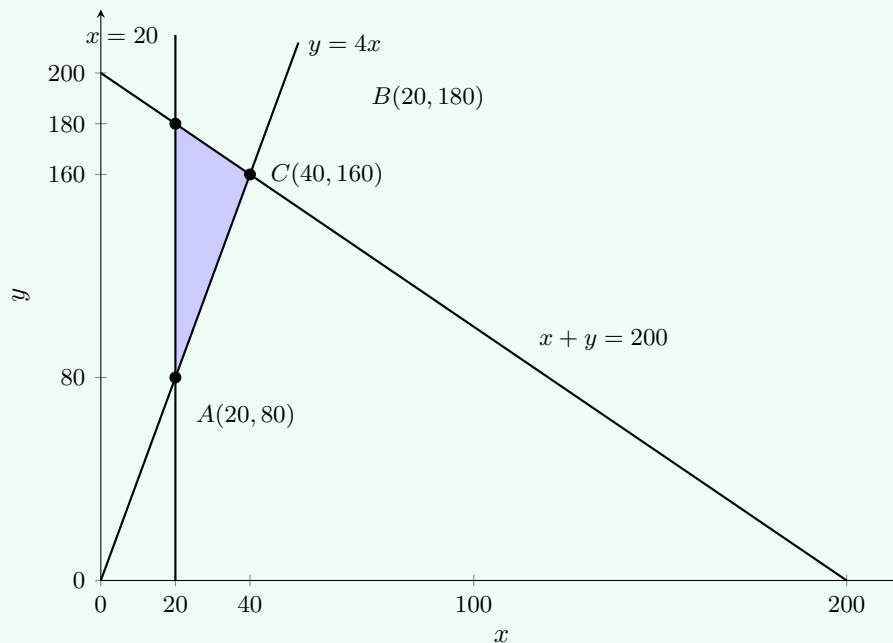
Concept:

- Each linear inequality represents a closed half-plane. Their common region in the first quadrant is the feasible region.
- For a nonempty bounded polygonal feasible region, a linear objective function attains its maximum at a vertex. Compare its value at all vertices.

Step 1: Draw the boundary lines and identify the feasible sides The constraints can be written as

$$x + y \leq 200, \quad x \geq 20, \quad y \geq 4x, \quad x, y \geq 0.$$

Draw $x + y = 200$ through $(0, 200)$ and $(200, 0)$, the vertical line $x = 20$, and $y = 4x$ through $(0, 0)$ and $(40, 160)$. The allowed points lie below or on $x + y = 200$, to the right of or on $x = 20$, and above or on $y = 4x$. Their common region is the shaded triangle ABC in the graph. All its boundary points satisfy the non-strict inequalities.



The shaded triangle is the feasible region for the LPP.

Step 2: Find all vertices of the feasible region At the intersection of $x = 20$ and $y = 4x$, we get $A = (20, 80)$. At the intersection of $x = 20$ and $x + y = 200$, we get $B = (20, 180)$. Finally, solving $y = 4x$ together with $x + y = 200$ gives

$$x + 4x = 200 \implies x = 40, \quad y = 160,$$

so $C = (40, 160)$. Each vertex has nonnegative coordinates and satisfies every inequality. Also, $y \geq 4x$ and $x + y \leq 200$ imply $5x \leq 200$, so $20 \leq x \leq 40$. Thus the feasible triangle is bounded.

Step 3: Evaluate the objective function at the vertices Compute $Z = 1000x + 600y$ at the three

Quick Tip to Solve: Intersect $y=4x$ with $x+y=200$ first; then compare this corner with the two corners on $x=20$.

corners:

Vertex	$Z = 1000x + 600y$
$A(20, 80)$	$20000 + 48000 = 68000$
$B(20, 180)$	$20000 + 108000 = 128000$
$C(40, 160)$	$40000 + 96000 = 136000$

By the corner-point theorem, comparing these values determines the maximum over the entire feasible region. The largest value occurs at C .

Step 4: Conclusion The maximum value is $Z_{\max} = 136000$, attained at $x = 40, y = 160$.

Quick Tip to Solve: Intersect $y = 4x$ with $x + y = 200$ first; then compare this corner with the two corners on $x = 20$.

Section D (36 marks)

Questions number 24 to 29 carry 6 marks each.

