

CBSE Class 12 Mathematics Question Paper 2019 with Solution PDF-Set 65-1-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) This question paper contains **29** questions divided into four sections A, B, C and D. Section A comprises of 4 questions of **one mark** each, Section B comprises of 8 questions of **two marks** each, Section C comprises of 11 questions of **four marks** each and Section D comprises of 6 questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt **only one** of the alternatives in all such questions.
- (v) Use of calculators is not permitted. You may ask logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. Find the order and the degree of the differential equation $x^2 \frac{d^2 y}{dx^2} = \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^4$. [1 mark]

Solution:

Concept:

- The order is the order of the highest derivative occurring in a differential equation.
- The degree is the power of the highest-order derivative after the equation is written as a polynomial in its derivatives.

Step 1: Identify the highest derivative The derivatives present are $\frac{dy}{dx}$ and $\frac{d^2 y}{dx^2}$. The second derivative is the highest-order derivative, so the order is determined by it, irrespective of the exponent on the right-hand side.

Step 2: Inspect its exponent The equation is already polynomial in its derivatives. The term $x^2 \frac{d^2 y}{dx^2}$ contains the highest derivative to the first power. Raising the expression involving the first derivative to the fourth power does not alter the degree of the second derivative.

Step 3: Conclusion The order is 2 and the degree is 1.

Quick Tip to Solve: Find the highest derivative first; the outer power 4 belongs to the first-derivative expression.

2. If $f(x) = x + 7$ and $g(x) = x - 7$, $x \in \mathbb{R}$, then find $\frac{d}{dx}(f \circ g)(x)$. [1 mark]

Solution:

Concept:

- Composition means $(f \circ g)(x) = f(g(x))$: apply g first and then f .
- The derivative of x is 1, while the derivative of a constant is zero.

Step 1: Form the composition Substitute $g(x) = x - 7$ into the input of f . Since $f(t) = t + 7$, we have

$$(f \circ g)(x) = f(x - 7) = (x - 7) + 7 = x.$$

The two constant shifts cancel for every real value of x .

Step 2: Differentiate the resulting function As the composition is the identity function on $\mathbb{R}$,

$$\frac{d}{dx}(f \circ g)(x) = \frac{d}{dx}x = 1.$$

Equivalently, the chain rule gives $f'(g(x))g'(x) = 1 \cdot 1$.

Step 3: Conclusion $\frac{d}{dx}(f \circ g)(x) = 1$ for every $x \in \mathbb{R}$.

Quick Tip to Solve: Simplify the composition before differentiating: the shifts -7 and $+7$ cancel.

3. Find the value of $x - y$, if $2 \begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$. [1 mark]

Solution:

Concept:

- Two matrices are equal when their corresponding entries are equal.
- Scalar multiplication multiplies every entry; matrix addition adds corresponding entries.

Step 1: Simplify the left-hand matrix Multiply the first matrix by 2, and add corresponding entries:

$$2 \begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix}.$$

The off-diagonal entries already agree with the given matrix.

Step 2: Compare the diagonal entries Equality of the top-left entries gives $2 + y = 5$, hence $y = 3$. Equality of the bottom-right entries gives $2x + 2 = 8$, hence $x = 3$. Both values satisfy the full matrix equation, so subtracting them gives $x - y = 3 - 3$.

Step 3: Conclusion $x - y = 0$.

Quick Tip to Solve: Use only the two diagonal entries to obtain x and y ; check the other entries afterward.

4(a). If a line makes angles $90^\circ, 135^\circ, 45^\circ$ with the x, y and z axes respectively, find its direction cosines. [1 mark]

Solution:

Concept:

- The direction cosines of a line are $l = \cos \alpha$, $m = \cos \beta$, and $n = \cos \gamma$, where the angles are measured with the positive coordinate axes.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$.

