

CBSE Class 12 Mathematics Question Paper 2017 with Solution PDF-Set 65-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 3 questions of four marks each and 3 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. Determine the value of k for which the following function is continuous at $x = 3$: $f(x) = \begin{cases} \frac{(x+3)^2 - 36}{x-3}, & x \neq 3 \\ k, & x = 3 \end{cases}$. [1 mark]

Solution:

Concept:

- Continuity at $x = a$ requires $\lim_{x \rightarrow a} f(x) = f(a)$. The limit uses nearby values, whereas the prescribed value determines the function at the point.
- The difference of squares identity is $u^2 - v^2 = (u - v)(u + v)$.

Step 1: Simplify the expression near the point For $x \neq 3$, factor the numerator before evaluating the limit:

$$f(x) = \frac{(x+3-6)(x+3+6)}{x-3} = \frac{(x-3)(x+9)}{x-3} = x+9.$$

Cancellation is valid for the nearby points used in the limit; direct substitution in the original quotient would give the indeterminate form $0/0$.

Step 2: Apply the condition of continuity The simplified expression is a polynomial, so its left and right limits agree:

$$\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} (x+9) = 12.$$

The piecewise definition gives $f(3) = k$. Equating this value to the common limit gives the required condition.

Step 3: Conclusion The function is continuous at $x = 3$ when $k = 12$.

Quick Tip to Solve: Factor the difference of squares first; cancel only for $x \neq 3$, then evaluate the limit.

2. If for any 2×2 square matrix A , $A(\text{adj } A) = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$, then write the value of $|A|$. [1 mark]

Solution:

Concept:

- For any square matrix A , the adjugate identity is $A \text{adj } A = |A|I$, where I is the identity matrix of the same order.
- The scalar multiplying the identity in this product is the determinant of A .

Step 1: Express the given product using the identity Since A has order two,

$$A \text{adj } A = |A| \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} |A| & 0 \\ 0 & |A| \end{pmatrix}.$$

Quick Tip to Solve: Read the common diagonal entry of $A \text{adj } A$ as $|A|$; do not take the determinant of the product as the answer.

This identity holds without needing to know the individual entries of A .

Step 2: Compare corresponding entries The given product is $\begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix} = 8I_2$. Comparison of either diagonal entry with the identity above therefore determines the scalar $|A|$. The off-diagonal zero entries are consistent with the same identity.

Step 3: Conclusion $|A| = 8$.

Quick Tip to Solve: Read the common diagonal entry of $A \operatorname{adj} A$ as $|A|$; do not take the determinant of the product as the answer.

3. Find the distance between the planes $2x - y + 2z = 5$ and $5x - 2.5y + 5z = 20$.

[1 mark]

Solution:

Concept:

- Parallel planes must first be written with identical coefficients of x, y, z .
- For $ax + by + cz = d_1$ and $ax + by + cz = d_2$, their distance is $D = |d_2 - d_1| / \sqrt{a^2 + b^2 + c^2}$.

Step 1: Put the planes in a common normal form Divide every term of the second equation by 2.5:

$$5x - 2.5y + 5z = 20 \implies 2x - y + 2z = 8.$$

The first plane is $2x - y + 2z = 5$. Thus both planes have the same normal vector $(2, -1, 2)$, so they are parallel.

Step 2: Calculate the perpendicular separation The magnitude of the common normal is $\sqrt{2^2 + (-1)^2 + 2^2} = 3$. Hence the distance formula gives

$$D = \frac{|8 - 5|}{3}.$$

Scaling the whole second equation, including its right-hand side, is essential before subtracting the constants.

Step 3: Conclusion The distance between the planes is 1 unit .

Quick Tip to Solve: Normalize the second plane by dividing all its terms by 2.5 before comparing the constants.

4. Find: $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$.

[1 mark]

Solution:

Concept:

- Split the quotient to express the integrand as $\tan x - \cot x$, on intervals where $\sin x \cos x \neq 0$.
- The standard integrals are $\int \tan x dx = -\ln |\cos x|$ and $\int \cot x dx = \ln |\sin x|$, each with an

Quick Tip to Solve: Split the numerator before using identities; the minus sign on $\cot x$ makes both logarithms negative.

arbitrary constant.

Step 1: Rewrite the rational trigonometric expression Cancel the common factors in each of the two terms:

$$\frac{\sin^2 x - \cos^2 x}{\sin x \cos x} = \frac{\sin x}{\cos x} - \frac{\cos x}{\sin x} = \tan x - \cot x.$$

The restriction on the denominator also ensures that these two ratios are defined.

Step 2: Integrate the two terms Using the standard logarithmic antiderivatives,

$$\int (\tan x - \cot x) dx = -\ln |\cos x| - \ln |\sin x| + C.$$

The sum of the logarithms can be combined using $\ln |a| + \ln |b| = \ln |ab|$. Differentiating this expression yields $\tan x - \cot x$, confirming its sign.

Step 3: Conclusion

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx = -\ln |\sin x \cos x| + C.$$

Quick Tip to Solve: Split the numerator before using identities; the minus sign on $\cot x$ makes both logarithms negative.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5. Find: $\int \frac{dx}{5 - 8x - x^2}.$

[2 marks]

Solution:

Concept:

- Completing the square converts a quadratic denominator into a standard integrable form: $5 - 8x - x^2 = 21 - (x + 4)^2$.
- For $a > 0$, $\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln \left| \frac{a+u}{a-u} \right| + C$, on any interval avoiding $u = \pm a$.

Step 1: Complete the square and substitute Write the denominator carefully, retaining the leading minus sign:

$$5 - 8x - x^2 = -(x^2 + 8x - 5) = 21 - (x + 4)^2.$$

Put $u = x + 4$, so $du = dx$. With $a = \sqrt{21}$, the required integral becomes

$$I = \int \frac{du}{a^2 - u^2}.$$

Step 2: Integrate using partial fractions Since $a^2 - u^2 = (a - u)(a + u)$,

$$\frac{1}{a^2 - u^2} = \frac{1}{2a} \left(\frac{1}{a + u} + \frac{1}{a - u} \right).$$

Quick Tip to Solve: Complete the square as $21 - (x + 4)^2$; keep the minus sign while integrating $1/(a - u)$.

Therefore

$$I = \frac{1}{2a} (\ln |a + u| - \ln |a - u|) + C.$$

The second logarithm has a minus sign because $d(a - u)/du = -1$. Combine the logarithms and substitute $u = x+4$ and $a = \sqrt{21}$. The formula applies on each interval where the original denominator is nonzero.

Step 3: Conclusion

$$\int \frac{dx}{5 - 8x - x^2} = \frac{1}{2\sqrt{21}} \ln \left| \frac{\sqrt{21} + x + 4}{\sqrt{21} - x - 4} \right| + C.$$

Quick Tip to Solve: Complete the square as $21 - (x + 4)^2$; keep the minus sign while integrating $1/(a - u)$.

6. Two tailors, A and B, earn ₹ 300 and ₹ 400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work and if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost, formulate this as an LPP. [2 marks]

Solution:

Concept:

- Decision variables measure the number of working days of the two tailors; total production and labour cost are linear functions of these variables.
- An "at least" production requirement gives a greater-than-or-equal-to constraint. Numbers of working days must be nonnegative.

Step 1: Define the variables and objective Let x and y denote the numbers of days worked by tailors A and B, respectively. Their daily labour costs are Rs. 300 and Rs. 400. Hence the total labour cost, in rupees, is

$$Z = 300x + 400y.$$

The objective is to minimize Z , subject to meeting both production requirements.

Step 2: Translate the production requirements Tailor A stitches $6x$ shirts and tailor B stitches $10y$ shirts. At least 60 shirts are required, giving

$$6x + 10y \geq 60.$$

Each tailor stitches four pairs of trousers per day, so their combined trouser production is $4x + 4y$ pairs. At least 32 pairs are required, giving

$$4x + 4y \geq 32.$$

Finally, $x \geq 0$ and $y \geq 0$, since negative working days have no meaning. Both inequalities are lower bounds; additional production is permitted. The question asks for formulation, so these variables, objective and constraints completely specify the LPP.

Quick Tip to Solve: Use days as the variables and translate each "at least" requirement into $\geq$, not $\leq$.

Step 3: Conclusion

$$\begin{array}{ll}\text{Minimize} & Z = 300x + 400y, \\ \text{subject to} & 6x + 10y \geq 60, \\ & 4x + 4y \geq 32, \\ & x \geq 0, \quad y \geq 0.\end{array}$$

Quick Tip to Solve: Use days as the variables and translate each "at least" requirement into $\geq$, not $\leq$.

7. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events. [2 marks]

Solution:**Concept:**

- For a fair die the six faces are equally likely, so $P(E) = n(E)/6$ for an event E .
- Two events A and B are independent exactly when $P(A \cap B) = P(A)P(B)$. Independence is different from mutual exclusivity.

Step 1: Identify the events and their intersection The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. The even-number event and red-number event are

$$A = \{2, 4, 6\}, \quad B = \{1, 2, 3\}.$$

A face belongs to both events only if it is both even and red. Of the three red numbers, only 2 is even, so $A \cap B = \{2\}$.

Step 2: Compare the intersection probability with the product Counting equally likely faces gives

$$P(A) = \frac{3}{6} = \frac{1}{2}, \quad P(B) = \frac{3}{6} = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{6}.$$

Their product is $P(A)P(B) = 1/4$, which differs from $1/6$. Equivalently, knowing that the face is red changes the probability of an even result from $1/2$ to $P(A | B) = 1/3$. Thus the colour information affects the chance of an even result, and the independence criterion fails.

Step 3: Conclusion A and B are not independent, since $P(A \cap B) = 1/6 \neq 1/4 = P(A)P(B)$.

Quick Tip to Solve: List the red even faces first: only 2 lies in the intersection, so its probability is $1/6$.

8. The x -coordinate of a point on the line joining the points $P(2, 2, 1)$ and $Q(5, 1, -2)$ is 4. Find its z -coordinate. [2 marks]

Solution:

Concept:

- A point on the line through P and Q can be written as $P + t(Q - P)$, where the real parameter t is the same for all three coordinates.
- The direction vector is obtained by subtracting the coordinates of P from those of Q ; coordinate differences must retain their signs.

Step 1: Write the coordinates of a general point For $P = (2, 2, 1)$ and $Q = (5, 1, -2)$, the direction vector is

$$Q - P = (5 - 2, 1 - 2, -2 - 1) = (3, -1, -3).$$

Thus a general point R on the line has coordinates

$$R = (2 + 3t, 2 - t, 1 - 3t), \quad t \in \mathbb{R}.$$

This expression gives P when $t = 0$ and Q when $t = 1$, confirming the direction and starting point.

