

CBSE Class 12 Mathematics Question Paper 2018 with Solution PDF-Set 65-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 3 questions of four marks each and 3 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. Find the value of $\tan^{-1} \sqrt{3} - \cot^{-1}(-\sqrt{3})$.

[1 mark]

Solution:

Concept:

- The principal range of $\tan^{-1} t$ is $(-\pi/2, \pi/2)$.
- The principal range of $\cot^{-1} t$ is $(0, \pi)$; choose an angle in this interval with the stated cotangent.

Step 1: Evaluate the inverse tangent Since $\tan(\pi/3) = \sqrt{3}$ and $\pi/3$ lies in the principal range, $\tan^{-1} \sqrt{3} = \pi/3$.

Step 2: Evaluate the inverse cotangent and subtract In the second quadrant cotangent is negative, and $\cot(5\pi/6) = -\sqrt{3}$. Therefore $\cot^{-1}(-\sqrt{3}) = 5\pi/6$, and the required difference is

$$\frac{\pi}{3} - \frac{5\pi}{6} = \frac{2\pi - 5\pi}{6} = -\frac{\pi}{2}.$$

Step 3: Conclusion The required value is $-\pi/2$.

Quick Tip to Solve: For a negative cotangent input, first place the principal angle in the second quadrant.

2. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew symmetric, find the values of a and b .

[1 mark]

Solution:

Concept:

- A real square matrix is skew symmetric when $A^T = -A$.
- Thus opposite entries satisfy $a_{ij} = -a_{ji}$, and all diagonal entries are zero.

Step 1: Compare the first and second positions The entries in positions $(1, 2)$ and $(2, 1)$ are a and 2 . Skew symmetry requires $a = -2$.

Step 2: Compare the first and third positions The entries in positions $(1, 3)$ and $(3, 1)$ are -3 and b , so $-3 = -b$, giving $b = 3$. The remaining pair, -1 and 1 , already consists of negatives of each other. The three diagonal zeros also meet the required condition.

Step 3: Conclusion Hence $a = -2, b = 3$.

Quick Tip to Solve: Compare entries reflected across the main diagonal; each must be the negative of its partner.

3. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60° and their scalar product is $\frac{9}{2}$. [1 mark]

Solution:

Concept:

- For vectors with angle θ , $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$.
- A vector magnitude is nonnegative, so only the nonnegative square root is admissible.

Step 1: Use the common magnitude Let $|\vec{a}| = |\vec{b}| = m$. Substituting the angle and scalar product gives

$$m^2 \cos 60^\circ = \frac{9}{2}.$$

Step 2: Solve for the magnitude As $\cos 60^\circ = 1/2$, we obtain $m^2/2 = 9/2$, hence $m^2 = 9$. Although the algebraic equation has roots ± 3 , a magnitude cannot be negative, so $m = 3$. Substitution gives $3 \times 3 \times (1/2) = 9/2$, as required.

Step 3: Conclusion Each vector has magnitude $\boxed{3}$.

Quick Tip to Solve: When both magnitudes are equal, replace their product by a square before substituting the angle.

4. If $a * b$ denotes the larger of a and b and if $a \circ b = (a * b) + 3$, then write the value of $(5) \circ (10)$, where $*$ and $\circ$ are binary operations. [1 mark]

Solution:

Concept:

- The operation $*$ selects the larger of its two inputs: $a * b = \max(a, b)$.
- The second operation is defined using the first: $a \circ b = (a * b) + 3$.

Step 1: Evaluate the inner operation For the inputs 5 and 10, the larger number is 10, so $5 * 10 = 10$. Here the star denotes the given operation and does not denote ordinary multiplication.

Step 2: Apply the definition of the second operation Substitute the value of $5 * 10$ into the definition:

$$5 \circ 10 = (5 * 10) + 3 = 10 + 3 = 13.$$

The parentheses around the inputs do not change either input or the operation; the addition of 3 is performed after choosing the larger number.

Step 3: Conclusion The required value is $\boxed{13}$.

Quick Tip to Solve: Apply the stated definition of each unfamiliar operation before using ordinary arithmetic.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5. **Prove that :** $3 \sin^{-1} x = \sin^{-1}(3x - 4x^3)$, $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$. [2 marks]

Solution:

Concept:

- The triple-angle identity is $\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$.
- The identity $\sin^{-1}(\sin u) = u$ holds when $u \in [-\pi/2, \pi/2]$, the principal range of inverse sine.

Step 1: Set an angle and check its range Let $\theta = \sin^{-1} x$. Then $x = \sin \theta$, with $\theta \in [-\pi/2, \pi/2]$. Since inverse sine is increasing and $-1/2 \leq x \leq 1/2$,

$$-\frac{\pi}{6} = \sin^{-1}(-1/2) \leq \theta \leq \sin^{-1}(1/2) = \frac{\pi}{6}.$$

Multiplying by 3 gives $-\pi/2 \leq 3\theta \leq \pi/2$.

Step 2: Apply the triple-angle identity and inverse sine Using $x = \sin \theta$,

$$3x - 4x^3 = 3 \sin \theta - 4 \sin^3 \theta = \sin 3\theta.$$

This expression belongs to $[-1, 1]$, so its inverse sine is defined. Moreover, the range check above permits

$$\sin^{-1}(3x - 4x^3) = \sin^{-1}(\sin 3\theta) = 3\theta = 3 \sin^{-1} x.$$

The endpoints are included because $3\theta = \pm\pi/2$ also lie in the principal range.

Step 3: Conclusion Thus $3 \sin^{-1} x = \sin^{-1}(3x - 4x^3)$ for every $x \in [-1/2, 1/2]$.

Quick Tip to Solve: Before cancelling sine with inverse sine, prove that the multiplied angle remains in the principal range.

6. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$, compute A^{-1} and show that $2A^{-1} = 9I - A$. [2 marks]

Solution:

Concept:

- For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$, $A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- The identity matrix of order two is $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; matrix equality is checked entry by entry.

Quick Tip to Solve: For a two by two inverse, compute the determinant first and carefully reverse both off-diagonal signs.

Step 1: Compute the determinant and inverse For the given matrix,

$$|A| = 2(7) - (-3)(-4) = 14 - 12 = 2 \neq 0.$$

Therefore the inverse exists and is

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7/2 & 3/2 \\ 2 & 1 \end{bmatrix}.$$

The diagonal entries are interchanged and both off-diagonal signs are reversed before dividing by the determinant.

Step 2: Compare the two sides of the required identity Multiplying the inverse by 2 gives

$$2A^{-1} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}.$$

On the other hand,

$$9I - A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}.$$

The two matrices agree in every entry, proving the identity.

Step 3: Conclusion $A^{-1} = \frac{1}{2} \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$, and hence $2A^{-1} = 9I - A$.

Quick Tip to Solve: For a two by two inverse, compute the determinant first and carefully reverse both off-diagonal signs.

7. Differentiate $\tan^{-1}\left(\frac{1 + \cos x}{\sin x}\right)$ **with respect to** x .

[2 marks]

Solution:

Concept:

- If $y = \tan^{-1} u(x)$, then $dy/dx = u'(x)/(1 + u(x)^2)$.
- The quotient rule states $(f/g)' = (f'g - fg')/g^2$, where $g \neq 0$.

Step 1: Differentiate the inner quotient Put $u = (1 + \cos x)/\sin x$; the original expression is defined only when $\sin x \neq 0$, that is, $x \neq n\pi$ for integers n . The quotient rule gives

$$u' = \frac{(-\sin x)\sin x - (1 + \cos x)\cos x}{\sin^2 x} = \frac{-1 - \cos x}{\sin^2 x}.$$

Here we used $\sin^2 x + \cos^2 x = 1$.

Quick Tip to Solve: Direct differentiation avoids principal-branch errors that can arise from replacing the expression by a half-angle.

Step 2: Apply the inverse tangent derivative and simplify By the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{-(1 + \cos x)/\sin^2 x}{1 + (1 + \cos x)^2/\sin^2 x} \\ &= -\frac{1 + \cos x}{\sin^2 x + (1 + \cos x)^2} \\ &= -\frac{1 + \cos x}{2(1 + \cos x)} = -\frac{1}{2}.\end{aligned}$$

Cancellation is legitimate on the original domain: $1 + \cos x = 0$ occurs only at odd multiples of π , where $\sin x = 0$ and the original function is excluded.

Step 3: Conclusion The derivative is $\boxed{-1/2}$ wherever the given function is defined, namely $x \neq n\pi$, $n \in \mathbb{Z}$.

Quick Tip to Solve: Direct differentiation avoids principal-branch errors that can arise from replacing the expression by a half-angle.

8. The total cost $C(x)$ associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output. [2 marks]

Solution:

Concept:

- The marginal cost at output x is $C'(x) = \frac{dC}{dx}$, the instantaneous change in total cost per additional unit of output.
- For a polynomial, differentiate each term separately using $\frac{d}{dx}(x^n) = nx^{n-1}$. A fixed cost is constant and therefore contributes zero to marginal cost.

Step 1: Differentiate the total cost The given cost function is $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Its derivative is

$$\begin{aligned}C'(x) &= 3(0.005)x^2 - 2(0.02)x + 30 + 0 \\ &= 0.015x^2 - 0.04x + 30.\end{aligned}$$

This derivative, rather than the total cost itself, measures the instantaneous rate specified in the question.

Step 2: Evaluate at the stated output When three units are produced, substitute $x = 3$ into the derivative:

$$\begin{aligned}C'(3) &= 0.015(3)^2 - 0.04(3) + 30 \\ &= 0.135 - 0.120 + 30 \\ &= 30.015.\end{aligned}$$

The constant fixed cost 5000 does not affect this rate. The result is expressed in the cost units of C , per unit of output; the question does not specify a currency.

Step 3: Conclusion The marginal cost when 3 units are produced is $\boxed{30.015}$ cost units per unit of

Quick Tip to Solve: Differentiate the cost function before substituting the output; the fixed-cost term always disappears.

output.

