

CBSE Class 12 Mathematics Question Paper 2017 with Solution PDF-Set 65-2-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 3 questions of four marks each and 3 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Questions number 1 to 4 carry 1 mark each.

1. Evaluate: $\int_0^{2\pi} \cos^5 x \, dx$.

[1 mark]

Solution:

Concept:

- For a continuous integrand, the fundamental theorem gives $\int_a^b f'(x) \, dx = F(b) - F(a)$.
- An odd power of cosine can be written as $\cos^5 x = \cos x(1 - \sin^2 x)^2$, using $\cos^2 x = 1 - \sin^2 x$.

Step 1: Find a convenient antiderivative Expanding the square and integrating term by term gives

$$\int \cos^5 x \, dx = \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C.$$

Indeed, differentiating this expression gives $\cos x(1 - 2 \sin^2 x + \sin^4 x) = \cos^5 x$.

Step 2: Apply the limits At both endpoints $x = 0$ and $x = 2\pi$, the value of $\sin x$ is zero. Therefore

$$\int_0^{2\pi} \cos^5 x \, dx = \left[\sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x \right]_0^{2\pi} = 0 - 0.$$

Step 3: Conclusion The required integral is $\boxed{0}$.

Quick Tip to Solve: Over a complete period, odd powers of cosine have equal positive and negative contributions.

2. Write the distance of the point $(3, -5, 12)$ from x -axis.

[1 mark]

Solution:

Concept:

- The x -axis consists of all points $(a, 0, 0)$, with $a \in \mathbb{R}$.
- The perpendicular distance of (x, y, z) from the x -axis is $\sqrt{y^2 + z^2}$; the coordinate along the axis does not contribute.

Step 1: Locate the perpendicular foot For $P = (3, -5, 12)$, the foot on the x -axis is $H = (3, 0, 0)$. The displacement $\overrightarrow{HP} = (0, -5, 12)$ has zero x -component, so it is perpendicular to the axis.

Step 2: Calculate the length Using the three-dimensional distance formula,

$$PH = \sqrt{(3-3)^2 + (-5-0)^2 + (12-0)^2} = \sqrt{25 + 144} = \sqrt{169} = 13.$$

This is the shortest distance because the segment ends at the perpendicular foot.

Step 3: Conclusion The distance from the x -axis is $\boxed{13 \text{ units}}$.

Quick Tip to Solve: For distance from an axis, square and add only the two coordinates perpendicular to that axis.

3. If $|A| = 3$ and $A^{-1} = \begin{pmatrix} 3 & -1 \\ 5 & 2 \\ -\frac{5}{3} & \frac{2}{3} \end{pmatrix}$, then write the $\text{adj } A$.

[1 mark]

Solution:

Concept:

- For an invertible square matrix A , $A^{-1} = \frac{\text{adj } A}{|A|}$.
- Hence $\text{adj } A = |A|A^{-1}$; scalar multiplication acts on every entry, preserving the arrangement of entries.

Step 1: Use the inverse-adjoint relation The given determinant is $|A| = 3 \neq 0$, so the inverse-adjoint identity applies. Rearranging it yields

$$\text{adj } A = 3 \begin{pmatrix} 3 & -1 \\ -\frac{5}{3} & \frac{2}{3} \end{pmatrix}.$$

There is no need to recover A first.

Step 2: Multiply the entries Multiply each of the four entries by 3: the first row becomes $(9, -3)$, and the second becomes $(-5, 2)$. Thus

$$3A^{-1} = \begin{pmatrix} 9 & -3 \\ -5 & 2 \end{pmatrix}.$$

The negative signs stay with their respective entries.

Step 3: Conclusion $\text{adj } A = \begin{pmatrix} 9 & -3 \\ -5 & 2 \end{pmatrix}$.

Quick Tip to Solve: When the inverse and determinant are supplied, multiply them directly to obtain the adjoint.

4. For what value of k is the function $f(x) = \begin{cases} \frac{\sin 5x}{3x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ continuous at $x = 0$? [1 mark]

Solution:

Concept:

- Continuity at $x = 0$ requires $\lim_{x \rightarrow 0} f(x) = f(0) = k$.
- The standard trigonometric limit is $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$, with the angle measured in radians.

Step 1: Evaluate the limit of the first term For $x \neq 0$, rewrite

$$\frac{\sin 5x}{3x} = \frac{5}{3} \frac{\sin 5x}{5x}.$$

As $x \rightarrow 0$, we also have $5x \rightarrow 0$, so the standard limit gives a value of $5/3$ for this term, from either

Quick Tip to Solve: First rewrite the denominator to match the angle inside sine, then equate the limit to the assigned value.

side.

Step 2: Impose continuity The cosine function is continuous, so $\lim_{x \rightarrow 0} \cos x = \cos 0 = 1$. Adding the two limits,

$$\lim_{x \rightarrow 0} f(x) = \frac{5}{3} + 1 = \frac{8}{3}.$$

The separately defined value $f(0) = k$ must equal this common two-sided limit.

Step 3: Conclusion The function is continuous at zero for $k = \frac{8}{3}$.

Quick Tip to Solve: First rewrite the denominator to match the angle inside sine, then equate the limit to the assigned value.

Section B (16 marks)

Questions number 5 to 12 carry 2 marks each.

5. Find $\frac{dy}{dx}$ at $t = \frac{2\pi}{3}$ when $x = 10(t - \sin t)$ and $y = 12(1 - \cos t)$.

[2 marks]

Solution:

Concept:

- For parametric equations $x = x(t)$ and $y = y(t)$, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ wherever $dx/dt \neq 0$.
- The derivatives are $\frac{d}{dt} \sin t = \cos t$ and $\frac{d}{dt} \cos t = -\sin t$, with t in radians.

Step 1: Differentiate both coordinates From $x = 10(t - \sin t)$ and $y = 12(1 - \cos t)$,

$$\frac{dx}{dt} = 10(1 - \cos t), \quad \frac{dy}{dt} = 12 \sin t.$$

The derivative of $1 - \cos t$ is positive $\sin t$, because the two minus signs cancel.

Step 2: Evaluate the ratio at the given parameter At $t = 2\pi/3$, $\sin t = \sqrt{3}/2$ and $\cos t = -1/2$. Therefore $dx/dt = 10(3/2) = 15$, which is nonzero, and $dy/dt = 12(\sqrt{3}/2) = 6\sqrt{3}$. The positive ratio indicates that y increases locally as x increases. The required slope is

$$\left. \frac{dy}{dx} \right|_{t=2\pi/3} = \frac{6\sqrt{3}}{15} = \frac{2\sqrt{3}}{5}.$$

Step 3: Conclusion $\left. \frac{dy}{dx} \right|_{t=2\pi/3} = \frac{2\sqrt{3}}{5}$.

Quick Tip to Solve: Evaluate both parameter derivatives at the same parameter value before taking their ratio.

6. Show that the function f given by $f(x) = \tan^{-1}(\sin x + \cos x)$ is decreasing for all $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. [2 marks]

Solution:

Concept:

- A differentiable function is strictly decreasing on an interval if its derivative is negative throughout that interval.
- For $u = u(x)$, $\frac{d}{dx} \tan^{-1} u = \frac{u'}{1 + u^2}$; the denominator is always positive for real u .

Step 1: Find the derivative Let $u = \sin x + \cos x$. Then $u' = \cos x - \sin x$, and the chain rule gives

$$f'(x) = \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}.$$

The function and its derivative are defined throughout the given interval, since the denominator cannot vanish.

Step 2: Determine its sign on the interval For $\pi/4 < x < \pi/2$, both sine and cosine are positive and $\sin x > \cos x$. For example, $\sin x - \cos x = \sqrt{2} \sin(x - \pi/4) > 0$, because $0 < x - \pi/4 < \pi/4$. Thus the numerator $\cos x - \sin x$ is negative, while the denominator is positive. Hence $f'(x) < 0$ at every point of the interval.

Step 3: Conclusion Therefore f is strictly decreasing on $(\pi/4, \pi/2)$.

Quick Tip to Solve: For an inverse tangent of an expression, the derivative sign comes entirely from the derivative of that expression.

7. A line passes through the point with position vector $2\hat{i} - 3\hat{j} + 4\hat{k}$ and is perpendicular to the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 5\hat{k}) = 7$. Find the equation of the line in cartesian and vector forms. [2 marks]

Solution:

Concept:

- In the plane equation $\vec{r} \cdot \vec{n} = d$, the coefficient vector $\vec{n}$ is a normal vector to the plane.
- A line through position vector $\vec{a}$ with direction vector $\vec{b} \neq \vec{0}$ has equation $\vec{r} = \vec{a} + \lambda \vec{b}$, where $\lambda \in \mathbb{R}$.

Step 1: Identify a direction for the line The plane is $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 5\hat{k}) = 7$, so a normal vector is $\vec{n} = 3\hat{i} + 4\hat{j} - 5\hat{k}$. Since the required line is perpendicular to this plane, its direction is parallel to $\vec{n}$. The given point has coordinates $(2, -3, 4)$.

Step 2: Write and convert the vector equation Taking the normal itself as the direction vector gives

$$\vec{r} = (2\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda(3\hat{i} + 4\hat{j} - 5\hat{k}).$$

Equating components yields $x = 2 + 3\lambda$, $y = -3 + 4\lambda$, and $z = 4 - 5\lambda$. Solving each for λ produces the cartesian symmetric form

$$\frac{x - 2}{3} = \frac{y + 3}{4} = \frac{z - 4}{-5}.$$

Quick Tip to Solve: For a line perpendicular to a plane, use the plane coefficients as direction ratios and retain all their signs.

At $\lambda = 0$ the line passes through the required point, and its direction is the plane normal.

Step 3: Conclusion The vector form is $\vec{r} = 2\hat{i} - 3\hat{j} + 4\hat{k} + \lambda(3\hat{i} + 4\hat{j} - 5\hat{k}), \lambda \in \mathbb{R}$. The cartesian form is $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-4}{-5}$.

Quick Tip to Solve: For a line perpendicular to a plane, use the plane coefficients as direction ratios and retain all their signs.

8. If $P(A) = 0.4$, $P(B) = p$, $P(A \cup B) = 0.6$ and A and B are given to be independent events, find the value of p . [2 marks]

Solution:

Concept:

- For any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. The intersection is subtracted because it occurs in both individual probabilities.
- For independent events, $P(A \cap B) = P(A)P(B)$. Independence does not mean that the intersection has probability zero.

Step 1: Use independence to express the intersection Let $P(B) = p$. Since the events are independent,

$$P(A \cap B) = 0.4p.$$

This expression must be used in the addition rule: simply adding the two probabilities would count outcomes common to both events twice.