Step 2: Use the specified x -coordinate The given coordinate is $x = 4$. Therefore

$$2 + 3t = 4 \implies t = \frac{2}{3}.$$

Substitute this same parameter into the third coordinate:

$$z = 1 - 3\left(\frac{2}{3}\right) = 1 - 2 = -1.$$

The corresponding y -coordinate is $2 - \frac{2}{3} = \frac{4}{3}$, so the point is $(4, \frac{4}{3}, -1)$, which lies on the specified line.

Step 3: Conclusion The required z -coordinate is $\boxed{-1}$.

Quick Tip to Solve: Find the line parameter from the known coordinate, then use exactly the same parameter in the required coordinate.

9. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on $\mathbb{R}$. [2 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive throughout that interval.
- Completing the square can show that a quadratic expression is positive for every real input, since a real square is non-negative.

Step 1: Differentiate the polynomial The given function $f(x) = x^3 - 3x^2 + 6x - 100$ is a polynomial,

Quick Tip to Solve: Complete the square in the derivative: the positive constant left after the square proves positivity on the whole real line.

so it is differentiable for every $x \in \mathbb{R}$. Differentiating each term gives

$$f'(x) = 3x^2 - 6x + 6.$$

The constant term contributes zero; the sign of the derivative, rather than the sign of the function itself, determines whether the function increases.

Step 2: Establish positivity for every real x Complete the square in the derivative:

$$\begin{aligned} f'(x) &= 3(x^2 - 2x + 2) \\ &= 3((x - 1)^2 + 1). \end{aligned}$$

For every real x , $(x - 1)^2 \geq 0$, hence $f'(x) \geq 3 > 0$. Thus the derivative has no zero or negative value on the real line. By the derivative criterion for monotonicity, the function increases throughout its domain.

Step 3: Conclusion Hence f is strictly increasing on $\mathbb{R}$.

Quick Tip to Solve: Complete the square in the derivative: the positive constant left after the square proves positivity on the whole real line.

10. Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $[-\sqrt{3}, 0]$. [2 marks]

Solution:

Concept:

- Rolle's theorem applies when f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$. It guarantees an interior point c with $f'(c) = 0$.
- Only roots of $f'(c) = 0$ that lie strictly inside the specified interval can be used.

Step 1: Check the hypotheses The function $f(x) = x^3 - 3x$ is a polynomial. It is therefore continuous on $[-\sqrt{3}, 0]$ and differentiable on $(-\sqrt{3}, 0)$. Its endpoint values are

$$\begin{aligned} f(-\sqrt{3}) &= (-\sqrt{3})^3 - 3(-\sqrt{3}) \\ &= -3\sqrt{3} + 3\sqrt{3} = 0, \\ f(0) &= 0. \end{aligned}$$

The endpoint values agree, so all the hypotheses of Rolle's theorem hold on this interval.

Step 2: Find the stationary point inside the interval Differentiate and set the derivative equal to zero:

$$f'(x) = 3x^2 - 3, \quad f'(c) = 0 \implies 3c^2 - 3 = 0.$$

Thus $c^2 = 1$, giving the two candidates $c = -1$ and $c = 1$. Since $-\sqrt{3} < -1 < 0$, the candidate -1 lies in the open interval. The other candidate, 1 , lies outside the interval and must be rejected.

Step 3: Conclusion The required value is $c = -1$.

Quick Tip to Solve: After solving the derivative equation, check every candidate against the open interval; Rolle's point cannot be an endpoint.

11. If A is a skew-symmetric matrix of order 3, then prove that $\det A = 0$.

[2 marks]

Solution:

Concept:

- A skew-symmetric matrix satisfies $A^T = -A$, where A^T denotes the transpose of A .
- Transposition preserves a determinant: $\det(A^T) = \det A$. For a matrix of order n , $\det(kA) = k^n \det A$, since each of its n rows is multiplied by k .

Step 1: Take determinants of the skew-symmetry identity Let $d = \det A$. Since A is skew-symmetric, $A^T = -A$. Taking determinants of both sides yields

$$\det(A^T) = \det(-A).$$

The determinant is unchanged on interchanging rows and columns, so the expression on the left equals d . This argument uses the defining matrix identity directly and does not require specifying individual entries of A .

Step 2: Use that the order is odd The matrix has order 3. In $-A$, each of the three rows of A is multiplied by -1 . Factoring these three scalars out of the determinant gives

$$\det(-A) = (-1)^3 \det A = -d.$$

Consequently the determinant identity becomes $d = -d$, or $2d = 0$. Over the real numbers this equality implies $d = 0$. The negative sign arises because the number of rows is odd; it is the essential reason for the result.

Step 3: Conclusion Therefore $\boxed{\det A = 0}$, as required.

Quick Tip to Solve: Use transpose invariance and factor one minus sign from every row; for odd order the determinant equals its own negative.

12. The volume of a sphere is increasing at the rate of $8 \text{ cm}^3/\text{s}$. Find the rate at which its surface area is increasing when the radius of the sphere is 12 cm.

[2 marks]

Solution:

Concept:

- For a sphere of radius r , its volume is $V = \frac{4}{3}\pi r^3$ and surface area is $S = 4\pi r^2$.
- When r changes with time t , differentiate with respect to t using the chain rule: $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ and $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$.

Step 1: Find the instantaneous rate of change of radius We are given $\frac{dV}{dt} = 8 \text{ cm}^3/\text{s}$. Differentiating the volume formula and substituting $r = 12 \text{ cm}$ gives

$$8 = 4\pi(12)^2 \frac{dr}{dt}.$$

Quick Tip to Solve: Divide the two time-derivative formulas to obtain $dS/dt = (2/r)(dV/dt)$, then substitute the instantaneous radius.

Hence at this instant,

$$\frac{dr}{dt} = \frac{8}{576\pi} = \frac{1}{72\pi} \text{ cm/s.}$$

The positive value agrees with the fact that the sphere is expanding.

Step 2: Calculate the rate of change of surface area The required quantity is the time derivative of $S = 4\pi r^2$, not merely its derivative with respect to radius. The chain rule gives

$$\begin{aligned}\frac{dS}{dt} &= 8\pi r \frac{dr}{dt} \\ &= 8\pi(12) \left(\frac{1}{72\pi} \right) \\ &= \frac{96}{72} = \frac{4}{3} \text{ cm}^2/\text{s}.\end{aligned}$$

Here π cancels, and the final units are area per unit time, as required.

Step 3: Conclusion The surface area is increasing at $\boxed{\frac{4}{3} \text{ cm}^2/\text{s}}$.

Quick Tip to Solve: Divide the two time-derivative formulas to obtain $dS/dt = (2/r)(dV/dt)$, then substitute the instantaneous radius.

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13. There are 4 cards numbered 1, 3, 5 and 7, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X . [4 marks]

Solution:

Concept:

- For equally likely outcomes, probability is the number of favourable outcomes divided by the total number of outcomes.
- The mean of a discrete random variable is $E(X) = \sum xP(X = x)$, where x runs over all possible values.
- The variance is $\text{Var}(X) = E(X^2) - [E(X)]^2$, where $E(X^2) = \sum x^2P(X = x)$.

Step 1: Enumerate the equally likely pairs The cards are drawn without replacement, so no card can occur twice. Since the order of drawing does not affect their sum, use unordered pairs. The total number of pairs is $\binom{4}{2} = 6$. Each unordered pair corresponds to two ordered draws among twelve equally likely ordered draws, so each has probability $1/6$.

Pair	(1, 3)	(1, 5)	(1, 7)	(3, 5)	(3, 7)	(5, 7)
X	4	6	8	8	10	12

Step 2: Form the probability distribution The sum 8 arises from two pairs, whereas every other sum

Quick Tip to Solve: List all six unordered pairs first: the sum 8 occurs twice and therefore has twice the probability of each other sum.

arises from just one pair. Combine the two outcomes with sum 8 to obtain

x	4	6	8	10	12
$P(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

The probabilities add to $(1 + 1 + 2 + 1 + 1)/6 = 1$, confirming that all possibilities have been included.

Step 3: Evaluate the mean and second moment Weight each possible sum by its probability:

$$E(X) = \frac{4 + 6 + 2(8) + 10 + 12}{6} = \frac{48}{6} = 8.$$

For the second moment, square each value of X , while keeping its probability unchanged:

$$E(X^2) = \frac{16 + 36 + 2(64) + 100 + 144}{6} = \frac{424}{6} = \frac{212}{3}.$$

Step 4: Evaluate the variance Subtract the square of the mean from the second moment:

$$\text{Var}(X) = \frac{212}{3} - 8^2 = \frac{212 - 192}{3} = \frac{20}{3}.$$

To check directly, the six equally likely sums have deviations $-4, -2, 0, 0, 2, 4$ from their mean 8. Averaging their squares gives $(16 + 4 + 0 + 0 + 4 + 16)/6 = 20/3$, the same value.

Step 5: Conclusion The mean is $\boxed{8}$, and the variance is $\boxed{\frac{20}{3}}$.

Quick Tip to Solve: List all six unordered pairs first: the sum 8 occurs twice and therefore has twice the probability of each other sum.

14. Show that the points A, B, C with position vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $3\hat{i} - 4\hat{j} - 4\hat{k}$ respectively, are the vertices of a right-angled triangle. Hence find the area of the triangle. [4 marks]

Solution:

Concept:

- The displacement vector from P to Q is $\overrightarrow{PQ} = \vec{q} - \vec{p}$, where $\vec{p}, \vec{q}$ are their position vectors.
- Two nonzero vectors $\vec{u}, \vec{v}$ are perpendicular when $\vec{u} \cdot \vec{v} = 0$. The dot product is the sum of the products of corresponding components.
- The area of a right-angled triangle is half the product of its perpendicular side lengths. The length of $a\hat{i} + b\hat{j} + c\hat{k}$ is $\sqrt{a^2 + b^2 + c^2}$.

Step 1: Calculate the side vectors The position vectors give the coordinates $A = (2, -1, 1)$, $B =$

Quick Tip to Solve: Compare the squared side lengths: the largest, 41, equals 35 plus 6, so the sides meeting at C are the perpendicular ones used for the area.

$(1, -3, -5)$ and $C = (3, -4, -4)$. Subtract coordinates in the same order to find

$$\begin{aligned}\overrightarrow{AB} &= (1 - 2)\hat{i} + (-3 + 1)\hat{j} + (-5 - 1)\hat{k} \\ &= -\hat{i} - 2\hat{j} - 6\hat{k},\end{aligned}$$

$$\begin{aligned}\overrightarrow{AC} &= (3 - 2)\hat{i} + (-4 + 1)\hat{j} + (-4 - 1)\hat{k} \\ &= \hat{i} - 3\hat{j} - 5\hat{k},\end{aligned}$$

$$\begin{aligned}\overrightarrow{BC} &= (3 - 1)\hat{i} + (-4 + 3)\hat{j} + (-4 + 5)\hat{k} \\ &= 2\hat{i} - \hat{j} + \hat{k}.\end{aligned}$$

Step 2: Identify the perpendicular sides To test the angle at C , use the two vectors starting from C :

$$\overrightarrow{CA} = -\hat{i} + 3\hat{j} + 5\hat{k}, \quad \overrightarrow{CB} = -2\hat{i} + \hat{j} - \hat{k}.$$

Their dot product is

$$\overrightarrow{CA} \cdot \overrightarrow{CB} = (-1)(-2) + (3)(1) + (5)(-1) = 2 + 3 - 5 = 0.$$

Both vectors are nonzero, so they are perpendicular. Thus $\angle ACB = 90^\circ$, proving that the three points form a right-angled triangle, with AB as its hypotenuse.

