

CBSE Class 12 Mathematics Question Paper 2017 with Solution PDF-Set 65-2-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of 4 questions of **one mark** each, Section B comprises of 8 questions of **two marks** each, Section C comprises of 11 questions of **four marks** each and Section D comprises of 6 questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 3 questions of four marks each and 3 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Questions number 1 to 4 carry 1 mark each.

1. Write the distance of the point $(3, -5, 12)$ from x -axis.

[1 mark]

Solution:

Concept:

- The distance of a point (x, y, z) from the x -axis is $\sqrt{y^2 + z^2}$, since a point on that axis has coordinates $(t, 0, 0)$.
- The perpendicular projection keeps the x -coordinate unchanged and makes the other two coordinates zero.

Step 1: Locate the perpendicular foot For $P = (3, -5, 12)$, the perpendicular foot on the x -axis is $H = (3, 0, 0)$. The displacement $\overrightarrow{HP} = (0, -5, 12)$ is perpendicular to the axis, so PH gives the shortest distance.

Step 2: Apply the distance formula Using the differences in the three coordinates,

$$PH = \sqrt{(3-3)^2 + (-5-0)^2 + (12-0)^2} = \sqrt{25 + 144} = \sqrt{169}.$$

Step 3: Conclusion The distance from the x -axis is $\boxed{13}$ units.

Quick Tip to Solve: For distance from the x -axis, ignore the x -coordinate and square the other two coordinates.

2. Evaluate: $\int_0^{2\pi} \cos^5 x \, dx$.

[1 mark]

Solution:

Concept:

- An odd power of cosine changes sign when its argument is increased by π : $\cos^5(x + \pi) = -\cos^5 x$.
- For a definite integral over one complete period, equal positive and negative contributions cancel.

Step 1: Split the interval into equal halves Let $I = \int_0^{2\pi} \cos^5 x \, dx$. Then

$$I = \int_0^{\pi} \cos^5 x \, dx + \int_{\pi}^{2\pi} \cos^5 x \, dx.$$

The integrand is continuous throughout this interval.

Step 2: Compare the two contributions In the second integral set $x = u + \pi$, so $dx = du$, and its limits become 0 and π . Hence

$$\int_{\pi}^{2\pi} \cos^5 x \, dx = \int_0^{\pi} \cos^5(u + \pi) \, du = - \int_0^{\pi} \cos^5 u \, du.$$

Adding the two integrals gives $I = 0$.

Quick Tip to Solve: On a full period, pair x with $x + \pi$ to cancel odd powers of sine or cosine.

Step 3: Conclusion $\int_0^{2\pi} \cos^5 x \, dx = 0$.

Quick Tip to Solve: On a full period, pair x with $x + \pi$ to cancel odd powers of sine or cosine.

3. For what value of k is the function $f(x) = \begin{cases} \frac{\sin 5x}{3x} + \cos x, & \text{if } x \neq 0 \\ k, & \text{if } x = 0 \end{cases}$ continuous at $x = 0$? [1 mark]

Solution:

Concept:

- Continuity at $x = 0$ requires $\lim_{x \rightarrow 0} f(x) = f(0) = k$.
- The standard limit is $\lim_{u \rightarrow 0} \sin u / u = 1$, with angles measured in radians.

Step 1: Evaluate the limit of the sine term For $x \neq 0$, rewrite

$$\frac{\sin 5x}{3x} = \frac{5}{3} \frac{\sin 5x}{5x}.$$

As $x \rightarrow 0$, $5x \rightarrow 0$, so the two-sided limit of this term is $5/3$.

Step 2: Match the limit with the assigned value Since cosine is continuous, $\lim_{x \rightarrow 0} \cos x = \cos 0 = 1$. Thus

$$\lim_{x \rightarrow 0} f(x) = \frac{5}{3} + 1 = \frac{8}{3}.$$

The left and right limits agree; continuity holds precisely when the value assigned at zero equals this limit.

Step 3: Conclusion The required value is $k = \frac{8}{3}$.

Quick Tip to Solve: Rewrite $\sin(5x)/(3x)$ as $(5/3)$ times $\sin(5x)/(5x)$ before applying the standard limit.

4. If $|A| = 3$ and $A^{-1} = \begin{pmatrix} 3 & -1 \\ -\frac{5}{3} & \frac{2}{3} \end{pmatrix}$, then write the $\text{adj } A$. [1 mark]

Solution:

Concept:

- For an invertible matrix, $A^{-1} = \text{adj } A / |A|$, so $\text{adj } A = |A| A^{-1}$.
- Multiplying a matrix by a scalar means multiplying every entry by that scalar.

Step 1: Use the inverse-adjoint identity The given determinant $|A| = 3$ is nonzero, so the inverse-

Quick Tip to Solve: When the inverse and determinant are given, multiply them directly; do not transpose the inverse.

adjoint identity applies. Multiplying it by $|A|$ yields

$$\text{adj } A = 3 \begin{pmatrix} 3 & -1 \\ -5/3 & 2/3 \end{pmatrix}.$$

Step 2: Multiply all four entries The first row becomes $(3 \cdot 3, 3 \cdot (-1)) = (9, -3)$, and the second becomes $(3 \cdot (-5/3), 3 \cdot (2/3)) = (-5, 2)$. The positions and signs of the entries stay the same under scalar multiplication.

Step 3: Conclusion $\text{adj } A = \begin{pmatrix} 9 & -3 \\ -5 & 2 \end{pmatrix}.$

Quick Tip to Solve: When the inverse and determinant are given, multiply them directly; do not transpose the inverse.

Section B (16 marks)

Questions number 5 to 12 carry 2 marks each.

5. Find: $\int \frac{dx}{\sqrt{3-2x-x^2}}.$

[2 marks]

Solution:

Concept:

- Complete the square to express a quadratic radical in the form $\sqrt{a^2 - u^2}$.
- The standard formula is $\int du/\sqrt{a^2 - u^2} = \sin^{-1}(u/a) + C$ for $|u| < a$, where C is an arbitrary constant.

Step 1: Complete the square in the denominator We have

$$3 - 2x - x^2 = 4 - (x^2 + 2x + 1) = 4 - (x + 1)^2.$$

Thus the given integral is $\int dx/\sqrt{4 - (x + 1)^2}$. For a real, nonzero denominator we require $|x + 1| < 2$, or $-3 < x < 1$.

Step 2: Shift the variable and apply the formula Put $u = x + 1$; then $du = dx$. Therefore

$$\int \frac{dx}{\sqrt{3-2x-x^2}} = \int \frac{du}{\sqrt{2^2 - u^2}} = \sin^{-1}\left(\frac{u}{2}\right) + C.$$

Substitution back gives the antiderivative in the original variable. As a check, differentiation yields

$$\frac{d}{dx} \sin^{-1}\left(\frac{x+1}{2}\right) = \frac{1/2}{\sqrt{1 - (x+1)^2/4}} = \frac{1}{\sqrt{4 - (x+1)^2}},$$

which is the original integrand on its real domain.

Quick Tip to Solve: Complete the square as 4 minus $(x+1)$ squared; the square root of 4 fixes the arcsine divisor as 2.

Step 3: Conclusion

$$\int \frac{dx}{\sqrt{3-2x-x^2}} = \sin^{-1} \left(\frac{x+1}{2} \right) + C, \text{ for } -3 < x < 1.$$

Quick Tip to Solve: Complete the square as 4 minus (x+1) squared; the square root of 4 fixes the arcsine divisor as 2.

6. A company produces two types of goods A and B, that require gold and silver. Each unit of type A requires 3 g of silver and 1 g of gold while that of type B requires 1 g of silver and 2 g of gold. The company can procure a maximum of 9 g of silver and 8 g of gold. If each unit of type A brings a profit of ₹ 40 and that of type B ₹ 50, formulate LPP to maximize profit. [2 marks]

Solution:

Concept:

- A linear programming formulation uses decision variables for the quantities produced and a linear objective for the total profit.
- For each limited resource, total consumption cannot exceed the available quantity. Production variables must be nonnegative.

Step 1: Define the production variables and profit Let x and y be the numbers of units of goods A and B, respectively. Their profits per unit are Rs. 40 and Rs. 50. Hence the total profit in rupees is

$$Z = 40x + 50y,$$

and this expression is to be maximized.

Step 2: Translate the resource limits Each unit of A uses 3 g of silver and each unit of B uses 1 g. The silver consumption is $3x + y$ g, which cannot exceed 9 g:

$$3x + y \leq 9.$$

Similarly, the gold consumption is $x + 2y$ g because the respective requirements are 1 g and 2 g; the gold limit gives

$$x + 2y \leq 8.$$

Since negative production is impossible, $x \geq 0$ and $y \geq 0$. These inequalities allow unused material; the stated maximum supplies do not force equality. The question asks for the formulation, so no graphical optimization is needed.

Step 3: Conclusion The required LPP is

$$\text{Maximize } Z = 40x + 50y$$

$$\text{subject to } 3x + y \leq 9, \quad x + 2y \leq 8, \quad x \geq 0, \quad y \geq 0.$$

Quick Tip to Solve: Write one inequality for silver and one for gold, keeping A and B coefficients in the same order.

7. If $P(A) = 0.4$, $P(B) = p$, $P(A \cup B) = 0.6$ and A and B are given to be independent events, find the value of p . [2 marks]

Solution:

Concept:

- For any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Independence gives $P(A \cap B) = P(A)P(B)$; independent events need not be mutually exclusive.

Step 1: Express the intersection probability The given probabilities are $P(A) = 0.4$ and $P(B) = p$. Since the events are independent,

$$P(A \cap B) = 0.4p.$$

This intersection must be subtracted once when forming the union, because its outcomes are counted in both individual probabilities.

Step 2: Use the union probability and solve Substituting $P(A \cup B) = 0.6$ in the addition rule gives

$$0.6 = 0.4 + p - 0.4p,$$

$$0.2 = 0.6p,$$

$$p = \frac{0.2}{0.6} = \frac{1}{3}.$$

This is a valid probability, lying between 0 and 1. For verification, the intersection is $0.4/3 = 2/15$, while $P(A) + P(B) = 2/5 + 1/3 = 11/15$. Their difference is $9/15 = 3/5 = 0.6$, exactly the given union probability.

Step 3: Conclusion The required value is $p = \frac{1}{3}$.

Quick Tip to Solve: For independent events, substitute the product into the union formula before collecting terms in p .

8. A line passes through the point with position vector $2\hat{i} - 3\hat{j} + 4\hat{k}$ and is perpendicular to the plane $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 5\hat{k}) = 7$. Find the equation of the line in cartesian and vector forms. [2 marks]

Solution:

Concept:

- A plane in the form $\vec{r} \cdot \vec{n} = d$ has normal vector $\vec{n}$; a line perpendicular to it has direction parallel to $\vec{n}$.
- A line through position vector $\vec{a}$ with direction $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\lambda \in \mathbb{R}$.