Step 1: Evaluate the three cosines Use the angles in the order of the x, y , and z axes:

$$l = \cos 90^\circ = 0, \quad m = \cos 135^\circ = -\frac{1}{\sqrt{2}}, \quad n = \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

The cosine of the obtuse angle is negative.

Step 2: Check the direction-cosine identity The computed values satisfy

$$l^2 + m^2 + n^2 = 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1.$$

This confirms that they form a valid direction-cosine triple with the prescribed angles.

Step 3: Conclusion The direction cosines are $\left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Quick Tip to Solve: Keep the axis order and the negative sign of $\cos 135^\circ$; then check the sum of squares.

4(b). Find the vector equation of the line which passes through the point $(3, 4, 5)$ and is parallel to the vector $2\hat{i} + 2\hat{j} - 3\hat{k}$. [1 mark]

Solution:

Concept:

- A line through position vector $\vec{a}$, parallel to nonzero vector $\vec{b}$, has equation $\vec{r} = \vec{a} + \lambda\vec{b}$, $\lambda \in \mathbb{R}$.
- The parameter λ describes all scalar displacements along the direction vector.

Step 1: Identify the point and direction vectors The given point $(3, 4, 5)$ has position vector $\vec{a} = 3\hat{i} + 4\hat{j} + 5\hat{k}$. The line is parallel to $\vec{b} = 2\hat{i} + 2\hat{j} - 3\hat{k}$, which is a nonzero vector and can therefore be used directly as its direction vector.

Step 2: Substitute into the equation of a line The position vector of a general point is

$$\vec{r} = (3\hat{i} + 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 2\hat{j} - 3\hat{k}).$$

At $\lambda = 0$ this gives the required point; changing λ moves parallel to the given vector.

Step 3: Conclusion $\vec{r} = 3\hat{i} + 4\hat{j} + 5\hat{k} + \lambda(2\hat{i} + 2\hat{j} - 3\hat{k})$, where $\lambda \in \mathbb{R}$.

Quick Tip to Solve: Put the point in the fixed vector and the parallel vector in the parameter term; retain its $-3\hat{k}$ sign.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5. Examine whether the operation $*$ defined on $\mathbb{R}$ by $a * b = ab + 1$ is (i) a binary or not. (ii) if a binary operation, is it associative or not? [2 marks]

Solution:

Concept:

- A binary operation on $\mathbb{R}$ assigns a unique real result to every ordered pair of real inputs (closure).
- Associativity requires $(a * b) * c = a * (b * c)$ for every $a, b, c \in \mathbb{R}$.

Step 1: Check closure on the real numbers For arbitrary real numbers a and b , their product ab is real, and adding the real number 1 keeps the result real. The formula $a * b = ab + 1$ specifies exactly one result for every ordered pair, with no excluded inputs. Thus it defines a map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$.

Step 2: Compare the two associations Applying the same operation at each stage gives

$$\begin{aligned}(a * b) * c &= (ab + 1)c + 1 = abc + c + 1, \\ a * (b * c) &= a(bc + 1) + 1 = abc + a + 1.\end{aligned}$$

These differ whenever $a \neq c$; equality for some special triples is insufficient.

Step 3: Give an explicit counterexample Take $a = 0$, $b = 0$, and $c = 1$. Then $(0 * 0) * 1 = 1 * 1 = 2$, whereas $0 * (0 * 1) = 0 * 1 = 1$. The two associations have unequal real results.

Quick Tip to Solve: Expand both bracketings using the operation formula; compare the extra terms c and a .

Step 4: Conclusion (i) It is a binary operation on $\mathbb{R}$. (ii) It is not associative.

Quick Tip to Solve: Expand both bracketings using the operation formula; compare the extra terms c and a .

6. If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then find $(A^2 - 5A)$.

[2 marks]

Solution:

Concept:

- For matrix multiplication, the (i, j) entry of A^2 is the dot product of row i of A with column j of A .
- Compute the scalar multiple $5A$ entry by entry and subtract it from A^2 .