Step 3: Calculate the perpendicular lengths and area The perpendicular sides are CA and CB ; their lengths are

$$CA = \sqrt{(-1)^2 + 3^2 + 5^2} = \sqrt{35}, \quad CB = \sqrt{(-2)^2 + 1^2 + (-1)^2} = \sqrt{6}.$$

Also, $AB^2 = 1 + 4 + 36 = 41 = 35 + 6 = CA^2 + CB^2$, which independently checks the identified right angle. Therefore the area is

$$\text{Area}(\triangle ABC) = \frac{1}{2}(CA)(CB) = \frac{1}{2}\sqrt{35}\sqrt{6} = \frac{\sqrt{210}}{2}.$$

Step 4: Conclusion The triangle is right-angled at C , and its area is $\boxed{\frac{\sqrt{210}}{2}}$ square units.

Quick Tip to Solve: Compare the squared side lengths: the largest, 41, equals 35 plus 6, so the sides meeting at C are the perpendicular ones used for the area.

15. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance? Is regularity required only in school? Justify your answer. [4 marks]

Solution:

Concept:

- Let R denote full attendance, I irregular attendance, and A attainment of an A grade. The events R and I partition the students.
- Total probability: $P(A) = P(R)P(A | R) + P(I)P(A | I)$.
- Bayes' theorem: $P(R | A) = P(R)P(A | R)/P(A)$. The observed grade determines the conditioning event.

Step 1: Translate the attendance and grade information The probabilities supplied by the question are

$$P(R) = 0.30, \quad P(I) = 0.70, \quad P(A | R) = 0.70, \quad P(A | I) = 0.10.$$

The last two probabilities refer to students within their respective attendance groups. Since an A grade is observed, we require $P(R | A)$.

Step 2: Find the total probability of an A grade An A grade may be obtained by a student from either group. As these groups are disjoint, their contributions add:

$$\begin{aligned} P(A) &= P(R \cap A) + P(I \cap A) \\ &= P(R)P(A | R) + P(I)P(A | I) \\ &= 0.30(0.70) + 0.70(0.10) \\ &= 0.21 + 0.07 = 0.28. \end{aligned}$$

Both attendance groups contribute to the A-grade event, so both belong in the denominator.

Step 3: Reverse the conditioning using Bayes' theorem Since $P(A) > 0$, Bayes' theorem applies:

$$\begin{aligned} P(R | A) &= \frac{P(R)P(A | R)}{P(A)} \\ &= \frac{0.30(0.70)}{0.28} \\ &= \frac{21}{28} = \frac{3}{4}. \end{aligned}$$

Step 4: Justify regularity outside school Regularity is necessary in everyday life too. Regular study and practice develop skills, regular exercise promotes health, and punctual work helps fulfil responsibilities. These habits build discipline and reliability and support sustained progress.

Step 5: Conclusion The required probability is $\frac{3}{4}$, or 75%. Regularity is also required in daily life, where consistent habits support discipline, health and dependable performance.

Quick Tip to Solve: Add the A-grade contributions of both attendance groups before dividing the full-attendance contribution by their total.

16. If $\tan^{-1}\left(\frac{x-3}{x-4}\right) + \tan^{-1}\left(\frac{x+3}{x+4}\right) = \frac{\pi}{4}$, then find the value of x .

[4 marks]

Solution:

Concept:

- Principal values of $\tan^{-1} t$ lie in $(-\pi/2, \pi/2)$.
- For angles α, β , $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$, when the denominator is nonzero.
- Taking tangents determines an angle modulo π ; check the principal-value sum.

Step 1: Introduce the two principal angles Put

$$\alpha = \tan^{-1}\left(\frac{x-3}{x-4}\right), \quad \beta = \tan^{-1}\left(\frac{x+3}{x+4}\right).$$

The arguments require $x \neq \pm 4$. Since $\alpha + \beta = \pi/4$, taking tangents gives $\tan(\alpha + \beta) = 1$.

Step 2: Simplify the tangent addition expression Combining the arguments gives

$$\begin{aligned}\tan \alpha + \tan \beta &= \frac{(x-3)(x+4) + (x+3)(x-4)}{x^2 - 16} \\ &= \frac{2x^2 - 24}{x^2 - 16}, \\ 1 - \tan \alpha \tan \beta &= 1 - \frac{(x-3)(x+3)}{(x-4)(x+4)} \\ &= 1 - \frac{x^2 - 9}{x^2 - 16} = \frac{-7}{x^2 - 16}.\end{aligned}$$

This denominator is nonzero for all allowed x .

Step 3: Obtain the algebraic candidates By the tangent addition formula,

$$\begin{aligned}1 &= \frac{24 - 2x^2}{7}, \\ 2x^2 &= 17, \quad x^2 = \frac{17}{2}.\end{aligned}$$

The candidates $x = \pm\sqrt{17/2}$ satisfy $x \neq \pm 4$. We must check principal values because tangent is periodic.

Step 4: Verify the principal-value sum for both candidates Since $17/2 < 9$, both candidates lie in $(-3, 3)$. Both inverse-tangent arguments are then positive, so $0 < \alpha, \beta < \pi/2$ and $0 < \alpha + \beta < \pi$. For both candidates, $\tan(\alpha + \beta) = 1$. Within $(0, \pi)$, this forces $\alpha + \beta = \pi/4$, because tangent is negative in the second quadrant. Both candidates therefore satisfy the original equation.

Step 5: Conclusion Both candidates satisfy the original equation. Hence

$$x = \pm\sqrt{\frac{17}{2}} = \pm\frac{\sqrt{34}}{2}.$$

Quick Tip to Solve: After solving the tangent equation, use the signs of both arguments to check the quadrant of their principal-angle sum.

17(a). Using properties of determinants, prove that $\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a - 1)^3$. [4 marks]

Solution:

Concept:

- Adding a multiple of one row to another row leaves the determinant unchanged. Subtracting the third row from the first two rows is therefore an allowed operation.
- A common factor in a row can be taken outside the determinant. Expansion along a column with two zeros reduces a determinant of order three to one of order two.

Step 1: Create zeros in the last column Let the given determinant be D . Apply $R_1 \rightarrow R_1 - R_3$ and $R_2 \rightarrow R_2 - R_3$. Neither operation changes its value:

$$D = \begin{vmatrix} a^2 + 2a - 3 & 2a - 2 & 0 \\ 2a - 2 & a - 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}.$$

Here $a^2 + 2a - 3 = (a - 1)(a + 3)$, while $2a - 2 = 2(a - 1)$. Consequently every entry of each of the first two rows has the common factor $a - 1$. The zero in each row also remains zero after this factor is extracted.

Step 2: Extract row factors and expand Taking $a - 1$ outside each of the first two rows gives

$$D = (a - 1)^2 \begin{vmatrix} a + 3 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}.$$

Expand along the third column. Only its bottom entry contributes, and its cofactor sign is $(-1)^{3+3} = 1$. Thus

$$D = (a - 1)^2 \begin{vmatrix} a + 3 & 2 \\ 2 & 1 \end{vmatrix} = (a - 1)^2 ((a + 3) \cdot 1 - 2 \cdot 2).$$

The quantity in parentheses is $a + 3 - 4 = a - 1$. This proof uses determinant properties throughout, with no division by $a - 1$. It therefore also covers $a = 1$: in that case the transformed first two rows vanish, and the determinant is zero on both sides of the identity.

Step 3: Conclusion Therefore $D = (a - 1)^3$, proving the required identity.

Quick Tip to Solve: For a column of identical entries, subtract one row from the others to create zeros before expanding the determinant.

17(b). Find matrix A such that $\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} A = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}$.

[4 marks]

Solution:**Concept:**

- If a matrix of order 3×2 multiplied by A gives a matrix of order 3×2 , then A has order 2×2 .
- Each entry of a matrix product is obtained by multiplying the corresponding row of the first matrix with the corresponding column of the second. Equal matrices have equal corresponding entries.

Step 1: Write the unknown matrix and multiply Let $A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. The product on the left is

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 2x - z & 2y - w \\ x & y \\ -3x + 4z & -3y + 4w \end{pmatrix}.$$

We compare this product with $\begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}$. The second row is particularly useful because it contains x and y directly, without either of the other unknown entries.

Step 2: Determine the entries from the first two rows Comparing the second rows gives $x = 1$ and $y = -2$. The first row then gives

$$\begin{aligned} 2x - z = -1 &\implies 2 - z = -1 \implies z = 3, \\ 2y - w = -8 &\implies -4 - w = -8 \implies w = 4. \end{aligned}$$

These equations uniquely determine all four entries of A . However, since the given product has a third row as well, we must check that the entries obtained satisfy that row; the first two rows alone would not verify the entire matrix equation.

Step 3: Verify the third row and the full product Substitution in the third row gives $-3x + 4z = -3 + 12 = 9$ and $-3y + 4w = 6 + 16 = 22$, exactly as required. In fact the full product is

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2 - 3 & -4 - 4 \\ 1 & -2 \\ -3 + 12 & 6 + 16 \end{pmatrix} = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}.$$

Step 4: Conclusion Hence the required matrix is $A = \begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix}$.

Quick Tip to Solve: Start with a row such as $(1, 0)$, which reads off a row of the unknown matrix directly, then verify every remaining row.

18(a). If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.

[4 marks]

Solution:

Concept:

- For a variable base and a variable exponent, write $u^v = e^{v \ln u}$. Both u and v must be differentiated; here we work on the usual real domain $x > 0, y > 0$.
- Implicit differentiation treats y as a differentiable function of x . Thus $\frac{d}{dx} \ln y = \frac{1}{y} \frac{dy}{dx}$, while a and b are constants.

Step 1: Differentiate the first variable power Put $y' = \frac{dy}{dx}$. Since $x^y = e^{y \ln x}$, the chain rule and product rule give

$$\frac{d}{dx}(x^y) = x^y \frac{d}{dx}(y \ln x) = x^y \left(y' \ln x + \frac{y}{x} \right).$$

The first term inside the brackets comes from differentiating the exponent y ; the second comes from differentiating the base through $\ln x$. Neither term may be dropped because both quantities vary with x .