Step 1: Identify a point and the direction The given point is $P(2, -3, 4)$, so its position vector is $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$. From the plane equation, its normal is $\vec{n} = 3\hat{i} + 4\hat{j} - 5\hat{k}$. Since the required line is perpendicular to the plane, we can use this normal as its direction vector. The constant 7 fixes the plane location; it does not change its normal.

Quick Tip to Solve: Use the plane normal as the line direction, and subtract the point coordinates carefully in the symmetric form.

Step 2: Write the two forms Using the point and direction, the vector equation is

$$\vec{r} = (2\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda(3\hat{i} + 4\hat{j} - 5\hat{k}), \quad \lambda \in \mathbb{R}.$$

Thus $x = 2 + 3\lambda$, $y = -3 + 4\lambda$ and $z = 4 - 5\lambda$. Eliminating the same parameter from these equations gives the cartesian form.

Step 3: Conclusion Vector form: $\vec{r} = (2\hat{i} - 3\hat{j} + 4\hat{k}) + \lambda(3\hat{i} + 4\hat{j} - 5\hat{k}), \lambda \in \mathbb{R}$. Cartesian form:

$$\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-4}{-5}.$$

Quick Tip to Solve: Use the plane normal as the line direction, and subtract the point coordinates carefully in the symmetric form.

9. Show that the function f given by $f(x) = \tan^{-1}(\sin x + \cos x)$ is decreasing for all $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.
[2 marks]

Solution:

Concept:

- A differentiable function is strictly decreasing on an interval if its derivative is negative throughout that interval.
- For a differentiable expression $u(x)$, $\frac{d}{dx} \tan^{-1} u = \frac{u'}{1+u^2}$.

Step 1: Differentiate using the chain rule Put $u(x) = \sin x + \cos x$. Its derivative is $u'(x) = \cos x - \sin x$. Hence

$$f'(x) = \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2}.$$

The denominator is positive for every real x , since it is 1 plus a square. Therefore the sign of the derivative is exactly the sign of its numerator.

Step 2: Check the sign on the entire interval For $\pi/4 < x < \pi/2$, both sine and cosine are positive, and $\tan x > \tan(\pi/4) = 1$. Multiplying by the positive number $\cos x$ gives $\sin x > \cos x$. Thus $\cos x - \sin x < 0$, and consequently $f'(x) < 0$ throughout the stated open interval. The derivative criterion therefore applies at every point of that interval.

Step 3: Conclusion Hence $f(x) = \tan^{-1}(\sin x + \cos x)$ is strictly decreasing on $(\pi/4, \pi/2)$.

Quick Tip to Solve: For an inverse tangent derivative, its denominator is always positive; check only the sign of the inner derivative.

10. Find $\frac{dy}{dx}$ at $t = \frac{2\pi}{3}$ when $x = 10(t - \sin t)$ and $y = 12(1 - \cos t)$.

[2 marks]

Solution:

Concept:

- For parametric equations $x = x(t)$, $y = y(t)$, the derivative is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided $dx/dt \neq 0$.
- Use $\frac{d}{dt} \sin t = \cos t$ and $\frac{d}{dt} \cos t = -\sin t$, with the parameter measured in radians.

Step 1: Differentiate both coordinates From $x = 10(t - \sin t)$, we obtain $\frac{dx}{dt} = 10(1 - \cos t)$. Similarly, $y = 12(1 - \cos t)$ gives $\frac{dy}{dt} = 12 \sin t$. Therefore, wherever the first derivative is nonzero,

$$\frac{dy}{dx} = \frac{12 \sin t}{10(1 - \cos t)}.$$

The constants 10 and 12 must stay attached to their respective derivatives.

Step 2: Evaluate at the given parameter At $t = 2\pi/3$, $\sin t = \sqrt{3}/2$ and $\cos t = -1/2$. In particular, $dx/dt = 10(1 + 1/2) = 15 \neq 0$, so the parametric derivative formula is valid here. Substitution gives

$$\left. \frac{dy}{dx} \right|_{t=2\pi/3} = \frac{12(\sqrt{3}/2)}{10(1 + 1/2)} = \frac{6\sqrt{3}}{15}.$$

Step 3: Conclusion The required derivative is $\boxed{\left. \frac{dy}{dx} \right|_{t=2\pi/3} = \frac{2\sqrt{3}}{5}}.$

Quick Tip to Solve: Evaluate dx/dt and dy/dt separately at the parameter value, then divide dy/dt by dx/dt in that order.

11. If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$, then find the value of $|2AB|$.

[2 marks]

Solution:

Concept:

- For square matrices of the same order, $|AB| = |A||B|$.
- For a square matrix M of order n , $|kM| = k^n|M|$, because every one of its n rows is multiplied by k .

Step 1: Find the determinant of the product The matrices A and B are both of order 3, so their product AB is also a square matrix of order 3. Applying the product property of determinants gives

$$|AB| = |A||B| = (-1)(3) = -3.$$

This property does not require the matrices to commute; their determinant values are sufficient to determine the product determinant.

Quick Tip to Solve: For a scalar outside an order-3 matrix, cube the scalar before multiplying by the determinant of the matrix.

Step 2: Account for scalar multiplication The expression $2AB$ means the matrix AB multiplied by the scalar 2. All three rows of AB are therefore multiplied by 2. By the row scaling property, each such scaling contributes a factor 2 to the determinant, producing

$$|2AB| = 2^3|AB| = 8(-3).$$

The exponent is the matrix order, so multiplying the determinant by just 2 would omit two row factors.

Step 3: Conclusion Hence the value is $|2AB| = -24$.

Quick Tip to Solve: For a scalar outside an order-3 matrix, cube the scalar before multiplying by the determinant of the matrix.

12. The radius r of a right circular cylinder is increasing uniformly at the rate of 0.3 cm/s and its height h is decreasing at the rate of 0.4 cm/s. When $r = 3.5$ cm and $h = 7$ cm, find the rate of change of the curved surface area of the cylinder. [Use $\pi = \frac{22}{7}$]. [2 marks]

Solution:

Concept:

- The curved surface area of a right circular cylinder is $S = 2\pi rh$, where r is radius and h is height.
- When both dimensions depend on time t , the product rule gives $\frac{dS}{dt} = 2\pi \left(h \frac{dr}{dt} + r \frac{dh}{dt} \right)$.

Step 1: Assign the signed rates and differentiate The radius increases, so $dr/dt = 0.3$ cm/s. The height decreases, so $dh/dt = -0.4$ cm/s. Both factors in rh vary with time; hence differentiating only the radius would not give the net rate of change. Differentiating $S = 2\pi rh$ by the product rule yields

$$\frac{dS}{dt} = 2\pi \left(h \frac{dr}{dt} + r \frac{dh}{dt} \right).$$

Step 2: Substitute the instantaneous dimensions At the specified instant, $r = 3.5$ cm and $h = 7$ cm. Using the prescribed $\pi = 22/7$,

$$\begin{aligned} \frac{dS}{dt} &= 2 \cdot \frac{22}{7} [7(0.3) + 3.5(-0.4)] \\ &= \frac{44}{7} (2.1 - 1.4) \\ &= \frac{44}{7} (0.7) = 4.4 \text{ cm}^2/\text{s}. \end{aligned}$$

The positive radius contribution exceeds the negative height contribution, so the net rate is positive.

Step 3: Conclusion The curved surface area is increasing at $4.4 \text{ cm}^2/\text{s}$.

Quick Tip to Solve: Give a decreasing dimension a negative rate, and include both product-rule terms before substituting the dimensions.

Section C (44 marks)

Questions number 13 to 23 carry 4 marks each.

13. There are 4 cards numbered 1 to 4, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X . [4 marks]

Solution:

Concept:

- The mean of a discrete random variable is $E(X) = \sum xP(X = x)$, the probability-weighted average of its possible values.
- The variance is $\text{Var}(X) = E(X^2) - [E(X)]^2$, where $E(X^2) = \sum x^2P(X = x)$.
- Without replacement the cards are distinct. The $\binom{4}{2} = 6$ unordered pairs are equally likely; exchanging their order does not change their sum.

Step 1: Construct the probability distribution The six unordered pairs are (1, 2), (1, 3), (1, 4), (2, 3), (2, 4) and (3, 4). Their respective sums are 3, 4, 5, 5, 6, 7. Each pair has probability $1/6$. The sum 5 occurs for two different pairs, so it has twice the probability of each other sum. Thus

x	3	4	5	6	7
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

All probabilities are non-negative and their sum is $(1 + 1 + 2 + 1 + 1)/6 = 1$, verifying that this is a probability distribution. Alternatively, counting ordered draws gives twelve outcomes, with two orders for every unordered pair, and therefore the same probabilities.

Step 2: Find the expectation Multiply every possible sum by its probability and add:

$$E(X) = \frac{3 + 4 + 2(5) + 6 + 7}{6} = \frac{30}{6} = 5.$$

As a check, the mean of a single draw is $(1 + 2 + 3 + 4)/4 = 5/2$. Both draw positions have this same marginal mean, so linearity of expectation gives $E(X) = 5/2 + 5/2 = 5$, even though the draws are dependent.

Step 3: Find the second moment and variance Using the same probabilities with the squared values,

$$\begin{aligned} E(X^2) &= \frac{3^2 + 4^2 + 2(5^2) + 6^2 + 7^2}{6} \\ &= \frac{9 + 16 + 50 + 36 + 49}{6} = \frac{80}{3}, \\ \text{Var}(X) &= \frac{80}{3} - 5^2 = \frac{5}{3}. \end{aligned}$$

Directly, the squared deviations from 5 are 4, 1, 0, 1, 4. Their probability-weighted average is $(4 + 1 + 0 + 1 + 4)/6 = 5/3$, confirming the variance calculation.

Step 4: Conclusion The mean of X is $\boxed{5}$, and its variance is $\boxed{\frac{5}{3}}$.

Quick Tip to Solve: List the six unordered pairs first: the sum 5 occurs twice, so the five distinct sums are not equally likely.

14. If $\vec{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\vec{b} = 4\hat{i} - 7\hat{j} + \hat{k}$, find a vector $\vec{c}$ such that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 6$.

[4 marks]

Solution:

Concept:

- Write the unknown vector in Cartesian components and compare coefficients of $\hat{i}, \hat{j}, \hat{k}$ in the prescribed cross product.
- For $\vec{a} = (a_1, a_2, a_3)$ and $\vec{c} = (x, y, z)$, the dot product is $a_1x + a_2y + a_3z$. This scalar condition fixes the component parallel to $\vec{a}$.
- A cross product is perpendicular to its first factor, so $\vec{a} \cdot \vec{b} = 0$ is a necessary consistency check for $\vec{a} \times \vec{c} = \vec{b}$.