Step 1: Calculate the matrix square The columns of A are $(2, 2, 1)^T$, $(0, 1, -1)^T$, and $(1, 3, 0)^T$. Using row-column products,

$$A^2 = \begin{bmatrix} 4+0+1 & 0+0-1 & 2+0+0 \\ 4+2+3 & 0+1-3 & 2+3+0 \\ 2-2+0 & 0-1+0 & 1-3+0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}.$$

Matrix squaring means multiplication by A , rather than squaring individual entries.

Step 2: Multiply by five and subtract The scalar multiple is

$$5A = \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix}.$$

Subtract corresponding entries:

$$A^2 - 5A = \begin{bmatrix} 5-10 & -1-0 & 2-5 \\ 9-10 & -2-5 & 5-15 \\ 0-5 & -1-(-5) & -2-0 \end{bmatrix}.$$

In particular, the bottom-middle entry is $-1 + 5 = 4$; subtracting a negative entry reverses its sign.

Step 3: Conclusion $A^2 - 5A = \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix}.$

Quick Tip to Solve: Compute A^2 by row-column products first, then check negative entries carefully when subtracting $5A$.

7(a). Find $\int \sqrt{1 - \sin 2x} \, dx$, $\frac{\pi}{4} < x < \frac{\pi}{2}$.

[2 marks]

Solution:

Concept:

- The identity $\sin 2x = 2 \sin x \cos x$ gives $1 - \sin 2x = (\sin x - \cos x)^2$.
- The principal square root satisfies $\sqrt{u^2} = |u|$; the stated interval determines the sign.
- Use $\int \sin x \, dx = -\cos x$ and $\int \cos x \, dx = \sin x$.

Step 1: Rewrite the expression under the radical Using $\sin^2 x + \cos^2 x = 1$,

$$1 - \sin 2x = \sin^2 x + \cos^2 x - 2 \sin x \cos x = (\sin x - \cos x)^2.$$

Therefore the integrand is $|\sin x - \cos x|$, rather than an expression with an arbitrary sign.

Step 2: Use the prescribed interval For $\pi/4 < x < \pi/2$, both sine and cosine are positive and $\sin x > \cos x$. Thus $|\sin x - \cos x| = \sin x - \cos x$ throughout this interval. Its sign is fixed, so no piecewise integration is needed on the stated domain.

Step 3: Integrate and verify Hence

$$\int \sqrt{1 - \sin 2x} \, dx = \int (\sin x - \cos x) \, dx = -\cos x - \sin x + C.$$

Differentiating the last expression gives $\sin x - \cos x$, which is the positive square root on the stated interval; C is an arbitrary constant.

Step 4: Conclusion $\int \sqrt{1 - \sin 2x} \, dx = -\cos x - \sin x + C$, for $\pi/4 < x < \pi/2$.

Quick Tip to Solve: Replace the radical by an absolute value first, then use $x > \pi/4$ to select $\sin x - \cos x$.

OR

7(b). Find $\int \sin^{-1}(2x) \, dx$.

[2 marks]

Solution:

Concept:

- Integration by parts gives $\int u \, dv = uv - \int v \, du$. Choose the inverse trigonometric function as u so that its derivative is algebraic.
- The chain rule gives $\frac{d}{dx} \sin^{-1}(2x) = \frac{2}{\sqrt{1-4x^2}}$, valid for $-\frac{1}{2} < x < \frac{1}{2}$.

Step 1: Apply integration by parts Let $I = \int \sin^{-1}(2x) \, dx$. Take $u = \sin^{-1}(2x)$ and $dv = dx$, so that $du = 2 \, dx / \sqrt{1-4x^2}$ and $v = x$. Thus

$$I = x \sin^{-1}(2x) - \int \frac{2x}{\sqrt{1-4x^2}} \, dx.$$

Quick Tip to Solve: Choose the inverse sine as the first factor in integration by parts, then use the expression under the square root as the substitution.