Step 2: Differentiate the second power and the given relation Similarly, express $y^x = e^{x \ln y}$. Differentiating gives

$$\frac{d}{dx}(y^x) = y^x \left(\ln y + \frac{x}{y} y' \right).$$

The right-hand side a^b is constant, so its derivative is zero. Therefore differentiating the entire equation yields

$$x^y \left(y' \ln x + \frac{y}{x} \right) + y^x \left(\ln y + \frac{x}{y} y' \right) = 0.$$

Logarithmic differentiation has been applied separately to each power, rather than to their sum, since the logarithm of a sum does not split into a sum of logarithms.

Step 3: Collect the derivative terms Move the terms independent of y' to the right and factor y' on the left:

$$y' \left(x^y \ln x + \frac{x}{y} y^x \right) = - \left(\frac{y}{x} x^y + y^x \ln y \right).$$

Divide by the coefficient of y' , wherever it is nonzero. The expression is valid at points of the implicit curve where y is differentiable as a function of x and this denominator is nonzero.

Step 4: Conclusion Hence,

$$\boxed{\frac{dy}{dx} = - \frac{yx^{y-1} + y^x \ln y}{x^y \ln x + xy^{x-1}}}.$$

Quick Tip to Solve: For each variable power, differentiate its exponential form separately, then collect every term containing dy/dx before dividing.

18(b). If $e^y(x+1) = 1$, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

[4 marks]

Solution:

Concept:

- The product rule is $\frac{d}{dx}(uv) = u'v + uv'$. For the composite function e^y , the chain rule gives $\frac{d}{dx}e^y = e^y \frac{dy}{dx}$.
- Differentiate the first derivative once more to find $\frac{d^2y}{dx^2}$. Because $e^y > 0$ for real y , it may be cancelled from an equation without changing its solutions.

Step 1: Differentiate the given equation implicitly The given relation is $e^y(x+1) = 1$. Since e^y is positive, this relation implies $x+1 > 0$, or $x > -1$. Differentiating both sides with respect to x , the product rule gives

$$e^y \frac{dy}{dx}(x+1) + e^y = 0.$$

Step 2: Isolate the first derivative The chain rule supplies dy/dx in the derivative of e^y , while the derivative of the constant right-hand side is zero. Cancel the nonzero factor e^y to obtain

$$(x+1) \frac{dy}{dx} + 1 = 0.$$

Since $x+1 > 0$ on the domain of the relation, division by $x+1$ is allowed. This gives

$$\frac{dy}{dx} = -\frac{1}{x+1} = -(x+1)^{-1}.$$

Step 3: Differentiate once more and compare Apply the power rule to the expression for the first derivative:

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}(x+1)^{-1} = (x+1)^{-2} = \frac{1}{(x+1)^2}.$$

Squaring the first derivative independently gives

$$\left(\frac{dy}{dx}\right)^2 = \left(-\frac{1}{x+1}\right)^2 = \frac{1}{(x+1)^2}.$$

Both expressions are defined throughout $x > -1$ and are identical.

Step 4: Conclusion Hence, on $x > -1$,

$$\boxed{\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2} = \left(\frac{dy}{dx}\right)^2}.$$

Quick Tip to Solve: Cancel e^y after the first differentiation; then differentiate $-1/(x+1)$ and compare it with its square.

19(a). Evaluate: $\int_0^\pi \frac{x \tan x}{\sec x + \tan x} dx$.

[4 marks]

Solution:

Concept:

- Reflection property: $\int_0^a g(x) dx = \int_0^a g(a-x) dx$, obtained by substituting $x = a - t$.
- Since $\sin(\pi - x) = \sin x$, a function of $\sin x$ has equal values at reflected points of $[0, \pi]$.
- The identity $1 - \sin^2 x = \cos^2 x$ permits rationalisation; also $\frac{d}{dx}(\tan x - \sec x) = \frac{1}{1 + \sin x}$ wherever the individual terms are defined.

Step 1: Simplify the integrand before applying symmetry Put $I = \int_0^\pi \frac{x \tan x}{\sec x + \tan x} dx$. Dividing the numerator and denominator of the trigonometric ratio by $\sec x$ gives

$$\frac{\tan x}{\sec x + \tan x} = \frac{\sin x}{1 + \sin x}.$$

Although the original separate trigonometric terms are undefined at $x = \pi/2$, the simplified ratio has the finite limit $1/2$ there. Thus this is a removable discontinuity, and the integral can be evaluated using its continuous extension. On the whole interval $1 + \sin x \geq 1$, so the simplified expression has no singularity.

Step 2: Pair the reflected integrals Apply the reflection property with $a = \pi$:

$$I = \int_0^\pi \frac{(\pi - x) \sin(\pi - x)}{1 + \sin(\pi - x)} dx = \int_0^\pi \frac{(\pi - x) \sin x}{1 + \sin x} dx.$$

Adding this expression to the original integral eliminates the factor x :

$$2I = \pi \int_0^\pi \frac{\sin x}{1 + \sin x} dx = \pi(\pi - J), \quad J = \int_0^\pi \frac{dx}{1 + \sin x}.$$

Step 3: Evaluate the remaining elementary integral Rationalise the denominator:

$$\frac{1}{1 + \sin x} = \frac{1 - \sin x}{\cos^2 x} = \sec^2 x - \sec x \tan x.$$

An antiderivative away from $\pi/2$ is $\tan x - \sec x$. To handle its apparent middle-point singularity correctly, rewrite it as

$$\tan x - \sec x = \frac{\sin x - 1}{\cos x} = -\frac{\cos x}{1 + \sin x}.$$

The last expression extends smoothly through $\pi/2$, where it is zero. It is therefore a valid antiderivative throughout the integration interval. Its values at 0 and π are -1 and 1 , respectively, so

$$J = 1 - (-1) = 2, \quad 2I = \pi(\pi - 2).$$

Step 4: Conclusion The required value is $\boxed{\frac{\pi(\pi - 2)}{2}}$.

Quick Tip to Solve: First convert the ratio to $\sin x/(1 + \sin x)$; its reflection symmetry removes the factor x when the two integrals are added.

19(b). Evaluate: $\int_1^4 \{|x-1| + |x-2| + |x-4|\} dx$.

[4 marks]

Solution:

Concept:

- The modulus satisfies $|u| = u$ when $u \geq 0$, and $|u| = -u$ when $u < 0$. Its expression must be chosen from the sign on each integration interval.
- A definite integral is additive: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, where $a \leq c \leq b$.
- Polynomial pieces are integrated using $\int x dx = x^2/2$ and $\int k dx = kx$, for a constant k .

Step 1: Locate the sign changes within the interval Let

$$I = \int_1^4 \{|x-1| + |x-2| + |x-4|\} dx.$$

The zeroes are 1, 2, 4; only 2 lies inside the interval. Throughout $[1, 4]$, $|x-1| = x-1$ and $|x-4| = 4-x$. Split at 2 to handle the sign change of $x-2$.

Step 2: Remove the moduli on each piece For $1 \leq x \leq 2$, we have $x-2 \leq 0$, giving

$$|x-1| + |x-2| + |x-4| = (x-1) + (2-x) + (4-x) = 5-x.$$

For $2 \leq x \leq 4$, we have $x-2 \geq 0$, giving

$$|x-1| + |x-2| + |x-4| = (x-1) + (x-2) + (4-x) = x+1.$$

Both formulas agree at $x = 2$. Therefore,

$$I = \int_1^2 (5-x) dx + \int_2^4 (x+1) dx.$$

Step 3: Evaluate and combine the two definite integrals Substitute the upper and lower limits:

$$\begin{aligned} \int_1^2 (5-x) dx &= \left[5x - \frac{x^2}{2} \right]_1^2 \\ &= (10-2) - \left(5 - \frac{1}{2} \right) = \frac{7}{2}, \\ \int_2^4 (x+1) dx &= \left[\frac{x^2}{2} + x \right]_2^4 \\ &= (8+4) - (2+2) = 8. \end{aligned}$$

Adding the two contributions yields $I = 7/2 + 8 = 23/2$. Check: the outer two moduli sum to 3, contributing 9, and the middle modulus contributes $5/2$.

Step 4: Conclusion The required value is $\boxed{\frac{23}{2}}$.

Quick Tip to Solve: On $[1, 4]$, combine $|x-1| + |x-4| = 3$ immediately; split only the remaining term $|x-2|$ at $x = 2$.

20. Solve the following linear programming problem graphically: Maximise $Z = 7x + 10y$ subject to the constraints $4x + 6y \leq 240$, $6x + 3y \leq 240$, $x \geq 10$, $x \geq 0$, $y \geq 0$. [4 marks]

Solution:

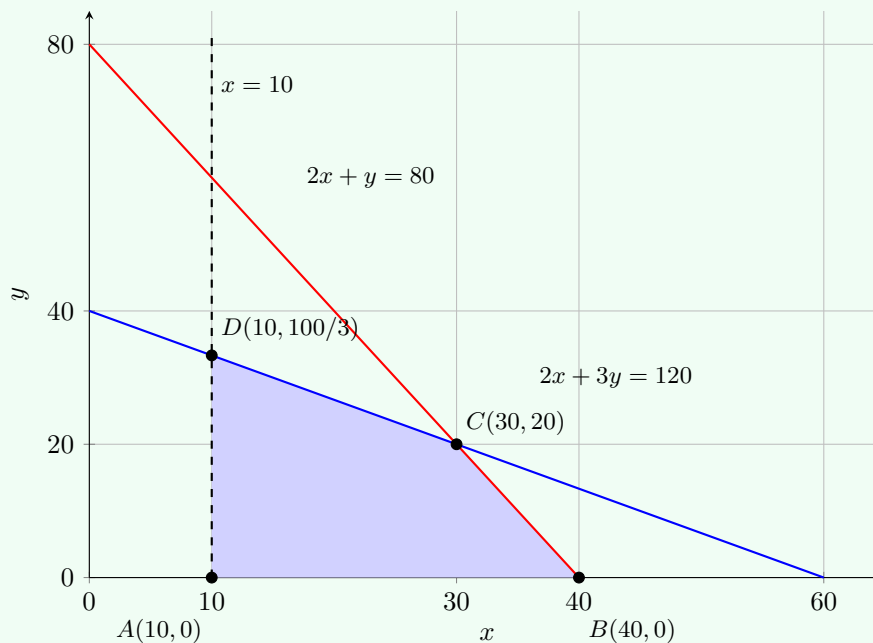
Concept:

- Each linear inequality represents a closed half-plane. The feasible region is the intersection of all the specified half-planes.
- A linear objective function on a nonempty bounded polygon attains its maximum at a vertex. Here $Z = 7x + 10y$, where x, y are the decision variables.