Step 1: Set up the unknown vector and check consistency Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$. First,

$$\vec{a} \cdot \vec{b} = 2(4) + 1(-7) + (-1)(1) = 8 - 7 - 1 = 0.$$

Thus the given target cross product is perpendicular to $\vec{a}$, as it must be. We now use the component equations together with the specified dot product to determine x, y, z .

Step 2: Expand the cross product Expanding the determinant carefully, including the negative sign on the middle cofactor, gives

$$\begin{aligned}\vec{a} \times \vec{c} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ x & y & z \end{vmatrix} \\ &= (y + z)\hat{i} - (2z + x)\hat{j} + (2y - x)\hat{k}.\end{aligned}$$

Comparing with $4\hat{i} - 7\hat{j} + \hat{k}$, we obtain

$$y + z = 4, \quad x + 2z = 7, \quad 2y - x = 1.$$

The first two equations give $y = 4 - z$ and $x = 7 - 2z$. Substitution in the third yields $2(4 - z) - (7 - 2z) = 1$, an identity. Hence the cross product alone leaves one free parameter.

Step 3: Use the dot product to determine the parameter The remaining condition is

$$\vec{a} \cdot \vec{c} = 2x + y - z = 6.$$

Substitute $x = 7 - 2z$ and $y = 4 - z$:

$$\begin{aligned}2(7 - 2z) + (4 - z) - z &= 6, \\ 18 - 6z &= 6, \\ z &= 2.\end{aligned}$$

Therefore $y = 4 - 2 = 2$ and $x = 7 - 4 = 3$. The dot product condition removes the freedom left by the cross product and produces a unique vector.

Quick Tip to Solve: Use two cross-product component equations to express x and y in terms of z , then use the dot product to fix z ; check the middle cofactor sign.

Step 4: Verify both prescribed products For the components (3, 2, 2), direct calculation gives

$$\begin{aligned}\vec{a} \times \vec{c} &= (2 + 2)\hat{i} - (4 + 3)\hat{j} + (4 - 3)\hat{k} \\ &= 4\hat{i} - 7\hat{j} + \hat{k} = \vec{b}, \\ \vec{a} \cdot \vec{c} &= 2(3) + 2 - 2 = 6.\end{aligned}$$

Thus the same vector satisfies both conditions, including the signs of all three components of the cross product.

Step 5: Conclusion The required vector is $\vec{c} = 3\hat{i} + 2\hat{j} + 2\hat{k}$.

Quick Tip to Solve: Use two cross-product component equations to express x and y in terms of z, then use the dot product to fix z; check the middle cofactor sign.

15(a). Evaluate: $\int_{-2}^1 |x^3 - x| dx$.

[4 marks]

Solution:

Concept:

- For a real expression $f(x)$, its modulus equals $f(x)$ where $f(x) \geq 0$, and equals $-f(x)$ where $f(x) < 0$. Split the definite integral at every zero where the expression changes sign.
- The factorisation $x^3 - x = x(x - 1)(x + 1)$ gives the zeros $-1, 0, 1$. An antiderivative is $F(x) = x^4/4 - x^2/2$; each definite integral is evaluated using $F(b) - F(a)$.

Step 1: Determine the sign on each interval On $(-2, -1)$, all three factors $x, x - 1, x + 1$ are negative, so their product is negative. On $(-1, 0)$, the first two are negative and the third is positive, so their product is positive. On $(0, 1)$, only $x - 1$ is negative, so the product is negative. At the splitting points the polynomial is zero; inclusion of endpoints does not change the value of the integral.

Step 2: Remove the modulus using the signs If I denotes the required integral, then

$$I = - \int_{-2}^{-1} (x^3 - x) dx + \int_{-1}^0 (x^3 - x) dx - \int_0^1 (x^3 - x) dx.$$

Thus each contribution is the nonnegative area associated with its interval. Using a single antiderivative over the whole interval without this sign adjustment would give cancellation and would not evaluate the modulus integral.

Step 3: Evaluate the endpoint contributions The antiderivative values are

$$F(-2) = 2, \quad F(-1) = -\frac{1}{4}, \quad F(0) = 0, \quad F(1) = -\frac{1}{4}.$$

Therefore

$$\begin{aligned}I &= - \left(-\frac{1}{4} - 2 \right) + \left(0 + \frac{1}{4} \right) - \left(-\frac{1}{4} - 0 \right) \\ &= \frac{9}{4} + \frac{1}{4} + \frac{1}{4}.\end{aligned}$$

Quick Tip to Solve: Factor the expression inside the modulus first, then split at its zeros and check the sign on each interval.

The two contributions from $[-1, 0]$ and $[0, 1]$ agree, as expected because the modulus of the odd polynomial is even.

Step 4: Conclusion Therefore, $\int_{-2}^1 |x^3 - x| dx = \boxed{\frac{11}{4}}$.

Quick Tip to Solve: Factor the expression inside the modulus first, then split at its zeros and check the sign on each interval.

OR

15(b). Find: $\int e^{2x} \sin(3x + 1) dx$.

[4 marks]

Solution:

Concept:

- Integration by parts states $\int u dv = uv - \int v du$. Applying it twice to an exponential multiplied by a sine brings back the original integral, which can then be isolated algebraically.
- The chain rule gives $\frac{d}{dx} \sin(3x + 1) = 3 \cos(3x + 1)$ and $\frac{d}{dx} \cos(3x + 1) = -3 \sin(3x + 1)$. Also, $\int e^{2x} dx = e^{2x}/2$.

Step 1: Apply integration by parts to the sine integral Let $I = \int e^{2x} \sin(3x + 1) dx$. Choose $u = \sin(3x + 1)$ and $dv = e^{2x} dx$, giving $du = 3 \cos(3x + 1) dx$ and $v = e^{2x}/2$. Thus

$$I = \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3}{2} J, \quad J = \int e^{2x} \cos(3x + 1) dx.$$

For the algebra below we choose compatible antiderivatives; all arbitrary constants will be absorbed into one final constant.

Step 2: Apply integration by parts to the cosine integral For J , choose $u = \cos(3x + 1)$ and $dv = e^{2x} dx$. Then $du = -3 \sin(3x + 1) dx$ and $v = e^{2x}/2$, so

$$J = \frac{e^{2x}}{2} \cos(3x + 1) + \frac{3}{2} I.$$

The positive sign before the last integral comes from subtracting a negative derivative of the cosine.

Step 3: Substitute and isolate the original integral Substituting the expression for J in the first equation gives

$$\begin{aligned} I &= \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3e^{2x}}{4} \cos(3x + 1) - \frac{9}{4} I, \\ \frac{13}{4} I &= \frac{e^{2x}}{2} \sin(3x + 1) - \frac{3e^{2x}}{4} \cos(3x + 1). \end{aligned}$$

Multiplication by $4/13$ isolates the desired antiderivative.

Quick Tip to Solve: After two integrations by parts, collect the repeated integral on one side; the coefficient becomes $1 + 9/4 = 13/4$.

Step 4: Check by differentiation Put $\theta = 3x + 1$. The derivative of the obtained expression is

$$\begin{aligned}\frac{d}{dx} \left[\frac{e^{2x}}{13} (2 \sin \theta - 3 \cos \theta) \right] &= \frac{e^{2x}}{13} [4 \sin \theta - 6 \cos \theta + 6 \cos \theta + 9 \sin \theta] \\ &= e^{2x} \sin \theta.\end{aligned}$$

This recovers the given integrand and checks both the coefficients and the sign.

Step 5: Conclusion

$$\int e^{2x} \sin(3x + 1) dx = \frac{e^{2x}}{13} [2 \sin(3x + 1) - 3 \cos(3x + 1)] + C,$$

where C is an arbitrary real constant.

Quick Tip to Solve: After two integrations by parts, collect the repeated integral on one side; the coefficient becomes $1 + 9/4 = 13/4$.

16. In a shop X , 30 tins of pure ghee and 40 tins of adulterated ghee which look alike, are kept for sale while in shop Y , similar 50 tins of pure ghee and 60 tins of adulterated ghee are there. One tin of ghee is purchased from one of the randomly selected shops and is found to be adulterated. Find the probability that it is purchased from shop Y . What measures should be taken to stop adulteration?

[4 marks]

Solution:

Concept:

- Bayes' theorem reverses a conditional probability: $P(Y | A) = P(Y)P(A | Y)/P(A)$, where A denotes the event that the purchased tin is adulterated.
- For two mutually exclusive and exhaustive choices of shop, the total probability is $P(A) = P(X)P(A | X) + P(Y)P(A | Y)$. A random choice between the two shops gives each shop probability $1/2$.

Step 1: Define the events and conditional probabilities Let X and Y denote purchases from the respective shops and let A denote an adulterated tin. The shop is selected first, with equal probability, so $P(X) = P(Y) = 1/2$. The tins are then selected within that shop. Shop X contains $30 + 40 = 70$ tins, of which 40 are adulterated. Shop Y contains $50 + 60 = 110$ tins, of which 60 are adulterated. Therefore,

$$P(A | X) = \frac{40}{70} = \frac{4}{7}, \quad P(A | Y) = \frac{60}{110} = \frac{6}{11}.$$

Step 2: Find the probability of observing an adulterated tin An adulterated tin can come from either shop, and these two possibilities are disjoint. Thus the total probability theorem gives

$$\begin{aligned}P(A) &= \frac{1}{2} \cdot \frac{4}{7} + \frac{1}{2} \cdot \frac{6}{11} \\ &= \frac{2}{7} + \frac{3}{11} = \frac{22 + 21}{77} = \frac{43}{77}.\end{aligned}$$

Quick Tip to Solve: Keep the two stages separate: first choose the shop with probability one-half, then use that shop's adulterated fraction before applying Bayes' theorem.

Also, the joint probability of buying an adulterated tin from Y is $P(Y \cap A) = \frac{1}{2} \cdot \frac{6}{11} = \frac{3}{11}$.

Step 3: Apply Bayes' theorem to the observed information Since the purchased tin is known to be adulterated, restrict the sample space to the event A . Then

$$P(Y | A) = \frac{P(Y \cap A)}{P(A)} = \frac{3/11}{43/77} = \frac{3 \times 7}{43} = \frac{21}{43}.$$

Step 4: State measures to prevent adulteration Food products should undergo regular quality testing and inspection. Authorities should enforce food-quality standards and take strict action against sellers who adulterate products. Consumers should prefer sealed, quality-certified products, check labels and report suspected adulteration.

Step 5: Conclusion The required probability is $\frac{21}{43}$. Regular testing, strict enforcement of food-quality standards and consumer awareness help prevent adulteration.

Quick Tip to Solve: Keep the two stages separate: first choose the shop with probability one-half, then use that shop's adulterated fraction before applying Bayes' theorem.

17. Find: $\int \frac{e^x dx}{(2 + e^x)(4 + e^{2x})}.$

[4 marks]

Solution:

Concept:

- Substitution $t = e^x$ gives $dt = e^x dx$, converting this exponential integral into a rational integral in t .
- A linear factor and an irreducible quadratic factor require partial fractions of the form $A/(t + 2) + (Bt + C)/(t^2 + 4)$, with constants A, B, C .
- Use $\int du/u = \ln |u| + C$ and $\int dt/(t^2 + a^2) = a^{-1} \tan^{-1}(t/a) + C$, where $a > 0$.