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9. Evaluate : $\int \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x} dx.$

[2 marks]

Solution:

Concept:

- The double-angle identity $\cos 2x = 1 - 2 \sin^2 x$ simplifies the numerator before integration.
- Where $\cos x \neq 0$, $1/\cos^2 x = \sec^2 x$, and $\frac{d}{dx}(\tan x) = \sec^2 x$. An indefinite integral includes an arbitrary constant.

Step 1: Simplify the integrand Let I denote the required integral. Using the double-angle identity gives

$$\cos 2x + 2 \sin^2 x = (1 - 2 \sin^2 x) + 2 \sin^2 x = 1.$$

Therefore, on every interval where the original denominator is nonzero,

$$I = \int \frac{1}{\cos^2 x} dx = \int \sec^2 x dx.$$

The cancellation occurs in the numerator; no division by $\sin x$ or other possibly zero expression is needed.

Step 2: Integrate and check Using the standard derivative of the tangent function,

$$\int \sec^2 x dx = \tan x + C,$$

where C is an arbitrary constant. To verify this antiderivative, differentiate it:

$$\frac{d}{dx}(\tan x + C) = \sec^2 x = \frac{\cos 2x + 2 \sin^2 x}{\cos^2 x}.$$

This equality holds wherever the given integrand is defined, namely where $\cos x \neq 0$.

Step 3: Conclusion The required indefinite integral is $\boxed{\tan x + C}$, on intervals where $\cos x \neq 0$.

Quick Tip to Solve: Before splitting the fraction, replace the double-angle cosine by $1 - 2 \sin^2 x$; the numerator becomes one.

10. Find the differential equation representing the family of curves $y = ae^{bx+5}$, where a and b are arbitrary constants. [2 marks]

Solution:

Concept:

- To form a differential equation for a family containing two arbitrary constants, differentiate twice and eliminate both constants.
- Write $y' = dy/dx$ and $y'' = d^2y/dx^2$. Since a and b are constants, differentiating ae^{bx+5} introduces the factor b .

Step 1: Differentiate the family twice Start with $y = ae^{bx+5}$. Differentiating with respect to x gives

$$y' = abe^{bx+5} = by.$$

Differentiating once more, keeping b constant, gives

$$y'' = by' = b^2y.$$

The constant a has already disappeared from these relations because the original exponential expression can be replaced by y .

Step 2: Eliminate the remaining constant Multiply $y'' = b^2y$ by y , and square $y' = by$:

$$yy'' = b^2y^2, \quad (y')^2 = b^2y^2.$$

Subtracting these expressions eliminates b :

$$yy'' - (y')^2 = 0.$$

This method avoids division by y , so it also preserves the member $y = 0$ arising when $a = 0$. The resulting equation has order two, as expected for two arbitrary constants.

Step 3: Conclusion The required differential equation is

$$\boxed{y \frac{d^2y}{dx^2} - \left(\frac{dy}{dx} \right)^2 = 0}.$$

Quick Tip to Solve: Use $y' = by$ and $y'' = b^2y$; multiply and square to eliminate b without dividing by y .

11. If θ is the angle between two vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$, find $\sin \theta$. [2 marks]

Solution:

Concept:

- For nonzero vectors $\vec{u}$ and $\vec{v}$, the angle $\theta \in [0, \pi]$ satisfies $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|}$.

Quick Tip to Solve: Find the dot product first; equal vector magnitudes make the cosine denominator simply 14.

- Use $\sin^2 \theta + \cos^2 \theta = 1$. Since the angle between vectors lies between 0 and π , its sine is nonnegative.

Step 1: Compute the scalar product and magnitudes Let $\vec{u} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{v} = 3\hat{i} - 2\hat{j} + \hat{k}$. Then

$$\vec{u} \cdot \vec{v} = (1)(3) + (-2)(-2) + (3)(1) = 10.$$

Their magnitudes are

$$|\vec{u}| = \sqrt{1 + 4 + 9} = \sqrt{14}, \quad |\vec{v}| = \sqrt{9 + 4 + 1} = \sqrt{14}.$$

Both are nonzero, so the formula for the angle applies.

Step 2: Find the sine of the angle The cosine is

$$\cos \theta = \frac{10}{\sqrt{14}\sqrt{14}} = \frac{5}{7}.$$

Hence

$$\begin{aligned} \sin \theta &= \sqrt{1 - \cos^2 \theta} \\ &= \sqrt{1 - \frac{25}{49}} \\ &= \sqrt{\frac{24}{49}} = \frac{2\sqrt{6}}{7}. \end{aligned}$$

We choose the positive square root because θ is the angle between two vectors; moreover its positive cosine shows that it is acute.

Step 3: Conclusion The required value is $\sin \theta = \frac{2\sqrt{6}}{7}$.

Quick Tip to Solve: Find the dot product first; equal vector magnitudes make the cosine denominator simply 14.

12. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4. [2 marks]

Solution:

Concept:

- For events A and B with $P(B) > 0$, conditional probability is $P(A | B) = \frac{P(A \cap B)}{P(B)}$.
- Because the dice have different colours, use ordered outcomes (b, r) , where b is the black-die number and r the red-die number. The 36 outcomes are equally likely.

Step 1: Count outcomes allowed by the condition Let A be the event that the sum is 8, and let B be the event that the red die shows a number less than 4. Under B , the red die can show 1, 2, 3; for each of these three values, the black die has six possible values. Thus

$$n(B) = 3 \times 6 = 18, \quad P(B) = \frac{18}{36}.$$

Quick Tip to Solve: Restrict the sample space to the eighteen outcomes with red value 1, 2, 3, then count pairs summing to eight.

The condition restricts only the red die, not both dice.

Step 2: Count favourable outcomes within the condition For sum 8, a red value of 1 would require black 7, which is impossible. Red values 2 and 3 require black values 6 and 5, respectively. Consequently

$$A \cap B = \{(6, 2), (5, 3)\}, \quad P(A \cap B) = \frac{2}{36}.$$

Applying the conditional probability formula gives

$$P(A | B) = \frac{2/36}{18/36} = \frac{2}{18} = \frac{1}{9}.$$

Step 3: Conclusion The conditional probability is $\boxed{\frac{1}{9}}$.

Quick Tip to Solve: Restrict the sample space to the eighteen outcomes with red value 1, 2, 3, then count pairs summing to eight.

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13. Using properties of determinants, prove that

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + yz + zx).$$

[4 marks]

Solution:

Concept:

- Subtracting one row from another preserves the value of a determinant. Choosing rows with matching entries creates zeros and simplifies the expansion.
- A common factor of a row can be taken outside the determinant. Expansion along the first row uses alternating signs +, −, +.

Step 1: Subtract the first row Denote the given determinant by D . Apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, keeping the first row unchanged:

$$D = \begin{vmatrix} 1 & 1 & 1+3x \\ 3y & 0 & -3x \\ 0 & 3z & -3x \end{vmatrix}.$$

The second row comes from $(1+3y) - 1 = 3y$, $1 - 1 = 0$, and $1 - (1+3x) = -3x$. Likewise the third row becomes $(0, 3z, -3x)$. Both are row replacement operations, so neither alters the determinant or its sign.

Step 2: Take out common factors Extract a factor 3 from each of the last two rows. Their combined

Quick Tip to Solve: Subtract the first row from the other two, then extract their factors of 3 before expanding.

contribution outside the determinant is $3 \times 3 = 9$, giving

$$D = 9 \begin{vmatrix} 1 & 1 & 1+3x \\ y & 0 & -x \\ 0 & z & -x \end{vmatrix}.$$

No variable is divided out, so the transformation is valid for all values of x, y, z , including zero. The two zero entries make the minors especially simple.

Step 3: Expand along the first row Use the cofactor signs $+, -, +$:

$$\begin{aligned} D &= 9\{0(-x) - (-x)z\} \\ &\quad - [y(-x) - (-x)0] + (1+3x)[yz - 0] \\ &= 9\{xz + xy + (1+3x)yz\} \\ &= 9\{xz + xy + yz + 3xyz\}. \end{aligned}$$

The middle term is positive because its minor equals $-xy$ and its cofactor has a negative sign. Distributing $1+3x$ over yz supplies both yz and $3xyz$. Rearranging the four terms matches the required expression.

Step 4: Conclusion Thus $\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix} = 9(3xyz + xy + yz + zx)$. Hence proved.

Quick Tip to Solve: Subtract the first row from the other two, then extract their factors of 3 before expanding.

14(a). If $(x^2 + y^2)^2 = xy$, find $\frac{dy}{dx}$.

[4 marks]

Solution:

Concept:

- For an implicitly defined curve, regard y as a function of x . The chain rule gives $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$.
- The chain rule differentiates a squared expression as twice that expression times its derivative. The product rule gives $\frac{d}{dx}(xy) = y + x \frac{dy}{dx}$.

Step 1: Differentiate the equation implicitly Start with $(x^2 + y^2)^2 = xy$. Let $y' = \frac{dy}{dx}$ at a point where the curve defines a differentiable function of x . Differentiating both sides with respect to x , apply the chain rule on the left and the product rule on the right:

$$2(x^2 + y^2)(2x + 2yy') = y + xy'.$$

The derivative inside the outer square is $2x + 2yy'$, since y varies with x . On the right, differentiating just x gives the term y , while differentiating y gives xy' . Treating y as constant would omit these necessary terms.

Quick Tip to Solve: Keep a factor of dy/dx whenever differentiating a term involving y , and collect those terms before dividing.

Step 2: Collect terms containing the derivative Expanding the preceding equation and moving the derivative terms to one side gives

$$\begin{aligned}4x(x^2 + y^2) + 4y(x^2 + y^2)y' &= y + xy', \\4y(x^2 + y^2)y' - xy' &= y - 4x(x^2 + y^2), \\[4y(x^2 + y^2) - x]y' &= y - 4x(x^2 + y^2).\end{aligned}$$

Factoring out y' makes its coefficient explicit. This algebra preserves the negative sign before x in that coefficient and the negative sign before $4x(x^2 + y^2)$ in the numerator.