Step 1: Plot the boundary lines and choose the half-planes Dividing the first two inequalities by 2 and 3, respectively, gives

$$2x + 3y \leq 120, \quad 2x + y \leq 80.$$

The line $2x + 3y = 120$ passes through $(60, 0)$ and $(0, 40)$; the line $2x + y = 80$ passes through $(40, 0)$ and $(0, 80)$. The origin satisfies each of these inequalities, so retain the side containing the origin for both. Next retain the region to the right of the vertical line $x = 10$, and above the x -axis because $y \geq 0$. The condition $x \geq 0$ is already ensured by $x \geq 10$. The shaded intersection in the graph is the feasible polygon.



The shaded polygon ABCD is the feasible region.

Step 2: Determine every vertex of the feasible region On $y = 0$, the permitted interval is $10 \leq x \leq 40$, giving $A = (10, 0)$ and $B = (40, 0)$. At the intersection of the two oblique boundaries,

$$\begin{aligned} 2x + 3y &= 120, \\ 2x + y &= 80. \end{aligned}$$

Subtraction yields $2y = 40$, hence $y = 20$, and then $2x = 60$, so $C = (30, 20)$. At $x = 10$, the first boundary gives $20 + 3y = 120$, hence $D = (10, 100/3)$. This point also satisfies $2x + y \leq 80$, since

Quick Tip to Solve: Keep the vertex on $x=10$ in the comparison: its objective value is only slightly smaller than the value at the intersection of the two oblique lines.

$20 + 100/3 = 160/3 < 80$. Thus all four vertices are feasible, and the graph shows the bounded polygon $ABCD$.

Step 3: Compare the objective function at the vertices Evaluate $Z = 7x + 10y$ at each vertex:

Vertex	Z
$A(10, 0)$	70
$B(40, 0)$	280
$C(30, 20)$	$210 + 200 = 410$
$D(10, 100/3)$	$70 + 1000/3 = 1210/3$

Since $1210/3 < 1230/3 = 410$, the value at C exceeds that at every other vertex. The corner-point theorem therefore identifies C as the maximizing point.

Step 4: Conclusion The maximum value is $Z_{\max} = 410$, attained at $x = 30, y = 20$.

Quick Tip to Solve: Keep the vertex on $x=10$ in the comparison: its objective value is only slightly smaller than the value at the intersection of the two oblique lines.

21. Find: $\int \frac{e^x dx}{(e^x - 1)^2(e^x + 2)}$.

[4 marks]

Solution:

Concept:

- The substitution $t = e^x$ gives $dt = e^x dx$, turning the integrand into a rational function of t .
- For a repeated linear factor $(t - 1)^2$, partial fractions require both $A/(t - 1)$ and $B/(t - 1)^2$, together with $C/(t + 2)$.
- Use $\int du/u = \ln |u|$ and $\int u^{-2} du = -u^{-1}$. The original expression is defined for real $x \neq 0$.

Step 1: Substitute to remove the exponential numerator Write the required integral as I . Put $t = e^x$, so $dt = e^x dx$. Then

$$I = \int \frac{dt}{(t - 1)^2(t + 2)}.$$

Here $t > 0$, and $t \neq 1$ because the original denominator must be nonzero. The substitution absorbs the entire numerator, leaving a rational function whose denominator has one repeated linear factor and one simple linear factor.

Step 2: Find the partial-fraction coefficients Let

$$\frac{1}{(t - 1)^2(t + 2)} = \frac{A}{t - 1} + \frac{B}{(t - 1)^2} + \frac{C}{t + 2}.$$

Multiplying by the common denominator gives the polynomial identity

$$1 = A(t - 1)(t + 2) + B(t + 2) + C(t - 1)^2.$$

At $t = 1$, this yields $1 = 3B$, hence $B = 1/3$. At $t = -2$, it yields $1 = 9C$, hence $C = 1/9$.

Quick Tip to Solve: Substitute $t = e^x$ first, then set $t = 1$ and $t = -2$ in the polynomial identity to obtain two partial-fraction coefficients immediately.

These substitutions are valid in the polynomial identity, even though the rational expression itself is undefined at these values. Comparing coefficients of t^2 now gives $A + C = 0$, so $A = -1/9$. Consequently,

$$I = \int \left[-\frac{1}{9(t-1)} + \frac{1}{3(t-1)^2} + \frac{1}{9(t+2)} \right] dt.$$

Step 3: Integrate and return to the original variable Integrating each term gives

$$I = -\frac{1}{9} \ln |t-1| - \frac{1}{3(t-1)} + \frac{1}{9} \ln |t+2| + K,$$

where K is the constant of integration. Since $t = e^x$, combine the logarithms and substitute back. To verify the coefficients, the numerator of the partial-fraction sum is

$$-\frac{1}{9}(t-1)(t+2) + \frac{1}{3}(t+2) + \frac{1}{9}(t-1)^2 = 1.$$

Thus differentiation of the integrated expression with respect to t , followed by multiplication by $dt/dx = e^x$, recovers precisely the given integrand. The antiderivative applies on each interval $x < 0$ or $x > 0$.

Step 4: Conclusion

$$I = \frac{1}{9} \ln \left| \frac{e^x + 2}{e^x - 1} \right| - \frac{1}{3(e^x - 1)} + K, \quad x \neq 0.$$

Quick Tip to Solve: Substitute $t = e^x$ first, then set $t = 1$ and $t = -2$ in the polynomial identity to obtain two partial-fraction coefficients immediately.

22. If $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 2\hat{j} - 3\hat{k}$, then express $\vec{b}$ in the form of $\vec{b} = \vec{b}_1 + \vec{b}_2$, where $\vec{b}_1$ is parallel to $\vec{a}$ and $\vec{b}_2$ is perpendicular to $\vec{a}$. [4 marks]

Solution:

Concept:

- The vector projection of $\vec{b}$ on a nonzero vector $\vec{a}$ is $\vec{b}_1 = (\vec{a} \cdot \vec{b})\vec{a}/(\vec{a} \cdot \vec{a})$.
- The remaining component is $\vec{b}_2 = \vec{b} - \vec{b}_1$. Two vectors are perpendicular when their dot product is zero.

Step 1: Determine the multiplier of the parallel component Since $\vec{b}_1$ is parallel to $\vec{a}$, write $\vec{b}_1 = \lambda\vec{a}$, where λ is a scalar. Then $\vec{b}_2 = \vec{b} - \lambda\vec{a}$. Its perpendicularity to $\vec{a}$ requires

$$(\vec{b} - \lambda\vec{a}) \cdot \vec{a} = 0, \quad \lambda = \frac{\vec{b} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}.$$

The denominator is positive because the given vector $\vec{a}$ is nonzero. Calculate both scalar products

Quick Tip to Solve: Calculate the parallel component by vector projection, subtract it from the original vector, and verify the remainder by a zero dot product.

from the given Cartesian components:

$$\vec{a} \cdot \vec{b} = 2(7) + (-1)(2) + (-2)(-3) = 18,$$
$$\vec{a} \cdot \vec{a} = 2^2 + (-1)^2 + (-2)^2 = 9.$$

Therefore $\lambda = 18/9 = 2$.

Step 2: Calculate the two component vectors Multiplying every component of $\vec{a}$ by the scalar just obtained gives

$$\vec{b}_1 = 2\vec{a} = 4\hat{i} - 2\hat{j} - 4\hat{k}.$$

Subtracting this from $\vec{b}$, component by component, gives

$$\begin{aligned}\vec{b}_2 &= (7\hat{i} + 2\hat{j} - 3\hat{k}) - (4\hat{i} - 2\hat{j} - 4\hat{k}) \\ &= 3\hat{i} + 4\hat{j} + \hat{k}.\end{aligned}$$

In particular, the second component is $2 - (-2) = 4$, and the third is $-3 - (-4) = 1$.

Step 3: Check the decomposition and perpendicularity The relation $\vec{b}_1 = 2\vec{a}$ directly establishes parallelism. Moreover,

$$\vec{a} \cdot \vec{b}_2 = 2(3) + (-1)(4) + (-2)(1) = 0,$$

establishing perpendicularity. Adding the vectors gives Cartesian components $(4 + 3, -2 + 4, -4 + 1) = (7, 2, -3)$, exactly those of $\vec{b}$. Thus both required directional conditions and the original sum are satisfied.

Step 4: Conclusion

$$\vec{b}_1 = 4\hat{i} - 2\hat{j} - 4\hat{k}, \quad \vec{b}_2 = 3\hat{i} + 4\hat{j} + \hat{k}.$$

Quick Tip to Solve: Calculate the parallel component by vector projection, subtract it from the original vector, and verify the remainder by a zero dot product.

23. Find the general solution of the differential equation $\frac{dy}{dx} - y = \sin x$.

[4 marks]

Solution:

Concept:

- A first-order linear equation has the form $y' + P(x)y = Q(x)$. Its integrating factor is $I(x) = e^{\int P(x) dx}$.
- Multiplication by the integrating factor gives $\frac{d}{dx}(Iy) = IQ$; integration introduces one arbitrary real constant for the general solution.

Step 1: Identify the integrating factor Compare the given equation $\frac{dy}{dx} - y = \sin x$ with the standard

Quick Tip to Solve: For this equation the integrating factor is e to the power minus x ; integrate twice by parts and substitute the repeated integral back into itself.

linear form. Here $P(x) = -1$ and $Q(x) = \sin x$, so the integrating factor is

$$I(x) = e^{\int -1 dx} = e^{-x}.$$

A nonzero constant multiplier in the integrating factor is unnecessary. Multiply the entire equation by e^{-x} :

$$e^{-x} \frac{dy}{dx} - e^{-x} y = e^{-x} \sin x.$$

By the product rule the left side is $\frac{d}{dx}(ye^{-x})$. Consequently,

$$ye^{-x} = \int e^{-x} \sin x dx + C.$$

Step 2: Evaluate the integral by integration by parts Write $J = \int e^{-x} \sin x dx$. Taking $\sin x$ as the first factor and integrating e^{-x} , integration by parts gives

$$J = -e^{-x} \sin x + \int e^{-x} \cos x dx.$$

Apply integration by parts once again to the remaining integral:

$$\int e^{-x} \cos x dx = -e^{-x} \cos x - J.$$

Combining these two relations yields

$$2J = -e^{-x}(\sin x + \cos x), \quad J = -\frac{1}{2}e^{-x}(\sin x + \cos x).$$

Any constant belonging to the antiderivative is absorbed into the arbitrary constant C already present in the equation.

Step 3: Obtain and verify the general solution Substitute the integral into the integrating-factor equation and multiply by e^x :

$$ye^{-x} = -\frac{1}{2}e^{-x}(\sin x + \cos x) + C, \quad y = Ce^x - \frac{1}{2}(\sin x + \cos x).$$

As a check, differentiating gives $y' = Ce^x - \frac{1}{2}(\cos x - \sin x)$. Hence

$$y' - y = -\frac{1}{2}(\cos x - \sin x) + \frac{1}{2}(\sin x + \cos x) = \sin x,$$

which is exactly the given equation. There is no initial condition to determine C ; it remains arbitrary. All coefficient functions are defined for every real x , so the solution is valid on the entire real line.