Step 1: Replace the exponential by a new variable Denote the given integral by I . Set $t = e^x$; then $t > 0$ for real x , $dt = e^x dx$ and $e^{2x} = t^2$. Therefore,

$$I = \int \frac{dt}{(t + 2)(t^2 + 4)}.$$

Step 2: Determine the partial fractions Write

$$\frac{1}{(t + 2)(t^2 + 4)} = \frac{A}{t + 2} + \frac{Bt + C}{t^2 + 4}.$$

Multiplying by the common denominator gives the polynomial identity

$$\begin{aligned} 1 &= A(t^2 + 4) + (Bt + C)(t + 2) \\ &= (A + B)t^2 + (2B + C)t + (4A + 2C). \end{aligned}$$

Quick Tip to Solve: Substitute $t = e^x$ before decomposing: the numerator becomes dt , and the quadratic term requires a linear numerator in the partial fractions.

Equating coefficients yields $A + B = 0$, $2B + C = 0$ and $4A + 2C = 1$. Thus $B = -A$, $C = 2A$ and $8A = 1$, giving $A = 1/8$, $B = -1/8$, $C = 1/4$.

Step 3: Integrate the three standard terms Using these constants, split the integral as

$$I = \frac{1}{8} \int \frac{dt}{t+2} - \frac{1}{8} \int \frac{t dt}{t^2+4} + \frac{1}{4} \int \frac{dt}{t^2+4}.$$

For the middle integral, use $u = t^2 + 4$, so $du = 2t dt$; hence its value is $\frac{1}{2} \ln(t^2 + 4)$. The last integral is $\frac{1}{2} \tan^{-1}(t/2)$. Thus,

$$I = \frac{1}{8} \ln(t+2) - \frac{1}{16} \ln(t^2+4) + \frac{1}{8} \tan^{-1}\left(\frac{t}{2}\right) + C.$$

Step 4: Restore the original variable Replace t by e^x in all three terms.

Step 5: Conclusion

$$I = \frac{1}{8} \ln(2 + e^x) - \frac{1}{16} \ln(4 + e^{2x}) + \frac{1}{8} \tan^{-1}\left(\frac{e^x}{2}\right) + C.$$

Quick Tip to Solve: Substitute $t = e^x$ before decomposing: the numerator becomes dt , and the quadratic term requires a linear numerator in the partial fractions.

18(a). If $xy = e^{(x-y)}$, then show that $\frac{dy}{dx} = \frac{y(x-1)}{x(y+1)}$.

[4 marks]

Solution:

Concept:

- In implicit differentiation, treat y as a differentiable function of x . Thus the product rule gives $\frac{d}{dx}(xy) = y + x \frac{dy}{dx}$.
- The chain rule gives $\frac{d}{dx}e^u = e^u \frac{du}{dx}$. For $u = x - y$, its derivative is $1 - \frac{dy}{dx}$.

Step 1: Differentiate the given relation implicitly Write $y' = \frac{dy}{dx}$. Differentiating both sides of $xy = e^{x-y}$ with respect to x , and remembering that y depends on x , gives

$$y + xy' = e^{x-y}(1 - y').$$

The term y on the left comes from differentiating x ; the factor $1 - y'$ on the right is essential because the exponent also contains y . The original equation now supplies $e^{x-y} = xy$, so this becomes

$$y + xy' = xy(1 - y') = xy - xy y'.$$

Step 2: Collect the derivative terms and isolate the derivative Bring both terms containing y' to the

Quick Tip to Solve: Replace the exponential by xy immediately after differentiation; then collect all terms containing dy/dx before dividing.

left and the remaining terms to the right:

$$\begin{aligned}xy' + xyy' &= xy - y, \\x(1 + y)y' &= y(x - 1).\end{aligned}$$

The factoring is the key algebraic step: the two derivative terms have the common factor xy' , and the two terms on the right have the common factor y . Thus the coefficient of the derivative is $x(1 + y)$. Division by this coefficient is valid at points where $x(1 + y) \neq 0$; the formula is understood on that domain. In particular, the original relation gives $xy = e^{x-y} > 0$, so $x \neq 0$. We retain the restriction $y \neq -1$ when dividing. The calculation establishes the required formula wherever this division is valid, without taking logarithms or assuming that x and y are individually positive.

Step 3: Conclusion Dividing $x(1 + y)y' = y(x - 1)$ by $x(1 + y)$, at points where the derivative exists, gives

$$\frac{dy}{dx} = \frac{y(x - 1)}{x(y + 1)}.$$

Quick Tip to Solve: Replace the exponential by xy immediately after differentiation; then collect all terms containing dy/dx before dividing.

OR

18(b). If $\log y = \tan^{-1} x$, then show that $(1 + x^2) \frac{d^2y}{dx^2} + (2x - 1) \frac{dy}{dx} = 0$.

[4 marks]

Solution:

Concept:

- For the natural logarithm, $\frac{d}{dx} \log y = \frac{1}{y} \frac{dy}{dx}$, with $y > 0$. The chain rule accounts for the dependence of y on x .
- The derivative of $\tan^{-1} x$ is $\frac{1}{1 + x^2}$. To find a second derivative relation, differentiate the first derivative equation using the product rule.

Step 1: Differentiate once and remove the denominator The given relation is $\log y = \tan^{-1} x$. Since the logarithm is defined, $y > 0$. Differentiation with respect to x yields

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{1 + x^2}.$$

Multiplying by $y(1 + x^2)$, which is nonzero for every real x in the relation, gives

$$(1 + x^2) \frac{dy}{dx} = y.$$

This form is convenient because the derivative appears multiplied by a polynomial. It permits a direct second differentiation without the additional algebra of the quotient rule. The logarithm derivative

Quick Tip to Solve: Rearrange the first derivative as $(1 + x^2)y' = y$ before differentiating again; this makes the desired coefficient $2x - 1$ appear immediately.

must include dy/dx ; treating y as a constant here would lose the required relation.

Step 2: Differentiate again using the product rule Differentiate both sides of the first derivative equation. The left side is a product of $1 + x^2$ and dy/dx , while the derivative of the right side is dy/dx . Thus

$$2x \frac{dy}{dx} + (1 + x^2) \frac{d^2y}{dx^2} = \frac{dy}{dx}.$$

Move the right-hand term to the left and combine the two first derivative terms:

$$(1 + x^2) \frac{d^2y}{dx^2} + (2x - 1) \frac{dy}{dx} = 0.$$

Both terms of the product rule are present: $2x dy/dx$ comes from differentiating the polynomial, and $(1 + x^2)d^2y/dx^2$ comes from differentiating dy/dx . The coefficient $2x - 1$ arises after subtracting the derivative of y , rather than from differentiating the inverse tangent a second time directly.

Step 3: Conclusion Hence the required identity is proved:

$$(1 + x^2) \frac{d^2y}{dx^2} + (2x - 1) \frac{dy}{dx} = 0.$$

Quick Tip to Solve: Rearrange the first derivative as $(1 + x^2)y' = y$ before differentiating again; this makes the desired coefficient $2x - 1$ appear immediately.

19(a). Using properties of determinants show that $\begin{vmatrix} 1 & 1 & 1+x \\ 1 & 1+y & 1 \\ 1+z & 1 & 1 \end{vmatrix} = xyz + yz + zx + xy.$ [4 marks]

Solution:

Concept:

- Adding a multiple of one row to another row leaves the determinant unchanged. Thus subtracting the first row from each of the other two rows is a useful way to introduce zeros.
- Expansion along the first row uses alternating cofactor signs $+, -, +$. For a two-by-two minor,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Step 1: Apply determinant-preserving row operations Denote the determinant on the left by D . Apply $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$, keeping the first row unchanged. The second row becomes $(0, y, -x)$, since $1 - (1 + x) = -x$; the third row becomes $(z, 0, -x)$. Hence

$$D = \begin{vmatrix} 1 & 1 & 1+x \\ 0 & y & -x \\ z & 0 & -x \end{vmatrix}.$$

Both operations preserve the value: no rows have been interchanged and no row has been multiplied

Quick Tip to Solve: Subtract the first row from the other two rows, then retain the alternating cofactor signs; checking $x=y=z=1$ quickly detects the printed sign error.

by a factor.

Step 2: Expand with the correct cofactor signs Expanding along the first row gives

$$\begin{aligned} D &= 1\{y(-x) - (-x)0\} \\ &\quad - 1\{0(-x) - (-x)z\} \\ &\quad + (1+x)\{0 \cdot 0 - yz\} \\ &= -xy - xz - (1+x)yz \\ &= -(xyz + yz + zx + xy). \end{aligned}$$

In particular, the second minor equals xz , and its cofactor sign is negative. The third minor equals $-yz$; these signs cannot be discarded.

Step 3: Check the printed identity The printed right-hand side has the opposite sign. To check that this is a real contradiction, take $x = y = z = 1$. The determinant is then

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 1(2 - 1) - 1(1 - 2) + 2(1 - 4) = -4,$$

whereas the stated right-hand side equals $1 + 1 + 1 + 1 = 4$. Thus the requested equality is false in general and cannot be proved as printed. The row-operation calculation identifies precisely the missing overall minus sign.

Step 4: Conclusion As printed, the identity contains a sign error. The correct evaluation is

$$D = -(xyz + yz + zx + xy).$$

Quick Tip to Solve: Subtract the first row from the other two rows, then retain the alternating cofactor signs; checking $x=y=z=1$ quickly detects the printed sign error.

OR

19(b). Find matrix X so that $X \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$.

[4 marks]

Solution:

Concept:

- For matrix multiplication, each entry is the dot product of a row of the left matrix and a column of the right matrix. The order of the factors must be preserved.
- Since the given right factor has order 2×3 and the product has order 2×3 , the unknown matrix X must have order 2×2 . Equality of matrices means equality of corresponding entries.

Step 1: Write the unknown entries and multiply Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Multiplication by the given matrix

Quick Tip to Solve: Solve each row of X from the first two product columns, then use the third column to check consistency of the overdetermined equations.

gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{pmatrix}.$$

Therefore the first row must equal $(-7, -8, -9)$, and the second row must equal $(2, 4, 6)$. Each row supplies three scalar equations for its two unknown entries.

Step 2: Determine the first row The first two entries give $a + 4b = -7$ and $2a + 5b = -8$. Subtract twice the first equation from the second:

$$(2a + 5b) - 2(a + 4b) = -8 - 2(-7), \quad -3b = 6.$$

Thus $b = -2$, and substitution in $a + 4b = -7$ gives $a = 1$. Check the third entry as well: $3a + 6b = 3 - 12 = -9$, as required. This check ensures that the extra equation is consistent rather than simply ignoring it.