Step 2: Evaluate the remaining integral Put $t = 1 - 4x^2$, so $dt = -8x dx$. Consequently,

$$\int \frac{2x}{\sqrt{1-4x^2}} dx = -\frac{1}{4} \int t^{-1/2} dt = -\frac{1}{2} \sqrt{1-4x^2}.$$

Substituting this into the expression for I changes the negative sign into a positive one. As a check, differentiating the two nonconstant terms produces $\sin^{-1}(2x) + 2x/\sqrt{1-4x^2} - 2x/\sqrt{1-4x^2}$, which is the given integrand.

Step 3: Conclusion $\int \sin^{-1}(2x) dx = x \sin^{-1}(2x) + \frac{1}{2} \sqrt{1-4x^2} + C$, where C is an arbitrary constant.

Quick Tip to Solve: Choose the inverse sine as the first factor in integration by parts, then use the expression under the square root as the substitution.

8. Form the differential equation representing the family of curves $y = e^{2x}(a + bx)$, where a and b are arbitrary constants. [2 marks]

Solution:

Concept:

- Forming a differential equation means eliminating all arbitrary constants from the family and its derivatives.
- Two independent arbitrary constants, a and b , require differentiation twice. Multiplying by e^{-2x} simplifies this family.

Step 1: Differentiate after removing the exponential The given family can be written as $ye^{-2x} = a + bx$. Differentiating with respect to x , using the product rule on the left, gives

$$e^{-2x}(y' - 2y) = b.$$

The constant a is now absent, but b remains; hence this first-order relation is not yet the required equation.

Step 2: Eliminate the remaining constant Differentiate the last equality again. Since b is constant, its derivative is zero. Thus

$$\begin{aligned} 0 &= -2e^{-2x}(y' - 2y) + e^{-2x}(y'' - 2y') \\ &= e^{-2x}(y'' - 4y' + 4y). \end{aligned}$$

For every real x , e^{-2x} is nonzero, so it can be divided out. This relation contains only x , y , and derivatives of y ; both parameters have been eliminated.

Step 3: Conclusion The required differential equation is $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

Quick Tip to Solve: Multiply the family by the reciprocal exponential before differentiating; the right side becomes a linear polynomial.

9(a). A die is thrown 6 times. If “getting an odd number” is a “success”, what is the probability of (i) 5 successes? (ii) at most 5 successes? [2 marks]

Solution:

Concept:

- For independent trials with success probability p , the number X of successes in n trials satisfies $P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$.
- An odd outcome on a fair die is one of 1, 3, 5, giving $p = 3/6 = 1/2$; an even outcome is a failure, also with probability $1/2$.

Step 1: Calculate the probability of five successes The six throws are independent, so X has the binomial distribution with $n = 6$ and $p = 1/2$. Exactly five successes means five odd outcomes and one even outcome. The failure can occur in any of six positions. Hence

$$P(X = 5) = \binom{6}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right) = \frac{6}{64} = \frac{3}{32}.$$

Step 2: Use the complement for at most five successes The phrase at most five includes all counts 0, 1, 2, 3, 4, 5. The only possible count outside this event is six successes, so these two events are complementary. All six throws must be odd for $X = 6$, which has probability $(1/2)^6 = 1/64$. Therefore

$$P(X \leq 5) = 1 - P(X = 6) = 1 - \frac{1}{64} = \frac{63}{64}.$$

This avoids adding six separate binomial terms and includes the possibility of zero successes.

Step 3: Conclusion (i) The probability of five successes is $\frac{3}{32}$. (ii) The probability of at most five successes is $\frac{63}{64}$.

Quick Tip to Solve: For at most five successes in six throws, subtract the probability of all six successes from one.

OR

9(b). The random variable X has a probability distribution $P(X)$ of the following form, where k is some number:

$$P(X = x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the value of k .