24. If $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$ are two square matrices, find AB and hence solve the system of linear equations $x - y = 3$, $2x + 3y + 4z = 17$ and $y + 2z = 7$. [6 marks]

Solution:

Concept:

- In a matrix product, entry (i, j) is the scalar product of row i of the first matrix and column j of the second matrix.
- If $AB = kI$, where $k \neq 0$, then $A^{-1} = B/k$. For a system $AX = C$, its unique solution is $X = A^{-1}C$.

Step 1: Calculate the product using row-column multiplication The first row of A is $(1, -1, 0)$, so its products with the three columns of B give $2 + 4 = 6$, $2 - 2 = 0$, and $-4 + 4 = 0$. Applying the same rule to the remaining rows,

$$AB = \begin{pmatrix} 2 + 4 + 0 & 2 - 2 + 0 & -4 + 4 + 0 \\ 4 - 12 + 8 & 4 + 6 - 4 & -8 - 12 + 20 \\ 0 - 4 + 4 & 0 + 2 - 2 & 0 - 4 + 10 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix}.$$

Each off-diagonal entry vanishes; all three diagonal entries are equal to 6. Thus the given second matrix supplies the inverse of the first after division by 6.

Step 2: Express the equations as a matrix equation Taking the variables in the order x, y, z , define

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad C = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}.$$

The coefficients of the three equations are exactly the rows of A , so the given system is $AX = C$. Since

Quick Tip to Solve: When a supplied matrix product is a nonzero multiple of the identity, divide the second matrix by that multiple to obtain the inverse directly.

$AB = 6I$ and A, B are square matrices, A is invertible and $A^{-1} = B/6$. Left-multiplying $AX = C$ by this inverse gives $X = (B/6)C$.

Step 3: Multiply the inverse by the constants and verify Substituting the given constants,

$$X = \frac{1}{6} \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 6 + 34 - 28 \\ -12 + 34 - 28 \\ 6 - 17 + 35 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}.$$

As a check, these values give $x - y = 2 - (-1) = 3$, $2x + 3y + 4z = 4 - 3 + 16 = 17$, and $y + 2z = -1 + 8 = 7$. They satisfy all three original equations, and invertibility guarantees uniqueness.

Step 4: Conclusion $AB = 6I_3$, and the unique solution is $x = 2, y = -1, z = 4$.

Quick Tip to Solve: When a supplied matrix product is a nonzero multiple of the identity, divide the second matrix by that multiple to obtain the inverse directly.

25(a). Let $A = \mathbb{R} - \{3\}$, $B = \mathbb{R} - \{1\}$. Let $f : A \rightarrow B$ be defined by $f(x) = \frac{x-2}{x-3} \forall x \in A$. Show that f is bijective. Also, find (i) x , if $f^{-1}(x) = 4$ (ii) $f^{-1}(7)$. [6 marks]

Solution:

Concept:

- A function is injective if equality of outputs implies equality of inputs. It is surjective if every element of the stated codomain has a preimage in the stated domain.
- A bijective function has an inverse. If $y = f(t)$, solving for t in terms of y gives $f^{-1}(y)$, with inverse domain equal to the original codomain.

Step 1: Check that the formula maps the stated domain into the codomain For $t \in A = \mathbb{R} \setminus \{3\}$, the denominator $t - 3$ is nonzero, so $f(t)$ is a real number. Moreover,

$$f(t) = \frac{t-2}{t-3} = 1 + \frac{1}{t-3}.$$

The second term cannot be zero, so $f(t) \neq 1$. Thus every output belongs to $B = \mathbb{R} \setminus \{1\}$, as required for the function $f : A \rightarrow B$.

Step 2: Prove that the function is one-one Let $u, v \in A$ and suppose $f(u) = f(v)$. Since both denominators are nonzero, cross-multiplication is valid:

$$\begin{aligned} \frac{u-2}{u-3} &= \frac{v-2}{v-3}, \\ (u-2)(v-3) &= (v-2)(u-3), \\ uv - 3u - 2v + 6 &= uv - 3v - 2u + 6, \\ -u + v &= 0. \end{aligned}$$

Therefore $u = v$. Equal outputs can arise only from equal inputs, and f is injective.