Step 2: Substitute in the union formula The given union probability gives

$$0.6 = 0.4 + p - 0.4p,$$

$$0.2 = 0.6p,$$

$$p = \frac{0.2}{0.6} = \frac{1}{3}.$$

The value lies between zero and one, as a probability must. Substitution gives $0.4 + \frac{1}{3} - \frac{0.4}{3} = 0.6$, verifying the required union probability.

Step 3: Conclusion The required value is $p = \frac{1}{3}$.

Quick Tip to Solve: For independent events, replace the intersection by the product before solving the union equation.

9. A company produces two types of goods A and B , that require gold and silver. Each unit of type A requires 3 g of silver and 1 g of gold while that of type B requires 1 g of silver and 2 g of gold. The company can procure a maximum of 9 g of silver and 8 g of gold. If each unit of type A brings a profit of ₹ 40 and that of type B ₹ 50, formulate LPP to maximize profit. [2 marks]

Solution:

Concept:

- Choose decision variables for the quantities to be produced: x units of type A and y units of type B . Production quantities satisfy $x \geq 0$ and $y \geq 0$.
- The objective function adds the profits from the two products. Each resource constraint adds the amounts consumed and cannot exceed the available amount.

Step 1: Form the objective function Each unit of type A contributes Rs. 40 and each unit of type B contributes Rs. 50. Therefore the total profit, in rupees, is

$$Z = 40x + 50y.$$

The company seeks to maximize this quantity by choosing feasible values of x and y .

Step 2: Form the resource constraints Type A uses 3 g of silver per unit and type B uses 1 g per unit. Since at most 9 g is available,

$$3x + y \leq 9.$$

The gold consumption is 1 g per unit of A and 2 g per unit of B . The gold limit therefore gives

$$x + 2y \leq 8.$$

Both inequalities are upper bounds because unused resources are allowed. Include $x \geq 0$ and $y \geq 0$, since negative production has no meaning.

Step 3: Conclusion The LPP is: maximize $Z = 40x + 50y$, subject to

$$3x + y \leq 9, \quad x + 2y \leq 8, \quad x \geq 0, \quad y \geq 0.$$

Quick Tip to Solve: Keep separate rows for silver and gold, multiplying each per-unit requirement by its production variable.

10. Find: $\int \frac{dx}{\sqrt{3 - 2x - x^2}}.$

[2 marks]

Solution:

Concept:

- Complete the square to put a quadratic radicand in the form $a^2 - u^2$, with $a > 0$.
- The standard antiderivative is $\int du/\sqrt{a^2 - u^2} = \sin^{-1}(u/a) + C$, valid for $|u| < a$.

Quick Tip to Solve: Complete the square first; the denominator in the inverse-sine argument is the square root of the constant.

Step 1: Complete the square The coefficient of x^2 is negative, so group the quadratic terms before completing the square:

$$\begin{aligned} 3 - 2x - x^2 &= 3 - (x^2 + 2x) \\ &= 3 - ((x + 1)^2 - 1) \\ &= 4 - (x + 1)^2. \end{aligned}$$

Thus the constant playing the role of a in the standard formula is 2, not 4.

Step 2: Use the standard inverse-sine integral Put $u = x + 1$. Then $du = dx$, so no additional factor arises from the substitution. Therefore

$$\begin{aligned} \int \frac{dx}{\sqrt{3 - 2x - x^2}} &= \int \frac{du}{\sqrt{2^2 - u^2}} \\ &= \sin^{-1} \left(\frac{u}{2} \right) + C. \end{aligned}$$

The real integrand is defined when $|x + 1| < 2$, that is, $-3 < x < 1$. Differentiating the expression gives $1/\sqrt{4 - (x + 1)^2}$, confirming the antiderivative.

Step 3: Conclusion

$$\int \frac{dx}{\sqrt{3 - 2x - x^2}} = \sin^{-1} \left(\frac{x + 1}{2} \right) + C.$$

Quick Tip to Solve: Complete the square first; the denominator in the inverse-sine argument is the square root of the constant.

11. If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$, then find the value of $|2AB|$. [2 marks]

Solution:

Concept:

- For square matrices of the same order, the determinant of their product is $|AB| = |A||B|$. This holds without requiring the matrices to commute.
- For an $n \times n$ matrix M , $|kM| = k^n|M|$, because each of the n rows is multiplied by k .

Step 1: Evaluate the determinant of the product Both matrices have order 3, so their product AB is also a square matrix of order 3. Using the product rule for determinants,

$$|AB| = |A||B| = (-1)(3) = -3.$$

The negative sign must be retained when subsequently multiplying by the scalar factor.

Step 2: Account for the scalar multiplying the matrix In $2AB$, the scalar 2 multiplies every entry of the matrix AB , and hence each of its three rows. A determinant is linear in each row separately; extracting a factor 2 from all three rows therefore contributes $2 \times 2 \times 2 = 2^3$. Thus

$$|2AB| = 2^3|AB| = 8(-3) = -24.$$

Quick Tip to Solve: When a scalar multiplies a whole matrix, raise it to the matrix order before multiplying the determinant.

Using only a single factor of 2 would be incorrect, because it would account for scaling just one row rather than the entire matrix.

Step 3: Conclusion The required determinant is $|2AB| = -24$.

Quick Tip to Solve: When a scalar multiplies a whole matrix, raise it to the matrix order before multiplying the determinant.

12. The radius r of the base of a right circular cone is decreasing at the rate of 2 cm/min and its height h is increasing at the rate of 3 cm/min. When $r = 3.5$ cm and $h = 6$ cm, find the rate of change of the volume of the cone. [Use $\pi = \frac{22}{7}$]. [2 marks]

Solution:

Concept:

- The volume of a right circular cone is $V = \frac{1}{3}\pi r^2 h$, where r is its base radius and h its height.
- When both dimensions depend on time t , the product and chain rules give $\frac{dV}{dt} = \frac{\pi}{3} \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right)$.

Step 1: Assign signs and differentiate the volume Take t in minutes. A decreasing radius has a negative rate, while an increasing height has a positive rate:

$$\frac{dr}{dt} = -2 \text{ cm/min}, \quad \frac{dh}{dt} = 3 \text{ cm/min}.$$

Differentiating $V = \frac{1}{3}\pi r^2 h$ with both r and h variable gives

$$\frac{dV}{dt} = \frac{\pi}{3} \left(2rh \frac{dr}{dt} + r^2 \frac{dh}{dt} \right).$$

Treating height as constant would omit its contribution to the volume change.

Step 2: Evaluate at the given instant Substitute $r = 3.5$, $h = 6$ and $\pi = 22/7$:

$$\begin{aligned} \frac{dV}{dt} &= \frac{22}{21} [2(3.5)(6)(-2) + (3.5)^2(3)] \\ &= \frac{22}{21} [-84 + 36.75] \\ &= \frac{22}{21} (-47.25) = -49.5 \text{ cm}^3/\text{min}. \end{aligned}$$

The radius contribution outweighs the height contribution, so the net volume change is negative despite the increasing height.

Step 3: Conclusion The rate of change is $\frac{dV}{dt} = -49.5 \text{ cm}^3/\text{min}$; the volume is decreasing at $49.5 \text{ cm}^3/\text{min}$.

Quick Tip to Solve: Write decreasing rates with a minus sign, and differentiate before inserting the instantaneous dimensions.

Section C (44 marks)

Questions number 13 to 23 carry 4 marks each.

13. There are 4 cards numbered 1 to 4, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X . [4 marks]

Solution:

Concept:

- For equally likely selections, a value's probability is the number of selections producing it divided by the total number of selections.
- The mean is $E(X) = \sum_x xP(X = x)$. The variance is $\text{Var}(X) = E(X^2) - [E(X)]^2$, with $E(X^2) = \sum_x x^2P(X = x)$.

Step 1: List the equally likely selections The cards are drawn without replacement, so the same card cannot occur twice. Since only the sum matters, the order of the two cards does not matter. There are $\binom{4}{2} = 6$ unordered selections. Each corresponds to two of the twelve equally likely ordered draws, and hence has probability $1/6$.

Cards	(1, 2)	(1, 3)	(1, 4)	(2, 3)	(2, 4)	(3, 4)
X	3	4	5	5	6	7

Thus the sum 5 occurs for two selections, whereas each other possible sum occurs for exactly one selection.

Step 2: Form the distribution and compute the mean The probability distribution is

x	3	4	5	6	7
$P(X = x)$	1/6	1/6	2/6	1/6	1/6

Its probabilities sum to $(1 + 1 + 2 + 1 + 1)/6 = 1$, confirming that every selection has been included. Therefore,

$$E(X) = \frac{3 + 4 + 2(5) + 6 + 7}{6} = \frac{30}{6} = 5.$$

The distribution is symmetric about 5, which provides an independent check of the mean.

Step 3: Calculate the variance using the second moment Square each value of X , retaining its original probability:

$$E(X^2) = \frac{3^2 + 4^2 + 2(5^2) + 6^2 + 7^2}{6} = \frac{9 + 16 + 50 + 36 + 49}{6} = \frac{80}{3}.$$

It follows that

$$\text{Var}(X) = \frac{80}{3} - 5^2 = \frac{5}{3}.$$

To check this directly, the squared deviations from the mean give $\{4 + 1 + 2(0) + 1 + 4\}/6 = 5/3$, agreeing with the second-moment method.

Step 4: Conclusion The mean is $\boxed{5}$, and the variance is $\boxed{\frac{5}{3}}$.

Quick Tip to Solve: Enumerate unordered pairs first: sum 5 occurs twice and must receive probability $2/6$, whereas the other sums each receive $1/6$.

14. Find the particular solution of the differential equation $2y e^{x/y} dx + (y - 2x e^{x/y}) dy = 0$, given that $x = 0$ when $y = 1$. [4 marks]

Solution:

Concept:

- An equation involving the ratio x/y can be simplified by treating x as a function of y and putting $v = x/y$, so $x = vy$ and $dx = v dy + y dv$.
- Separation of variables reduces the equation to integrals in v and y . An initial condition determines the constant of integration; $\int dy/y = \ln |y|$ for $y \neq 0$.