[2 marks]

Solution:

Concept:

- A discrete probability distribution must satisfy $P(X = x) \geq 0$ for every value of x .

Quick Tip to Solve: Add the probability weights k , $2k$ and $3k$ and set their sum equal to one; do not multiply them by the values of X .

- The sum of probabilities over all possible values equals one: $\sum_x P(X = x) = 1$. Values with probability zero do not change this sum.

Step 1: Apply the normalization condition The only values with possibly nonzero probabilities are $X = 0, 1, 2$. They have probabilities $k, 2k, 3k$, respectively. These events are mutually exclusive, and together account for the entire probability because every other value is assigned probability zero. Therefore

$$P(X = 0) + P(X = 1) + P(X = 2) = 1, \quad k + 2k + 3k = 1.$$

The probabilities themselves are added; the values 0, 1, 2 are not factors in this normalization condition.

Step 2: Solve and check the probability conditions Combining the like terms gives $6k = 1$, hence $k = 1/6$. With this value the three probabilities become

$$P(X = 0) = \frac{1}{6}, \quad P(X = 1) = \frac{1}{3}, \quad P(X = 2) = \frac{1}{2}.$$

Each probability is nonnegative and at most one, and their sum is $1/6 + 1/3 + 1/2 = 1$. The probabilities assigned to all other values remain zero. Thus the value obtained satisfies both requirements for a valid probability distribution.

Step 3: Conclusion The required value is $k = \frac{1}{6}$.

Quick Tip to Solve: Add the probability weights $k, 2k$ and $3k$ and set their sum equal to one; do not multiply them by the values of X .

10. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event “number is even” and B be the event “number is marked red”. Find whether the events A and B are independent or not. [2 marks]

Solution:

Concept:

- Two events A and B are independent exactly when $P(A \cap B) = P(A)P(B)$.
- For a fair die, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$, with each outcome having probability $1/6$.

Step 1: Identify the events and their intersection The even-number event is $A = \{2, 4, 6\}$. The red-number event is $B = \{1, 2, 3\}$, since precisely the first three numbers are marked red. The only outcome that is both even and red is 2. Hence

$$A \cap B = \{2\}, \quad P(A) = \frac{3}{6} = \frac{1}{2}, \quad P(B) = \frac{3}{6} = \frac{1}{2}.$$

It follows that $P(A \cap B) = 1/6$. The colouring does not alter the equal likelihood of the six numbered faces.

Quick Tip to Solve: List the common outcomes first; independence depends on the intersection probability, not on equal event sizes.

Step 2: Test independence The product of the individual probabilities is

$$P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Since $1/6 \neq 1/4$, the independence condition fails. Equivalently, among the three red numbers only one is even, so $P(A | B) = 1/3$, whereas without the information about colour $P(A) = 1/2$. Knowing that the outcome is red therefore changes the probability that it is even. Equal individual probabilities alone do not make two events independent.

Step 3: Conclusion The events A and B are not independent, because $P(A \cap B) = \frac{1}{6} \neq \frac{1}{4} = P(A)P(B)$.

Quick Tip to Solve: List the common outcomes first; independence depends on the intersection probability, not on equal event sizes.

11(a). If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$. [2 marks]

Solution:

Concept:

- For any vectors $\vec{u}$ and $\vec{v}$, $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v}$.
- Similarly, $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v}$. A unit vector has magnitude one.

Step 1: Use the condition on the sum Let the two unit vectors be $\vec{u}$ and $\vec{v}$. Then $|\vec{u}| = |\vec{v}| = 1$, and the given condition is $|\vec{u} + \vec{v}| = 1$. Squaring the latter equality and expanding by the dot product gives

$$1 = 1 + 1 + 2\vec{u} \cdot \vec{v}.$$

Thus $2\vec{u} \cdot \vec{v} = -1$, or $\vec{u} \cdot \vec{v} = -1/2$. This uses the magnitude of the sum, which is specified to be one, rather than adding the two magnitudes.