Quick Tip to Solve: Solve the output equation for the input, then check the excluded domain value explicitly; this establishes surjectivity and supplies the inverse in the same calculation.

Step 3: Prove that the function is onto and obtain its inverse Take any $y \in B$, so $y \neq 1$. To find its preimage, solve

$$y = \frac{t-2}{t-3} \implies (y-1)t = 3y-2 \implies t = \frac{3y-2}{y-1}.$$

This is real, and

$$t-3 = \frac{3y-2-3(y-1)}{y-1} = \frac{1}{y-1} \neq 0.$$

Hence $t \in A$. Also $t-2 = y/(y-1)$, so $f(t) = (t-2)/(t-3) = y$. Every element of B therefore has a preimage in A , proving surjectivity. Together with injectivity, this proves bijectivity and yields

$$f^{-1} : B \rightarrow A, \quad f^{-1}(y) = \frac{3y-2}{y-1}.$$

Step 4: Evaluate the two requested quantities (i) Since $4 \in A$, applying f to $f^{-1}(x) = 4$ gives

$$x = f(4) = \frac{4-2}{4-3} = 2.$$

The result 2 belongs to B , and $f^{-1}(2) = (6-2)/(2-1) = 4$, which checks the inverse equation. For (ii), $7 \in B$, and substitution into the inverse formula gives

$$f^{-1}(7) = \frac{3(7)-2}{7-1} = \frac{19}{6}.$$

This value is different from 3, hence lies in A . Substituting it in f gives $(7/6)/(1/6) = 7$, verifying the evaluation.

Step 5: Conclusion The function f is bijective, with $f^{-1}(y) = \frac{3y-2}{y-1}$ for $y \neq 1$. The requested values are $\boxed{\text{(i) } x = 2, \quad \text{(ii) } f^{-1}(7) = \frac{19}{6}}.$

Quick Tip to Solve: Solve the output equation for the input, then check the excluded domain value explicitly; this establishes surjectivity and supplies the inverse in the same calculation.

OR

25(b). Let $A = \mathbb{R} \times \mathbb{R}$ and let $*$ be a binary operation on A defined by $(a, b) * (c, d) = (ad + bc, bd)$ for all $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$. (i) Show that $*$ is commutative on A . (ii) Show that $*$ is associative on A . (iii) Find the identity element of $*$ in A . [6 marks]

Solution:

Concept:

- A binary operation is commutative if $u * v = v * u$ for every pair of elements, and associative if $(u * v) * w = u * (v * w)$ for every triple.
- An identity E must satisfy $u * E = E * u = u$ for every $u \in A$. Equality of ordered pairs means

Quick Tip to Solve: For associativity, retain the intermediate ordered pair explicitly before substituting it into the operation again; this prevents swapping its coordinates.

equality of both corresponding coordinates.

Step 1: (i) Verify commutativity Take arbitrary $u = (a, b)$ and $v = (c, d)$ in A . The given operation gives

$$u * v = (ad + bc, bd), \quad v * u = (cb + da, db).$$

Since multiplication and addition of real numbers are commutative, $cb + da = ad + bc$ and $db = bd$. Thus both coordinates agree and $u * v = v * u$. As the elements were arbitrary, the operation is commutative on all of A .

Step 2: (ii) Compute the first grouping Let $w = (e, f)$ be a third arbitrary element of A . In applying the operation a second time, use the two coordinates of $u * v$ as the new first ordered pair. Therefore

$$\begin{aligned}(u * v) * w &= (ad + bc, bd) * (e, f) \\ &= ((ad + bc)f + (bd)e, (bd)f) \\ &= (adf + bcf + bde, bdf).\end{aligned}$$

All these coordinates are real, so the products remain in A .

Step 3: Compare with the second grouping First $v * w = (cf + de, df)$. Consequently,

$$\begin{aligned}u * (v * w) &= (a, b) * (cf + de, df) \\ &= (a(df) + b(cf + de), b(df)) \\ &= (adf + bcf + bde, bdf).\end{aligned}$$

The two groupings have identical first and second coordinates for every choice of the six real numbers. Hence $(u * v) * w = u * (v * w)$, establishing associativity.