Step 3: Isolate the slope At any regular point where $4y(x^2 + y^2) - x \neq 0$, divide by this coefficient:

$$y' = \frac{y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}.$$

Equivalently, this equals $\frac{4x(x^2 + y^2) - y}{x - 4y(x^2 + y^2)}$, obtained by multiplying both numerator and denominator by -1 . At a point where the coefficient vanishes, division is invalid and the curve's local behaviour must be examined separately; the displayed quotient gives the ordinary finite slope wherever its denominator is nonzero.

Step 4: Conclusion $\frac{dy}{dx} = \frac{y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}$, wherever the denominator is nonzero.

Quick Tip to Solve: Keep a factor of dy/dx whenever differentiating a term involving y , and collect those terms before dividing.

OR

14(b). If $x = a(2\theta - \sin 2\theta)$ and $y = a(1 - \cos 2\theta)$, find $\frac{dy}{dx}$ when $\theta = \frac{\pi}{3}$.

[4 marks]

Solution:

Concept:

- For parametric equations $x = x(\theta)$, $y = y(\theta)$, the chain rule gives $dy/dx = (dy/d\theta)/(dx/d\theta)$, provided $dx/d\theta \neq 0$.
- Use $\sin 2\theta = 2 \sin \theta \cos \theta$ and $1 - \cos 2\theta = 2 \sin^2 \theta$. The parameter a is a constant.

Step 1: Differentiate both coordinates Differentiate both given expressions with respect to the same parameter θ . The chain rule contributes a factor 2 when differentiating the angle 2θ :

$$\begin{aligned}\frac{dx}{d\theta} &= a(2 - 2 \cos 2\theta) = 2a(1 - \cos 2\theta), \\ \frac{dy}{d\theta} &= 2a \sin 2\theta.\end{aligned}$$

These are the rates of change of the two coordinates with the parameter. Their ratio gives the rate of change of y with x , as required by the question.

Step 2: Form the ratio and simplify Apply parametric differentiation and cancel the common factor

Quick Tip to Solve: Cancel $2a$ first, then use $\sin 2\theta/(1 - \cos 2\theta) = \cot \theta$ before substituting the angle.

2a:

$$\frac{dy}{dx} = \frac{2a \sin 2\theta}{2a(1 - \cos 2\theta)} = \frac{\sin 2\theta}{1 - \cos 2\theta}.$$

The double-angle identities express the numerator and denominator in terms of θ :

$$\frac{dy}{dx} = \frac{2 \sin \theta \cos \theta}{2 \sin^2 \theta} = \cot \theta.$$

This simplification is valid at $\theta = \pi/3$ since $\sin(\pi/3) \neq 0$. As usual for this parametrized curve, $a \neq 0$; otherwise both coordinates are constant and there is no slope of a curve to compute.

Step 3: Substitute the specified angle At $\theta = \pi/3$, the original denominator $dx/d\theta$ equals $2a[1 - \cos(2\pi/3)] = 3a$, so the slope exists. The standard trigonometric values now give

$$\left. \frac{dy}{dx} \right|_{\theta=\pi/3} = \frac{\cos(\pi/3)}{\sin(\pi/3)} = \frac{1/2}{\sqrt{3}/2}.$$

Thus the parameter a cancels and the slope depends only on the specified angle.

Step 4: Conclusion Therefore, $\left. \frac{dy}{dx} \right|_{\theta=\pi/3} = \frac{1}{\sqrt{3}}.$

Quick Tip to Solve: Cancel $2a$ first, then use $\sin 2\theta/(1 - \cos 2\theta) = \cot \theta$ before substituting the angle.

15. If $y = \sin(\sin x)$, prove that $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0$.

[4 marks]

Solution:

Concept:

- The chain rule differentiates a composite function: if $y = \sin u$, then $dy/dx = \cos u (du/dx)$. Here the inner function is $u = \sin x$.
- The product rule is $\frac{d}{dx}(fg) = f'g + fg'$. Use it when differentiating the product obtained for the first derivative.
- Since $\tan x = \sin x / \cos x$, the proposed identity is considered at points where $\cos x \neq 0$, so that every term is defined.

Step 1: Calculate the first derivative The sine function is applied to $\sin x$, so we differentiate the outer sine and then multiply by the derivative of its argument. This gives

$$\frac{dy}{dx} = \cos(\sin x) \cos x.$$

Keep the two factors separate because both depend on x . In particular, $\cos(\sin x)$ must itself be differentiated by the chain rule in the next step.

Step 2: Calculate the second derivative Apply the product rule to $\cos(\sin x) \cos x$. The derivative of

Quick Tip to Solve: After finding $y' = \cos(\sin x) \cos x$, simplify $\tan x y'$ first; it cancels the product-rule term containing $\sin x$.

the first factor is $-\sin(\sin x) \cos x$, while that of the second is $-\sin x$. Hence

$$\begin{aligned}\frac{d^2y}{dx^2} &= [-\sin(\sin x) \cos x] \cos x + \cos(\sin x)(-\sin x) \\ &= -\sin(\sin x) \cos^2 x - \cos(\sin x) \sin x.\end{aligned}$$

The two negative signs arise from differentiating cosine in each factor. Recalling $y = \sin(\sin x)$, the first term is also $-y \cos^2 x$.

Step 3: Substitute and cancel matching terms First simplify the middle term of the required expression using the first derivative:

$$\tan x \frac{dy}{dx} = \frac{\sin x}{\cos x} \cos(\sin x) \cos x = \sin x \cos(\sin x).$$

Substituting the expressions for y , its first derivative and its second derivative yields

$$\begin{aligned}\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x &= -\sin(\sin x) \cos^2 x - \cos(\sin x) \sin x \\ &\quad + \sin x \cos(\sin x) + \sin(\sin x) \cos^2 x \\ &= 0.\end{aligned}$$

Each negative term has an equal positive counterpart, proving the required cancellation.

Step 4: Conclusion Hence $\frac{d^2y}{dx^2} + \tan x \frac{dy}{dx} + y \cos^2 x = 0$, wherever $\cos x \neq 0$, as required.

Quick Tip to Solve: After finding $y' = \cos(\sin x) \cos x$, simplify $\tan x y'$ first; it cancels the product-rule term containing $\sin x$.

16(a). Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1) , where $x_1 = 2$ and $y_1 > 0$. [4 marks]

Solution:

Concept:

- A point of contact must satisfy the curve equation. A sign condition selects the required coordinate when substitution gives two possible values.
- Implicit differentiation treats y as a function of x . The derivative dy/dx evaluated at the point of contact gives the tangent slope.
- A line through (x_1, y_1) with slope m is $y - y_1 = m(x - x_1)$. The normal slope is the negative reciprocal of a finite nonzero tangent slope.

Step 1: Find the point of contact Substituting $x_1 = 2$ in the ellipse equation gives

$$16(2)^2 + 9y_1^2 = 145 \implies 64 + 9y_1^2 = 145.$$

Therefore $9y_1^2 = 81$, or $y_1^2 = 9$. The two roots are $y_1 = \pm 3$. Because the question requires $y_1 > 0$,

Quick Tip to Solve: Select the positive ordinate before finding slopes, then take the negative reciprocal of the tangent slope to obtain the normal slope.

select $y_1 = 3$. Thus the required point is $P = (2, 3)$, and the slopes must be calculated at this point.

Step 2: Calculate the tangent slope Differentiate both sides with respect to x , applying the chain rule to y^2 :

$$32x + 18y \frac{dy}{dx} = 0, \quad \frac{dy}{dx} = -\frac{16x}{9y}.$$

At P , this gives $m_t = -16(2)/(9(3)) = -32/27$. The denominator is nonzero, so a finite tangent slope exists at the specified point.

Step 3: Write the tangent equation Apply the point-slope form using $P = (2, 3)$:

$$y - 3 = -\frac{32}{27}(x - 2).$$

Multiplication by 27 gives $27y - 81 = -32x + 64$, and hence $32x + 27y = 145$. Substitution of the contact point gives $32(2) + 27(3) = 64 + 81 = 145$, confirming that the line passes through P .

Step 4: Write and check the normal equation The normal passes through the same point and is perpendicular to the tangent. Thus

$$m_n = -\frac{1}{m_t} = \frac{27}{32}, \quad y - 3 = \frac{27}{32}(x - 2).$$

On multiplying by 32, we get $32y - 96 = 27x - 54$, so the normal equation is $27x - 32y + 42 = 0$. Its slope product with the tangent is $(-32/27)(27/32) = -1$, confirming perpendicularity. Substitution of P gives $54 - 96 + 42 = 0$, so this line also passes through the required point.

Step 5: Conclusion The tangent is $\boxed{32x + 27y = 145}$, and the normal is $\boxed{27x - 32y + 42 = 0}$, both at $(2, 3)$.

Quick Tip to Solve: Select the positive ordinate before finding slopes, then take the negative reciprocal of the tangent slope to obtain the normal slope.

OR

16(b). Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is (a) strictly increasing, (b) strictly decreasing. [4 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval where its derivative is positive, and strictly decreasing on an interval where its derivative is negative.
- The real zeros of the derivative divide the real line into intervals. Test the sign on each interval; a continuous derivative cannot change sign there without a zero.
- Factoring the derivative makes its sign transparent. Each simple linear factor changes sign at its root, so the product changes sign across each distinct simple root.

Step 1: Differentiate the given polynomial The function is a polynomial and is differentiable for all

Quick Tip to Solve: Factor the derivative as $(x + 3)(x - 2)(x - 4)$; start with a positive sign to the right of 4 and alternate across its three simple roots.

real x . Differentiating term by term,

$$\begin{aligned}f'(x) &= \frac{d}{dx} \left(\frac{x^4}{4} - x^3 - 5x^2 + 24x + 12 \right) \\&= x^3 - 3x^2 - 10x + 24.\end{aligned}$$

The constant term contributes zero. Monotonicity is determined by the sign of this cubic, rather than by the sign of the original function.