Step 4: Conclusion The general solution is

$$y = Ce^x - \frac{\sin x + \cos x}{2}, \quad C \in \mathbb{R}.$$

Quick Tip to Solve: For this equation the integrating factor is e to the power minus x ; integrate twice by parts and substitute the repeated integral back into itself.

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24(a). Using the method of integration, find the area of the triangle ABC , coordinates of whose vertices are $A(4, 1)$, $B(6, 6)$ and $C(8, 4)$. [6 marks]

Solution:

Concept:

- The area between an upper boundary $y = u(x)$ and a lower boundary $y = l(x)$, over $a \leq x \leq b$, is $\int_a^b [u(x) - l(x)] dx$. The integrand is the vertical length of a thin strip.
- A line through (x_1, y_1) and (x_2, y_2) has slope $m = (y_2 - y_1)/(x_2 - x_1)$ and equation $y - y_1 = m(x - x_1)$. For a triangle, split the integral wherever its upper or lower edge changes.

Step 1: Find the equations of the three sides For AB , the slope is $(6 - 1)/(6 - 4) = 5/2$, so

$$y - 1 = \frac{5}{2}(x - 4), \quad y_{AB} = \frac{5}{2}x - 9.$$

For BC , the slope is $(4 - 6)/(8 - 6) = -1$, giving

$$y - 6 = -(x - 6), \quad y_{BC} = 12 - x.$$

Finally, the slope of AC is $(4 - 1)/(8 - 4) = 3/4$, hence

$$y - 1 = \frac{3}{4}(x - 4), \quad y_{AC} = \frac{3}{4}x - 2.$$

Each equation passes through both of its stated endpoints, so these lines give the exact triangle boundaries.

Step 2: Choose the boundaries of the vertical strips The triangle extends from $x = 4$ to $x = 8$. Its bottom boundary throughout is AC . From $x = 4$ to $x = 6$, the top boundary is AB ; from $x = 6$ to $x = 8$, it is BC . Thus the change of upper boundary at vertex B requires two integrals:

$$\begin{aligned} \mathcal{A} &= \int_4^6 \left[\left(\frac{5}{2}x - 9 \right) - \left(\frac{3}{4}x - 2 \right) \right] dx \\ &\quad + \int_6^8 \left[(12 - x) - \left(\frac{3}{4}x - 2 \right) \right] dx. \end{aligned}$$

The two heights simplify to $\frac{7}{4}x - 7$ and $14 - \frac{7}{4}x$, respectively. They are nonnegative on their respective intervals and vanish at A and C .

Step 3: Evaluate the two integrals The left-hand portion has area

$$\begin{aligned} \mathcal{A}_1 &= \int_4^6 \left(\frac{7}{4}x - 7 \right) dx = \left[\frac{7}{8}x^2 - 7x \right]_4^6 \\ &= \left(\frac{63}{2} - 42 \right) - (14 - 28) = \frac{7}{2}. \end{aligned}$$

Quick Tip to Solve: Split at the middle vertex's x-coordinate, $x=6$: the lower side AC stays fixed while the upper side switches from AB to BC .

The right-hand portion has area

$$\begin{aligned}\mathcal{A}_2 &= \int_6^8 \left(14 - \frac{7}{4}x\right) dx = \left[14x - \frac{7}{8}x^2\right]_6^8 \\ &= (112 - 56) - \left(84 - \frac{63}{2}\right) = \frac{7}{2}.\end{aligned}$$

Adding these integrals accounts for every vertical strip of the triangle exactly once.

Step 4: Conclusion The required area of triangle ABC , obtained by integration, is

$$\mathcal{A} = \frac{7}{2} + \frac{7}{2} = 7 \text{ square units}.$$

Quick Tip to Solve: Split at the middle vertex's x-coordinate, $x=6$: the lower side AC stays fixed while the upper side switches from AB to BC.

OR

24(b). Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

[6 marks]

Solution:

Concept:

- The area enclosed between two curves is the integral of their vertical separation: $\mathcal{A} = \int_a^b [u(x) - l(x)] dx$, where u and l are the upper and lower boundaries and a, b are the intersection abscissae.
- Intersection points satisfy both curve equations. Their coordinates determine the limits of integration, while the sign of the difference between the functions determines which curve is above the other.

Step 1: Express both boundaries as functions of x The parabola $4y = 3x^2$ has equation

$$y_P = \frac{3}{4}x^2.$$

The straight line $3x - 2y + 12 = 0$ gives

$$2y = 3x + 12, \quad y_L = \frac{3}{2}x + 6.$$

The parabola opens upwards, while the line cuts it at two points. We first find those points so that the integral covers precisely the enclosed region.

Step 2: Calculate the intersection points At an intersection the two expressions for y are equal.

Quick Tip to Solve: Factor line minus parabola as $(3/4)(4-x)(x+2)$; this gives both intersection limits and confirms the correct subtraction order.

Therefore

$$\begin{aligned}\frac{3}{4}x^2 &= \frac{3}{2}x + 6, \\ 3x^2 &= 6x + 24, \\ x^2 - 2x - 8 &= 0, \\ (x - 4)(x + 2) &= 0.\end{aligned}$$

Thus $x = -2$ or $x = 4$. Substitution into $y = 3x^2/4$ gives $y = 3$ and $y = 12$, respectively. The two intersection points are consequently $(-2, 3)$ and $(4, 12)$, and the enclosed region extends over $-2 \leq x \leq 4$.

Step 3: Identify the upper boundary and form the area integral The vertical difference between the line and the parabola factors as

$$y_L - y_P = \frac{3}{2}x + 6 - \frac{3}{4}x^2 = \frac{3}{4}(4 - x)(x + 2).$$

Both factors $4 - x$ and $x + 2$ are nonnegative on $[-2, 4]$, so the line is the upper boundary throughout the enclosed region. This establishes the sign of the strip height, without relying only on a test point. The area is therefore

$$\mathcal{A} = \int_{-2}^4 \left(\frac{3}{2}x + 6 - \frac{3}{4}x^2 \right) dx.$$

Step 4: Integrate and evaluate at the limits Integrating each term gives

$$\begin{aligned}\mathcal{A} &= \left[\frac{3}{4}x^2 + 6x - \frac{1}{4}x^3 \right]_{-2}^4 \\ &= (12 + 24 - 16) - (3 - 12 + 2) \\ &= 20 - (-7) = 27.\end{aligned}$$

The lower-limit value is negative, so subtracting it adds 7 to the upper-limit value 20. Since the integrand is a nonnegative vertical length over the entire interval, this value is the geometric area itself.

Step 5: Conclusion The area enclosed by $4y = 3x^2$ and $3x - 2y + 12 = 0$ is 27 square units.

Quick Tip to Solve: Factor line minus parabola as $(3/4)(4-x)(x+2)$; this gives both intersection limits and confirms the correct subtraction order.

25. Find the particular solution of the differential equation $(x - y)\frac{dy}{dx} = (x + 2y)$, given that $y = 0$ when $x = 1$. [6 marks]

Solution:

Concept:

- A first-order equation is homogeneous when its right-hand side depends only on y/x . The substitution $y = vx$ gives $dy/dx = v + x dv/dx$.

Quick Tip to Solve: After setting $v = y/x$, split $1 - v$ into $-\frac{1}{2}(2v + 1) + \frac{3}{2}$; this immediately separates the logarithm and inverse tangent integrals.

- Separate the variables after this substitution. Complete the square in the quadratic denominator to integrate its reciprocal using the inverse tangent formula.
- An initial condition determines the integration constant. Here $x = 1$ and $y = 0$ imply $v = 0$; use the continuous inverse tangent branch around this point.

Step 1: Reduce the homogeneous equation The printed differential equation is

$$\frac{dy}{dx} = \frac{x + 2y}{x - y}.$$

In a neighbourhood of the initial point, $x \neq 0$ and $x - y \neq 0$. Put $y = vx$, so that $dy/dx = v + x dv/dx$. Then

$$\begin{aligned} v + x \frac{dv}{dx} &= \frac{1 + 2v}{1 - v}, \\ x \frac{dv}{dx} &= \frac{1 + 2v - v(1 - v)}{1 - v} = \frac{v^2 + v + 1}{1 - v}. \end{aligned}$$

Since $v^2 + v + 1 = (v + 1/2)^2 + 3/4 > 0$, division by this quadratic loses no real solution. Thus the separated equation is

$$\frac{1 - v}{v^2 + v + 1} dv = \frac{dx}{x}.$$

Step 2: Integrate by splitting the numerator Write $1 - v = -\frac{1}{2}(2v + 1) + \frac{3}{2}$. The first term integrates to a logarithm, while completing the square handles the second:

$$\begin{aligned} \int \frac{1 - v}{v^2 + v + 1} dv &= -\frac{1}{2} \ln(v^2 + v + 1) + \frac{3}{2} \int \frac{dv}{(v + 1/2)^2 + 3/4} \\ &= -\frac{1}{2} \ln(v^2 + v + 1) + \sqrt{3} \tan^{-1} \left(\frac{2v + 1}{\sqrt{3}} \right). \end{aligned}$$

Therefore

$$-\frac{1}{2} \ln(v^2 + v + 1) + \sqrt{3} \tan^{-1} \left(\frac{2v + 1}{\sqrt{3}} \right) = \ln |x| + C.$$

Step 3: Apply the given initial condition and restore the variables At $(x, y) = (1, 0)$, we have $v = 0$. Since $\tan^{-1}(1/\sqrt{3}) = \pi/6$, the preceding equation gives

$$C = \sqrt{3} \cdot \frac{\pi}{6} = \frac{\pi}{2\sqrt{3}}.$$

Substitute $v = y/x$ and use

$$v^2 + v + 1 = \frac{x^2 + xy + y^2}{x^2}.$$

The logarithmic contribution $\ln |x|$ on the left cancels the same term on the right. Rearranging the resulting equation gives the particular integral below, on its continuous branch containing $(1, 0)$.

Step 4: Verify the differential equation Let $D = x^2 + xy + y^2$ and let $y' = dy/dx$. Differentiating the implicit relation yields

$$\frac{2x + y + (x + 2y)y'}{D} = \frac{3(xy' - y)}{D}.$$

Thus $2x + 4y + 2(y - x)y' = 0$, or $(x - y)y' = x + 2y$, exactly the original equation. At $(1, 0)$, both

Quick Tip to Solve: After setting $v = y/x$, split $1 - v$ into $-\frac{1}{2}(2v + 1) + \frac{3}{2}$; this immediately separates the logarithm and inverse tangent integrals.

sides of the final logarithmic relation are zero, which also checks the initial condition.