Step 4: (iii) Determine and verify the identity Suppose $E = (r, s)$ is an identity. For arbitrary (a, b) , the equation $(a, b) * E = (a, b)$ requires

$$as + br = a, \quad bs = b.$$

Putting $b = 1$ in the second equation forces $s = 1$. The first equation then becomes $a + br = a$ for every b , so $r = 0$. Verify this candidate on both sides:

$$\begin{aligned}(a, b) * (0, 1) &= (a \cdot 1 + b \cdot 0, b \cdot 1) = (a, b), \\ (0, 1) * (a, b) &= (0 \cdot b + 1 \cdot a, 1 \cdot b) = (a, b).\end{aligned}$$

Thus the candidate belongs to A and satisfies the identity condition for every element.

Step 5: Conclusion The operation $*$ is commutative and associative on $\mathbb{R} \times \mathbb{R}$, and its identity element is $\boxed{(0, 1)}$.

Quick Tip to Solve: For associativity, retain the intermediate ordered pair explicitly before substituting it into the operation again; this prevents swapping its coordinates.

26. A wire of length 34 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a rectangle whose length is twice its breadth. What should be the lengths of the two pieces, so that the combined area of the square and the rectangle is minimum? [6 marks]

Solution:

Concept:

- The perimeter of a square of side s is $4s$, while a rectangle of breadth b and length $2b$ has perimeter $6b$.
- Express the combined area in one variable using the fixed total wire length. An interior stationary point satisfies $A'(x) = 0$, and $A''(x) > 0$ confirms a minimum.

Step 1: Assign the lengths of the two wire pieces Let x m be the length of the piece used for the square. The piece used for the rectangle then has length $(34 - x)$ m, where $0 < x < 34$ ensures that both shapes have positive dimensions. These wire lengths are the perimeters, not the sides of the shapes. The square therefore has side $x/4$ m. If the rectangle's breadth is b m, its length is $2b$ m, so

$$2(2b + b) = 34 - x \implies b = \frac{34 - x}{6}.$$

Step 2: Express the combined area The area of the square is the square of its side. The area of the rectangle is its length multiplied by its breadth. Hence the combined area, in square metres, is

$$\begin{aligned} A(x) &= \left(\frac{x}{4}\right)^2 + 2\left(\frac{34 - x}{6}\right)^2 \\ &= \frac{x^2}{16} + \frac{(34 - x)^2}{18}. \end{aligned}$$

The factor 2 in the rectangle term comes from its prescribed length-to-breadth ratio; omitting this factor would change the optimal allocation of wire.

Step 3: Find the stationary allocation Differentiate with respect to x , remembering the negative derivative of $34 - x$:

$$A'(x) = \frac{x}{8} - \frac{34 - x}{9}.$$

At a stationary point,

$$\begin{aligned} \frac{x}{8} &= \frac{34 - x}{9}, \\ 9x &= 8(34 - x), \\ 17x &= 272, \quad x = 16. \end{aligned}$$

This value is inside the permitted interval, so it produces two genuine shapes.

Step 4: Confirm the minimum and check the dimensions The second derivative is constant and positive:

$$A''(x) = \frac{1}{8} + \frac{1}{9} = \frac{17}{72} > 0.$$

Thus A' is strictly increasing throughout $(0, 34)$, and the stationary point gives the unique global

Quick Tip to Solve: Treat each wire piece as a perimeter: divide the square's piece by 4 and the rectangle's piece by 6 to obtain its breadth before forming the area.

minimum on that interval. In fact, the area can also be written as

$$A(x) = 34 + \frac{17}{144}(x - 16)^2,$$

which confirms directly that the least possible area is 34 m^2 . At $x = 16$, the square's side is 4 m; the rectangle's remaining perimeter is 18 m, giving breadth 3 m and length 6 m. Their areas are 16 m^2 and 18 m^2 , respectively, and their perimeters total the given 34 m.

Step 5: Conclusion Cut the wire into pieces of $\boxed{16 \text{ m}}$ for the square and $\boxed{18 \text{ m}}$ for the rectangle. The minimum combined area is 34 m^2 .

Quick Tip to Solve: Treat each wire piece as a perimeter: divide the square's piece by 4 and the rectangle's piece by 6 to obtain its breadth before forming the area.