Step 3: Determine the second row The first two equations are $c + 4d = 2$ and $2c + 5d = 4$. Subtracting twice the first equation from the second gives $-3d = 0$, so $d = 0$. Then $c = 2$, and the third entry is $3c + 6d = 6$, again agreeing with the given product.

Step 4: Verify the matrix equation Substitute all four values into the original product:

$$\begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 1-8 & 2-10 & 3-12 \\ 2 & 4 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}.$$

All six entries agree. The two independent equations used for each row have coefficient determinant $5 - 8 = -3 \neq 0$, so the entries are uniquely determined.

Step 5: Conclusion The required matrix is $X = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$.

Quick Tip to Solve: Solve each row of X from the first two product columns, then use the third column to check consistency of the overdetermined equations.

20. Solve the following LPP graphically: Maximise $Z = 105x + 90y$ subject to the constraints $x + y \leq 50$, $2x + y \leq 80$, $x \geq 0$, $y \geq 0$. [4 marks]

Solution:

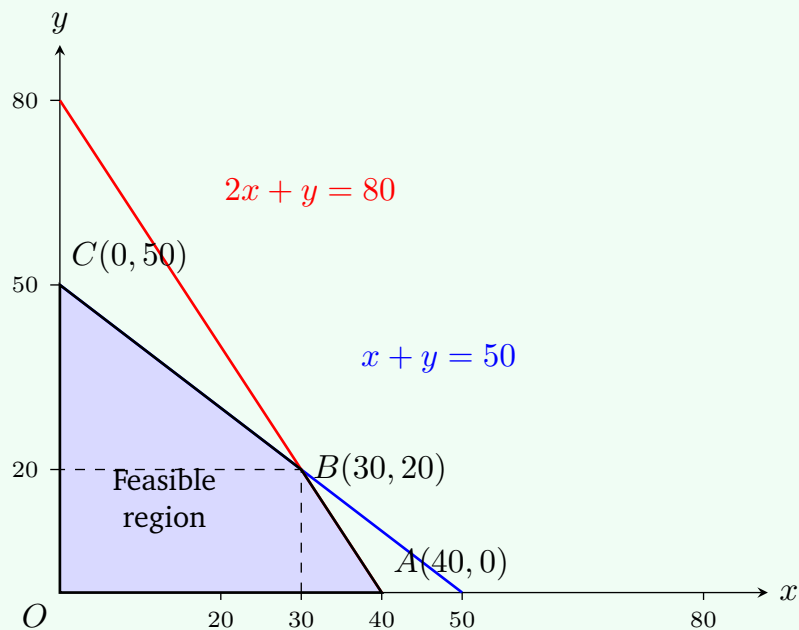
Concept:

- The feasible region is the common part of all constraint half-planes, including the first quadrant imposed by $x \geq 0$ and $y \geq 0$.
- For a linear objective function on a nonempty bounded polygonal feasible region, the maximum occurs at a corner point. Thus evaluate $Z = 105x + 90y$ at every vertex.

Step 1: Plot the constraint lines and select their half-planes The line $x + y = 50$ passes through $(50, 0)$ and $(0, 50)$. The line $2x + y = 80$ passes through $(40, 0)$ and $(0, 80)$. Plot both using these intercepts. Since $(0, 0)$ satisfies each inequality, retain the side containing the origin for each line. Intersect these

Quick Tip to Solve: Find the intersection of the two constraint lines, but compare its objective value with both axis vertices before selecting the maximum.

sides with $x \geq 0, y \geq 0$. The shaded polygon in the graph is therefore the feasible region; its boundary is included because the inequalities are non-strict.



The shaded feasible region is the polygon $OABC$.

Step 2: Find all vertices of the feasible region On the x -axis, the tighter bound is $2x \leq 80$, giving $A = (40, 0)$. On the y -axis, the tighter bound is $y \leq 50$, giving $C = (0, 50)$. The two slant boundaries meet at B , obtained from

$$\begin{aligned} x + y &= 50, \\ 2x + y &= 80. \end{aligned}$$

Subtracting gives $x = 30$, and then $y = 20$. Thus $B = (30, 20)$, and the fourth vertex is $O = (0, 0)$. Each of these points satisfies both original inequalities. The feasible region $OABC$ is nonempty and bounded.

Step 3: Compare the objective function at the corner points Evaluate the given objective without changing its coefficients:

Corner point (x, y)	$Z = 105x + 90y$
$(0, 0)$	0
$(40, 0)$	4200
$(30, 20)$	$3150 + 1800 = 4950$
$(0, 50)$	4500

The largest listed value is attained at the intersection of the two slant boundaries. No edge has constant objective value because its slope differs from that of the objective lines $105x + 90y = k$. Hence this maximizing corner is unique.

Step 4: Conclusion The maximum value is $Z_{\max} = 4950$, attained at $x = 30, y = 20$.

Quick Tip to Solve: Find the intersection of the two constraint lines, but compare its objective value with both axis vertices before selecting the maximum.

21. Find the general solution of the differential equation $x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x$.

[4 marks]

Solution:

Concept:

- A homogeneous differential equation depending on the ratio y/x is reduced by the substitution $v = y/x$, or $y = vx$. The product rule then gives $dy/dx = v + x dv/dx$.
- Separation of variables puts all terms involving v with dv and all terms involving x with dx . Here $\int \cos v dv = \sin v$ and $\int dx/x = \ln|x|$ on any interval not containing zero.

Step 1: Substitute the homogeneous variable The given equation contains the ratio y/x , so it is defined only for $x \neq 0$. Put $y = vx$, where v is a function of x . Differentiating gives

$$\frac{dy}{dx} = v + x \frac{dv}{dx}.$$

Substitute both $y = vx$ and this derivative into the equation as printed:

$$x \cos v \left(v + x \frac{dv}{dx} \right) = vx \cos v + x.$$

The terms $xv \cos v$ on the two sides are equal. Cancel them to obtain

$$x^2 \cos v \frac{dv}{dx} = x.$$

Step 2: Separate the variables and integrate Dividing by x^2 , which is nonzero on the solution interval, yields

$$\cos v \frac{dv}{dx} = \frac{1}{x}, \quad \cos v dv = \frac{dx}{x}.$$

Integrating both sides gives

$$\sin v = \ln|x| + C,$$

where C is an arbitrary real constant. Replacing v by y/x gives the implicit family

$$\sin\left(\frac{y}{x}\right) - \ln|x| = C.$$

The absolute value permits either a positive or a negative interval of x . No solution interval can cross $x = 0$, where the original ratio is undefined.

Step 3: Check the result in the original equation Differentiating the implicit relation gives

$$\cos\left(\frac{y}{x}\right) \frac{x dy/dx - y}{x^2} = \frac{1}{x}.$$

Multiplication by x^2 and rearrangement yield

$$x \cos\left(\frac{y}{x}\right) \frac{dy}{dx} = y \cos\left(\frac{y}{x}\right) + x,$$

which is exactly the given equation. For a differentiable solution with $x \neq 0$, a point where $\cos(y/x) =$

Quick Tip to Solve: Substitute directly into the printed equation before dividing by the cosine; the two terms containing $v \cos v$ cancel immediately.

0 cannot occur: substituting such a point in the original equation would give $0 = x$. The implicit family is therefore read on intervals where it defines a differentiable branch and $\cos(y/x) \neq 0$.

Step 4: Conclusion The general solution is $\boxed{\sin\left(\frac{y}{x}\right) = \ln|x| + C}$, on its differentiable solution intervals with $x \neq 0$.

Quick Tip to Solve: Substitute directly into the printed equation before dividing by the cosine; the two terms containing $v \cos v$ cancel immediately.

22. Prove that: $\tan^{-1} \left[\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} \right] = \frac{\pi}{4} + \frac{1}{2} \cos^{-1} x^2, -1 < x < 1.$ [4 marks]

Solution:

Concept:

- The principal value of $\cos^{-1} t$ lies in $[0, \pi]$, whereas that of $\tan^{-1} t$ lies in $(-\pi/2, \pi/2)$. These ranges determine whether an inverse-tangent identity can be used directly.
- For $\theta \in [0, \pi]$, the half-angle identities give $\sqrt{1 + \cos \theta} = \sqrt{2} \cos(\theta/2)$ and $\sqrt{1 - \cos \theta} = \sqrt{2} \sin(\theta/2)$; both half-angle functions are nonnegative.
- The tangent addition identity is $\tan(\pi/4 + u) = (1 + \tan u)/(1 - \tan u)$, whenever the denominator is nonzero.

Step 1: Introduce an angle and identify the domain For $-1 < x < 1$ with $x \neq 0$, put $\theta = \cos^{-1}(x^2)$. Since $0 < x^2 < 1$, we have $0 < \theta < \pi/2$ and $x^2 = \cos \theta$. Thus $0 < \theta/2 < \pi/4$. The positive square roots in the question become

$$\sqrt{1+x^2} = \sqrt{2} \cos \frac{\theta}{2}, \quad \sqrt{1-x^2} = \sqrt{2} \sin \frac{\theta}{2}.$$

In this range, $\cos(\theta/2) > \sin(\theta/2)$, so the denominator of the given quotient is strictly positive.

Step 2: Apply the tangent addition formula Substituting the two square roots and cancelling their common factor gives

$$\begin{aligned} \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}} &= \frac{\cos(\theta/2) + \sin(\theta/2)}{\cos(\theta/2) - \sin(\theta/2)} \\ &= \frac{1 + \tan(\theta/2)}{1 - \tan(\theta/2)} \\ &= \tan \left(\frac{\pi}{4} + \frac{\theta}{2} \right). \end{aligned}$$

Division by $\cos(\theta/2)$ is valid because this cosine is positive.

Step 3: Check the inverse-tangent branch and the zero case The angle $\pi/4 + \theta/2$ lies strictly between $\pi/4$ and $\pi/2$, hence inside the principal range of inverse tangent. Taking $\tan^{-1}$ therefore recovers exactly this angle. Substitution of $\theta = \cos^{-1}(x^2)$ produces the required right-hand side.

The printed interval also includes $x = 0$. At this value the quotient has denominator $1 - 1 = 0$, so the left-hand side is undefined as an ordinary real expression. As $x \rightarrow 0$, the quotient tends to $+\infty$ and

Quick Tip to Solve: Set $x^2 = \cos \theta$, replace both radicals using half-angle identities, and check the principal range before taking inverse tangent.

its inverse tangent tends to $\pi/2$; the right-hand side at zero also equals $\pi/2$. Thus equality at zero is understood only through this continuous extension.

Step 4: Conclusion Hence $\tan^{-1}\left(\frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}\right) = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}(x^2)$ for $0 < |x| < 1$. At $x = 0$, both sides agree at $\pi/2$ after extending the left-hand side by its limit.

Quick Tip to Solve: Set $x^2 = \cos \theta$, replace both radicals using half-angle identities, and check the principal range before taking inverse tangent.