Step 2: Find the squared magnitude of the difference Now expand the square of the difference using the same dot product:

$$\begin{aligned} |\vec{u} - \vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2\vec{u} \cdot \vec{v} \\ &= 1 + 1 - 2\left(-\frac{1}{2}\right) = 3. \end{aligned}$$

The magnitude of a vector is nonnegative. Consequently the positive square root, rather than either signed root, gives the required magnitude. All three unit-vector conditions have been used, and no special choice of coordinates is needed.

Step 3: Conclusion Hence the magnitude of the difference of the two unit vectors is $\boxed{\sqrt{3}}$, as required.

Quick Tip to Solve: First expand the square of the sum to find the dot product, then reverse its sign in the square of the difference.

11(b). If $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$, find $[\vec{a} \vec{b} \vec{c}]$.

[2 marks]

Solution:

Concept:

- The scalar triple product is $[\vec{a} \vec{b} \vec{c}] = \vec{a} \cdot (\vec{b} \times \vec{c})$. It is a signed scalar, so the order of the vectors must be preserved.
- In a right-handed Cartesian basis, the product equals the determinant whose successive rows are the components of $\vec{a}, \vec{b}, \vec{c}$.

Step 1: Form the determinant in the stated order The given component triples are $(2, 3, 1)$, $(1, -2, 1)$ and $(-3, 1, 2)$. Therefore

$$[\vec{a} \vec{b} \vec{c}] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix}.$$

We do not take an absolute value, since the question asks for the scalar triple product, rather than the volume of a parallelepiped.

Step 2: Expand along the first row Applying the alternating cofactor signs $+, -, +$,

$$\begin{aligned} [\vec{a} \vec{b} \vec{c}] &= 2\{(-2)(2) - (1)(1)\} \\ &\quad - 3\{(1)(2) - (1)(-3)\} \\ &\quad + 1\{(1)(1) - (-2)(-3)\} \\ &= 2(-5) - 3(5) + 1(-5) = -30. \end{aligned}$$

As a check, $\vec{b} \times \vec{c} = -5\hat{i} - 5\hat{j} - 5\hat{k}$, whose dot product with $2\hat{i} + 3\hat{j} + \hat{k}$ gives the same value.

Step 3: Conclusion Hence, $[\vec{a} \vec{b} \vec{c}] = -30$.

Quick Tip to Solve: Keep the vectors in the printed order and use the first-row cofactor signs $+, -, +$; interchanging two vectors reverses the answer's sign.

12. Find $\int \frac{\tan^2 x \sec^2 x}{1 - \tan^6 x} dx$.

[2 marks]

Solution:

Concept:

- Substitution converts an integral into a simpler variable when the numerator contains the derivative of an expression in the denominator.
- Since $\frac{d}{dx}(\tan^3 x) = 3 \tan^2 x \sec^2 x$, take $t = \tan^3 x$. The standard integral is $\int \frac{dt}{1 - t^2} = \frac{1}{2} \log \left| \frac{1+t}{1-t} \right| + C$.

Quick Tip to Solve: Spot $\tan^2 x \sec^2 x$ as one-third of the derivative of $\tan^3 x$; this makes the denominator $1 - t^2$ immediately.

Step 1: Substitute the cube of the tangent Write I for the given indefinite integral. Put $t = \tan^3 x$, so that $dt = 3 \tan^2 x \sec^2 x dx$ and $\tan^6 x = t^2$. Thus

$$I = \frac{1}{3} \int \frac{dt}{1 - t^2}.$$

This substitution absorbs the entire numerator and changes the sixth power in the denominator to a square.