Step 1: Use the homogeneous substitution The given equation is

$$2y e^{x/y} dx + (y - 2x e^{x/y}) dy = 0.$$

Put $x = vy$; this is valid since $y \neq 0$ on the domain of the given equation. Differentiating $x = vy$ gives $dx = v dy + y dv$. Substitute both expressions into the equation:

$$2y e^v (v dy + y dv) + (y - 2vy e^v) dy = 0.$$

The two terms involving $2vy e^v dy$ cancel, leaving

$$2y^2 e^v dv + y dy = 0.$$

Step 2: Separate and integrate Divide by y^2 , which is nonzero on the domain of the equation:

$$2e^v dv = -\frac{dy}{y}.$$

Integrating the two sides gives

$$2e^v = -\ln |y| + C, \quad 2e^{x/y} + \ln |y| = C.$$

This is the implicit general integral.

Step 3: Apply the initial condition and check the integral At $x = 0$, $y = 1$, so

$$C = 2e^{0/1} + \ln 1 = 2.$$

The solution through this point lies on the positive- y domain, where $\ln |y| = \ln y$. To verify the result, differentiate $2e^{x/y} + \ln y = 2$:

$$2e^{x/y} \left(\frac{dx}{y} - \frac{x dy}{y^2} \right) + \frac{dy}{y} = 0.$$

Multiplying by y^2 restores $2y e^{x/y} dx + (y - 2x e^{x/y}) dy = 0$, exactly the original equation. The initial point also satisfies the integral.

Step 4: Conclusion The particular solution is $\boxed{2e^{x/y} + \ln y = 2}$, with $y > 0$.

Quick Tip to Solve: Put $x = vy$: the two terms containing v cancel immediately, leaving the separable equation $2e^v dv = -dy/y$.

15. In a shop X , 30 tins of pure ghee and 40 tins of adulterated ghee which look alike, are kept for sale while in shop Y , similar 50 tins of pure ghee and 60 tins of adulterated ghee are there. One tin of ghee is purchased from one of the randomly selected shops and is found to be adulterated. Find the probability that it is purchased from shop Y . What measures should be taken to stop adulteration?

[4 marks]

Solution:

Concept:

- Bayes' theorem reverses conditional probability: $P(Y | A) = \frac{P(Y)P(A | Y)}{P(X)P(A | X) + P(Y)P(A | Y)}$, where A means that the purchased tin is adulterated.
- The shops are selected at random, so $P(X) = P(Y) = \frac{1}{2}$. Each tin within the chosen shop is equally likely to be purchased.

Step 1: Find the adulterated fraction in each shop Shop X has $30 + 40 = 70$ tins, including 40 adulterated tins. Shop Y has $50 + 60 = 110$ tins, including 60 adulterated tins. Thus

$$P(A | X) = \frac{40}{70} = \frac{4}{7}, \quad P(A | Y) = \frac{60}{110} = \frac{6}{11}.$$

These fractions are conditional on the selected shop. We must retain each shop's own total: pooling the tins would incorrectly give the larger shop a greater probability of being selected.

Step 2: Use total probability and Bayes' theorem Selecting X and selecting Y are mutually exclusive and exhaustive possibilities. Consequently,

$$\begin{aligned} P(A) &= P(X)P(A | X) + P(Y)P(A | Y) \\ &= \frac{1}{2} \cdot \frac{4}{7} + \frac{1}{2} \cdot \frac{6}{11} = \frac{2}{7} + \frac{3}{11} = \frac{43}{77}. \end{aligned}$$

The joint probability of selecting Y and obtaining an adulterated tin is $P(Y \cap A) = \frac{1}{2} \cdot \frac{6}{11} = \frac{3}{11}$. Given the information that the tin is adulterated, the required probability is therefore

$$P(Y | A) = \frac{P(Y \cap A)}{P(A)} = \frac{3/11}{43/77} = \frac{21}{43}.$$

The result is slightly less than one-half, consistent with the adulterated fraction in Y being slightly smaller than that in X .

Step 3: State measures to stop adulteration Food safety authorities should carry out regular inspections and laboratory testing of random samples. Strict penalties should be imposed on sellers found adulterating food. Consumers should purchase sealed, properly labelled products from reliable sellers, check quality certification and expiry details, and report suspected adulteration. Public awareness about adulteration and its health effects can also discourage dishonest practices.

Step 4: Conclusion The probability that the adulterated tin was purchased from shop Y is $\frac{21}{43}$. Regular testing, strict enforcement and informed purchasing help prevent adulteration.

Quick Tip to Solve: Assign each shop probability one-half first, then use its own adulterated fraction; pooling all tins would describe a different random experiment.

16. If $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b} = 4\hat{i} - 7\hat{j} + \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 6$.

[4 marks]

Solution:

Concept:

- For $\vec{a} = (a_1, a_2, a_3)$ and $\vec{c} = (x, y, z)$, the cross product is $(a_2z - a_3y)\hat{i} + (a_3x - a_1z)\hat{j} + (a_1y - a_2x)\hat{k}$.
- The dot product is $\vec{a} \cdot \vec{c} = a_1x + a_2y + a_3z$. Equating components of equal vectors gives simultaneous scalar equations.

Step 1: Express the unknown vector and use the cross product Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$. With $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$, expansion of the determinant gives

$$\begin{aligned}\vec{a} \times \vec{c} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ x & y & z \end{vmatrix} \\ &= (y + z)\hat{i} - (x + 2z)\hat{j} + (2y - x)\hat{k}.\end{aligned}$$

Comparing this with $\vec{b} = 4\hat{i} - 7\hat{j} + \hat{k}$, we obtain

$$y + z = 4, \quad x + 2z = 7, \quad 2y - x = 1.$$

The minus sign in the middle component is essential: its comparison with -7 gives $x + 2z = 7$.

Step 2: Combine the dot-product condition with the component equations The remaining condition is

$$\vec{a} \cdot \vec{c} = 2x + y - z = 6.$$

The first two cross-product equations yield $y = 4 - z$ and $x = 7 - 2z$. Substituting these expressions into the dot-product equation,

$$\begin{aligned}2(7 - 2z) + (4 - z) - z &= 6, \\ 18 - 6z &= 6, \\ z &= 2.\end{aligned}$$

Hence $y = 4 - 2 = 2$ and $x = 7 - 2(2) = 3$. The third cross-product equation is consistent because $2y - x = 4 - 3 = 1$. The cross-product condition alone leaves a free component along $\vec{a}$; the dot-product condition fixes that component and determines the vector uniquely.

Step 3: Check both prescribed products For $\vec{c} = 3\hat{i} + 2\hat{j} + 2\hat{k}$, direct substitution gives

$$\begin{aligned}\vec{a} \times \vec{c} &= (2 + 2)\hat{i} - (3 + 4)\hat{j} + (4 - 3)\hat{k} \\ &= 4\hat{i} - 7\hat{j} + \hat{k} = \vec{b}, \\ \vec{a} \cdot \vec{c} &= 2(3) + 1(2) - 1(2) = 6.\end{aligned}$$

Thus the vector satisfies both requirements, including the direction and sign of every component of the cross product.

Step 4: Conclusion The required vector is $\boxed{\vec{c} = 3\hat{i} + 2\hat{j} + 2\hat{k}}$.

Quick Tip to Solve: Solve two cross-product component equations together with the dot-product equation, then use the remaining cross-product component as a check.

17(a). If $xy = e^{(x-y)}$, then show that $\frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$.

[4 marks]

Solution:

Concept:

- In implicit differentiation, y is a differentiable function of x . The product rule gives $\frac{d}{dx}(xy) = y + x \frac{dy}{dx}$.
- The chain rule gives $\frac{d}{dx}e^u = e^u \frac{du}{dx}$. For $u = x - y$, the inner derivative is $1 - \frac{dy}{dx}$.
- Use the original equation after differentiation to replace e^{x-y} by xy , obtaining an equation linear in dy/dx .

Step 1: Differentiate both sides Starting with $xy = e^{x-y}$, differentiate with respect to x , remembering that y depends on x . Apply the product rule to the left side and the chain rule to the right side:

$$y + x \frac{dy}{dx} = e^{x-y} \left(1 - \frac{dy}{dx} \right).$$

The derivative of $-y$ is $-dy/dx$, which accounts for the minus sign inside the parentheses.

Step 2: Substitute and collect the derivative terms The given relation states that $e^{x-y} = xy$. Replacing the exponential by this product gives

$$\begin{aligned} y + x \frac{dy}{dx} &= xy - xy \frac{dy}{dx}, \\ x \frac{dy}{dx} + xy \frac{dy}{dx} &= xy - y, \\ x(1+y) \frac{dy}{dx} &= y(x-1). \end{aligned}$$

Both derivative terms have been brought to the left side. Factoring their common factor $x dy/dx$ gives $x(1+y) dy/dx$, and factoring the right side gives $y(x-1)$.

Step 3: Isolate the derivative Since $xy = e^{x-y} > 0$, $x \neq 0$. At points on a differentiable branch with $y \neq -1$, divide by $x(1+y)$. This produces the desired ratio. If $y = -1$, positivity of xy implies $x < 0$. The differentiated equation would then have zero on its left side and $1 - x \neq 0$ on its right side, so a finite derivative cannot exist at such a point. Thus this division is valid wherever the required derivative exists.

Step 4: Conclusion Hence, wherever y is differentiable as a function of x ,

$$\boxed{\frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}}.$$

Quick Tip to Solve: Replace the exponential by xy immediately after differentiating, then collect all dy/dx terms on one side.

17(b). If $\log y = \tan^{-1} x$, then show that $(1 + x^2) \frac{d^2 y}{dx^2} + (2x - 1) \frac{dy}{dx} = 0$.

[4 marks]

Solution:

Concept:

- In calculus, $\log y$ denotes the natural logarithm, with $y > 0$. For a differentiable function $y(x)$, the chain rule gives $\frac{d}{dx} \log y = \frac{1}{y} \frac{dy}{dx}$.
- The derivative of the principal inverse tangent is $\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$, for every real x .
- The product rule applied to $(1 + x^2)y'$ gives $(1 + x^2)y'' + 2xy'$, where $y' = dy/dx$ and $y'' = d^2y/dx^2$.

Step 1: Differentiate the given logarithmic relation We have $\log y = \tan^{-1} x$. Differentiate both sides with respect to x . On the left, use the chain rule because y depends on x ; on the right, use the derivative of inverse tangent:

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{1 + x^2}.$$

The original relation requires $y > 0$, so multiplication by y is valid. Also, $1 + x^2 > 0$ for all real x , so there is no zero denominator.

Step 2: Write an equation suitable for a second differentiation Multiplying the first derivative equation by $y(1 + x^2)$ yields

$$(1 + x^2) \frac{dy}{dx} = y.$$

This form is convenient because the expression on the left is a product of two differentiable functions. It lets us obtain the second derivative without expanding a quotient. The original relation also gives $y = e^{\tan^{-1} x}$, so y has the derivatives needed in the following step for every real x .