23. Using vectors, find the area of triangle ABC , with vertices $A(1, 2, 3)$, $B(2, -1, 4)$ and $C(4, 5, -1)$.

[4 marks]

Solution:

Concept:

- The vector from a point A to a point B is $\vec{AB} = \vec{OB} - \vec{OA}$, obtained by subtracting corresponding coordinates.
- The magnitude $|\mathbf{u} \times \mathbf{v}|$ equals the area of the parallelogram formed by two adjacent vectors $\mathbf{u}$ and $\mathbf{v}$. The triangle with these sides has half that area.
- For $\mathbf{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$, its magnitude is $|\mathbf{w}| = \sqrt{w_1^2 + w_2^2 + w_3^2}$.

Step 1: Form two sides of the triangle from the same vertex Using $A(1, 2, 3)$ as the common initial point and subtracting its coordinates from those of $B(2, -1, 4)$ and $C(4, 5, -1)$, respectively, gives

$$\begin{aligned}\vec{AB} &= (2-1)\hat{i} + (-1-2)\hat{j} + (4-3)\hat{k} \\ &= \hat{i} - 3\hat{j} + \hat{k}, \\ \vec{AC} &= (4-1)\hat{i} + (5-2)\hat{j} + (-1-3)\hat{k} \\ &= 3\hat{i} + 3\hat{j} - 4\hat{k}.\end{aligned}$$

These two vectors span the required triangle; their cross product supplies its area without needing to calculate its angles or an altitude.

Step 2: Evaluate the cross product Expanding the determinant along its first row, remembering the negative sign of the middle cofactor, gives

$$\begin{aligned}\vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 3 & 3 & -4 \end{vmatrix} \\ &= (12-3)\hat{i} - (-4-3)\hat{j} + (3+9)\hat{k} \\ &= 9\hat{i} + 7\hat{j} + 12\hat{k}.\end{aligned}$$

As a check, this vector is perpendicular to both sides: its scalar products with $(1, -3, 1)$ and $(3, 3, -4)$ are $9 - 21 + 12 = 0$ and $27 + 21 - 48 = 0$, respectively.

Quick Tip to Solve: Use two side vectors with the same starting vertex, check the middle cofactor sign, and halve the cross-product magnitude for the triangle.

Step 3: Obtain the magnitude and apply the triangle-area formula The cross-product magnitude is

$$|\vec{AB} \times \vec{AC}| = \sqrt{9^2 + 7^2 + 12^2} = \sqrt{81 + 49 + 144} = \sqrt{274}.$$

This is the area of the parallelogram built on $\vec{AB}$ and $\vec{AC}$. Its diagonal divides it into two congruent triangles, so the required area is obtained by multiplying this magnitude by $1/2$.

Step 4: Conclusion The area of triangle ABC is $\boxed{\frac{\sqrt{274}}{2} \text{ square units}}.$

Quick Tip to Solve: Use two side vectors with the same starting vertex, check the middle cofactor sign, and halve the cross-product magnitude for the triangle.

Section D (36 marks)

Questions number 24 to 29 carry 6 marks each.

24(a). Using the method of integration, find the area of the triangle ABC , coordinates of whose vertices are $A(1, 2)$, $B(2, 0)$ and $C(4, 3)$. [6 marks]

Solution:

Concept:

- The area between an upper boundary $y = u(x)$ and a lower boundary $y = l(x)$, for $a \leq x \leq b$, is $\int_a^b [u(x) - l(x)] dx$.
- A line through two points has slope $m = (y_2 - y_1)/(x_2 - x_1)$ and equation $y - y_1 = m(x - x_1)$. The integral must be split wherever the lower or upper side of the triangle changes.

Step 1: Obtain the equations of the three sides The vertices are $A(1, 2)$, $B(2, 0)$ and $C(4, 3)$. For AB , the slope is $(0 - 2)/(2 - 1) = -2$, so $y - 2 = -2(x - 1)$, or $y = 4 - 2x$. For BC , the slope is $(3 - 0)/(4 - 2) = 3/2$, giving $y = \frac{3}{2}(x - 2) = \frac{3}{2}x - 3$. For AC , the slope is $(3 - 2)/(4 - 1) = 1/3$, giving $y - 2 = \frac{1}{3}(x - 1)$, or $y = \frac{x+5}{3}$. These equations recover the given endpoint coordinates, so they represent the required sides.

Step 2: Identify the vertical boundaries and split the integral The triangle extends from $x = 1$ to $x = 4$. Its upper boundary throughout this interval is AC . The lower boundary is AB from $x = 1$ to $x = 2$, and BC from $x = 2$ to $x = 4$. In particular, the vertical strip at $x = 2$ starts at B and ends on AC . Thus the area S is

$$S = \int_1^2 \left(\frac{x+5}{3} - (4-2x) \right) dx + \int_2^4 \left(\frac{x+5}{3} - \left(\frac{3}{2}x - 3 \right) \right) dx.$$

The two strip heights simplify to $7(x-1)/3$ and $7(4-x)/6$, respectively. Both are non-negative on their intervals and vanish at the outer vertices.

Quick Tip to Solve: Sort the vertices by their x-coordinates and split at the middle value, $x=2$, where the lower side changes from AB to BC .

Step 3: Evaluate the two definite integrals Integrating the first strip height gives

$$S_1 = \frac{7}{3} \int_1^2 (x-1) dx = \frac{7}{3} \left[\frac{(x-1)^2}{2} \right]_1^2 = \frac{7}{6}.$$

For the second part,

$$S_2 = \frac{7}{6} \int_2^4 (4-x) dx = \frac{7}{6} \left[4x - \frac{x^2}{2} \right]_2^4 = \frac{7}{6}(8-6) = \frac{7}{3}.$$

Adding the contributions of these adjacent intervals gives $S = \frac{7}{6} + \frac{7}{3} = \frac{21}{6}$. Each contribution is positive, as an area must be.

Step 4: Conclusion Hence, using integration, the area of triangle ABC is $\boxed{\frac{7}{2} \text{ square units}}$.

Quick Tip to Solve: Sort the vertices by their x-coordinates and split at the middle value, $x=2$, where the lower side changes from AB to BC .

OR

24(b). Using integration, find the area of the region $\{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$.

[6 marks]

Solution:

Concept:

- The inequality $x^2 + y^2 \leq 1$ describes the interior and boundary of the unit circle. The inequality $x + y \geq 1$ selects the half-plane above the line $y = 1 - x$.
- A region bounded above by $u(x)$ and below by $l(x)$ has area $\int_a^b [u(x) - l(x)] dx$. For the upper semicircle, $u(x) = \sqrt{1 - x^2}$.

Step 1: Find the points where the line meets the circle The chained inequality means that both $x^2 + y^2 \leq 1$ and $1 \leq x + y$ must hold. The straight boundary is therefore $y = 1 - x$. Substituting it in the circle equation gives

$$x^2 + (1 - x)^2 = 1 \implies 2x^2 - 2x = 0 \implies x = 0 \text{ or } x = 1.$$

The intersections are $(0, 1)$ and $(1, 0)$. The desired region is the circular segment between the chord joining these points and the corresponding arc in the first quadrant.

Step 2: Determine the integration interval and strip height Within the unit disk, $-1 \leq x \leq 1$. If $x < 0$, then the condition $y \geq 1 - x$ would give $y > 1$, which is impossible inside the disk. Consequently, the selected region has $0 \leq x \leq 1$. In this interval, the line has $1 - x \geq 0$, so the lower circular branch cannot bound the selected region. The upper boundary is $y = \sqrt{1 - x^2}$ and the lower boundary is $y = 1 - x$. Moreover,

$$1 - x^2 - (1 - x)^2 = 2x(1 - x) \geq 0,$$

Quick Tip to Solve: Read the chained inequality as two simultaneous conditions; integrate the upper semicircle minus the line from $x=0$ to $x=1$.

which confirms that the upper semicircle lies above the line throughout the interval. Hence the area is

$$S = \int_0^1 \left(\sqrt{1-x^2} - (1-x) \right) dx.$$

Step 3: Evaluate the circular contribution by substitution Put $x = \sin \theta$, where $0 \leq \theta \leq \pi/2$. Then $dx = \cos \theta d\theta$ and $\sqrt{1-x^2} = \cos \theta$, since cosine is non-negative on this interval. Thus

$$\begin{aligned} \int_0^1 \sqrt{1-x^2} dx &= \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4} \right]_0^{\pi/2} = \frac{\pi}{4}. \end{aligned}$$

The line contribution is

$$\int_0^1 (1-x) dx = \left[x - \frac{x^2}{2} \right]_0^1 = \frac{1}{2}.$$

Subtracting this contribution from the circular contribution yields $S = \frac{\pi}{4} - \frac{1}{2}$. The result is positive because the circle lies above the chord between their intersections.

Step 4: Conclusion The area of the required region is $\boxed{\frac{\pi - 2}{4} \text{ square units}}$.

Quick Tip to Solve: Read the chained inequality as two simultaneous conditions; integrate the upper semicircle minus the line from $x=0$ to $x=1$.

25. A wire of length 34 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a rectangle whose length is twice its breadth. What should be the lengths of the two pieces, so that the combined area of the square and the rectangle is minimum? [6 marks]

Solution:

Concept:

- Express both areas in terms of one wire length using the perimeter formulas. A square of side s has perimeter $4s$ and area s^2 .
- A rectangle with breadth b and length $2b$ has perimeter $6b$ and area $2b^2$. An interior critical point is a strict minimum when the second derivative is positive throughout the feasible interval.

Step 1: Choose the variable and express the dimensions Let x m of the wire be used for the square. The other piece then has length $(34 - x)$ m. Since both pieces form shapes, $0 < x < 34$. The side of the square is $s = x/4$. If the rectangle has breadth b , its length is $2b$, so its perimeter is $2(2b + b) = 6b$. Hence

$$b = \frac{34-x}{6}, \quad 2b = \frac{34-x}{3}.$$

These lengths use exactly the available wire and preserve the required ratio of the rectangle's dimensions.

Step 2: Form the combined area and locate its stationary point The combined area, in square metres,

Quick Tip to Solve: Use the square's wire length as the single variable; the rectangle's breadth is the remaining wire length divided by six.

is

$$A(x) = \left(\frac{x}{4}\right)^2 + 2\left(\frac{34-x}{6}\right)^2 = \frac{x^2}{16} + \frac{(34-x)^2}{18}.$$

Differentiating with respect to x ,

$$A'(x) = \frac{x}{8} - \frac{34-x}{9}.$$

For a stationary point, set $A'(x) = 0$. Thus $9x = 8(34-x)$, giving $17x = 272$ and $x = 16$. This lies inside the allowed interval. The length left for the rectangle is then $34 - 16 = 18$ m.