Step 2: Factor the derivative and locate its zeros Substitution of $x = 2$ in the derivative gives $8 - 12 - 20 + 24 = 0$, so $x - 2$ is a factor. Dividing the cubic by $x - 2$ and factoring the quotient gives

$$f'(x) = (x - 2)(x^2 - x - 12) = (x + 3)(x - 2)(x - 4).$$

Thus the critical numbers, in increasing order, are $-3, 2, 4$. They divide the real line into four intervals on each of which every factor has a fixed nonzero sign.

Step 3: Determine the derivative sign on each interval For $x < -3$, all three factors are negative, so their product is negative. For $-3 < x < 2$, the signs of $x + 3, x - 2, x - 4$ are respectively positive, negative and negative; their product is positive. For $2 < x < 4$, the signs are positive, positive and negative, giving a negative product. For $x > 4$, all three factors are positive and the product is positive. The complete sign calculation is

x	$(-\infty, -3)$	$(-3, 2)$	$(2, 4)$	$(4, \infty)$
$x + 3$	−	+	+	+
$x - 2$	−	−	+	+
$x - 4$	−	−	−	+
$f'(x)$	−	+	−	+

Step 4: Apply the monotonicity criterion Since $f'(x) > 0$ throughout $(-3, 2)$ and $(4, \infty)$, the function is strictly increasing on each of these intervals. Since $f'(x) < 0$ throughout $(-\infty, -3)$ and $(2, 4)$, it is strictly decreasing on each of those intervals. At the critical numbers themselves the derivative equals zero. The usual derivative-sign answer lists open intervals; the isolated zero slopes do not constitute additional intervals of constant behaviour.

Step 5: Conclusion (a) Strictly increasing on $(-3, 2)$ and $(4, \infty)$. (b) Strictly decreasing on $(-\infty, -3)$ and $(2, 4)$.

Quick Tip to Solve: Factor the derivative as $(x + 3)(x - 2)(x - 4)$; start with a positive sign to the right of 4 and alternate across its three simple roots.

17. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question? [4 marks]

Solution:

Concept:

- A square-based tank of width $x > 0$ and depth $h > 0$ holds volume $V = x^2h$, where $V > 0$ is fixed.
- An open tank has one square base and four rectangular sides, so its sheet area is $A = x^2 + 4xh$. Uniform cost per unit area makes minimum cost equivalent to minimum area.

Step 1: Use the volume constraint Let $k > 0$ be the constant cost per unit area of sheet. Since the quantity of water to be held is fixed, $x^2h = V$, and hence $h = V/x^2$. The tank has no lid. Its base requires area x^2 , while each of its four sides requires area xh . Therefore

$$A(x) = x^2 + 4x \frac{V}{x^2} = x^2 + \frac{4V}{x}, \quad x > 0.$$

The total material cost is $kA(x)$. Multiplication by the positive constant k leaves the minimizing dimensions unchanged.

Step 2: Minimize the required area Differentiating the area with respect to the width gives

$$A'(x) = 2x - \frac{4V}{x^2}.$$

A stationary point therefore satisfies

$$2x - \frac{4V}{x^2} = 0 \implies x^3 = 2V.$$

Further,

$$A''(x) = 2 + \frac{8V}{x^3} > 0 \quad \text{for every } x > 0.$$

Thus the stationary point is a minimum. It is the absolute minimum: the derivative is negative for $x^3 < 2V$ and positive for $x^3 > 2V$, so the area decreases up to this width and increases thereafter. The area also tends to infinity at both ends of the permissible domain.

Step 3: Obtain the depth and explain the value At the minimizing width, $V = x^3/2$. Substitution into the volume constraint gives

$$h = \frac{V}{x^2} = \frac{x^3/2}{x^2} = \frac{x}{2}.$$

Choosing these dimensions avoids unnecessary material expenditure for the nearby lower income families who bear the tank's cost. It illustrates social concern for economically weaker people, economy in construction, and responsible use of their limited financial resources. Water can be provided with the least material expense for the stipulated capacity.

Step 4: Conclusion The cost of material is least when $h = x/2$, so the depth is half the width. The hidden value is concern for lower income families through economical use of resources and avoidance of waste.

Quick Tip to Solve: Count one base and four sides for an open tank, then eliminate the depth using the fixed volume before differentiating.

18. Find : $\int \frac{2 \cos x}{(1 - \sin x)(1 + \sin^2 x)} dx.$

[4 marks]

Solution:

Concept:

- When an integrand contains $\sin x$ together with $\cos x dx$, use $t = \sin x$, so $dt = \cos x dx$.
- A rational function with denominator $(1 - t)(1 + t^2)$ can be split into a constant numerator over the linear factor and a linear numerator over the irreducible quadratic.
- The standard antiderivatives are $\int du/u = \ln |u|$ and $\int dt/(1 + t^2) = \tan^{-1} t$.

Step 1: Substitute the sine function Denote the requested integral by I . Set $t = \sin x$, so that $dt = \cos x dx$. Then

$$I = \int \frac{2}{(1 - t)(1 + t^2)} dt.$$

Work on an interval where $\sin x \neq 1$, so the integrand is defined.

Step 2: Find the partial fractions Write

$$\frac{2}{(1 - t)(1 + t^2)} = \frac{A}{1 - t} + \frac{Bt + D}{1 + t^2}.$$

Clearing denominators and expanding gives

$$\begin{aligned} 2 &= A(1 + t^2) + (Bt + D)(1 - t) \\ &= (A - B)t^2 + (B - D)t + (A + D). \end{aligned}$$

Equating coefficients yields $A - B = 0$, $B - D = 0$ and $A + D = 2$. Hence $A = B = D = 1$. Therefore

$$I = \int \left(\frac{1}{1 - t} + \frac{t}{1 + t^2} + \frac{1}{1 + t^2} \right) dt.$$

Step 3: Integrate and return to the original variable Since $d(1 - t)/dt = -1$ and $d(1 + t^2)/dt = 2t$, integration gives

$$I = -\ln |1 - t| + \frac{1}{2} \ln(1 + t^2) + \tan^{-1} t + C.$$

Check the antiderivative by differentiation with respect to t :

$$\frac{1}{1 - t} + \frac{t + 1}{1 + t^2} = \frac{1 + t^2 + (t + 1)(1 - t)}{(1 - t)(1 + t^2)} = \frac{2}{(1 - t)(1 + t^2)}.$$

Multiplying by $dt/dx = \cos x$ recovers the original integrand. Finally substitute $t = \sin x$.

Step 4: Conclusion

$$I = -\ln |1 - \sin x| + \frac{1}{2} \ln(1 + \sin^2 x) + \tan^{-1}(\sin x) + C.$$

Quick Tip to Solve: Use the numerator cosine to substitute sine, then check the partial fractions by combining their numerators to get exactly 2.

19(a). Find the particular solution of the differential equation $e^x \tan y \, dx + (2 - e^x) \sec^2 y \, dy = 0$, given that $y = \frac{\pi}{4}$ when $x = 0$. [4 marks]

Solution:

Concept:

- If $v = \tan y$, then $dv = \sec^2 y \, dy$. This substitution simplifies the two trigonometric expressions appearing together.
- In a separable first-order equation, collect the dependent variable on one side and integrate. The initial condition then determines the arbitrary constant.

Step 1: Transform the equation The given equation is

$$e^x \tan y \, dx + (2 - e^x) \sec^2 y \, dy = 0.$$

Put $v = \tan y$, so that $dv = \sec^2 y \, dy$. We obtain

$$e^x v \, dx + (2 - e^x) \, dv = 0.$$

At the prescribed initial point, $v = \tan(\pi/4) = 1$, while $2 - e^0 = 1$. These quantities are nonzero near that point, so we may divide by them to separate the variables.

Step 2: Separate and integrate Rearranging gives

$$\frac{dv}{v} = -\frac{e^x}{2 - e^x} \, dx.$$

Set $u = 2 - e^x$ in the right-hand integral. Since $du = -e^x \, dx$, integration gives

$$\ln |v| = \ln |2 - e^x| + C_1.$$

Exponentiating and absorbing the constant and its sign into C , we get

$$v = C(2 - e^x), \quad \tan y = C(2 - e^x).$$

The negative sign in the separated equation is crucial: it matches the derivative of the denominator $2 - e^x$. The integrated relation is obtained on an interval around the initial point.

Step 3: Determine the constant and verify Apply $x = 0$ and $y = \pi/4$:

$$1 = C(2 - 1), \quad C = 1.$$

Differentiate the resulting relation to check it:

$$\sec^2 y \frac{dy}{dx} = -e^x.$$

On substitution in the original differential equation divided by dx , its left-hand side becomes

$$e^x(2 - e^x) + (2 - e^x)(-e^x) = 0.$$

Quick Tip to Solve: Substitute $v = \tan y$ at once: $\sec^2 y \, dy$ becomes dv , and the initial condition simplifies to $v(0) = 1$.

The relation therefore satisfies the differential equation. On the branch through the specified point, its value at $x = 0$ is $y = \pi/4$, so it also satisfies the initial condition. Although separation excluded $2 - e^x = 0$, the resulting explicit solution remains differentiable there and satisfies the original equation by the same check.

Step 4: Conclusion The required particular solution is

$$\tan y = 2 - e^x.$$

Equivalently, on the branch containing the initial point,

$$y = \tan^{-1}(2 - e^x).$$

Quick Tip to Solve: Substitute $v = \tan y$ at once: $\sec^2 y \, dy$ becomes dv , and the initial condition simplifies to $v(0) = 1$.

OR

19(b). Find the particular solution of the differential equation $\frac{dy}{dx} + 2y \tan x = \sin x$, given that $y = 0$ when $x = \frac{\pi}{3}$. [4 marks]

Solution:

Concept:

- For a linear differential equation $y' + P(x)y = Q(x)$, the integrating factor is $I = e^{\int P(x) \, dx}$, and $(Iy)' = IQ(x)$.
- Here $P(x) = 2 \tan x$ and $Q(x) = \sin x$. Use $\int \tan x \, dx = -\ln |\cos x|$ and $(\sec x)' = \sec x \tan x$.