Step 5: Conclusion The required particular solution, on the branch through $(1, 0)$, is

$$\ln(x^2 + xy + y^2) = 2\sqrt{3} \tan^{-1} \left(\frac{x + 2y}{\sqrt{3}x} \right) - \frac{\pi}{\sqrt{3}}.$$

Quick Tip to Solve: After setting $v = y/x$, split $1 - v$ into $-\frac{1}{2}(2v + 1) + \frac{3}{2}$; this immediately separates the logarithm and inverse tangent integrals.

26(a). Find the coordinates of the point where the line through the points $(3, -4, -5)$ and $(2, -3, 1)$, crosses the plane determined by the points $(1, 2, 3)$, $(4, 2, -3)$ and $(0, 4, 3)$. [6 marks]

Solution:

Concept:

- A line through points P and Q has vector equation $\mathbf{r} = \mathbf{P} + t(\mathbf{Q} - \mathbf{P})$, where t is a real parameter.
- Two nonparallel vectors in a plane determine a normal vector through their cross product. If $\mathbf{n}$ is this normal and A lies in the plane, then $\mathbf{n} \cdot (\mathbf{r} - \mathbf{A}) = 0$.
- At an intersection, the point satisfies both equations. Substitute the parametric coordinates of the line into the plane equation, solve for the parameter, and check the resulting coordinates.

Step 1: Form the equation of the line Denote the two points on the line by $P = (3, -4, -5)$ and $Q = (2, -3, 1)$. Their difference supplies the direction vector:

$$\overrightarrow{PQ} = (2 - 3, -3 + 4, 1 + 5) = (-1, 1, 6).$$

Therefore the coordinates of an arbitrary point of this line are

$$(x, y, z) = (3, -4, -5) + t(-1, 1, 6),$$

or

$$x = 3 - t, \quad y = -4 + t, \quad z = -5 + 6t.$$

The values $t = 0$ and $t = 1$ recover P and Q , respectively, confirming the chosen direction and starting point.

Step 2: Determine a normal to the plane Let the three points determining the plane be

$$A = (1, 2, 3), \quad B = (4, 2, -3), \quad C = (0, 4, 3).$$

Two vectors lying in the plane are

$$\overrightarrow{AB} = (3, 0, -6), \quad \overrightarrow{AC} = (-1, 2, 0).$$

Quick Tip to Solve: Reduce the cross-product normal $(12, 6, 6)$ to $(2, 1, 1)$ before forming the plane equation; the substitution then becomes the simple equation $-3 + 5t = 7$.

Their cross product is

$$\begin{aligned}\vec{AB} \times \vec{AC} &= (0 \cdot 0 - (-6) \cdot 2, (-6)(-1) - 3 \cdot 0, 3 \cdot 2 - 0(-1)) \\ &= (12, 6, 6) = 6(2, 1, 1).\end{aligned}$$

The nonzero result shows that the three points are not collinear. We may consequently take $(2, 1, 1)$ as a normal vector.

Step 3: Write the plane equation Using the point $A = (1, 2, 3)$ and the normal vector just found, the point-normal equation gives

$$2(x - 1) + (y - 2) + (z - 3) = 0,$$

which simplifies to

$$2x + y + z = 7.$$

This equation also holds at B , because $2(4) + 2 - 3 = 7$, and at C , because $2(0) + 4 + 3 = 7$. Thus it is the required plane through all three printed points.

Step 4: Find and verify the intersection parameter Substitute the line coordinates into the plane equation:

$$\begin{aligned}2(3 - t) + (-4 + t) + (-5 + 6t) &= 7, \\ -3 + 5t &= 7, \\ t &= 2.\end{aligned}$$

The coefficient of t is nonzero, so the line meets this plane at exactly one point. At $t = 2$, its coordinates are

$$x = 3 - 2 = 1, \quad y = -4 + 2 = -2, \quad z = -5 + 12 = 7.$$

As an independent check, the displacement from P to this point is $(-2, 2, 12) = 2(-1, 1, 6)$, so it lies on the line. Also, substituting into the plane gives $2(1) + (-2) + 7 = 7$, verifying that it lies on the plane.

Step 5: Conclusion The line crosses the given plane at the unique point

$$\boxed{(1, -2, 7)}.$$

Quick Tip to Solve: Reduce the cross-product normal $(12, 6, 6)$ to $(2, 1, 1)$ before forming the plane equation; the substitution then becomes the simple equation $-3 + 5t = 7$.

26(b). A variable plane which remains at a constant distance $3p$ from the origin cuts the coordinate axes at A, B, C . Show that the locus of the centroid of triangle ABC is $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}$. [6 marks]

Solution:

Concept:

- A plane with nonzero finite intercepts a, b, c on the coordinate axes has equation $X/a + Y/b + Z/c = 1$.
- The perpendicular distance of $AX + BY + CZ + D = 0$ from the origin is $|D|/\sqrt{A^2 + B^2 + C^2}$.
- The centroid of a triangle is obtained by averaging the three coordinates of its vertices separately.

Step 1: Represent the intercepts of the variable plane Let the plane meet the coordinate axes at $A(a, 0, 0)$, $B(0, b, 0)$ and $C(0, 0, c)$, where a, b, c are finite and nonzero. These intercepts may be positive or negative. Use capital letters X, Y, Z for a general point on the plane, so that its equation is

$$\frac{X}{a} + \frac{Y}{b} + \frac{Z}{c} - 1 = 0.$$

Its normal vector is $(1/a, 1/b, 1/c)$. Thus the distance formula can be applied directly without first finding a unit normal.

Step 2: Use the fixed perpendicular distance The origin has coordinates $(0, 0, 0)$. Substituting these coordinates into the numerator of the point-to-plane distance formula gives $|-1| = 1$. Therefore the given distance $3p$ yields

$$\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = 3p.$$

Since this is a positive distance, $p > 0$. Squaring and taking reciprocals gives the restriction on the intercepts:

$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{9p^2}.$$

The squared terms ensure that the same relation holds whichever octants contain the three intercepts.

Step 3: Express the intercepts through the centroid Let $G(x, y, z)$ be the centroid of $\triangle ABC$. By the centroid formula,

$$(x, y, z) = \left(\frac{a + 0 + 0}{3}, \frac{0 + b + 0}{3}, \frac{0 + 0 + c}{3} \right) = \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3} \right).$$

Hence $a = 3x$, $b = 3y$ and $c = 3z$. Substitute these three relations into the distance restriction:

$$\frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2} = \frac{1}{9p^2}.$$

Multiplication by 9 eliminates the intercept variables and expresses the required restriction solely in the coordinates of the centroid.

Step 4: Verify that the restriction describes the entire locus Conversely, take any point (x, y, z) with

Quick Tip to Solve: Use intercept form first: each intercept is three times the matching centroid coordinate, so the factor of nine cancels immediately.

nonzero coordinates satisfying the resulting restriction. The plane with intercepts $3x, 3y, 3z$ has equation $X/(3x) + Y/(3y) + Z/(3z) = 1$. Its distance from the origin is

$$\frac{1}{\sqrt{\frac{1}{9x^2} + \frac{1}{9y^2} + \frac{1}{9z^2}}} = 3p.$$

Its intercept triangle has centroid (x, y, z) . Thus every admissible point on the obtained surface actually arises from a plane in the stated family.

Step 5: Conclusion The locus of the centroid is $\boxed{\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{p^2}}$, with $x, y, z \neq 0$.

Quick Tip to Solve: Use intercept form first: each intercept is three times the matching centroid coordinate, so the factor of nine cancels immediately.

27(a). Consider $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \rightarrow \mathbb{R} - \left\{\frac{4}{3}\right\}$ given by $f(x) = \frac{4x+3}{3x+4}$. Show that f is bijective. Find the inverse of f and hence find $f^{-1}(0)$ and x such that $f^{-1}(x) = 2$. [6 marks]

Solution:

Concept:

- A function is injective if equality of two outputs implies equality of the corresponding inputs.
- A function is surjective if every element of the stated codomain has a preimage in its domain; a bijection is both injective and surjective.
- For a bijection, solving $y = f(x)$ for x gives $x = f^{-1}(y)$. The domain and codomain of the inverse are interchanged.

Step 1: Check that the stated function is well defined Write $D = \mathbb{R} \setminus \{-4/3\}$ and $C = \mathbb{R} \setminus \{4/3\}$. For $x \in D$, the denominator $3x + 4$ is nonzero, so $f(x)$ exists. Also, if $f(x) = 4/3$, cross multiplication would give $3(4x + 3) = 4(3x + 4)$, that is, $12x + 9 = 12x + 16$, which is impossible. Hence $f(x) \in C$ for every allowed input.

Step 2: Prove that the function is injective Let $u, v \in D$ and suppose $f(u) = f(v)$. Both denominators are nonzero, so cross multiplication is valid:

$$\begin{aligned}\frac{4u+3}{3u+4} &= \frac{4v+3}{3v+4}, \\ (4u+3)(3v+4) &= (4v+3)(3u+4), \\ 12uv + 16u + 9v + 12 &= 12uv + 16v + 9u + 12, \\ 7u &= 7v.\end{aligned}$$

Therefore $u = v$, proving that distinct allowed inputs cannot have the same image.

Step 3: Prove surjectivity by finding a preimage Take any $y \in C$, so $y \neq 4/3$. Solve $y = (4x+3)/(3x+4)$

Quick Tip to Solve: To solve an equation of the form $f^{-1}(x) = a$, calculate $x = f(a)$, after checking that a lies in the original domain.

4):

$$\begin{aligned}3xy + 4y &= 4x + 3, \\(3y - 4)x &= 3 - 4y, \\x &= \frac{3 - 4y}{3y - 4}.\end{aligned}$$

This value is defined because $3y - 4 \neq 0$. It belongs to D , since

$$x + \frac{4}{3} = \frac{3(3 - 4y) + 4(3y - 4)}{3(3y - 4)} = \frac{-7}{3(3y - 4)} \neq 0.$$

Moreover, substitution confirms the desired image:

$$4x + 3 = \frac{-7y}{3y - 4}, \quad 3x + 4 = \frac{-7}{3y - 4}, \quad f(x) = y.$$

Thus every codomain element has a preimage in the domain. Together with injectivity, this proves bijectivity.

Step 4: Obtain the inverse and evaluate it at zero The preceding solution for the preimage gives the inverse function

$$f^{-1} : C \longrightarrow D, \quad f^{-1}(t) = \frac{3 - 4t}{3t - 4}, \quad t \neq \frac{4}{3}.$$

Since $0 \in C$, substitute $t = 0$:

$$f^{-1}(0) = \frac{3 - 0}{0 - 4} = -\frac{3}{4}.$$

This is admissible in D , and $f(-3/4) = 0$ checks the evaluation directly.