Step 3: Differentiate again using the product rule Differentiate the last equation with respect to x . The derivative of $1 + x^2$ is $2x$, and the derivative of dy/dx is d^2y/dx^2 . Therefore,

$$(1 + x^2) \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} = \frac{dy}{dx}.$$

Subtract the derivative on the right side from both sides. The coefficient of dy/dx becomes $2x - 1$; the second derivative term retains the factor $1 + x^2$.

Step 4: Conclusion Thus the required identity is

$$(1 + x^2) \frac{d^2 y}{dx^2} + (2x - 1) \frac{dy}{dx} = 0.$$

Quick Tip to Solve: Rearrange the first derivative as $(1 + x^2)y' = y$ before differentiating again to avoid a quotient-rule calculation.

18(a). Evaluate: $\int_{-2}^1 |x^3 - x| dx$.

[4 marks]

Solution:

Concept:

- For a real function $f(x)$, its modulus equals $f(x)$ where $f(x) \geq 0$ and equals $-f(x)$ where $f(x) < 0$. Thus a definite integral involving a modulus must be split wherever the function changes sign.
- The fundamental theorem of calculus gives $\int_a^b f(x) dx = F(b) - F(a)$, where $F'(x) = f(x)$. Here an antiderivative of $x^3 - x$ is $F(x) = x^4/4 - x^2/2$.

Step 1: Locate the zeros and determine the signs Factor the expression inside the modulus:

$$x^3 - x = x(x - 1)(x + 1).$$

Its zeros in the interval of integration are $-1, 0, 1$. For $-2 < x < -1$, all three factors are negative, so their product is negative. For $-1 < x < 0$, the factors x and $x - 1$ are negative while $x + 1$ is positive, so the product is positive. For $0 < x < 1$, only $x - 1$ is negative, so the product is negative. At each zero the integrand is zero; these endpoints do not affect the integral.

Step 2: Remove the modulus interval by interval Denote the required integral by I . Using the signs just found,

$$I = - \int_{-2}^{-1} (x^3 - x) dx + \int_{-1}^0 (x^3 - x) dx - \int_0^1 (x^3 - x) dx.$$

Each term now integrates a polynomial. The minus signs on the first and last intervals ensure that the original nonnegative modulus is preserved.

Step 3: Evaluate the three contributions With $F(x) = x^4/4 - x^2/2$, the endpoint values are

$$F(-2) = 2, \quad F(-1) = -\frac{1}{4}, \quad F(0) = 0, \quad F(1) = -\frac{1}{4}.$$

Therefore,

$$I = - \left(-\frac{1}{4} - 2 \right) + \left(0 + \frac{1}{4} \right) - \left(-\frac{1}{4} - 0 \right) = \frac{9}{4} + \frac{1}{4} + \frac{1}{4}.$$

All three contributions are positive, as expected for an integral of an absolute value.

Step 4: Conclusion Hence, $\int_{-2}^1 |x^3 - x| dx = \boxed{\frac{11}{4}}$.

Quick Tip to Solve: Factor the expression before removing its modulus; split at every interior root and check the sign on each resulting interval.

18(b). Find: $\int e^{2x} \sin(3x + 1) dx$.

[4 marks]

Solution:

Concept:

- Integration by parts states $\int u dv = uv - \int v du$. Choosing a trigonometric factor as u and the exponential factor as dv produces a second integral that returns to the original one after another application.
- The chain rule gives $\frac{d}{dx} \sin(3x + 1) = 3 \cos(3x + 1)$ and $\frac{d}{dx} \cos(3x + 1) = -3 \sin(3x + 1)$, while $\int e^{2x} dx = e^{2x}/2$.

Step 1: Apply integration by parts to the given integral Write $I = \int e^{2x} \sin(3x + 1) dx$. Choose

$$u = \sin(3x + 1), \quad dv = e^{2x} dx, \quad du = 3 \cos(3x + 1) dx, \quad v = \frac{e^{2x}}{2}.$$

Integration by parts then gives

$$I = \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3}{2} J, \quad J = \int e^{2x} \cos(3x + 1) dx.$$

Include the arbitrary constant after isolating I .

Step 2: Evaluate the auxiliary integral by parts For J , take $u = \cos(3x + 1)$ and $dv = e^{2x} dx$. Then $du = -3 \sin(3x + 1) dx$ and $v = e^{2x}/2$. Hence

$$\begin{aligned} J &= \frac{e^{2x}}{2} \cos(3x + 1) - \int \frac{e^{2x}}{2} [-3 \sin(3x + 1)] dx \\ &= \frac{e^{2x}}{2} \cos(3x + 1) + \frac{3}{2} I. \end{aligned}$$

The two negative signs give the positive coefficient of I .

Step 3: Substitute and collect the original integral Substituting the expression for J into the first equation yields

$$\begin{aligned} I &= \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3e^{2x}}{4} \cos(3x + 1) - \frac{9}{4} I, \\ \frac{13}{4} I &= \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3e^{2x}}{4} \cos(3x + 1). \end{aligned}$$

Multiplying by $4/13$ isolates I . To verify, differentiate the resulting numerator using the product rule. The cosine terms cancel, and the sine coefficient is $2 \cdot 2 + 3 \cdot 3 = 13$, leaving exactly $e^{2x} \sin(3x + 1)$.

Step 4: Conclusion Therefore,

$$\int e^{2x} \sin(3x + 1) dx = \frac{e^{2x}}{13} [2 \sin(3x + 1) - 3 \cos(3x + 1)] + C,$$

where C is an arbitrary real constant.

Quick Tip to Solve: Apply integration by parts twice and bring the returned original integral to the left; retain the factor 3 from differentiating the angle.

19(a). Using properties of determinants show that $\begin{vmatrix} 1 & 1 & 1+x \\ 1 & 1+y & 1 \\ 1+z & 1 & 1 \end{vmatrix} = xyz + yz + zx + xy.$ [4 marks]

Solution:

Concept:

- Subtracting one row from another does not change the determinant. Thus the operations $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ simplify the given determinant without changing its value.
- Cofactor expansion along the first row has signs $+, -, +$. The determinant of a two-by-two matrix is $ad - bc$; retaining these signs is essential here.

Step 1: Simplify by row operations Let the determinant on the left be D . Apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$. The first row remains unchanged, while the second and third rows become $(0, y, -x)$ and $(z, 0, -x)$, respectively. Therefore

$$D = \begin{vmatrix} 1 & 1 & 1+x \\ 0 & y & -x \\ z & 0 & -x \end{vmatrix}.$$

In particular, the negative signs in the third column are forced by subtracting $1+x$ from 1.

Step 2: Expand the simplified determinant Expanding along the first row with its alternating cofactor signs gives

$$\begin{aligned} D &= 1\{y(-x) - (-x)0\} - 1\{0(-x) - (-x)z\} \\ &\quad + (1+x)\{0 \cdot 0 - yz\} \\ &= -xy - xz - (1+x)yz \\ &= -xy - xz - yz - xyz. \end{aligned}$$

This calculation uses determinant properties and is valid for arbitrary real x, y, z , including zero values, since no division has been used.

Step 3: Check the printed equality The right side printed in the question is $xyz + yz + zx + xy$, which is the negative of the value just obtained. For an independent check, put $x = y = z = 1$. Then

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 1(2-1) - 1(1-2) + 2(1-4) = -4,$$

whereas the printed right side is $1 + 1 + 1 + 1 = 4$. Thus the requested equality is false as printed and cannot be proved for general x, y, z .

Step 4: Conclusion The determinant is $\boxed{-(xyz + yz + zx + xy)}$. The printed identity has a sign error on its right side.

Quick Tip to Solve: Subtract the first row from the other two, then track the minus sign in each third-column entry before expanding.

19(b). Find matrix X so that $X \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$.

[4 marks]

Solution:**Concept:**

- For matrix multiplication, the number of columns in the first matrix equals the number of rows in the second. Each product entry is the scalar product of a row of the first matrix with a column of the second.
- Two matrices are equal when their orders agree and their corresponding entries are equal. This converts the given matrix equation into simultaneous linear equations.

Step 1: Determine the order and multiply The matrix multiplying X is of order 2×3 , and the product has order 2×3 . Thus X must have two columns and two rows. Write

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Multiplying row by column gives

$$X \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{pmatrix}.$$

Equate corresponding entries of the given matrices.

Step 2: Find the entries in the first row From the first two columns of the first row,

$$a + 4b = -7, \quad 2a + 5b = -8.$$

Subtract twice the first equation from the second to obtain $-3b = -8 - 2(-7) = 6$, so $b = -2$. Substituting in $a + 4b = -7$ gives $a - 8 = -7$, hence $a = 1$. The third entry is also correct because $3a + 6b = 3 - 12 = -9$.

Step 3: Find the second row and verify the full product Similarly, the second row gives

$$c + 4d = 2, \quad 2c + 5d = 4.$$

Subtracting twice the first equation from the second yields $-3d = 0$, so $d = 0$ and $c = 2$. The third entry is $3c + 6d = 6$, as required. Direct substitution now verifies every entry:

$$\begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1-8 & 2-10 & 3-12 \\ 2 & 4 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}.$$

Step 4: Conclusion The required matrix is $X = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$.

Quick Tip to Solve: Use the first two columns to determine each row of X , then check the third column to confirm consistency.

20. Find: $\int \frac{x \, dx}{(2+x^2)(4+x^4)}.$

[4 marks]

Solution:

Concept:

- Use the substitution $t = x^2$, so that $dt = 2x \, dx$. This changes the even powers of x in the denominator into a linear and a quadratic factor in t .
- For distinct factors, resolve the rational integrand into partial fractions. The standard antiderivatives needed are $\int du/u = \ln |u|$ and $\int du/(u^2 + a^2) = a^{-1} \tan^{-1}(u/a)$.

Step 1: Reduce the powers by substitution Put $t = x^2$. Then $x \, dx = dt/2$ and $x^4 = t^2$. Thus the given integral becomes

$$I = \frac{1}{2} \int \frac{dt}{(t+2)(t^2+4)}.$$

The substitution is especially useful because the numerator already contains the factor x , which is proportional to the derivative of x^2 .