Step 1: Find the integrating factor The equation

$$\frac{dy}{dx} + 2y \tan x = \sin x$$

is already in standard linear form. Its coefficient of y is $2 \tan x$; it is not necessary to separate the variables. The integrating factor is

$$I = e^{\int 2 \tan x \, dx} = e^{-2 \ln |\cos x|} = \sec^2 x.$$

Work on the interval $-\pi/2 < x < \pi/2$, which contains the initial point $x = \pi/3$ and on which the coefficients and integrating factor are defined.

Step 2: Multiply and integrate Multiplying the equation by $\sec^2 x$ gives

$$\sec^2 x \frac{dy}{dx} + 2y \sec^2 x \tan x = \sin x \sec^2 x.$$

Quick Tip to Solve: After using the integrating factor $\sec^2 x$, recognise $\sin x \sec^2 x$ as the derivative of $\sec x$.

Since $(\sec^2 x)' = 2 \sec^2 x \tan x$, the product rule identifies the left-hand side as $(y \sec^2 x)'$. Thus

$$y \sec^2 x = \int \frac{\sin x}{\cos^2 x} dx = \sec x + C.$$

For the integral one may set $u = \cos x$, $du = -\sin x dx$: then $-\int u^{-2} du = u^{-1}$. This confirms the positive sign of $\sec x$.

Step 3: Apply the initial condition At $x = \pi/3$, $y = 0$ and $\sec(\pi/3) = 2$. Hence

$$0 = 2 + C, \quad C = -2.$$

The relation becomes $y \sec^2 x = \sec x - 2$. Multiplication by $\cos^2 x$ yields $y = \cos x - 2 \cos^2 x$.

Step 4: Check the differential equation Differentiation of this expression gives

$$y' = -\sin x + 4 \sin x \cos x.$$

Also,

$$2y \tan x = 2 \sin x - 4 \sin x \cos x.$$

Adding these two expressions gives $y' + 2y \tan x = \sin x$, exactly as required. At $x = \pi/3$, the expression for y gives $1/2 - 2(1/4) = 0$, verifying the initial condition as well.

Step 5: Conclusion The particular solution is $y = \cos x - 2 \cos^2 x$, on the interval containing $\pi/3$ where $\tan x$ is defined.

Quick Tip to Solve: After using the integrating factor $\sec^2 x$, recognise $\sin x \sec^2 x$ as the derivative of $\sec x$.

20. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$, $\vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d} \cdot \vec{a} = 21$. [4 marks]

Solution:

Concept:

- The cross product of two non-parallel vectors is perpendicular to both. Thus any vector perpendicular to both $\vec{c}$ and $\vec{b}$ is a scalar multiple of $\vec{c} \times \vec{b}$.
- For $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$ and $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$, their dot product is $u_1v_1 + u_2v_2 + u_3v_3$. The given dot product fixes the scalar multiple.

Step 1: Construct a common perpendicular vector Using the given components, expand the cross product along the first row:

$$\begin{aligned} \vec{c} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -1 \\ 1 & -4 & 5 \end{vmatrix} \\ &= \hat{i}(5 - 4) - \hat{j}(15 + 1) + \hat{k}(-12 - 1) \\ &= \hat{i} - 16\hat{j} - 13\hat{k}. \end{aligned}$$

Quick Tip to Solve: Find a cross product first, then determine its scalar multiple from the dot product; check the middle-component sign carefully.

This cross product is nonzero, so the two given vectors are non-parallel and their common perpendicular direction is well defined. Put $\vec{d} = t(\hat{i} - 16\hat{j} - 13\hat{k})$, where t is a real scalar.

Step 2: Use the prescribed dot product Substitute $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k}$ into the condition:

$$\begin{aligned}\vec{d} \cdot \vec{a} &= t\{4 - 80 + 13\} = -63t, \\ -63t &= 21, \quad t = -\frac{1}{3}.\end{aligned}$$

The negative scalar reverses the chosen cross-product direction and sets the required magnitude. No unit-vector normalization is needed because the question specifies a dot product, not a unit vector.

Step 3: Verify all three requirements For the resulting components, direct substitution gives

$$\begin{aligned}\vec{d} \cdot \vec{c} &= \frac{-3 + 16 - 13}{3} = 0, \\ \vec{d} \cdot \vec{b} &= \frac{-1 - 64 + 65}{3} = 0, \\ \vec{d} \cdot \vec{a} &= \frac{-4 + 80 - 13}{3} = 21.\end{aligned}$$

Thus both perpendicularity conditions and the specified dot product hold simultaneously.

Step 4: Conclusion The required vector is $\boxed{\vec{d} = -\frac{1}{3}\hat{i} + \frac{16}{3}\hat{j} + \frac{13}{3}\hat{k}}.$

Quick Tip to Solve: Find a cross product first, then determine its scalar multiple from the dot product; check the middle-component sign carefully.

21. Find the shortest distance between the lines

$$\vec{r} = (4\hat{i} - \hat{j}) + \lambda(\hat{i} + 2\hat{j} - 3\hat{k}) \text{ and } \vec{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(2\hat{i} + 4\hat{j} - 5\hat{k}).$$

[4 marks]

Solution:

Concept:

- For lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ with non-parallel direction vectors, a common perpendicular has direction $\vec{b}_1 \times \vec{b}_2$.
- The shortest distance is the absolute projection of the joining vector $\vec{a}_2 - \vec{a}_1$ onto that common perpendicular: $D = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}.$

Step 1: Identify points and direction vectors Reading the constant and parameter coefficients separately gives

$$\begin{aligned}\vec{a}_1 &= (4, -1, 0), \quad \vec{b}_1 = (1, 2, -3), \\ \vec{a}_2 &= (1, -1, 2), \quad \vec{b}_2 = (2, 4, -5).\end{aligned}$$

The displacement from the chosen point on the first line to the chosen point on the second line is

$$\vec{a}_2 - \vec{a}_1 = (-3, 0, 2).$$

Quick Tip to Solve: Subtract the position vectors before taking the scalar triple product, and divide its absolute value by the cross-product magnitude.

The direction vectors are not proportional: their first two components have ratio 2, but the third components do not have that ratio.

Step 2: Compute the common perpendicular direction Expanding the determinant gives

$$\begin{aligned}\vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} \\ &= \hat{i}(-10 + 12) - \hat{j}(-5 + 6) + \hat{k}(4 - 4) \\ &= 2\hat{i} - \hat{j}.\end{aligned}$$

Its magnitude is $\sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}$. The nonzero cross product confirms that the distance formula for non-parallel lines applies.

Step 3: Project the joining vector onto the perpendicular The scalar triple product in the numerator is

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = (-3)(2) + 0(-1) + 2(0) = -6.$$

Its nonzero value also shows that the lines are skew, rather than intersecting. The component of any joining vector along a common perpendicular is unchanged when either endpoint moves along its own line, since both line directions have zero dot product with that perpendicular. Therefore this projection gives the shortest separation:

$$D = \frac{|-6|}{\sqrt{5}} = \frac{6}{\sqrt{5}}.$$

Taking the absolute value is necessary because distance is nonnegative and does not depend on the chosen order of the lines.

Step 4: Conclusion The shortest distance is $\boxed{\frac{6}{\sqrt{5}} = \frac{6\sqrt{5}}{5} \text{ units}}.$

Quick Tip to Solve: Subtract the position vectors before taking the scalar triple product, and divide its absolute value by the cross-product magnitude.

22. Suppose a girl throws a die. If she gets 1 or 2, she tosses a coin three times and notes the number of tails. If she gets 3, 4, 5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3, 4, 5 or 6 with the die? [4 marks]

Solution:

Concept:

- Bayes' theorem gives $P(B | T) = P(B)P(T | B)/P(T)$, the probability of a branch after an observation.
- In three independent tosses of a fair coin, each sequence has probability $(1/2)^3$; exactly one tail occurs in three sequences.

Quick Tip to Solve: Weight each coin likelihood by its die probability before conditioning on the observed tail.

Step 1: Identify the branches and the observed event Let A mean that the die shows 1 or 2, and let B mean that it shows 3, 4, 5 or 6. Let T mean that exactly one tail is obtained in the coin tosses performed. In the standard fair die-and-coin experiment,

$$P(A) = \frac{2}{6} = \frac{1}{3}, \quad P(B) = \frac{4}{6} = \frac{2}{3}.$$

These branches are mutually exclusive and exhaustive. The information about the tail is already given, so the required probability is $P(B | T)$, rather than the original probability $P(B)$.

Step 2: Find the conditional coin probabilities In branch A , three coin tosses are made. The eight equally likely sequences include exactly three with one tail: THH, HTH and HHT. Thus

$$P(T | A) = \binom{3}{1} \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^2 = \frac{3}{8}.$$

In branch B , only one toss is made. Exactly one tail means that this toss shows a tail, so

$$P(T | B) = \frac{1}{2}.$$

The different numbers of tosses explain why the two likelihoods must be calculated separately before combining them.

Step 3: Use total probability and then condition The event T can arise through either branch. The theorem of total probability yields

$$P(T) = P(A)P(T | A) + P(B)P(T | B) = \frac{1}{3} \cdot \frac{3}{8} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{8} + \frac{1}{3} = \frac{11}{24}.$$

Also, $P(B \cap T) = P(B)P(T | B) = (2/3)(1/2) = 1/3$. By Bayes' theorem,

$$P(B | T) = \frac{P(B \cap T)}{P(T)} = \frac{1/3}{11/24} = \frac{8}{11}.$$

Dividing the desired branch's contribution by the total probability of the observation accounts for every possible way that exactly one tail could have occurred.

Step 4: Conclusion The required probability is $\boxed{\frac{8}{11}}$.

Quick Tip to Solve: Weight each coin likelihood by its die probability before conditioning on the observed tail.