Step 5: Solve the prescribed inverse-function equation In $f^{-1}(x) = 2$, the input x must lie in C . Substituting the inverse formula and cross multiplying gives

$$\begin{aligned}\frac{3 - 4x}{3x - 4} &= 2, \\3 - 4x &= 6x - 8, \\10x &= 11, \\x &= \frac{11}{10}.\end{aligned}$$

This value is different from $4/3$, so the denominator does not vanish. Equivalently, applying f to the inverse equation gives $x = f(2) = (8 + 3)/(6 + 4) = 11/10$, which independently confirms the calculation.

Step 6: Conclusion The function f is bijective, and $f^{-1}(x) = \frac{3 - 4x}{3x - 4}$ for $x \neq 4/3$. Also, $f^{-1}(0) = -3/4$, and $f^{-1}(x) = 2$ gives $x = 11/10$.

Quick Tip to Solve: To solve an equation of the form $f^{-1}(x) = a$, calculate $x = f(a)$, after checking that a lies in the original domain.

27(b). Let $A = \mathbb{Q} \times \mathbb{Q}$ and let $*$ be a binary operation on A defined by $(a, b) * (c, d) = (ac, b + ad)$ for $(a, b), (c, d) \in A$. Determine, whether $*$ is commutative and associative. Then, with respect to $*$ on A :
 (i) Find the identity element in A . (ii) Find the invertible elements of A . [6 marks]

Solution:

Concept:

- A binary operation is commutative if $u * v = v * u$ for every pair of elements, and associative if $(u * v) * w = u * (v * w)$ for every triple.
- An identity e must satisfy both $u * e = u$ and $e * u = u$. An inverse v of u must satisfy both $u * v = e$ and $v * u = e$.

Step 1: Check closure and commutativity If a, b, c, d are rational, then ac and $b + ad$ are rational. Thus the prescribed operation maps $A \times A$ into A . Reversing the factors gives

$$(a, b) * (c, d) = (ac, b + ad), \quad (c, d) * (a, b) = (ac, d + cb).$$

The second coordinates need not agree. For example, take $u = (2, 0)$ and $v = (1, 1)$. Then $u * v = (2, 2)$, whereas $v * u = (2, 1)$. Since these ordered pairs are unequal, the operation is not commutative.

Step 2: Verify associativity for arbitrary elements Let $(a, b), (c, d), (r, s) \in A$. Computing the two possible bracketings directly gives

$$\begin{aligned} ((a, b) * (c, d)) * (r, s) &= (ac, b + ad) * (r, s) \\ &= (acr, b + ad + acs), \\ (a, b) * ((c, d) * (r, s)) &= (a, b) * (cr, d + cs) \\ &= (acr, b + a(d + cs)) \\ &= (acr, b + ad + acs). \end{aligned}$$

Both coordinates agree for every choice of the three elements, proving associativity.

Step 3: Find and verify the identity element Suppose the identity is $e = (p, q)$. The right-identity requirement gives

$$(a, b) * (p, q) = (ap, b + aq) = (a, b)$$

for all rational a, b . Taking $a = 1$ forces $p = 1$ and $q = 0$. Conversely,

$$(a, b) * (1, 0) = (a, b), \quad (1, 0) * (a, b) = (a, b).$$

Hence the candidate satisfies both identity requirements, and is the identity of the operation.

Step 4: Determine exactly which elements are invertible For an inverse (c, d) of (a, b) , the equation $(a, b) * (c, d) = (1, 0)$ requires

$$ac = 1, \quad b + ad = 0.$$

If $a = 0$, the first equation is impossible. If $a \neq 0$, these equations uniquely give $c = 1/a$ and $d = -b/a$,

Quick Tip to Solve: Compare second coordinates to test commutativity; for an inverse, solve the first-coordinate equation first and then check the reverse product.

both rational. Check the reverse product as well:

$$\left(\frac{1}{a}, -\frac{b}{a}\right) * (a, b) = \left(1, -\frac{b}{a} + \frac{b}{a}\right) = (1, 0).$$

Thus every pair with nonzero first coordinate has a two-sided inverse, and no pair with zero first coordinate has one.

Step 5: Conclusion The operation is not commutative but is associative. Its identity is $(1, 0)$. The invertible elements are precisely $\{(a, b) \in \mathbb{Q} \times \mathbb{Q} : a \neq 0\}$, with $(a, b)^{-1} = (1/a, -b/a)$.

Quick Tip to Solve: Compare second coordinates to test commutativity; for an inverse, solve the first-coordinate equation first and then check the reverse product.

28. If $A = \begin{pmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{pmatrix}$, find A^{-1} . Hence using A^{-1} solve the system of equations $2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$. [6 marks]

Solution:

Concept:

- A square matrix A has an inverse when $|A| \neq 0$; then $A^{-1} = \text{adj } A / |A|$. The adjugate is the transpose of the cofactor matrix.
- A linear system written as $AX = B$, where X is the column of unknowns, has the unique solution $X = A^{-1}B$ when A is invertible.

Step 1: Calculate the determinant Expanding along the first row of the given matrix,

$$\begin{aligned} |A| &= 2\{2(-2) - (-4)(1)\} - (-3)\{3(-2) - (-4)(1)\} \\ &\quad + 5\{3(1) - 2(1)\} \\ &= 2(0) + 3(-2) + 5(1) = -1. \end{aligned}$$

As the determinant is nonzero, the inverse exists, and the corresponding system has exactly one solution.

Step 2: Compute the cofactors and adjugate The cofactor C_{ij} equals $(-1)^{i+j}$ times the determinant

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant, and keep the constants in the same row order as the coefficient matrix.

obtained by deleting row i and column j . Thus

$$\begin{aligned}C_{11} &= 2(-2) - (-4)(1) = 0, \\C_{12} &= -\{3(-2) - (-4)(1)\} = 2, \\C_{13} &= 3(1) - 2(1) = 1, \\C_{21} &= -\{(-3)(-2) - 5(1)\} = -1, \\C_{22} &= 2(-2) - 5(1) = -9, \\C_{23} &= -\{2(1) - (-3)(1)\} = -5, \\C_{31} &= (-3)(-4) - 5(2) = 2, \\C_{32} &= -\{2(-4) - 5(3)\} = 23, \\C_{33} &= 2(2) - (-3)(3) = 13.\end{aligned}$$

Hence the cofactor matrix and its transpose are

$$C = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -9 & -5 \\ 2 & 23 & 13 \end{pmatrix}, \quad \text{adj } A = \begin{pmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{pmatrix}.$$

Step 3: Obtain the inverse Divide the adjugate by the determinant -1 :

$$A^{-1} = \frac{1}{-1} \text{adj } A = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix}.$$

The division changes the sign of each adjugate entry. As a check, multiplication by the original matrix gives

$$AA^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Step 4: Use the inverse to solve the system Set $X = (x, y, z)^T$ and $B = (11, -5, -3)^T$. The three printed equations are exactly $AX = B$, so multiplying on the left by the inverse gives

$$\begin{aligned}X &= A^{-1}B \\&= \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix} \begin{pmatrix} 11 \\ -5 \\ -3 \end{pmatrix} \\&= \begin{pmatrix} -5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.\end{aligned}$$

Substitution verifies every equation: $2(1) - 3(2) + 5(3) = 11$, $3(1) + 2(2) - 4(3) = -5$, and $1 + 2 - 2(3) = -3$.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant, and keep the constants in the same row order as the coefficient matrix.

Step 5: Conclusion

$$A^{-1} = \begin{pmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{pmatrix}.$$

Using this inverse, the unique solution is

$$x = 1, y = 2, z = 3.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant, and keep the constants in the same row order as the coefficient matrix.

29. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening. [6 marks]

Solution:

Concept:

- For a uniformly open window, the light admitted is proportional to its area. Thus maximizing light means maximizing the combined area of its rectangular and semicircular portions.
- A semicircle of radius r has area $\frac{1}{2}\pi r^2$ and curved boundary length πr . Its diameter $2r$ is the width of the rectangle.
- For a differentiable area function, an interior maximum satisfies $A'(r) = 0$. A negative second derivative establishes that the stationary value is a maximum.

Step 1: Define the dimensions and use the perimeter Let the radius of the semicircular opening be r m, and let the height of the rectangular portion be h m. The rectangular width is consequently $2r$ m. The outer perimeter consists of the bottom edge, two vertical sides and the semicircular arc. The common diameter between the two portions is inside the window and is not included in this perimeter. Hence

$$2r + 2h + \pi r = 10, \quad h = 5 - \frac{\pi + 2}{2}r.$$

Positive dimensions require $0 < r < \frac{10}{\pi + 2}$.

Step 2: Express the whole opening's area in one variable Adding the rectangle's area and the semicircle's area gives

$$\begin{aligned} A(r) &= 2rh + \frac{1}{2}\pi r^2 \\ &= 2r \left(5 - \frac{\pi + 2}{2}r \right) + \frac{1}{2}\pi r^2 \\ &= 10r - \frac{\pi + 4}{2}r^2. \end{aligned}$$

This area includes both portions of the opening, as required; maximizing only the rectangular portion would give different dimensions.

Step 3: Determine and verify the maximizing radius Differentiating with respect to r ,

$$A'(r) = 10 - (\pi + 4)r.$$

Quick Tip to Solve: Count only the outer boundary in the perimeter: the semicircle's diameter is shared internally with the rectangle, so it must not be added again.

Therefore the stationary radius is $r = \frac{10}{\pi+4}$. It lies in the permissible interval since $\pi + 4 > \pi + 2$. Moreover,

$$A''(r) = -(\pi + 4) < 0.$$

The area function is strictly concave throughout the permissible interval, so this stationary point gives its unique global maximum. Equivalently, its completed-square form is

$$A(r) = \frac{50}{\pi + 4} - \frac{\pi + 4}{2} \left(r - \frac{10}{\pi + 4} \right)^2.$$

The square term cannot be negative, directly confirming the maximum.

Step 4: Obtain all dimensions of the window Substituting the maximizing radius into the perimeter relation,

$$\begin{aligned} h &= 5 - \frac{\pi + 2}{2} \cdot \frac{10}{\pi + 4} \\ &= \frac{5(\pi + 4) - 5(\pi + 2)}{\pi + 4} = \frac{10}{\pi + 4}. \end{aligned}$$

Thus the rectangle's height equals the semicircle's radius. The width is $2r = \frac{20}{\pi+4}$ m, and the total height of the whole window is $h + r = \frac{20}{\pi+4}$ m. Checking the original constraint gives $2r + 2h + \pi r = (\pi + 4)r = 10$ m.

Step 5: Conclusion The rectangular portion must have width $\frac{20}{\pi + 4}$ m and height $\frac{10}{\pi + 4}$ m. The semicircle's radius is $\frac{10}{\pi + 4}$ m. Hence the whole window has width and total height each equal to $\frac{20}{\pi+4}$ m, approximately 2.80 m.

Quick Tip to Solve: Count only the outer boundary in the perimeter: the semicircle's diameter is shared internally with the rectangle, so it must not be added again.