Step 2: Determine the partial fractions Write

$$\frac{1}{(t+2)(t^2+4)} = \frac{A}{t+2} + \frac{Bt+C}{t^2+4}.$$

Multiplication by the common denominator gives

$$1 = A(t^2+4) + (Bt+C)(t+2).$$

Comparing coefficients yields $A+B=0$, $2B+C=0$, and $4A+2C=1$. Therefore $B=-A$, $C=2A$, and $8A=1$, giving

$$A = \frac{1}{8}, \quad B = -\frac{1}{8}, \quad C = \frac{1}{4}.$$

Step 3: Integrate and return to the original variable Substituting these coefficients,

$$\begin{aligned} I &= \frac{1}{16} \int \frac{dt}{t+2} - \frac{1}{16} \int \frac{t \, dt}{t^2+4} + \frac{1}{8} \int \frac{dt}{t^2+4} \\ &= \frac{1}{16} \ln |t+2| - \frac{1}{32} \ln(t^2+4) + \frac{1}{16} \tan^{-1}\left(\frac{t}{2}\right) + C. \end{aligned}$$

In the second term, the derivative of t^2+4 is $2t$; in the third term, use $a=2$ in the arctangent formula. Since $t = x^2$, both logarithm arguments are positive for every real x . Hence no absolute value is needed after back-substitution.

Step 4: Conclusion

$$\int \frac{x \, dx}{(2+x^2)(4+x^4)} = \frac{1}{16} \ln(x^2+2) - \frac{1}{32} \ln(x^4+4) + \frac{1}{16} \tan^{-1}\left(\frac{x^2}{2}\right) + C.$$

Quick Tip to Solve: Whenever even powers of x occur with $x \, dx$, try $t = x^2$ before attempting partial fractions.

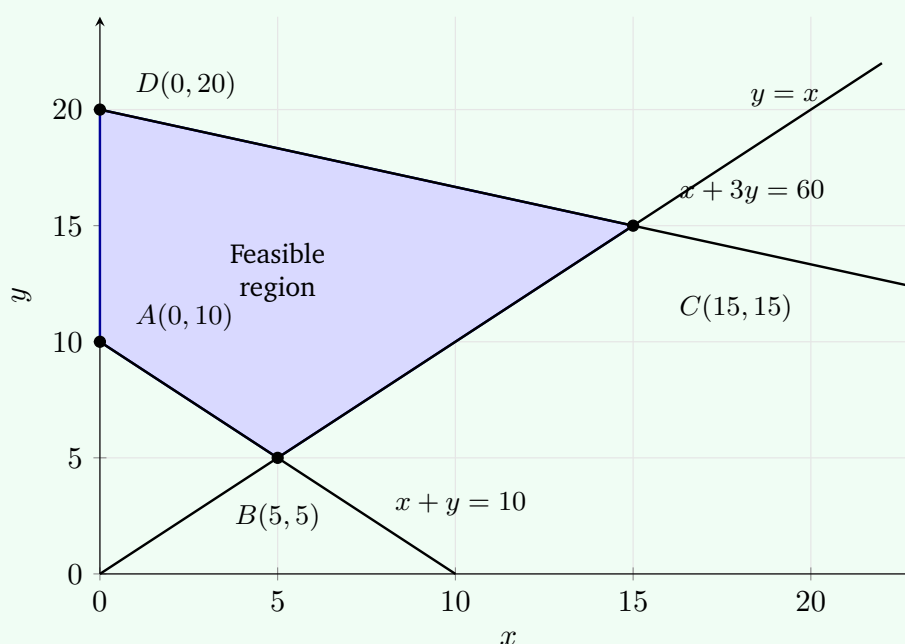
21. Solve the following LPP graphically: Minimise $Z = 3x + 9y$ subject to the constraints $x + 3y \leq 60$, $x + y \geq 10$, $x \leq y$, $x \geq 0$, $y \geq 0$. [4 marks]

Solution:

Concept:

- The feasible region is the intersection of the half-planes given by all the constraints. Here $x, y \geq 0$ restrict the graph to the first quadrant.
- For a nonempty bounded feasible polygon, a linear objective attains a minimum at a vertex. Evaluate $Z = 3x + 9y$ at every vertex to identify it.

Step 1: Draw the boundary lines and select the half-planes The line $x + 3y = 60$ passes through $(0, 20)$ and $(60, 0)$; retain the side containing the origin because $x + 3y \leq 60$. The line $x + y = 10$ passes through $(0, 10)$ and $(10, 0)$; retain the side away from the origin because $x + y \geq 10$. Finally, $x \leq y$ means $y \geq x$, the region on or above the line $y = x$. Intersect these three regions with the first quadrant. The shaded quadrilateral in the graph is their common feasible region.



Feasible region ABCD and the three boundary lines

Step 2: Find all vertices of the feasible polygon On the y -axis, the two sloping boundaries give $A = (0, 10)$ and $D = (0, 20)$. Intersecting $x + y = 10$ with $y = x$ gives $2x = 10$, so $B = (5, 5)$. Intersecting $x + 3y = 60$ with $y = x$ gives $4x = 60$, so $C = (15, 15)$. Thus the feasible polygon has vertices A, B, C, D . The intersections on the positive x -axis are excluded by $x \leq y$. Each of the four listed vertices satisfies every original inequality.

Quick Tip to Solve: Draw $y = x$ first: it excludes the positive x -axis and reduces the feasible-region search to the part above the diagonal.

Step 3: Compare the objective values At the vertices,

Vertex (x, y)	$Z = 3x + 9y$
$A(0, 10)$	90
$B(5, 5)$	60
$C(15, 15)$	180
$D(0, 20)$	180

The smallest entry is 60 at B . The feasible region is closed and bounded, so the corner-point method applies. Moreover, the objective decreases along AB towards B and increases along BC away from B ; hence the minimising vertex is unique.

Step 4: Conclusion

$$Z_{\min} = 60 \quad \text{at} \quad x = 5, y = 5.$$

Quick Tip to Solve: Draw $y = x$ first: it excludes the positive x -axis and reduces the feasible-region search to the part above the diagonal.

22. Find the area of a parallelogram $ABCD$ whose side AB and the diagonal DB are given by the vectors $5\hat{i} + 7\hat{k}$ and $2\hat{i} + 2\hat{j} + 3\hat{k}$ respectively. [4 marks]

Solution:

Concept:

- The area of a parallelogram is $|\vec{a} \times \vec{b}|$, where $\vec{a}$ and $\vec{b}$ are its two adjacent side vectors with the same initial point.
- Directed segments satisfy the triangle law. Here $\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB}$, so $\overrightarrow{AD} = \overrightarrow{AB} - \overrightarrow{DB}$.
- The magnitude of a vector $p\hat{i} + q\hat{j} + r\hat{k}$ is $\sqrt{p^2 + q^2 + r^2}$. Its magnitude, rather than its signed components, gives the area.

Step 1: Express the adjacent side using the diagonal Let $\vec{a} = \overrightarrow{AB} = 5\hat{i} + 7\hat{k}$ and $\vec{d} = \overrightarrow{DB} = 2\hat{i} + 2\hat{j} + 3\hat{k}$. The diagonal is directed from D to B , so we must preserve that direction when applying the triangle law. Following the path from D through A to B ,

$$\overrightarrow{DB} = \overrightarrow{DA} + \overrightarrow{AB} = -\overrightarrow{AD} + \overrightarrow{AB}.$$

Consequently, the side adjacent to AB , starting at A , is

$$\overrightarrow{AD} = \vec{a} - \vec{d} = (5 - 2)\hat{i} + (0 - 2)\hat{j} + (7 - 3)\hat{k} = 3\hat{i} - 2\hat{j} + 4\hat{k}.$$

Thus we have obtained both side vectors required for the parallelogram area formula.

Step 2: Calculate the cross product and its magnitude Using the two adjacent sides in the order

Quick Tip to Solve: For a side and a diagonal of a parallelogram, take the magnitude of their cross product directly: subtracting the side from the diagonal does not change that magnitude.

$AB, AD,$

$$\begin{aligned}\vec{AB} \times \vec{AD} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 0 & 7 \\ 3 & -2 & 4 \end{vmatrix} \\ &= \hat{i}\{0(4) - 7(-2)\} - \hat{j}\{5(4) - 7(3)\} + \hat{k}\{5(-2) - 0(3)\} \\ &= 14\hat{i} + \hat{j} - 10\hat{k}.\end{aligned}$$

The area is therefore

$$|\vec{AB} \times \vec{AD}| = \sqrt{14^2 + 1^2 + (-10)^2} = \sqrt{297} = \sqrt{9 \cdot 33}.$$

As a check, $\vec{a} \times (\vec{a} - \vec{d}) = -\vec{a} \times \vec{d}$; using the given side and diagonal directly gives the same magnitude. No factor of one-half is required because the requested figure is the whole parallelogram, not one of its triangular halves.

Step 3: Conclusion The area of parallelogram $ABCD$ is $\boxed{3\sqrt{33} \text{ square units}}$.

Quick Tip to Solve: For a side and a diagonal of a parallelogram, take the magnitude of their cross product directly: subtracting the side from the diagonal does not change that magnitude.

23. Prove that: $\tan^{-1} 2x + \tan^{-1} \frac{4x}{1-4x^2} = \tan^{-1} \frac{6x-8x^3}{1-12x^2}, |x| < \frac{1}{2\sqrt{3}}.$

[4 marks]

Solution:

Concept:

- The principal value $\tan^{-1} t$ belongs to $(-\pi/2, \pi/2)$. Equality of tangents implies equality of angles only after checking this interval.
- The double-angle formula is $\tan 2\theta = 2 \tan \theta / (1 - \tan^2 \theta)$, and the addition formula is $\tan(A+B) = (\tan A + \tan B) / (1 - \tan A \tan B)$.
- The function $\tan \theta$ is strictly increasing on its principal interval; in particular $\tan(\pi/6) = 1/\sqrt{3}$.

Step 1: Use the bound on x to control the principal angles Put $\theta = \tan^{-1}(2x)$, so that $\tan \theta = 2x$. The given strict inequality implies

$$-\frac{1}{\sqrt{3}} < 2x < \frac{1}{\sqrt{3}}, \quad -\frac{\pi}{6} < \theta < \frac{\pi}{6}.$$

Therefore $2\theta \in (-\pi/3, \pi/3)$ and $3\theta \in (-\pi/2, \pi/2)$. These interval checks will ensure that taking the inverse tangent returns precisely the angles we need. Also $4x^2 < 1/3$ and $12x^2 < 1$, so the two denominators in the printed identity are positive and nonzero throughout its stated domain.

Step 2: Identify the second inverse tangent Applying the double-angle formula gives

$$\tan 2\theta = \frac{2(2x)}{1 - (2x)^2} = \frac{4x}{1 - 4x^2}.$$

Quick Tip to Solve: Set theta equal to the first inverse tangent. Recognise the second term as twice theta, then use the given bound to place three theta in the principal interval.