23. Two numbers are selected at random (without replacement) from the first five positive integers. Let X denote the larger of the two numbers obtained. Find the mean and variance of X . [4 marks]

Solution:

Concept:

- For a discrete random variable, $E(X) = \sum_x xP(X = x)$ and $E(X^2) = \sum_x x^2P(X = x)$.
- The variance is $\text{Var}(X) = E(X^2) - [E(X)]^2$. Sampling without replacement gives two distinct

Quick Tip to Solve: For maximum k in a pair without replacement, count the $k-1$ possible smaller partners.

integers.

Step 1: Construct the equally likely sample space The first five positive integers are 1, 2, 3, 4, 5. Since the two selected integers are distinct and their order does not affect the larger number, we use unordered pairs. There are $\binom{5}{2} = 10$ equally likely pairs:

$$(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), \\ (2, 4), (2, 5), (3, 4), (3, 5), (4, 5).$$

If selection is viewed as successive draws, each unordered pair corresponds to two equally likely ordered draws, so it still has probability $2/(5 \cdot 4) = 1/10$. Because replacement is excluded, the larger number cannot be 1.

Step 2: Obtain the probability distribution of the larger number For $X = k$, the pair must contain k and one integer from $1, 2, \dots, k - 1$. There are $k - 1$ such pairs. Hence $P(X = k) = (k - 1)/10$ for $k = 2, 3, 4, 5$. The distribution is

x	2	3	4	5
$P(X = x)$	1/10	2/10	3/10	4/10

The probabilities sum to $(1 + 2 + 3 + 4)/10 = 1$, confirming that all possibilities have been included. The values of X are not equally likely: a larger maximum allows more choices for the other integer.

Step 3: Calculate the mean and the second moment Weighting each possible value by its probability gives

$$E(X) = \frac{2 \cdot 1 + 3 \cdot 2 + 4 \cdot 3 + 5 \cdot 4}{10} = \frac{2 + 6 + 12 + 20}{10} = 4.$$

For the second moment, square the value before taking its weighted average:

$$E(X^2) = \frac{2^2 \cdot 1 + 3^2 \cdot 2 + 4^2 \cdot 3 + 5^2 \cdot 4}{10} = \frac{4 + 18 + 48 + 100}{10} = 17.$$

Step 4: Compute the variance Apply the variance identity using the mean just found:

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 17 - 4^2 = 1.$$

As a check, the weighted squared deviations from the mean are $[1(2 - 4)^2 + 2(3 - 4)^2 + 3(4 - 4)^2 + 4(5 - 4)^2]/10 = (4 + 2 + 0 + 4)/10 = 1$, agreeing with the moment calculation.

Step 5: Conclusion The mean of X is $\boxed{4}$ and its variance is $\boxed{1}$.

Quick Tip to Solve: For maximum k in a pair without replacement, count the $k-1$ possible smaller partners.

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24(a). Let $A = \{x \in \mathbb{Z} : 0 \leq x \leq 12\}$. Show that $R = \{(a, b) : a, b \in A, |a - b| \text{ is divisible by } 4\}$ is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class $[2]$. [6 marks]

Solution:

Concept:

- An equivalence relation on a set is a relation that is reflexive, symmetric and transitive. Each property must hold for every choice of the relevant elements of the set.
- For integers a, b , divisibility of $|a - b|$ by 4 is equivalent to $a - b = 4k$ for some integer k . Allowing negative integer multiples makes the transitivity proof direct.
- The equivalence class $[r]$ consists of all $x \in A$ such that $(x, r) \in R$; here these are precisely the elements having the same remainder as r on division by 4.

Step 1: Specify the underlying set and prove reflexivity The set is $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. Take any $a \in A$. Since $|a - a| = 0 = 4 \cdot 0$, the difference is divisible by 4. Thus $(a, a) \in R$ for every $a \in A$, so R is reflexive. In particular, zero is divisible by four; it is not an exception to the given rule.

Step 2: Prove symmetry Let $a, b \in A$ and suppose $(a, b) \in R$. By definition, $|a - b|$ is divisible by 4. But $|b - a| = |a - b|$, so $|b - a|$ is also divisible by 4. Hence $(b, a) \in R$. This implication holds for every related pair, proving that R is symmetric.

Step 3: Prove transitivity Let $a, b, c \in A$, with $(a, b) \in R$ and $(b, c) \in R$. There exist integers m, n such that $a - b = 4m$ and $b - c = 4n$; their signs account for either ordering of the elements. Adding gives

$$a - c = (a - b) + (b - c) = 4(m + n).$$

Since $m + n$ is an integer, $a - c$, and therefore $|a - c|$, is divisible by 4. Thus $(a, c) \in R$. Consequently R is transitive. Having verified all three properties, R is an equivalence relation on A .

Step 4: Determine the requested equivalence classes An element $x \in A$ is related to 1 exactly when $x - 1$ is a multiple of 4. Thus $x = 1 + 4k$, where $k \in \mathbb{Z}$ and $0 \leq 1 + 4k \leq 12$. The permitted integer values are $k = 0, 1, 2$, giving $x = 1, 5, 9$. Similarly, for $[2]$, write $x = 2 + 4k$. The restriction $0 \leq 2 + 4k \leq 12$ again gives $k = 0, 1, 2$, hence $x = 2, 6, 10$. Values outside these lists either have a different remainder modulo four or lie outside A .

Step 5: Conclusion R is reflexive, symmetric and transitive, hence an equivalence relation. The set of all elements related to 1 is $\{1, 5, 9\}$, and $[2] = \{2, 6, 10\}$.

Quick Tip to Solve: Group the elements of A by their remainder on division by 4; each group is one equivalence class, but still prove all three relation properties.

24(b). Show that the function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x}{x^2 + 1}$, $\forall x \in \mathbb{R}$ is neither one-one nor onto. Also, if $g : \mathbb{R} \rightarrow \mathbb{R}$ is defined as $g(x) = 2x - 1$, find $f \circ g(x)$. [6 marks]

Solution:

Concept:

- A function is one-one if equal outputs imply equal inputs. To disprove this property, it is sufficient to exhibit two distinct domain elements with the same image.
- A function onto $\mathbb{R}$ must attain every real number. One real number in the codomain having no preimage proves that the function is not onto.
- Composition means $(f \circ g)(x) = f(g(x))$: substitute the entire expression for $g(x)$ into every occurrence of the input of f .

Step 1: Check the domain and disprove the one-one property For every real x , $x^2 + 1 > 0$, so the displayed formula defines f on all of $\mathbb{R}$. Choose the distinct inputs 2 and $\frac{1}{2}$. Their images are

$$f(2) = \frac{2}{4+1} = \frac{2}{5}, \quad f\left(\frac{1}{2}\right) = \frac{\frac{1}{2}}{\frac{1}{4}+1} = \frac{2}{5}.$$

Thus $f(2) = f(\frac{1}{2})$ although $2 \neq \frac{1}{2}$. The defining implication for a one-one function therefore fails.

Step 2: Disprove the onto property The real number 1 belongs to the stated codomain $\mathbb{R}$. If it had a preimage, some real x would satisfy

$$\frac{x}{x^2 + 1} = 1 \implies x^2 - x + 1 = 0.$$

The discriminant is $(-1)^2 - 4(1)(1) = -3 < 0$, so this equation has no real solution. Therefore 1 is not in the image of f , and f is not onto $\mathbb{R}$.

Step 3: Substitute the inner function to obtain the composition Since $g(x) = 2x - 1$ is real for every real x , its output always lies in the domain of f . Put $u = g(x) = 2x - 1$ in $f(u) = u/(u^2 + 1)$. This gives

$$\begin{aligned} (f \circ g)(x) &= f(2x - 1) \\ &= \frac{2x - 1}{(2x - 1)^2 + 1} \\ &= \frac{2x - 1}{4x^2 - 4x + 2}. \end{aligned}$$

The denominator is $(2x - 1)^2 + 1 \geq 1$, so the composition is defined for every real x .

Step 4: Conclusion The function f is neither one-one nor onto $\mathbb{R}$. For all $x \in \mathbb{R}$, the composition is

$$(f \circ g)(x) = \frac{2x - 1}{4x^2 - 4x + 2}.$$

Quick Tip to Solve: For $x/(x^2 + 1)$, try reciprocal nonzero inputs to find equal images; to test onto $\mathbb{R}$, check whether the codomain value 1 is attained.

25(a). If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, find A^{-1} . Use it to solve the system of equations

$$2x - 3y + 5z = 11,$$

$$3x + 2y - 4z = -5,$$

$$x + y - 2z = -3.$$

[6 marks]

Solution:

Concept:

- A square matrix A is invertible when $|A| \neq 0$. Its inverse is $A^{-1} = \frac{\text{adj } A}{|A|}$, where the adjugate is the transpose of the cofactor matrix.
- For $AX = B$, the column vector of unknowns is $X = A^{-1}B$. The entries in each row of A must follow the same order x, y, z as in X .

Step 1: Calculate the determinant Expanding along the first row, retain the alternating signs of the cofactors:

$$\begin{aligned} |A| &= 2\{2(-2) - (-4)(1)\} + 3\{3(-2) - (-4)(1)\} \\ &\quad + 5\{3(1) - 2(1)\} \\ &= 2(0) + 3(-2) + 5(1) = -1. \end{aligned}$$

Since the determinant is nonzero, the inverse exists and the system has a unique solution.

Step 2: Obtain the cofactors and the inverse If $C_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the minor obtained by deleting row i and column j , the nine cofactors are

$$\begin{aligned} C_{11} &= 0, & C_{12} &= 2, & C_{13} &= 1, \\ C_{21} &= -1, & C_{22} &= -9, & C_{23} &= -5, \\ C_{31} &= 2, & C_{32} &= 23, & C_{33} &= 13. \end{aligned}$$

For example, $C_{32} = -\{2(-4) - 5(3)\} = 23$, and $C_{33} = 2(2) - (-3)(3) = 13$. Transposing the cofactor matrix and dividing by the determinant gives

$$\text{adj } A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}.$$

Step 3: Apply the inverse to the system The given equations are $AX = B$, with $X = \begin{bmatrix} x & y & z \end{bmatrix}^T$ and

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; then check the resulting values in all three original equations.