27(a). Using the method of integration, find the area of the triangle ABC , coordinates of whose vertices are $A(1, 2)$, $B(2, 0)$ and $C(4, 3)$. [6 marks]

Solution:

Concept:

- For vertical strips between an upper boundary $y = u(x)$ and a lower boundary $y = l(x)$, the area is $\int [u(x) - l(x)] dx$. The strip height must be nonnegative on each interval.
- The equation of a line through two given points follows from $y - y_1 = m(x - x_1)$, where $m = (y_2 - y_1)/(x_2 - x_1)$. A triangle may need two integrals when its lower boundary changes at a vertex.

Step 1: Find the equations of all three sides For $A(1, 2)$ and $B(2, 0)$, the slope is $(0 - 2)/(2 - 1) = -2$. Hence

$$AB : y - 2 = -2(x - 1), \quad y = 4 - 2x.$$

For $A(1, 2)$ and $C(4, 3)$, the slope is $(3 - 2)/(4 - 1) = 1/3$, giving

$$AC : y - 2 = \frac{x - 1}{3}, \quad y = \frac{x + 5}{3}.$$

Finally, the slope of BC is $(3 - 0)/(4 - 2) = 3/2$. Its equation is

$$BC : y = \frac{3}{2}(x - 2) = \frac{3x}{2} - 3.$$

These equations reproduce the given endpoint coordinates and determine the boundaries of every vertical strip inside the triangle.

Step 2: Set up the area integral in two intervals The horizontal extent of the triangle is $1 \leq x \leq 4$. Its upper boundary throughout this interval is AC . Between $x = 1$ and $x = 2$, the lower boundary is AB ;

Quick Tip to Solve: Split the vertical-strip integral at $x = 2$, the middle vertex's abscissa: AC stays above, while the lower side changes from AB to BC .

between $x = 2$ and $x = 4$, it is BC . Therefore, denoting the area by S ,

$$\begin{aligned} S &= \int_1^2 \left(\frac{x+5}{3} - (4-2x) \right) dx \\ &\quad + \int_2^4 \left(\frac{x+5}{3} - \left(\frac{3x}{2} - 3 \right) \right) dx \\ &= \int_1^2 \frac{7(x-1)}{3} dx + \int_2^4 \frac{7(4-x)}{6} dx. \end{aligned}$$

Both heights are nonnegative on their corresponding intervals. At $x = 2$, both expressions give height $7/3$, so the split has neither a gap nor an overlap.

Step 3: Evaluate both definite integrals The first contribution is

$$\frac{7}{3} \left[\frac{(x-1)^2}{2} \right]_1^2 = \frac{7}{6}.$$

The second contribution is

$$\begin{aligned} \frac{7}{6} \left[4x - \frac{x^2}{2} \right]_2^4 &= \frac{7}{6} [(16-8) - (8-2)] \\ &= \frac{7}{6}(2) = \frac{7}{3}. \end{aligned}$$

Adding these two nonoverlapping portions gives $S = 7/6 + 7/3$. The calculation uses the side equations as boundaries, so it obtains the requested area directly by integration.

Step 4: Conclusion The area of triangle ABC is $\boxed{\frac{7}{2} \text{ square units}}$.

Quick Tip to Solve: Split the vertical-strip integral at $x = 2$, the middle vertex's abscissa: AC stays above, while the lower side changes from AB to BC .

OR

27(b). Using integration, find the area of the region $\{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$.

[6 marks]

Solution:

Concept:

- The inequality $x^2 + y^2 \leq 1$ describes the closed unit disk, while $x + y \geq 1$ selects the half-plane above the line $y = 1 - x$. The required region is their intersection.
- For vertical strips, area equals the integral of upper boundary minus lower boundary. The circle's upper semicircle is $y = \sqrt{1 - x^2}$, and $\int \sqrt{1 - x^2} dx = \frac{1}{2}(x\sqrt{1 - x^2} + \sin^{-1} x) + C$ for $-1 \leq x \leq 1$.

Step 1: Locate the intersection points and the relevant interval On the boundary line, $y = 1 - x$.

Quick Tip to Solve: First intersect $y = 1 - x$ with the unit circle; on $[0, 1]$, integrate the upper arc minus the chord, rather than the entire semicircle.