Step 2: Resolve into partial fractions and integrate Since

$$\frac{1}{1 - t^2} = \frac{1}{2} \left(\frac{1}{1 + t} + \frac{1}{1 - t} \right),$$

integration gives

$$\begin{aligned} I &= \frac{1}{6} \{ \log |1 + t| - \log |1 - t| \} + C \\ &= \frac{1}{6} \log \left| \frac{1 + \tan^3 x}{1 - \tan^3 x} \right| + C. \end{aligned}$$

The minus sign in the second logarithm comes from integrating $1/(1 - t)$. The formula applies on any interval where $\tan x$ is defined and $\tan^6 x \neq 1$.

Step 3: Conclusion Therefore, $\int \frac{\tan^2 x \sec^2 x}{1 - \tan^6 x} dx = \frac{1}{6} \log \left| \frac{1 + \tan^3 x}{1 - \tan^3 x} \right| + C$.

Quick Tip to Solve: Spot $\tan^2 x \sec^2 x$ as one-third of the derivative of $\tan^3 x$; this makes the denominator $1 - t^2$ immediately.

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13. Solve: $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$.

[4 marks]

Solution:

Concept:

- The principal value of $\tan^{-1} u$ lies in $(-\pi/2, \pi/2)$, has the same sign as u , and is strictly increasing in u .
- For angles A, B with $1 - \tan A \tan B \neq 0$, $\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$. Taking tangents gives necessary candidates, which must be checked against the original principal-value equation.

Step 1: Establish the sign and introduce the angles Let $A = \tan^{-1}(2x)$ and $B = \tan^{-1}(3x)$. The given equation is $A + B = \pi/4$. If $x \leq 0$, both inverse tangents are nonpositive, so their sum cannot be $\pi/4 > 0$. Hence any solution must satisfy $x > 0$. In particular both angles are then positive principal angles.

Step 2: Take tangents and obtain the quadratic Since the prescribed sum has finite tangent, the

Quick Tip to Solve: First note that the positive right-hand side forces $x > 0$; after taking tangents, check the surviving root lies in the correct principal-angle branch.

addition formula yields

$$\frac{2x + 3x}{1 - (2x)(3x)} = \tan \frac{\pi}{4} = 1.$$

Thus $1 - 6x^2 \neq 0$, and multiplication gives

$$5x = 1 - 6x^2, \quad 6x^2 + 5x - 1 = 0.$$

Factorising,

$$(6x - 1)(x + 1) = 0,$$

so the algebraic candidates are $x = 1/6$ and $x = -1$.

Step 3: Reject the negative candidate and verify the positive one The candidate $x = -1$ is impossible because $\tan^{-1}(-2) + \tan^{-1}(-3) < 0$. Equal tangents alone do not imply equal angles, so this candidate cannot be retained. For $x = 1/6$, the arguments become $1/3$ and $1/2$. Each is between 0 and 1, so $0 < A < \pi/4$ and $0 < B < \pi/4$. Consequently $0 < A + B < \pi/2$. Furthermore,

$$\tan(A + B) = \frac{1/3 + 1/2}{1 - (1/3)(1/2)} = \frac{5/6}{5/6} = 1.$$

On $(0, \pi/2)$, tangent is one-to-one; hence $A + B = \pi/4$, exactly as required. This verifies the candidate in the original equation and completes the branch check.

Step 4: Conclusion The unique solution is $x = \frac{1}{6}$.

Quick Tip to Solve: First note that the positive right-hand side forces $x > 0$; after taking tangents, check the surviving root lies in the correct principal-angle branch.

14(a). If $\log(x^2 + y^2) = 2 \tan^{-1} \left(\frac{y}{x} \right)$, show that $\frac{dy}{dx} = \frac{x + y}{x - y}$.