$B = \begin{bmatrix} 11 & -5 & -3 \end{bmatrix}^T$. Multiplication yields

$$\begin{aligned} X &= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} -5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}. \end{aligned}$$

As a check, substituting into the original left sides gives $2 - 6 + 15 = 11$, $3 + 4 - 12 = -5$ and $1 + 2 - 6 = -3$, respectively, matching all three right sides.

Step 4: Conclusion $A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$, and the unique solution is $x = 1, y = 2, z = 3$.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; then check the resulting values in all three original equations.

OR

25(b). Using elementary row transformations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$.
[6 marks]

Solution:

Concept:

- To find an inverse by elementary row transformations, augment the matrix with the identity matrix and reduce $[A \mid I]$ to $[I \mid A^{-1}]$.
- Apply each row operation to the entire augmented row, including the right block. This keeps the right block equal to the cumulative elementary transformations applied to A .

Step 1: Form the augmented matrix Start with the given matrix on the left and the identity on the right:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 7 & 0 & 1 & 0 \\ -2 & -4 & -5 & 0 & 0 & 1 \end{array} \right].$$

The leading entry in the first row is already 1. Use this row to remove the first-column entries below it. There is no need to divide a row or introduce fractions.

Step 2: Eliminate entries beneath the first pivot Apply $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 + 2R_1$. These

Quick Tip to Solve: Here the first elimination immediately gives a third-row pivot of 1; use that row to clear the third column before clearing the second column.

operations leave the first row unchanged and give

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right].$$

In particular, the third row in the left block is $(0, 0, 1)$, since $-5 + 2(3) = 1$. Its right block is $(2, 0, 1)$, obtained by applying the very same operation to the identity block. All three pivot entries are now 1.

Step 3: Clear the third column above its pivot Now apply $R_2 \rightarrow R_2 - R_3$ and $R_1 \rightarrow R_1 - 3R_3$. Both use the current third row, which is unchanged by these operations. The augmented matrix becomes

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & -5 & 0 & -3 \\ 0 & 1 & 0 & -4 & 1 & -1 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right].$$

For example, the right block of the first row is $(1, 0, 0) - 3(2, 0, 1) = (-5, 0, -3)$. The third column in the left block now agrees with that of the identity matrix.

Step 4: Complete the reduction and check the inverse The only remaining off-diagonal entry is 2 in the first row and second column. Using $R_1 \rightarrow R_1 - 2R_2$ gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -2 & -1 \\ 0 & 1 & 0 & -4 & 1 & -1 \\ 0 & 0 & 1 & 2 & 0 & 1 \end{array} \right].$$

Hence the right block is the required inverse. To verify the calculation independently, multiplication by the original matrix gives

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus the row transformations have indeed produced an inverse.

Step 5: Conclusion The required inverse is $A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$.

Quick Tip to Solve: Here the first elimination immediately gives a third-row pivot of 1; use that row to clear the third column before clearing the second column.

26. Using integration, find the area of the region in the first quadrant enclosed by the x -axis, the line $y = x$ and the circle $x^2 + y^2 = 32$. [6 marks]

Solution:

Concept:

- The area between an upper curve $y = f(x)$ and the x -axis over $[a, b]$ is $\int_a^b f(x) dx$, provided $f(x) \geq 0$.
- In the first quadrant, the circle $x^2 + y^2 = 32$ has upper boundary $y = \sqrt{32 - x^2}$. Where the boundary changes from the line to the circle, the area integral must be split.
- For $a > 0$, $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}(x/a) + C$.

Step 1: Locate the boundary intersections The line $y = x$ meets the x -axis at $O = (0, 0)$. Its first-quadrant intersection with the circle satisfies

$$x^2 + x^2 = 32 \implies x = 4, \quad y = 4.$$

Denote this point by $P = (4, 4)$. The positive x -axis meets the circle at $Q = (4\sqrt{2}, 0)$. Thus the enclosed boundary consists of the line segment OP , the circular arc PQ , and the axis segment QO .

Step 2: Express the required area as two integrals For $0 \leq x \leq 4$, a vertical strip in the enclosed region extends from $y = 0$ to the line $y = x$. For $4 \leq x \leq 4\sqrt{2}$, it extends from the axis to the circular arc. Consequently,

$$A = \int_0^4 x dx + \int_4^{4\sqrt{2}} \sqrt{32 - x^2} dx.$$

The first contribution is $[x^2/2]_0^4 = 8$. Using the circular curve throughout the whole interval would incorrectly include the region above the line for $0 < x < 4$.

Step 3: Integrate the circular contribution Here $a^2 = 32$, so an antiderivative of the circular boundary is

$$F(x) = \frac{x}{2} \sqrt{32 - x^2} + 16 \sin^{-1}\left(\frac{x}{4\sqrt{2}}\right).$$

At the upper limit, the square-root term vanishes and the inverse sine is $\pi/2$, giving $F(4\sqrt{2}) = 8\pi$. At the lower limit, $\sqrt{32 - 16} = 4$ and $\sin^{-1}(1/\sqrt{2}) = \pi/4$, giving $F(4) = 8 + 4\pi$. Hence

$$\int_4^{4\sqrt{2}} \sqrt{32 - x^2} dx = 8\pi - (8 + 4\pi) = 4\pi - 8.$$

Adding the two nonnegative strip contributions gives $A = 8 + (4\pi - 8)$.

Step 4: Conclusion Therefore, the required area is 4π square units.

Quick Tip to Solve: Find the line-circle intersection first and split the vertical-strip integral at its x -coordinate, which here is 4.

27(a). Evaluate : $\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9 \sin 2x} dx.$

[6 marks]

Solution:

Concept:

- In a definite integral, a substitution $t = g(x)$ transforms both the differential and the limits: $dt = g'(x) dx$.
- The identity $(\sin x - \cos x)^2 = 1 - \sin 2x$ converts the denominator into a quadratic in a variable whose derivative is the numerator.
- For positive a, b and $|bt| < a$, $\int \frac{dt}{a^2 - b^2 t^2} = \frac{1}{2ab} \log \left| \frac{a+bt}{a-bt} \right| + C$, where $\log$ denotes the natural logarithm.

Step 1: Choose a substitution that matches the numerator Let the required integral be I . Set

$$t = \sin x - \cos x, \quad dt = (\cos x + \sin x) dx.$$

This substitution absorbs the entire numerator. Moreover,

$$t^2 = \sin^2 x + \cos^2 x - 2 \sin x \cos x = 1 - \sin 2x,$$

$$\text{so } 16 + 9 \sin 2x = 16 + 9(1 - t^2) = 25 - 9t^2.$$

Step 2: Transform the limits and the integral At $x = 0$, $t = -1$. At $x = \pi/4$, $t = 1/\sqrt{2} - 1/\sqrt{2} = 0$. Thus

$$I = \int_{-1}^0 \frac{dt}{25 - 9t^2}.$$

For $-1 \leq t \leq 0$, $25 - 9t^2 \geq 16 > 0$; hence the transformed integrand is continuous and all evaluations below are valid. Factoring the denominator gives $(5 + 3t)(5 - 3t)$.

Step 3: Resolve the rational function into partial fractions The useful decomposition is

$$\frac{1}{25 - 9t^2} = \frac{1}{10} \left(\frac{1}{5 + 3t} + \frac{1}{5 - 3t} \right).$$

Integrating the first term contributes $\frac{1}{3} \log(5 + 3t)$, while the second contributes $-\frac{1}{3} \log(5 - 3t)$. Both logarithm arguments are positive throughout the interval. Therefore,

$$I = \frac{1}{30} \left[\log \left(\frac{5 + 3t}{5 - 3t} \right) \right]_{-1}^0.$$

Step 4: Evaluate at the two endpoints At $t = 0$, the ratio inside the logarithm is $5/5 = 1$. At $t = -1$, it is $(5 - 3)/(5 + 3) = 2/8 = 1/4$. Thus the upper endpoint contributes zero, and subtracting the lower endpoint gives

$$I = \frac{1}{30} \left(0 - \log \frac{1}{4} \right) = \frac{\log 4}{30}.$$

Since $\log 4 = 2 \log 2$, this expression simplifies to $\frac{\log 2}{15}$.

Step 5: Conclusion Hence, $\boxed{\int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9 \sin 2x} dx = \frac{\log 2}{15}}.$

Quick Tip to Solve: Pair the numerator $\sin x + \cos x$ with $t = \sin x - \cos x$; its square replaces $\sin 2x$ by $1 - t$ squared.

27(b). Evaluate $\int_1^3 (x^2 + 3x + e^x) dx$ as the limit of the sum.

[6 marks]

Solution:

Concept:

- For a continuous function f on $[a, b]$, its definite integral is the limit of right-endpoint sums: $\int_a^b f(x) dx = \lim_{n \rightarrow \infty} h \sum_{r=1}^n f(a + rh)$, where $h = (b - a)/n$.
- The standard sums are $\sum_{r=1}^n r = n(n+1)/2$ and $\sum_{r=1}^n r^2 = n(n+1)(2n+1)/6$. Exponential terms form a geometric progression, whose sum can be evaluated before taking the limit.

Step 1: Form the sum on the interval Let $f(x) = x^2 + 3x + e^x$. This function is continuous on $[1, 3]$, so the required limit exists. Divide the interval into n equal parts of width $h = 2/n$. Their right endpoints are $x_r = 1 + 2r/n$, for $r = 1, 2, \dots, n$. Thus, if I denotes the given integral,

$$I = \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{r=1}^n \left[\left(1 + \frac{2r}{n}\right)^2 + 3 \left(1 + \frac{2r}{n}\right) + e^{1+2r/n} \right].$$

Expanding only the polynomial part gives $4 + 10r/n + 4r^2/n^2$. We evaluate its sum and the exponential sum separately; this preserves the required limit-of-sum method.