Substituting this into the circle gives

$$\begin{aligned}x^2 + (1 - x)^2 &= 1, \\2x^2 - 2x &= 0, \\2x(x - 1) &= 0.\end{aligned}$$

Thus the line meets the circle at $(0, 1)$ and $(1, 0)$. To identify the correct side, observe that $x + y \geq 1$ means $y \geq 1 - x$. Inside the disk, $x \leq 1$, so this also gives $y \geq 0$; similarly $y \leq 1$ implies $x \geq 0$. Consequently the entire required region lies in the first quadrant, and its horizontal limits are $0 \leq x \leq 1$.

Step 2: Express the region as vertical strips For any x between 0 and 1, the lower boundary is the chord $y = 1 - x$ and the upper boundary is the circular arc $y = \sqrt{1 - x^2}$. Indeed,

$$(1 - x^2) - (1 - x)^2 = 2x(1 - x) \geq 0,$$

and both boundary ordinates are nonnegative. Thus the arc lies above the chord throughout the chosen interval. If S denotes the required area, the integration formula is

$$S = \int_0^1 \left(\sqrt{1 - x^2} - (1 - x) \right) dx.$$

This subtracts the area beneath the straight chord from the area beneath the first-quadrant arc. It selects precisely the points satisfying both printed inequalities.

Step 3: Evaluate the circular part by substitution Put $x = \sin \theta$, so $dx = \cos \theta d\theta$. The limits $x = 0, 1$ become $\theta = 0, \pi/2$. On this interval $\cos \theta \geq 0$, hence $\sqrt{1 - \sin^2 \theta} = \cos \theta$. Therefore

$$\begin{aligned}\int_0^1 \sqrt{1 - x^2} dx &= \int_0^{\pi/2} \cos^2 \theta d\theta \\&= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\pi/2} = \frac{\pi}{4}.\end{aligned}$$

The part below the chord is

$$\int_0^1 (1 - x) dx = \left[x - \frac{x^2}{2} \right]_0^1 = \frac{1}{2}.$$

Subtracting these evaluated integrals yields $S = \pi/4 - 1/2$, which is positive as expected for the segment between the chord and the arc.

Step 4: Conclusion The required area is $\boxed{\frac{\pi - 2}{4} \text{ square units}}.$

Quick Tip to Solve: First intersect $y = 1 - x$ with the unit circle; on $[0, 1]$, integrate the upper arc minus the chord, rather than the entire semicircle.

28. Find the particular solution of the differential equation $\frac{dy}{dx} - 3y \cot x = \sin 2x$, given that $y = 2$ when $x = \frac{\pi}{2}$. [6 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $y' + P(x)y = Q(x)$. Its integrating factor is $I(x) = e^{\int P(x) dx}$, which changes the left-hand side into $\frac{d}{dx}(Iy)$.
- The identity $\sin 2x = 2 \sin x \cos x$ and the derivative $\frac{d}{dx}(\csc x) = -\frac{\cos x}{\sin^2 x}$ simplify the resulting integral.
- The given initial point lies in $0 < x < \pi$, where $\sin x > 0$; this is the interval containing the initial point on which the coefficient $\cot x$ is continuous.

Step 1: Identify the linear equation and its integrating factor Comparing $\frac{dy}{dx} - 3y \cot x = \sin 2x$ with the linear standard form gives $P(x) = -3 \cot x$ and $Q(x) = \sin 2x$. Since $\int \cot x dx = \log |\sin x|$, on the interval containing $\pi/2$ we may choose

$$I(x) = e^{-3 \log(\sin x)} = \sin^{-3} x = \csc^3 x.$$

A nonzero constant multiple of this integrating factor would give the same family of solutions.

Step 2: Multiply through and integrate Multiplying every term by $\csc^3 x$, and using the product rule, gives

$$\begin{aligned} \frac{d}{dx}(y \csc^3 x) &= \sin 2x \csc^3 x \\ &= \frac{2 \cos x}{\sin^2 x}. \end{aligned}$$

Thus

$$y \csc^3 x = \int \frac{2 \cos x}{\sin^2 x} dx = -2 \csc x + C.$$

Multiplication by $\sin^3 x$ now gives the general solution on this interval as $y = -2 \sin^2 x + C \sin^3 x$.