Step 3: Verify that the stationary point gives the minimum The second derivative is

$$A''(x) = \frac{1}{8} + \frac{1}{9} = \frac{17}{72} > 0.$$

Hence the area function is strictly convex on the whole feasible interval, so its only stationary point gives the global minimum. Directly, completing the square also gives

$$A(x) = \frac{17}{144}(x-16)^2 + 34 \geq 34.$$

Equality is possible exactly at $x = 16$. At this value the square has side 4 m, and the rectangle has breadth 3 m and length 6 m. Their areas are 16 m^2 and 18 m^2 , respectively, confirming the computed minimum.

Step 4: Conclusion Cut the wire into pieces of 16 m for the square and 18 m for the rectangle. The minimum combined area is 34 m^2 .

Quick Tip to Solve: Use the square's wire length as the single variable; the rectangle's breadth is the remaining wire length divided by six.

26(a). Let $A = \mathbb{R} - \{3\}$, $B = \mathbb{R} - \{1\}$. Let $f : A \rightarrow B$ be defined by $f(x) = \frac{x-2}{x-3} \forall x \in A$. Show that f is bijective. Also, find (i) x , if $f^{-1}(x) = 4$ (ii) $f^{-1}(7)$. [6 marks]

Solution:

Concept:

- A function is injective if equal output values imply equal inputs. It is surjective if every element of its stated codomain has a preimage in its domain; both properties together give bijectivity.
- To find the inverse, solve $y = f(t)$ for t , observing the excluded values in the domain and codomain. For a bijection, $f^{-1}(y) = t$ exactly when $f(t) = y$.

Step 1: Check that the function maps into the stated codomain For every $t \in A = \mathbb{R} \setminus \{3\}$, the denominator $t - 3$ is nonzero, so $f(t)$ is defined. Moreover,

$$f(t) = \frac{t-2}{t-3} = 1 + \frac{1}{t-3}.$$

The second term cannot be zero, so $f(t) \neq 1$. Thus each output lies in $B = \mathbb{R} \setminus \{1\}$, as required.

Quick Tip to Solve: For $f^{-1}(x) = 4$, compute $f(4)$ immediately; check the excluded value when deriving the general inverse.

Step 2: Prove injectivity Let $u, v \in A$ and suppose $f(u) = f(v)$. Since both denominators are nonzero, cross multiplication is valid:

$$\begin{aligned}\frac{u-2}{u-3} &= \frac{v-2}{v-3}, \\ (u-2)(v-3) &= (v-2)(u-3), \\ uv - 3u - 2v + 6 &= uv - 3v - 2u + 6.\end{aligned}$$

Cancelling equal terms gives $-u + v = 0$, hence $u = v$. Therefore no two distinct inputs have the same output, and f is one-one.

Step 3: Prove surjectivity and obtain the inverse Take any $y \in B$, so $y \neq 1$. Solving $y = (t-2)/(t-3)$ yields

$$yt - 3y = t - 2, \quad (y-1)t = 3y - 2, \quad t = \frac{3y-2}{y-1}.$$

This input belongs to A , because

$$t - 3 = \frac{3y - 2 - 3(y - 1)}{y - 1} = \frac{1}{y - 1} \neq 0.$$

Also $t - 2 = y/(y - 1)$, so substitution gives $f(t) = y$. Every element of B therefore has a preimage in A , proving that f is onto. Combined with injectivity, this proves bijectivity. Its inverse, with domain B and codomain A , is

$$f^{-1}(y) = \frac{3y-2}{y-1}, \quad y \neq 1.$$

Step 4: Evaluate the two requested quantities (i) If $f^{-1}(x) = 4$, then applying f to both sides gives $x = f(4)$. Since $4 \in A$, this is legitimate, and

$$x = \frac{4-2}{4-3} = 2.$$

Equivalently, $(3x-2)/(x-1) = 4$ gives $3x-2 = 4x-4$, hence $x = 2$, which lies in B . (ii) As $7 \neq 1$, the inverse formula applies:

$$f^{-1}(7) = \frac{3(7)-2}{7-1} = \frac{19}{6}.$$

This value is not 3, and substituting it in the original function gives $(7/6)/(1/6) = 7$, verifying the result.

Step 5: Conclusion The function f is bijective, with $f^{-1}(y) = \frac{3y-2}{y-1}$ for $y \neq 1$. The requested values are (i) $x = 2$ and (ii) $f^{-1}(7) = \frac{19}{6}$.

Quick Tip to Solve: For $f^{-1}(x) = 4$, compute $f(4)$ immediately; check the excluded value when deriving the general inverse.

26(b). Let $A = \mathbb{R} \times \mathbb{R}$ and let $*$ be a binary operation on A defined by $(a, b) * (c, d) = (ad + bc, bd)$ for all $(a, b), (c, d) \in \mathbb{R} \times \mathbb{R}$. (i) Show that $*$ is commutative on A . (ii) Show that $*$ is associative on A . (iii) Find the identity element of $*$ in A . [6 marks]

Solution:

Concept:

- A binary operation is commutative if $u * v = v * u$ for every pair of elements u, v in its domain.
- Associativity requires $(u * v) * w = u * (v * w)$ for every triple; it is checked by evaluating both bracketings using the given rule.
- An identity E must satisfy $u * E = E * u = u$ for every $u \in A$. Equality of ordered pairs means equality of their corresponding coordinates.

Step 1: (i) Verify commutativity Take arbitrary $u = (a, b)$ and $v = (c, d)$ in $A = \mathbb{R} \times \mathbb{R}$. The given rule gives

$$u * v = (ad + bc, bd), \quad v * u = (cb + da, db).$$

Since multiplication and addition of real numbers are commutative, $cb + da = ad + bc$ and $db = bd$. Thus both corresponding coordinates agree for all choices of a, b, c, d , so $u * v = v * u$. Therefore $*$ is commutative on A .

Step 2: (ii) Evaluate the two bracketings Let a third arbitrary element be $w = (e, f)$. First calculate $u * v = (ad + bc, bd)$, and apply the operation again:

$$\begin{aligned} (u * v) * w &= ((ad + bc)f + (bd)e, (bd)f) \\ &= (adf + bcf + bde, bdf). \end{aligned}$$

In the other order, $v * w = (cf + de, df)$. Hence

$$\begin{aligned} u * (v * w) &= (a(df) + b(cf + de), b(df)) \\ &= (adf + bcf + bde, bdf). \end{aligned}$$

The two results have identical first and second coordinates. Because the three ordered pairs were arbitrary, this proves associativity throughout A , including elements whose second coordinate is zero.

Step 3: (iii) Determine and verify the identity Suppose $E = (h, k)$ is an identity. For every $(a, b) \in A$, we need

$$(a, b) * (h, k) = (ak + bh, bk) = (a, b).$$

The second coordinate requires $bk = b$ for every real b ; taking $b = 1$ gives $k = 1$. Substituting this in the first coordinate gives $a + bh = a$ for every b ; taking $b = 1$ gives $h = 0$. The only possible identity is therefore $(0, 1)$. Verify it directly on both sides:

$$\begin{aligned} (a, b) * (0, 1) &= (a \cdot 1 + b \cdot 0, b \cdot 1) = (a, b), \\ (0, 1) * (a, b) &= (0 \cdot b + 1 \cdot a, 1 \cdot b) = (a, b). \end{aligned}$$

Both equations hold for every element of A , which establishes existence as well as the required

Quick Tip to Solve: For associativity, compute both coordinates in each bracketing; for the identity, start with the second coordinate to obtain $k=1$ without dividing by a possibly zero b .

identity property.

Step 4: Conclusion The operation $*$ is commutative and associative on $\mathbb{R} \times \mathbb{R}$, and its identity element is $(0, 1)$.

Quick Tip to Solve: For associativity, compute both coordinates in each bracketing; for the identity, start with the second coordinate to obtain $k=1$ without dividing by a possibly zero b .

27(a). Find the vector equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$ which is perpendicular to the plane $x - y + z = 0$. Hence find whether the plane thus obtained contains the line $\frac{x+2}{5} = \frac{y-3}{4} = \frac{z}{5}$ or not. [6 marks]

Solution:

Concept:

- Every plane in the family $P_1 + \lambda P_2 = 0$ contains the intersection line of the distinct planes $P_1 = 0$ and $P_2 = 0$.
- Two planes are perpendicular when their normal vectors have zero dot product. The vector equation $\vec{r} \cdot \vec{n} = d$ has normal $\vec{n}$.
- A line is contained in a plane if all its points satisfy the plane equation; substituting a parametric representation tests the entire line.

Step 1: Form the family through the intersection line Write the given planes as $P_1 = x + y + z - 1 = 0$ and $P_2 = 2x + 3y + 4z - 5 = 0$. A required member of their family has equation

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0.$$

Thus its normal vector is

$$\vec{n} = (1 + 2\lambda)\hat{i} + (1 + 3\lambda)\hat{j} + (1 + 4\lambda)\hat{k}.$$

This construction ensures that every point common to the two original planes also lies on the required plane, because both expressions vanish at that point.

Step 2: Apply the perpendicularity condition The plane $x - y + z = 0$ has normal $\vec{m} = \hat{i} - \hat{j} + \hat{k}$. Perpendicularity requires $\vec{n} \cdot \vec{m} = 0$, giving

$$\begin{aligned}(1 + 2\lambda) - (1 + 3\lambda) + (1 + 4\lambda) &= 0, \\ 1 + 3\lambda &= 0, \quad \lambda = -\frac{1}{3}.\end{aligned}$$

Substitute this value in the family equation and multiply by three:

$$3(x + y + z - 1) - (2x + 3y + 4z - 5) = 0.$$

On collecting terms, the plane equation is $x - z + 2 = 0$. Its normal $\hat{i} - \hat{k}$ indeed has dot product $1 - 1 = 0$ with $\vec{m}$, which checks the perpendicularity independently.

Step 3: Express the plane in vector form For a general point with position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$,

Quick Tip to Solve: Use the normal of the plane family to find the parameter, then substitute the line's common ratio into the final plane equation and check that the parameter cancels identically.

the scalar product $\vec{r} \cdot (\hat{i} - \hat{k}) = x - z$. Therefore the vector equation of the required plane is

$$\vec{r} \cdot (\hat{i} - \hat{k}) = -2.$$

The absence of a y term means that points can move parallel to the y -axis while remaining in this plane.

Step 4: Test whether the complete line lies in the plane Put the common ratio in the printed line equal to a real parameter t :

$$\frac{x+2}{5} = \frac{y-3}{4} = \frac{z}{5} = t.$$

Every point on the line therefore has coordinates

$$x = 5t - 2, \quad y = 4t + 3, \quad z = 5t.$$

Substitute into the plane equation:

$$x - z + 2 = (5t - 2) - 5t + 2 = 0$$

for every real t . Hence the whole line is contained in the plane. Equivalently, its point $(-2, 3, 0)$ satisfies the plane equation and its direction $(5, 4, 5)$ has dot product $5 - 5 = 0$ with the plane normal; both necessary containment conditions hold.