Step 2: Evaluate the polynomial contribution Apply the two standard finite-sum identities:

$$\begin{aligned} P_n &= \frac{2}{n} \sum_{r=1}^n \left(4 + \frac{10r}{n} + \frac{4r^2}{n^2} \right) \\ &= 8 + 10 \frac{n+1}{n} + \frac{4}{3} \frac{(n+1)(2n+1)}{n^2}. \end{aligned}$$

As $n \rightarrow \infty$, the ratios $(n+1)/n$ and $(n+1)(2n+1)/n^2$ tend to 1 and 2, respectively. Consequently, $\lim P_n = 8 + 10 + 8/3 = 62/3$.

Step 3: Sum the exponential progression and take its limit Put $q = e^{2/n}$. The exponential contribution is

$$\begin{aligned} E_n &= \frac{2e}{n} \sum_{r=1}^n q^r = \frac{2e}{n} \frac{q(q^n - 1)}{q - 1} \\ &= e(e^2 - 1) e^{2/n} \frac{2/n}{e^{2/n} - 1}, \end{aligned}$$

because $q^n = e^2$. With $t = 2/n$, use the standard limit $\lim_{t \rightarrow 0} (e^t - 1)/t = 1$. Also $e^{2/n} \rightarrow 1$, and hence $\lim E_n = e(e^2 - 1) = e^3 - e$. Both component limits are finite, so the limit of their sum is their sum.

Step 4: Conclusion Therefore, evaluating directly as the limit of the sum gives

$$\int_1^3 (x^2 + 3x + e^x) dx = \frac{62}{3} + e^3 - e.$$

Quick Tip to Solve: For this limit-of-sum integral, expand the polynomial at $1 + 2r/n$, but keep the exponential terms together as a geometric progression.

28. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$. [6 marks]

Solution:

Concept:

- A point on a line $\vec{r} = \vec{a} + \lambda\vec{b}$ has coordinates obtained by equating its components. Substituting these coordinates into the plane equation determines the parameter of the intersection.
- The plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$ has Cartesian equation $x - y + z = 5$. For points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Step 1: Express a general point of the line Let Q be the point where the given line intersects the given plane. The position vector on the line is

$$\vec{r} = (2 + 3\lambda)\hat{i} + (-1 + 4\lambda)\hat{j} + (2 + 2\lambda)\hat{k}.$$

Therefore a general point of the line has coordinates $(2 + 3\lambda, -1 + 4\lambda, 2 + 2\lambda)$. All three coordinates use the same parameter λ ; the value that makes this point satisfy the plane equation identifies Q .

Step 2: Determine the parameter at the intersection Since Q lies on the plane, its coordinates satisfy $x - y + z = 5$. Substitution yields

$$\begin{aligned}(2 + 3\lambda) - (-1 + 4\lambda) + (2 + 2\lambda) &= 5, \\ 2 + 3\lambda + 1 - 4\lambda + 2 + 2\lambda &= 5, \\ 5 + \lambda &= 5, \\ \lambda &= 0.\end{aligned}$$

Notice the sign change when subtracting the y -coordinate. The coefficient of λ is $3 - 4 + 2 = 1$, so this equation determines one unique intersection; the line is not parallel to the plane.

Step 3: Identify and check the intersection point Substituting $\lambda = 0$ into the line gives $Q = (2, -1, 2)$, whose position vector is $2\hat{i} - \hat{j} + 2\hat{k}$. This point also satisfies the plane, since $2 - (-1) + 2 = 5$. Thus both equations agree with the point found, and no second intersection needs to be considered.

Step 4: Calculate the distance from the given point The supplied point is $P = (-1, -5, -10)$. The displacement from P to Q is

$$\overrightarrow{PQ} = (2 - (-1))\hat{i} + (-1 - (-5))\hat{j} + (2 - (-10))\hat{k} = 3\hat{i} + 4\hat{j} + 12\hat{k}.$$

Its magnitude is therefore

$$PQ = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{9 + 16 + 144} = \sqrt{169}.$$

This is the distance to the intersection point specified in the question. We use the point-to-point distance formula after finding that intersection.

Step 5: Conclusion The line meets the plane at $(2, -1, 2)$. Hence the required distance from $(-1, -5, -10)$ is **13 units**.

Quick Tip to Solve: First test the line's base point in the plane: here $2 - (-1) + 2 = 5$, immediately giving the intersection at $\lambda = 0$.

29. A factory manufactures two types of screws A and B, each type requiring the use of two machines, an automatic and a hand-operated. It takes 4 minutes on the automatic and 6 minutes on the hand-operated machines to manufacture a packet of screws 'A' while it takes 6 minutes on the automatic and 3 minutes on the hand-operated machine to manufacture a packet of screws 'B'. Each machine is available for at most 4 hours on any day. The manufacturer can sell a packet of screws 'A' at a profit of 70 paise and screws 'B' at a profit of ₹1. Assuming that he can sell all the screws he manufactures, how many packets of each type should the factory owner produce in a day in order to maximize his profit? Formulate the above LPP and solve it graphically and find the maximum profit. [6 marks]

Solution:

Concept:

- A linear programming problem has a linear objective function and linear constraints. Here the decision variables count packets produced, and each machine's available time gives an upper bound on production.
- In the graphical method, the feasible region is the intersection of all the constraint half-planes. For a nonempty bounded feasible polygon, a linear objective attains its maximum at a corner point.

Step 1: Define the variables and the objective Let x and y be the numbers of packets of screws A and B respectively manufactured in one day. Since a negative number of packets is impossible, $x \geq 0$ and $y \geq 0$. Convert both profits to the same unit: 70 paise equals ₹0.70, while each packet of B yields ₹1. Thus the daily profit in rupees is

$$Z = 0.70x + y.$$

The objective is to maximize Z . The assumption that every packet can be sold means no additional sales constraint is needed.

Step 2: Express the machine-time constraints Each machine is available for $4 \times 60 = 240$ minutes per day. Producing x packets of A and y packets of B uses $4x + 6y$ minutes on the automatic machine and $6x + 3y$ minutes on the hand-operated machine. Hence the complete LPP is

$$\begin{aligned} \text{Maximize } Z &= 0.70x + y, \\ \text{subject to } 4x + 6y &\leq 240, \\ 6x + 3y &\leq 240, \\ x &\geq 0, \quad y \geq 0. \end{aligned}$$

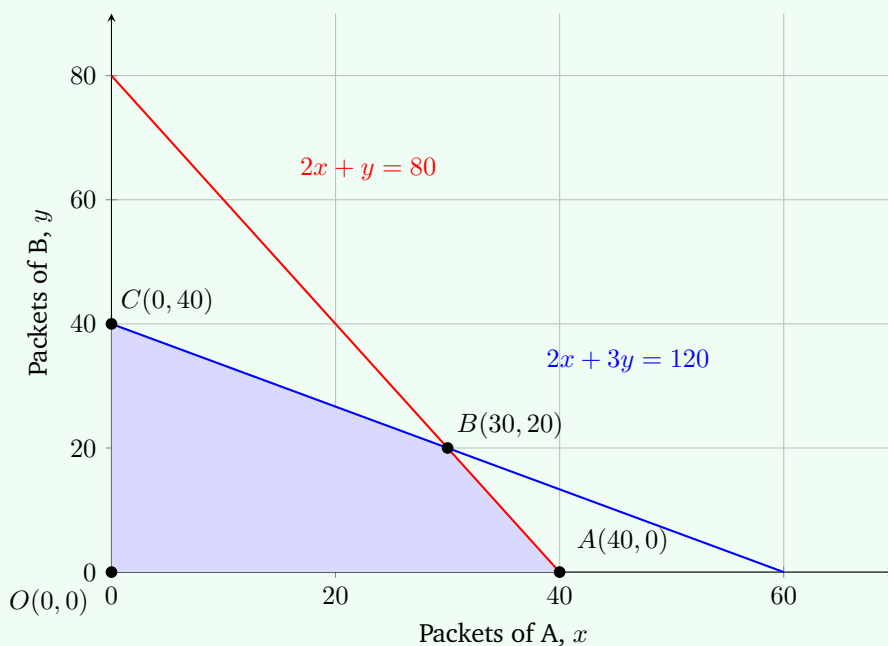
Dividing the first machine constraint by 2 and the second by 3 gives $2x + 3y \leq 120$ and $2x + y \leq 80$. These equivalent inequalities are convenient for drawing the graph.

Step 3: Draw the boundary lines and identify the feasible polygon The line $2x + 3y = 120$ passes through $(60, 0)$ and $(0, 40)$. The line $2x + y = 80$ passes through $(40, 0)$ and $(0, 80)$. Both inequalities are satisfied at the origin, so shade the common region below these two lines in the first quadrant. Their intersection is obtained by subtraction:

$$\begin{aligned} (2x + 3y) - (2x + y) &= 120 - 80, \\ 2y &= 40, \quad y = 20, \\ 2x + 20 &= 80, \quad x = 30. \end{aligned}$$

Quick Tip to Solve: Convert hours to minutes before forming the constraints, and compare profits at all four corners; the y-axis corner gives 40 rupees, just below the optimal 41 rupees.

Therefore the feasible polygon has corner points $O(0,0)$, $A(40,0)$, $B(30,20)$ and $C(0,40)$. The intercepts $(60,0)$ and $(0,80)$ lie outside the common feasible region.



The shaded polygon OABC is the common feasible region.

Step 4: Evaluate the objective at every corner Since the feasible polygon is bounded, compare the profit at all its vertices:

Corner point (x, y)	$Z = 0.70x + y$ (rupees)
$(0, 0)$	0
$(40, 0)$	28
$(30, 20)$	$21 + 20 = 41$
$(0, 40)$	40

The greatest value in this comparison is 41, attained at $(30, 20)$. This point corresponds to whole packets and is therefore a physically possible production plan. As a check, it uses $4(30) + 6(20) = 240$ minutes of automatic-machine time and $6(30) + 3(20) = 240$ minutes of hand-operated-machine time, exactly the available time on each machine.

Step 5: Conclusion The factory should produce 30 packets of A and 20 packets of B per day. Its maximum daily profit is ₹ 41.

Quick Tip to Solve: Convert hours to minutes before forming the constraints, and compare profits at all four corners; the y-axis corner gives 40 rupees, just below the optimal 41 rupees.