Step 3: Apply the initial value At $x = \pi/2$, the sine is 1 and the specified value of y is 2. Substituting in the general solution yields

$$2 = -2 + C, \quad C = 4.$$

Therefore the member of the solution family selected by the initial condition is $y = 4 \sin^3 x - 2 \sin^2 x$.

Step 4: Verify the differential equation and the initial condition Differentiating the selected expression gives $y' = 12 \sin^2 x \cos x - 4 \sin x \cos x$. Consequently,

$$\begin{aligned} y' - 3y \cot x &= 12 \sin^2 x \cos x - 4 \sin x \cos x \\ &\quad - 12 \sin^2 x \cos x + 6 \sin x \cos x \\ &= 2 \sin x \cos x = \sin 2x. \end{aligned}$$

Also, its value at $\pi/2$ is $4 - 2 = 2$, so both requirements are satisfied.

Quick Tip to Solve: After using the integrating factor $\csc^3 x$, replace $\sin 2x$ by $2 \sin x \cos x$; the remaining integrand is exactly -2 times the derivative of $\csc x$.

Step 5: Conclusion The particular solution is $y = 4 \sin^3 x - 2 \sin^2 x$, on the interval $0 < x < \pi$ containing the initial point.

Quick Tip to Solve: After using the integrating factor $\csc^3 x$, replace $\sin 2x$ by $2 \sin x \cos x$; the remaining integrand is exactly -2 times the derivative of $\csc x$.

29(a). Find the vector equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. Hence find whether the plane thus obtained contains the line $\frac{x+2}{5} = \frac{y-3}{4} = \frac{z}{5}$ or not. [6 marks]

Solution:

Concept:

- Every plane of the family $P_1 + \lambda P_2 = 0$ contains the intersection of $P_1 = 0$ and $P_2 = 0$, since both expressions vanish at each point of that common line.
- For a plane $ax + by + cz = d$, a normal vector is $a\hat{i} + b\hat{j} + c\hat{k}$. Two planes are perpendicular precisely when the dot product of their normals is zero.
- A line lies in a plane when a point on it belongs to the plane and its direction vector is perpendicular to the plane's normal. Equivalently, substitution of the line's parametric equations must satisfy the plane equation for every parameter value.

Step 1: Write the family of planes containing the common line Let the required plane be

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0.$$

On expanding, this becomes

$$(1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z = 1 + 5\lambda.$$

Its normal is therefore $\vec{n} = (1 + 2\lambda)\hat{i} + (1 + 3\lambda)\hat{j} + (1 + 4\lambda)\hat{k}$. The second given plane alone, which is not represented by finite λ , has normal $(2, 3, 4)$; its dot product with $(1, -1, 1)$ is 3, so it cannot be the required perpendicular plane.

Step 2: Impose perpendicularity and simplify The normal of $x - y + z = 0$ is $\vec{m} = \hat{i} - \hat{j} + \hat{k}$. The perpendicularity condition is

$$\begin{aligned}\vec{n} \cdot \vec{m} &= 0, \\ (1 + 2\lambda) - (1 + 3\lambda) + (1 + 4\lambda) &= 0, \\ 1 + 3\lambda &= 0.\end{aligned}$$

Hence $\lambda = -1/3$. Substituting this in the family and multiplying by 3 yields

$$3(x + y + z - 1) - (2x + 3y + 4z - 5) = 0,$$

or $x - z + 2 = 0$. Its normal $\hat{i} - \hat{k}$ indeed has zero dot product with $\hat{i} - \hat{j} + \hat{k}$.

Quick Tip to Solve: Use the dot product of plane normals to determine the family parameter; then substitute the line using a single parameter and check that the plane equation holds for every value.

Step 3: Express the plane in vector form For the position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we have $\vec{r} \cdot (\hat{i} - \hat{k}) = x - z$. Thus the vector equation is

$$\vec{r} \cdot (\hat{i} - \hat{k}) = -2.$$

As a further check on containment of the original intersection, subtracting its second plane equation from three times its first equation gives $x - z = -2$; every point common to those planes therefore lies in the plane just obtained.

Step 4: Test whether the entire given line lies in the plane Set the three equal ratios of the printed line equal to t . Its coordinates are

$$x = -2 + 5t, \quad y = 3 + 4t, \quad z = 5t.$$

Substitution into the required plane gives

$$x - z + 2 = (-2 + 5t) - 5t + 2 = 0$$

for every real t . This proves that the whole line is contained in the plane. In vector terms its point $(-2, 3, 0)$ satisfies the plane equation, and its direction $5\hat{i} + 4\hat{j} + 5\hat{k}$ has dot product $5 - 5 = 0$ with the plane normal.