Since 2θ lies inside the principal interval of inverse tangent,

$$\tan^{-1}\left(\frac{4x}{1-4x^2}\right) = 2\theta.$$

Thus the entire left-hand side of the desired identity equals $\theta + 2\theta = 3\theta$, with no additional multiple of π .

Step 3: Compute the tangent of the sum and invert on the correct branch By the tangent addition formula,

$$\begin{aligned}\tan 3\theta &= \frac{2x + \frac{4x}{1-4x^2}}{1 - (2x)\frac{4x}{1-4x^2}} \\ &= \frac{2x(1-4x^2) + 4x}{1-4x^2-8x^2} \\ &= \frac{6x-8x^3}{1-12x^2}.\end{aligned}$$

We have already established that 3θ is strictly between $-\pi/2$ and $\pi/2$. Hence

$$\tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right) = \tan^{-1}(\tan 3\theta) = 3\theta.$$

Both sides therefore represent the same principal angle for every permitted positive, negative or zero value of x .

Step 4: Conclusion Hence, for $|x| < 1/(2\sqrt{3})$,

$$\boxed{\tan^{-1}(2x) + \tan^{-1}\left(\frac{4x}{1-4x^2}\right) = \tan^{-1}\left(\frac{6x-8x^3}{1-12x^2}\right)}.$$

Quick Tip to Solve: Set theta equal to the first inverse tangent. Recognise the second term as twice theta, then use the given bound to place three theta in the principal interval.

Section D (36 marks)

Questions number 24 to 29 carry 6 marks each.

24(a). Find the vector equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. Hence find whether the plane thus obtained contains the line $\frac{x+2}{5} = \frac{y-3}{4} = \frac{z}{5}$ or not. [6 marks]

Solution:

Concept:

- A plane containing the intersection of two non-parallel planes has equation $F_1 + \lambda F_2 = 0$, where $F_1 = 0$ and $F_2 = 0$ are their equations. Every point of the common line satisfies each member of this family.
- Two planes are perpendicular when their normal vectors have zero scalar product. A line lies in a plane when one point of the line satisfies the plane equation and its direction vector is perpendicular to the plane's normal.

Step 1: Form the family through the intersection The required plane passes through the intersection of $x + y + z - 1 = 0$ and $2x + 3y + 4z - 5 = 0$. Its equation can therefore be written as

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0.$$

Collecting coefficients gives

$$(1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z = 1 + 5\lambda.$$

Thus a normal vector to this member of the family is $\vec{n} = (1 + 2\lambda)\hat{i} + (1 + 3\lambda)\hat{j} + (1 + 4\lambda)\hat{k}$. The excluded second plane itself has normal $(2, 3, 4)$, whose scalar product with $(1, -1, 1)$ is 3, so it cannot be the required perpendicular plane.

Step 2: Apply the perpendicularity condition The normal to $x - y + z = 0$ is $\vec{m} = \hat{i} - \hat{j} + \hat{k}$. Perpendicularity requires

$$\begin{aligned}\vec{n} \cdot \vec{m} &= 0, \\ (1 + 2\lambda) - (1 + 3\lambda) + (1 + 4\lambda) &= 0, \\ 1 + 3\lambda &= 0, \quad \lambda = -\frac{1}{3}.\end{aligned}$$

Substituting this value in the plane family and multiplying by 3 gives

$$3(x + y + z - 1) - (2x + 3y + 4z - 5) = 0, \quad x - z + 2 = 0.$$

If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, its vector equation is $\vec{r} \cdot (\hat{i} - \hat{k}) = -2$. The new normal $(1, 0, -1)$ has zero scalar product with $(1, -1, 1)$, verifying the condition.

Step 3: Test the whole given line Set the common value of the three ratios equal to t . The printed

Quick Tip to Solve: After finding the plane, substitute the parametric line: an identity in the parameter proves containment immediately.

line is then

$$x = 5t - 2, \quad y = 4t + 3, \quad z = 5t.$$

Substituting these coordinates into the plane's left side gives

$$x - z + 2 = (5t - 2) - 5t + 2 = 0$$

for every real t . Thus every point of the line satisfies the plane equation. Equivalently, its point $(-2, 3, 0)$ lies in the plane and its direction $(5, 4, 5)$ has scalar product $5 - 5 = 0$ with the plane normal. Both checks are necessary: parallelism alone would not establish containment.

Step 4: Conclusion The required plane is $\boxed{\vec{r} \cdot (\hat{i} - \hat{k}) + 2 = 0}$, and it contains the entire given line.

Quick Tip to Solve: After finding the plane, substitute the parametric line: an identity in the parameter proves containment immediately.

OR

24(b). Find the image P' of the point P having position vector $\hat{i} + 3\hat{j} + 4\hat{k}$ in the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$. Hence find the length of PP' . [6 marks]

Solution:

Concept:

- Reflection in a plane sends a point P to P' such that PP' is perpendicular to the plane and its midpoint H lies on the plane. The segment is therefore parallel to a normal vector of the plane.
- For a plane $\vec{r} \cdot \vec{n} + d = 0$, the distance of a point with position vector $\vec{p}$ is $|\vec{p} \cdot \vec{n} + d| / \|\vec{n}\|$. The reflected point is twice this distance away from the original point.

Step 1: Write the normal line through the given point The given point has position vector $\vec{p} = \hat{i} + 3\hat{j} + 4\hat{k}$, so $P = (1, 3, 4)$. The plane equation is

$$2x - y + z + 3 = 0.$$

A normal vector to the plane is $\vec{n} = 2\hat{i} - \hat{j} + \hat{k}$. Since the foot H of the perpendicular from P must lie on the normal line through P , write

$$\vec{h} = \vec{p} + t\vec{n} = (1 + 2t)\hat{i} + (3 - t)\hat{j} + (4 + t)\hat{k}.$$

This parameterization ensures that PH is perpendicular to the plane before we impose the condition that H lies on it.

Step 2: Find the foot of the perpendicular Substitute the coordinates of H into the plane equation:

$$\begin{aligned} 2(1 + 2t) - (3 - t) + (4 + t) + 3 &= 0, \\ 6 + 6t &= 0, \\ t &= -1. \end{aligned}$$

Quick Tip to Solve: Find the perpendicular foot first and use $P' = 2H - P$; double the point-to-plane distance to check the final length.

Consequently $H = (-1, 4, 3)$, with position vector $\vec{h} = -\hat{i} + 4\hat{j} + 3\hat{k}$. Its coordinates give $2(-1) - 4 + 3 + 3 = 0$, verifying that the perpendicular foot actually lies on the plane.

Step 3: Use the midpoint property to obtain the image Reflection makes H the midpoint of PP' , so

$$\begin{aligned}\vec{p}' &= 2\vec{h} - \vec{p} \\ &= 2(-\hat{i} + 4\hat{j} + 3\hat{k}) - (\hat{i} + 3\hat{j} + 4\hat{k}) \\ &= -3\hat{i} + 5\hat{j} + 2\hat{k}.\end{aligned}$$

The displacement $\vec{p}' - \vec{p} = -4\hat{i} + 2\hat{j} - 2\hat{k} = -2\vec{n}$ is parallel to the plane normal. Its midpoint is precisely H , so the result satisfies both defining properties of the image. Also the plane expression at P is 6, while at P' it is -6 ; the points are on opposite sides at equal distances.

Step 4: Calculate the length of the joining segment The coordinate differences between P and P' are $(-4, 2, -2)$. Therefore

$$PP' = \sqrt{(-4)^2 + 2^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}.$$

As an independent check, the distance of P from the plane is

$$PH = \frac{|2(1) - 3 + 4 + 3|}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{6}{\sqrt{6}} = \sqrt{6}.$$

Since reflection doubles this perpendicular distance, it again gives $PP' = 2\sqrt{6}$.

Step 5: Conclusion The image is $\boxed{\vec{p}' = -3\hat{i} + 5\hat{j} + 2\hat{k}}$, or $P' = (-3, 5, 2)$, and $\boxed{PP' = 2\sqrt{6}}$ units.

Quick Tip to Solve: Find the perpendicular foot first and use $P' = 2H - P$; double the point-to-plane distance to check the final length.

25(a). Using the method of integration, find the area of the triangle ABC , coordinates of whose vertices are $A(1, 2)$, $B(2, 0)$ and $C(4, 3)$. [6 marks]

Solution:

Concept:

- The area between an upper boundary $y = u(x)$ and a lower boundary $y = l(x)$, from $x = a$ to $x = b$, is $\int_a^b [u(x) - l(x)] dx$. Each vertical strip has positive height equal to upper ordinate minus lower ordinate.
- A straight line through (x_1, y_1) and (x_2, y_2) has slope $m = (y_2 - y_1)/(x_2 - x_1)$ and equation $y - y_1 = m(x - x_1)$. The lower side of a triangle can change at the middle vertex, so the integral must then be split there.

Step 1: Obtain the equations of the three sides For $A(1, 2)$, $B(2, 0)$ and $C(4, 3)$, the slopes of AB , BC

Quick Tip to Solve: Sort the vertices by their x-coordinates and split the integral at the middle x-coordinate, where one bounding side changes.

and AC are respectively -2 , $3/2$ and $1/3$. Consequently,

$$AB : y - 2 = -2(x - 1), \quad y = 4 - 2x,$$

$$BC : y = \frac{3}{2}(x - 2), \quad y = \frac{3}{2}x - 3,$$

$$AC : y - 2 = \frac{1}{3}(x - 1), \quad y = \frac{x + 5}{3}.$$

These equations reproduce both endpoints of each side. In particular, the side AC extends across the entire horizontal interval from $x = 1$ to $x = 4$.

Step 2: Identify the boundaries for vertical strips Over $1 \leq x \leq 2$, the upper boundary is AC and the lower boundary is AB . Their vertical separation is

$$\frac{x + 5}{3} - (4 - 2x) = \frac{7(x - 1)}{3} \geq 0.$$

Over $2 \leq x \leq 4$, the upper boundary is again AC , while the lower boundary becomes BC . Here the separation is

$$\frac{x + 5}{3} - \left(\frac{3}{2}x - 3\right) = \frac{7(4 - x)}{6} \geq 0.$$

Thus both integrals represent ordinary positive areas, and the change of lower boundary at B is accounted for without counting any exterior region.

Step 3: Integrate the two strip heights Denoting the area by S , we obtain

$$\begin{aligned} S &= \int_1^2 \frac{7(x - 1)}{3} dx + \int_2^4 \frac{7(4 - x)}{6} dx \\ &= \frac{7}{3} \left[\frac{(x - 1)^2}{2} \right]_1^2 + \frac{7}{6} \left[4x - \frac{x^2}{2} \right]_2^4 \\ &= \frac{7}{6} + \frac{7}{6} [(16 - 8) - (8 - 2)] \\ &= \frac{7}{6} + \frac{7}{3}. \end{aligned}$$

The two pieces meet along the vertical line through B , which has zero area, so adding them gives exactly the area of the whole triangle.