Step 5: Conclusion The required plane is $\vec{r} \cdot (\hat{i} - \hat{k}) + 2 = 0$. It contains the given line, since all points of that line satisfy $x - z + 2 = 0$.

Quick Tip to Solve: Use the normal of the plane family to find the parameter, then substitute the line's common ratio into the final plane equation and check that the parameter cancels identically.

OR

27(b). Find the image P' of the point P having position vector $\hat{i} + 3\hat{j} + 4\hat{k}$ in the plane $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) + 3 = 0$. Hence find the length of PP' . [6 marks]

Solution:

Concept:

- A reflection in a plane sends a point to the opposite side along the normal. The plane bisects the segment joining the point and its image at right angles.
- For the plane $2x - y + z + 3 = 0$, a normal vector is $\vec{n} = 2\hat{i} - \hat{j} + \hat{k}$, with $\|\vec{n}\| = \sqrt{6}$.
- If M is the perpendicular foot from P and P' is its reflection, then $\vec{OP'} = 2\vec{OM} - \vec{OP}$.

Step 1: Write the normal through the given point The position vector of P is $\hat{i} + 3\hat{j} + 4\hat{k}$, so its coordinates are $(1, 3, 4)$. A line through this point perpendicular to the plane has vector equation

$$\vec{r} = \hat{i} + 3\hat{j} + 4\hat{k} + t(2\hat{i} - \hat{j} + \hat{k}), \quad t \in \mathbb{R}.$$

Quick Tip to Solve: Find the perpendicular foot first, then double its position vector and subtract the original point; check that the signed plane values reverse sign.

Thus any point on this normal has coordinates $(1 + 2t, 3 - t, 4 + t)$. The required foot M lies on this line and also satisfies the plane equation; these two conditions determine its position uniquely.

Step 2: Find the perpendicular foot on the plane Substitute the normal-line coordinates into $2x - y + z + 3 = 0$:

$$\begin{aligned} 2(1 + 2t) - (3 - t) + (4 + t) + 3 &= 0, \\ 6 + 6t &= 0, \\ t &= -1. \end{aligned}$$

Therefore $M = (-1, 4, 3)$. As a direct check, $2(-1) - 4 + 3 + 3 = 0$, so M is on the plane. Also $\overrightarrow{PM} = (-2, 1, -1) = -\vec{n}$, confirming that the segment from P to M is perpendicular to the plane.

Step 3: Use the midpoint condition to obtain the image The plane of reflection passes through the midpoint M of PP' . Hence

$$\begin{aligned} \overrightarrow{OP'} &= 2\overrightarrow{OM} - \overrightarrow{OP} \\ &= 2(-\hat{i} + 4\hat{j} + 3\hat{k}) - (\hat{i} + 3\hat{j} + 4\hat{k}) \\ &= -3\hat{i} + 5\hat{j} + 2\hat{k}. \end{aligned}$$

The signed plane expressions at P and P' are respectively 6 and -6 . This confirms that the two points lie on opposite sides at equal perpendicular distances.

Step 4: Calculate the distance between the point and its image The displacement from P to its image is $(-4, 2, -2)$, so

$$PP' = \sqrt{(-4)^2 + 2^2 + (-2)^2} = \sqrt{24} = 2\sqrt{6}.$$

Independently, the distance of P from the plane is $|2(1) - 3 + 4 + 3| / \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$. Reflection doubles this distance, agreeing with the coordinate calculation.

Step 5: Conclusion The image is $P' = (-3, 5, 2)$, with position vector $-3\hat{i} + 5\hat{j} + 2\hat{k}$, and $PP' = 2\sqrt{6}$.

Quick Tip to Solve: Find the perpendicular foot first, then double its position vector and subtract the original point; check that the signed plane values reverse sign.

28. If $A = \begin{pmatrix} 1 & -2 & 0 \\ 2 & 1 & 3 \\ 0 & -2 & 1 \end{pmatrix}$, find A^{-1} and hence solve the system of equations $x - 2y = 10$, $2x + y + 3z = 8$ and $-2y + z = 7$. [6 marks]

Solution:

Concept:

- A square matrix has an inverse when its determinant is nonzero. The adjugate formula is $A^{-1} = \text{adj } A / |A|$.
- The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$ uses the minor M_{ij} , and the adjugate is the transpose of the cofactor matrix.

Quick Tip to Solve: Keep the variable order fixed and transpose the cofactor matrix before multiplying the right-hand-side column; finally substitute into all three equations.

- For a system $AX = B$, multiplying by A^{-1} gives the unique solution $X = A^{-1}B$, with the variables ordered as the columns of A .

Step 1: Calculate the determinant For the given matrix, expansion along its first row gives

$$|A| = 1 \begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} - (-2) \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} \\ = (1 + 6) + 2(2 - 0) = 11.$$

Since $11 \neq 0$, the inverse exists. Consequently the given system has a unique solution for its right-hand side.

Step 2: Evaluate the cofactors Deleting the corresponding row and column and applying alternating signs gives

$$\begin{aligned} C_{11} &= 1 + 6 = 7, & C_{12} &= -(2 - 0) = -2, & C_{13} &= -4 - 0 = -4, \\ C_{21} &= -(-2 - 0) = 2, & C_{22} &= 1 - 0 = 1, & C_{23} &= -(-2 - 0) = 2, \\ C_{31} &= -6 - 0 = -6, & C_{32} &= -(3 - 0) = -3, & C_{33} &= 1 + 4 = 5. \end{aligned}$$

Thus the cofactor matrix is $C = \begin{pmatrix} 7 & -2 & -4 \\ 2 & 1 & 2 \\ -6 & -3 & 5 \end{pmatrix}$. The signs here follow the pattern $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

Step 3: Transpose to obtain the adjugate and inverse Transposing the cofactor matrix interchanges its rows and columns:

$$\text{adj } A = C^T = \begin{pmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{pmatrix}.$$

Therefore

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{pmatrix}.$$

For example, the first row of A multiplied by these three adjugate columns gives $(11, 0, 0)$, consistent with $A \text{adj } A = 11I$.

Step 4: Apply the inverse to the system The three equations, with variables in the order x, y, z , have matrix form

$$A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix}.$$

Quick Tip to Solve: Keep the variable order fixed and transpose the cofactor matrix before multiplying the right-hand-side column; finally substitute into all three equations.

Hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{pmatrix} \begin{pmatrix} 10 \\ 8 \\ 7 \end{pmatrix} \\ = \frac{1}{11} \begin{pmatrix} 70 + 16 - 42 \\ -20 + 8 - 21 \\ -40 + 16 + 35 \end{pmatrix} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}.$$

Substitution verifies all three equations: $4 - 2(-3) = 10$, $2(4) - 3 + 3(1) = 8$, and $-2(-3) + 1 = 7$.

Step 5: Conclusion

$$A^{-1} = \frac{1}{11} \begin{pmatrix} 7 & 2 & -6 \\ -2 & 1 & -3 \\ -4 & 2 & 5 \end{pmatrix}$$

The unique solution is $x = 4, y = -3, z = 1$.

Quick Tip to Solve: Keep the variable order fixed and transpose the cofactor matrix before multiplying the right-hand-side column; finally substitute into all three equations.

29. Find the particular solution of the differential equation $(1 + y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$, given that $y = 0$ when $x = 1$. [6 marks]

Solution:

Concept:

- When an equation contains dy/dx but is linear in x , regard x as a function of y . Its linear form is $dx/dy + P(y)x = Q(y)$.
- The integrating factor is $I(y) = e^{\int P(y) dy}$. Multiplication by this factor makes the left side the derivative $d(xI)/dy$.
- The substitution $t = \tan^{-1} y$ gives $dt = dy/(1 + y^2)$, where the inverse tangent has its usual principal values. The given condition determines the integration constant.

Step 1: Write a linear equation with dependent variable x The given equation is

$$(1 + y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0.$$

Writing it in differential form gives

$$(1 + y^2) dx + (x - e^{\tan^{-1} y}) dy = 0.$$

Since $1 + y^2 > 0$ for every real y , we can divide by this expression and treat x as a function of y :

$$\frac{dx}{dy} + \frac{x}{1 + y^2} = \frac{e^{\tan^{-1} y}}{1 + y^2}.$$

Quick Tip to Solve: Treat x as the dependent variable first; the factor $1/(1 + y^2)$ then suggests the substitution $t = \tan^{-1} y$ immediately.

Thus $P(y) = 1/(1 + y^2)$ and $Q(y) = e^{\tan^{-1} y}/(1 + y^2)$. This reversal of the dependent variable is what turns the equation into a first-order linear equation.

Step 2: Calculate and apply the integrating factor The integrating factor is

$$I(y) = e^{\int \frac{dy}{1+y^2}} = e^{\tan^{-1} y}.$$

Multiplying the linear equation by it, we obtain

$$e^{\tan^{-1} y} \frac{dx}{dy} + \frac{x e^{\tan^{-1} y}}{1 + y^2} = \frac{e^{2 \tan^{-1} y}}{1 + y^2}.$$

By the product rule, this is

$$\frac{d}{dy} (x e^{\tan^{-1} y}) = \frac{e^{2 \tan^{-1} y}}{1 + y^2}.$$

Step 3: Integrate using the inverse tangent substitution Integrating both sides with respect to y gives

$$x e^{\tan^{-1} y} = \int \frac{e^{2 \tan^{-1} y}}{1 + y^2} dy.$$

Put $t = \tan^{-1} y$, so that $dt = dy/(1 + y^2)$. The integral becomes

$$\int e^{2t} dt = \frac{1}{2} e^{2t} + C.$$

Consequently, the family of integral curves is

$$x e^{\tan^{-1} y} = \frac{1}{2} e^{2 \tan^{-1} y} + C.$$

Step 4: Use the given point and verify the relation At $x = 1$, $y = 0$, we have $\tan^{-1} 0 = 0$. Substitution gives $1 = 1/2 + C$, and therefore $C = 1/2$. To check the resulting relation, differentiate it with respect to y :

$$e^{\tan^{-1} y} \frac{dx}{dy} + \frac{x e^{\tan^{-1} y}}{1 + y^2} = \frac{e^{2 \tan^{-1} y}}{1 + y^2}.$$

Dividing by the integrating factor and multiplying by $1 + y^2$ recovers $(1 + y^2)dx/dy + x - e^{\tan^{-1} y} = 0$, which is the differential form of the printed equation. The relation also gives $x = 1$ at $y = 0$.

Step 5: Conclusion The required particular integral curve is

$$\boxed{2x e^{\tan^{-1} y} = e^{2 \tan^{-1} y} + 1}.$$

Equivalently, $x = \frac{1}{2}(e^{\tan^{-1} y} + e^{-\tan^{-1} y})$.

Quick Tip to Solve: Treat x as the dependent variable first; the factor $1/(1 + y^2)$ then suggests the substitution $t = \tan^{-1} y$ immediately.