Step 5: Conclusion The required vector equation is $\vec{r} \cdot (\hat{i} - \hat{k}) = -2$. The given line $\frac{x+2}{5} = \frac{y-3}{4} = \frac{z}{5}$ is wholly contained in this plane.

Quick Tip to Solve: Use the dot product of plane normals to determine the family parameter; then substitute the line using a single parameter and check that the plane equation holds for every value.

OR

29(b). Find the image P' of the point P having position vector $\hat{i} + 3\hat{j} + 4\hat{k}$ in the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$. Hence find the length of PP' . [6 marks]

Solution:

Concept:

- Reflection in a plane: the perpendicular from a point to the plane meets it at the midpoint of the segment joining the point to its image.
- For the plane $ax + by + cz + d = 0$, the vector $a\hat{i} + b\hat{j} + c\hat{k}$ is normal to the plane. A line along this vector is perpendicular to the plane.
- The distance of (x_0, y_0, z_0) from this plane is $\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$; the distance from the point to its image is twice this distance.

Step 1: Write the perpendicular through the given point The position vector of P is $\hat{i} + 3\hat{j} + 4\hat{k}$, so

Quick Tip to Solve: Find the perpendicular foot first, then use the coordinate identity $P' = 2H - P$; doubling the point-to-plane distance provides a quick check of the length.

$P = (1, 3, 4)$. The given plane has Cartesian equation

$$2x - y + z + 3 = 0.$$

Its normal vector is $\vec{n} = 2\hat{i} - \hat{j} + \hat{k}$. Therefore the perpendicular through P has vector equation

$$\vec{r} = (\hat{i} + 3\hat{j} + 4\hat{k}) + t(2\hat{i} - \hat{j} + \hat{k}), \quad t \in \mathbb{R}.$$

A point on this perpendicular has coordinates $(1 + 2t, 3 - t, 4 + t)$. Both the foot of the perpendicular and the reflected point must lie on this line.

Step 2: Determine the foot of the perpendicular Let H be the point where the perpendicular meets the plane. Substitute its coordinates into the plane equation:

$$\begin{aligned} 2(1 + 2t) - (3 - t) + (4 + t) + 3 &= 0, \\ 6 + 6t &= 0, \\ t &= -1. \end{aligned}$$

Thus $H = (-1, 4, 3)$. Substitution verifies that $2(-1) - 4 + 3 + 3 = 0$, so H does belong to the reflecting plane.

Step 3: Reflect the point using the midpoint property Write the image as $P' = (x', y', z')$. Since H is the midpoint of PP' ,

$$\left(\frac{1 + x'}{2}, \frac{3 + y'}{2}, \frac{4 + z'}{2} \right) = (-1, 4, 3).$$

Solving these three equations gives $x' = -3$, $y' = 5$, and $z' = 2$. Also,

$$\overrightarrow{PP'} = (-4, 2, -2) = -2(2, -1, 1),$$

which confirms that the segment joining the original point to its image is perpendicular to the plane. The plane expression at P is 6, whereas at P' it is -6 , confirming that the two points lie on opposite sides at equal distances.

Step 4: Calculate the length of the reflection segment Using the distance formula between the two points,

$$\begin{aligned} PP' &= \sqrt{(-3 - 1)^2 + (5 - 3)^2 + (2 - 4)^2} \\ &= \sqrt{16 + 4 + 4} = \sqrt{24}. \end{aligned}$$

As a check, the perpendicular distance from P to the plane is $6/\sqrt{6} = \sqrt{6}$. The image is equally far on the other side, so the segment has twice this length.

Step 5: Conclusion The image is $P' = (-3, 5, 2)$, with position vector $\boxed{-3\hat{i} + 5\hat{j} + 2\hat{k}}$, and $\boxed{PP' = 2\sqrt{6} \text{ units}}$.

Quick Tip to Solve: Find the perpendicular foot first, then use the coordinate identity $P' = 2H - P$; doubling the point-to-plane distance provides a quick check of the length.