Step 4: Conclusion Therefore, using integration, the area of triangle ABC is $\boxed{\frac{7}{2} \text{ square units}}$.

Quick Tip to Solve: Sort the vertices by their x -coordinates and split the integral at the middle x -coordinate, where one bounding side changes.

25(b). Using integration, find the area of the region $\{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$.

[6 marks]

Solution:

Concept:

- The inequality $x^2 + y^2 \leq 1$ describes the closed unit disk, while $x + y \geq 1$ describes the half-plane on or above $y = 1 - x$. Their common region is bounded by a circular arc and a chord.
- For vertical strips, the area is the integral of upper ordinate minus lower ordinate. The upper semicircle is $y = \sqrt{1 - x^2}$, and the standard antiderivative is $\int \sqrt{1 - x^2} dx = \frac{1}{2}[x\sqrt{1 - x^2} + \sin^{-1} x] + C$, for $-1 < x < 1$, extended to the endpoints by continuity.

Step 1: Find the intersection points of the boundaries The straight boundary is $x + y = 1$, so $y = 1 - x$. Substitution into the circular boundary gives

$$x^2 + (1 - x)^2 = 1 \implies 2x^2 - 2x = 0 \implies x = 0 \text{ or } x = 1.$$

The corresponding intersection points are therefore $(0, 1)$ and $(1, 0)$. The chord between these points and the first-quadrant arc of the unit circle enclose the required region.

Step 2: Establish the limits and the strip height Every point of the disk has $-1 \leq x, y \leq 1$. If $x < 0$, the condition $x + y \geq 1$ would require $y > 1$, which is impossible inside the disk. Similarly, $y < 0$ would require $x > 1$. Hence the required region lies in the first quadrant. For $0 \leq x \leq 1$, its lower boundary is the line $y = 1 - x$ and its upper boundary is the arc $y = \sqrt{1 - x^2}$. To check their order, both quantities are nonnegative and

$$(1 - x^2) - (1 - x)^2 = 2x(1 - x) \geq 0.$$

Thus the strip height is $\sqrt{1 - x^2} - (1 - x)$, and the full required area S is

$$S = \int_0^1 [\sqrt{1 - x^2} - (1 - x)] dx.$$

Step 3: Evaluate the area integral For completeness, put $x = \sin \theta$ in the circular term. Then $dx = \cos \theta d\theta$, the limits become 0 and $\pi/2$, and $\sqrt{1 - x^2} = \cos \theta$. Therefore,

$$\begin{aligned} \int_0^1 \sqrt{1 - x^2} dx &= \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\pi/2} = \frac{\pi}{4}. \end{aligned}$$

The straight-line term is

$$\int_0^1 (1 - x) dx = \left[x - \frac{x^2}{2} \right]_0^1 = \frac{1}{2}.$$

Subtracting this term removes exactly the portion below the chord within the first-quadrant circular region. Boundary points are included by the non-strict inequalities, but they contribute zero additional area.

Quick Tip to Solve: First intersect the line with the circle; on the interval between the intersections integrate the upper arc minus the line.

Step 4: Conclusion The area of the given region is $\frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}$ square units.

Quick Tip to Solve: First intersect the line with the circle; on the interval between the intersections integrate the upper arc minus the line.

26(a). Let $A = \mathbb{R} - \{3\}$, $B = \mathbb{R} - \{1\}$. Let $f : A \rightarrow B$ be defined by $f(x) = \frac{x-2}{x-3} \forall x \in A$. Show that f is bijective. Also, find (i) x , if $f^{-1}(x) = 4$ (ii) $f^{-1}(7)$. [6 marks]

Solution:

Concept:

- A function is injective if equal images imply equal inputs, and surjective if every element of its codomain has a preimage in its domain. A function having both properties is bijective.
- For a bijection, the inverse reverses the correspondence: $f^{-1}(y) = x$ exactly when $f(x) = y$. Its domain is the codomain of f .

Step 1: Check that the images belong to the stated codomain For every $x \in A = \mathbb{R} \setminus \{3\}$, the denominator $x - 3$ is nonzero, so $f(x)$ is a real number. Furthermore,

$$f(x) - 1 = \frac{x-2}{x-3} - 1 = \frac{1}{x-3} \neq 0.$$

Thus $f(x) \neq 1$, and every image lies in $B = \mathbb{R} \setminus \{1\}$. This also explains why the exclusions in the domain and codomain are necessary.

Step 2: Prove that the function is one-one Let $u, v \in A$ and suppose $f(u) = f(v)$. Since both denominators are nonzero, cross multiplication is valid:

$$\begin{aligned} \frac{u-2}{u-3} &= \frac{v-2}{v-3}, \\ (u-2)(v-3) &= (v-2)(u-3), \\ uv - 3u - 2v + 6 &= uv - 3v - 2u + 6, \\ u &= v. \end{aligned}$$

Hence different elements of A cannot have the same image, and f is injective.

Step 3: Prove that the function is onto and obtain its inverse Take an arbitrary $y \in B$. To find its preimage, solve

$$y = \frac{x-2}{x-3} \implies (y-1)x = 3y-2 \implies x = \frac{3y-2}{y-1}.$$

This is a real number because $y \neq 1$, and

$$x-3 = \frac{3y-2-3(y-1)}{y-1} = \frac{1}{y-1} \neq 0.$$

Therefore $x \in A$. Also $x-2 = y/(y-1)$, so substituting this preimage into f gives $f(x) = y$. Every

Quick Tip to Solve: Rewrite $f(x) = 1 + 1/(x-3)$ to see the excluded output immediately; solve $y = f(x)$ and check $x \neq 3$ to establish surjectivity.

element of B is attained; hence f is surjective. Together with injectivity this proves bijectivity, and the inverse formula is

$$f^{-1}(y) = \frac{3y - 2}{y - 1}, \quad y \in B.$$

Step 4: Evaluate the two requested quantities (i) The equation $f^{-1}(x) = 4$ means that x is the image of 4 under f . Since $4 \in A$,

$$x = f(4) = \frac{4 - 2}{4 - 3} = 2.$$

Checking with the inverse gives $f^{-1}(2) = (6 - 2)/(2 - 1) = 4$, as required. (ii) Since $7 \in B$, the inverse formula applies:

$$f^{-1}(7) = \frac{3(7) - 2}{7 - 1} = \frac{19}{6}.$$

This value belongs to A ; substituting it into f gives $(7/6)/(1/6) = 7$, verifying the result.

Step 5: Conclusion The function $f : A \rightarrow B$ is bijective, with $f^{-1}(y) = \frac{3y - 2}{y - 1}$. The requested answers are (i) $x = 2$, and (ii) $f^{-1}(7) = \frac{19}{6}$.

Quick Tip to Solve: Rewrite $f(x) = 1 + 1/(x - 3)$ to see the excluded output immediately; solve $y = f(x)$ and check $x \neq 3$ to establish surjectivity.

OR

26(b). Let $A = \mathbb{R} \times \mathbb{R}$ and let $*$ be a binary operation on A defined by $(a, b) * (c, d) = (ad + bc, bd)$ for all $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$. (i) Show that $*$ is commutative on A . (ii) Show that $*$ is associative on A . (iii) Find the identity element of $*$ in A . [6 marks]

Solution:

Concept:

- A binary operation on A assigns an element of A to each pair of elements of A . Commutativity means $P * Q = Q * P$; associativity means $(P * Q) * R = P * (Q * R)$.
- An identity $E \in A$ must satisfy $P * E = E * P = P$ for every $P \in A$. Ordered pairs are equal precisely when their corresponding coordinates are equal.

Step 1: Verify closure and prove commutativity Let $P = (a, b)$ and $Q = (c, d)$ be arbitrary elements of $A = \mathbb{R} \times \mathbb{R}$. The operation gives

$$P * Q = (ad + bc, bd).$$

Both coordinates are real, so this pair belongs to A . Interchanging the two inputs gives

$$Q * P = (cb + da, db) = (ad + bc, bd) = P * Q,$$

because multiplication and addition of real numbers are commutative. Both coordinates agree for every choice of a, b, c, d . Therefore $*$ is commutative on A .

Step 2: Compute the left-associated product For associativity, take an additional arbitrary element

Quick Tip to Solve: In the identity equations, choose the second coordinate $b = 1$ to obtain $k = 1$ and $h = 0$; always verify both ordered products afterward.

$R = (e, f) \in A$. First combine P and Q , and then apply the same operation to their resulting pair and R :

$$\begin{aligned}(P * Q) * R &= (ad + bc, bd) * (e, f) \\ &= ((ad + bc)f + (bd)e, (bd)f) \\ &= (adf + bcf + bde, bdf).\end{aligned}$$

The first coordinate uses the first coordinate of the left input times the second coordinate of the right input, plus the second coordinate of the left input times the first coordinate of the right input. Keeping that order prevents confusing this operation with coordinatewise multiplication.

Step 3: Compute the right-associated product and compare Now combine Q and R first:

$$Q * R = (cf + de, df).$$

Consequently,

$$\begin{aligned}P * (Q * R) &= (a, b) * (cf + de, df) \\ &= (a(df) + b(cf + de), b(df)) \\ &= (adf + bcf + bde, bdf).\end{aligned}$$

This is exactly the pair obtained for $(P * Q) * R$. The equality holds for every three elements of A , with no restrictions on their coordinates. Therefore $*$ is associative on A .

Step 4: Determine and verify the identity element Let a possible identity be $E = (h, k)$. For every $(a, b) \in A$, the condition $(a, b) * E = (a, b)$ requires

$$(ak + bh, bk) = (a, b).$$

From $bk = b$ for all real b , taking $b = 1$ gives $k = 1$. The first-coordinate condition then becomes $a + bh = a$ for all a, b ; taking $b = 1$ gives $h = 0$. Thus the only candidate is $E = (0, 1)$. Verify both sides directly:

$$\begin{aligned}(a, b) * (0, 1) &= (a \cdot 1 + b \cdot 0, b \cdot 1) = (a, b), \\ (0, 1) * (a, b) &= (0 \cdot b + 1 \cdot a, 1 \cdot b) = (a, b).\end{aligned}$$

Hence $(0, 1) \in A$ is indeed the two-sided identity for the operation.

Step 5: Conclusion (i) The operation $*$ is commutative. (ii) It is associative, since both grouped products equal $(adf + bcf + bde, bdf)$. (iii) Its identity element is $\boxed{(0, 1)}$.

Quick Tip to Solve: In the identity equations, choose the second coordinate $b = 1$ to obtain $k = 1$ and $h = 0$; always verify both ordered products afterward.

27. A wire of length 34 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a rectangle whose length is twice its breadth. What should be the lengths of the two pieces, so that the combined area of the square and the rectangle is minimum? [6 marks]

Solution:

Concept:

- The wire lengths are the perimeters of the two shapes. For a square of side s , perimeter is $4s$ and area is s^2 .
- For a rectangle of breadth b and length $2b$, perimeter is $6b$ and area is $2b^2$.
- An interior minimum of a differentiable area function satisfies $A'(x) = 0$. A strictly positive second derivative throughout the feasible interval establishes a unique minimum.

Step 1: Express the dimensions in terms of one wire length Let x m be the length of wire used for the square. The remaining $(34 - x)$ m is used for the rectangle, with $0 < x < 34$ because both pieces must have positive length. The square has side $x/4$. If the rectangle has breadth b , its length is $2b$, and its perimeter gives

$$2(2b + b) = 34 - x, \quad b = \frac{34 - x}{6}.$$

Thus the rectangle's length is $(34 - x)/3$. It is essential to use the perimeter condition, since the cut lengths are not the sides of the shapes.

Step 2: Form the combined area and locate its stationary point The combined area, in square metres, is

$$A(x) = \left(\frac{x}{4}\right)^2 + 2\left(\frac{34 - x}{6}\right)^2 = \frac{x^2}{16} + \frac{(34 - x)^2}{18}.$$

Differentiating with respect to x , we obtain

$$\begin{aligned} A'(x) &= \frac{x}{8} - \frac{34 - x}{9}, \\ A'(x) = 0 &\implies 9x = 8(34 - x), \\ 17x &= 272, \quad x = 16. \end{aligned}$$

This value lies within the allowed interval. The other piece then has length $34 - 16 = 18$ m.

Step 3: Establish that the stationary point gives the minimum The second derivative is

$$A''(x) = \frac{1}{8} + \frac{1}{9} = \frac{17}{72} > 0.$$

Hence the area function is strictly convex throughout $0 < x < 34$, so its stationary point is the unique global minimum. As an exact confirmation, completing the square gives

$$A(x) = 34 + \frac{17}{144}(x - 16)^2 \geq 34.$$

Equality holds precisely at $x = 16$. The resulting square has side 4 m, and the rectangle has breadth 3 m and length 6 m. Their perimeters are 16 m and 18 m, adding to the given 34 m, while their areas add to $16 + 18 = 34 \text{ m}^2$.

Quick Tip to Solve: Use the square's wire length as the variable: the two area terms become $x^2/16$ and $(34 - x)^2/18$, so only one derivative is needed.

Step 4: Conclusion Cut the wire into $\boxed{16 \text{ m}}$ for the square and $\boxed{18 \text{ m}}$ for the rectangle. The minimum combined area is 34 m^2 .

Quick Tip to Solve: Use the square's wire length as the variable: the two area terms become $x^2/16$ and $(34 - x)^2/18$, so only one derivative is needed.

28. If $A = \begin{pmatrix} 1 & 2 & 5 \\ 1 & -1 & -1 \\ 2 & 3 & -1 \end{pmatrix}$, find A^{-1} and hence solve the system of equations $x + 2y + 5z = 10$, $x - y - z = -2$ and $2x + 3y - z = -11$. [6 marks]

Solution:

Concept:

- A square matrix A is invertible when $|A| \neq 0$, and $A^{-1} = \text{adj}(A)/|A|$.
- The cofactor C_{ij} equals $(-1)^{i+j}$ times the minor formed by deleting row i and column j . The adjugate is the transpose of the cofactor matrix.
- A system $AX = B$ has the unique solution $X = A^{-1}B$ when A is invertible, with X the column of unknowns and B the column of constants.

Step 1: Evaluate the determinant Expanding the determinant along the first row, we obtain

$$\begin{aligned} |A| &= 1\{(-1)(-1) - (-1)(3)\} \\ &\quad - 2\{1(-1) - (-1)(2)\} \\ &\quad + 5\{1(3) - (-1)(2)\} \\ &= 4 - 2 + 25 = 27. \end{aligned}$$

Since the determinant is nonzero, the inverse exists and the system has a unique solution. The alternating signs in this expansion correspond to the cofactor signs $+, -, +$ in the first row.

Step 2: Calculate the cofactors and transpose their matrix Evaluating the nine signed minors gives

$$\begin{aligned} C_{11} &= 1 + 3 = 4, & C_{12} &= -(-1 + 2) = -1, \\ C_{13} &= 3 + 2 = 5, & C_{21} &= -(-2 - 15) = 17, \\ C_{22} &= -1 - 10 = -11, & C_{23} &= -(3 - 4) = 1, \\ C_{31} &= -2 + 5 = 3, & C_{32} &= -(-1 - 5) = 6, \\ C_{33} &= -1 - 2 = -3. \end{aligned}$$

Therefore the cofactor matrix and its transpose are

$$C = \begin{pmatrix} 4 & -1 & 5 \\ 17 & -11 & 1 \\ 3 & 6 & -3 \end{pmatrix}, \quad \text{adj}(A) = \begin{pmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{pmatrix}.$$

The transpose is necessary: the cofactor matrix itself is not the adjugate.

Quick Tip to Solve: Keep the cofactor matrix and adjugate distinct by explicitly transposing; then multiply the inverse by the constants in the original equation order.

Step 3: Find the inverse and apply it to the system Using the adjugate formula,

$$A^{-1} = \frac{1}{27} \begin{pmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{pmatrix}.$$

The equations in the printed order have the matrix form

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ -2 \\ -11 \end{pmatrix}.$$

Premultiplication by A^{-1} yields

$$\begin{aligned} \begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{1}{27} \begin{pmatrix} 4(10) + 17(-2) + 3(-11) \\ -1(10) - 11(-2) + 6(-11) \\ 5(10) + 1(-2) - 3(-11) \end{pmatrix} \\ &= \frac{1}{27} \begin{pmatrix} -27 \\ -54 \\ 81 \end{pmatrix}. \end{aligned}$$

This evaluates all three unknowns using the inverse requested in the question.

Step 4: Check the result in every original equation Substitution of $x = -1$, $y = -2$ and $z = 3$ gives

$$x + 2y + 5z = -1 - 4 + 15 = 10,$$

$$x - y - z = -1 + 2 - 3 = -2,$$

$$2x + 3y - z = -2 - 6 - 3 = -11.$$

Each right-hand side agrees with the printed system. This also checks the signs of the constants used in the matrix multiplication.

Step 5: Conclusion

$$A^{-1} = \frac{1}{27} \begin{pmatrix} 4 & 17 & 3 \\ -1 & -11 & 6 \\ 5 & 1 & -3 \end{pmatrix}, \quad x = -1, y = -2, z = 3.$$

Quick Tip to Solve: Keep the cofactor matrix and adjugate distinct by explicitly transposing; then multiply the inverse by the constants in the original equation order.

29. Find the particular solution of the differential equation $dy = \cos x (2 - y \operatorname{cosec} x) dx$, given that $y = 2$ when $x = \frac{\pi}{2}$. [6 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$. Its integrating factor is $IF = e^{\int P(x) dx}$, which converts the left side into the derivative of $y IF$.
- Here $\cos x \operatorname{cosec} x = \cot x$. Since the equation contains $\operatorname{cosec} x$, it is defined only where $\sin x \neq 0$. The interval containing the prescribed initial point $x = \pi/2$ is $0 < x < \pi$.
- The initial condition determines the arbitrary constant after integration. We use $\int \cot x dx = \ln |\sin x|$ and $\frac{d}{dx}(\sin^2 x) = 2 \sin x \cos x$.

Step 1: Put the equation in linear form Dividing the given equation by dx and expanding its right side gives

$$\frac{dy}{dx} = 2 \cos x - y \cos x \operatorname{cosec} x.$$

Therefore, moving the term containing y to the left,

$$\frac{dy}{dx} + y \cot x = 2 \cos x.$$

Comparing with the standard linear form, we identify $P(x) = \cot x$ and $Q(x) = 2 \cos x$. The coefficient of y has a positive sign after transposition.

Step 2: Find the integrating factor and integrate On $0 < x < \pi$, $\sin x$ is positive, so an integrating factor is

$$IF = e^{\int \cot x dx} = e^{\ln(\sin x)} = \sin x.$$

Multiplying the linear equation by $\sin x$ yields

$$\sin x \frac{dy}{dx} + y \cos x = 2 \sin x \cos x.$$

By the product rule, the left side equals $\frac{d}{dx}(y \sin x)$. Thus

$$\begin{aligned} \frac{d}{dx}(y \sin x) &= 2 \sin x \cos x, \\ y \sin x &= \int 2 \sin x \cos x dx = \sin^2 x + C. \end{aligned}$$

This gives the one-parameter family before applying the initial condition.

Step 3: Apply the initial condition Substituting $x = \pi/2$, $y = 2$, and $\sin(\pi/2) = 1$ into the integrated relation, we obtain

$$2 \cdot 1 = 1^2 + C, \quad C = 1.$$

Consequently, the required relation is $y \sin x = \sin^2 x + 1$. Division by $\sin x$ is valid throughout the interval containing the given initial point.

Quick Tip to Solve: Rewrite $\cos x \operatorname{cosec} x$ as $\cot x$; multiplying by $\sin x$ immediately exposes the product derivative $(y \sin x)'$.

Step 4: Verify the equation and initial value For $y = \sin x + \operatorname{cosec} x$, differentiation gives

$$\frac{dy}{dx} = \cos x - \operatorname{cosec} x \cot x.$$

The right side of the original equation, after division by dx , is

$$\begin{aligned}\cos x(2 - y \operatorname{cosec} x) &= \cos x(1 - \operatorname{cosec}^2 x) \\ &= \cos x - \operatorname{cosec} x \cot x.\end{aligned}$$

Thus both sides agree. At $x = \pi/2$, the same expression gives $y = 1 + 1 = 2$, as required.

Step 5: Conclusion The particular solution is $y = \sin x + \operatorname{cosec} x$, equivalently $y \sin x = \sin^2 x + 1$, on $0 < x < \pi$.

Quick Tip to Solve: Rewrite $\cos x \operatorname{cosec} x$ as $\cot x$; multiplying by $\sin x$ immediately exposes the product derivative $(y \sin x)'$.