[4 marks]

Solution:

Concept:

- Implicit differentiation treats y as a function of x . Thus $\frac{d}{dx}(y^2) = 2y \frac{dy}{dx}$; the derivative of y must be included wherever it occurs.
- The chain rule gives $\frac{d}{dx} \log u = \frac{u'}{u}$ and $\frac{d}{dx} \tan^{-1} u = \frac{u'}{1 + u^2}$. Here $\log$ denotes the natural logarithm.
- The quotient rule gives $\frac{d}{dx}(y/x) = \frac{x dy/dx - y}{x^2}$, on the domain $x \neq 0$ required by the given expression.

Step 1: Differentiate both sides of the given relation Put $p = \frac{dy}{dx}$. Starting from

$$\log(x^2 + y^2) = 2 \tan^{-1} \left(\frac{y}{x} \right),$$

Quick Tip to Solve: Simplify $1 + (y/x)^2$ before clearing denominators; both sides then share $x^2 + y^2$, making the collection of derivative terms immediate.

apply the chain rule on each side. On the left, the derivative of the inner function is $2x + 2yp$. On the right, first differentiate the inverse tangent, then its argument y/x . Consequently,

$$\frac{2x + 2yp}{x^2 + y^2} = \frac{2}{1 + y^2/x^2} \left(\frac{xp - y}{x^2} \right).$$

Both sides have been differentiated with respect to the same independent variable x , so every occurrence of y contributes its derivative p .

Step 2: Simplify and collect the derivative terms Since $1 + y^2/x^2 = (x^2 + y^2)/x^2$, the right side simplifies to $2(xp - y)/(x^2 + y^2)$. The original equation requires $x \neq 0$, so $x^2 + y^2 > 0$. We may therefore multiply by $(x^2 + y^2)/2$ to obtain

$$\begin{aligned} x + yp &= xp - y, \\ x + y &= xp - yp, \\ x + y &= (x - y)p. \end{aligned}$$

Collecting the derivative terms before dividing avoids losing the sign of the y term. Wherever the curve has a finite derivative and $x - y \neq 0$, division by $x - y$ now isolates p .

Step 3: Conclusion Hence, $\frac{dy}{dx} = \frac{x + y}{x - y}$, as required, wherever the derivative exists as a finite value.

Quick Tip to Solve: Simplify $1 + (y/x)^2$ before clearing denominators; both sides then share $x^2 + y^2$, making the collection of derivative terms immediate.

OR

14(b). If $x^y - y^x = a^b$, find $\frac{dy}{dx}$.

[4 marks]

Solution:

Concept:

- In implicit differentiation y is a function of x , while a and b are constants. Therefore $\frac{d}{dx}(a^b) = 0$.
- For a variable base and exponent, rewrite $u^v = e^{v \log u}$. Its derivative is $u^v \left(v' \log u + v \frac{u'}{u} \right)$ on the positive-base domain.
- The standard real logarithmic calculation here uses $x > 0$ and $y > 0$. Apply the formula separately to x^y and y^x , since a logarithm does not separate a difference.

Step 1: Differentiate each variable power Put $p = \frac{dy}{dx}$. For the first term, use $x^y = e^{y \log x}$. The product rule applied to its exponent gives

$$\frac{d}{dx}(x^y) = x^y \left(p \log x + \frac{y}{x} \right).$$

For the second term, use $y^x = e^{x \log y}$. Now the exponent has derivative $\log y + xp/y$, since both x and

Quick Tip to Solve: Rewrite each power as an exponential separately, and collect all dy/dx terms before dividing; never take the logarithm of the difference term by term.

y vary. Hence

$$\frac{d}{dx}(y^x) = y^x \left(\log y + \frac{x}{y} p \right).$$

In particular, x^y cannot be differentiated as if its exponent were constant, and y^x cannot be differentiated as if its base were constant.

Step 2: Differentiate the relation and solve for the derivative Differentiate $x^y - y^x = a^b$ and substitute the two derivatives above. Since the right side is constant,

$$x^y \left(p \log x + \frac{y}{x} \right) - y^x \left(\log y + \frac{x}{y} p \right) = 0.$$

Expand the brackets and move the terms without p to the right side:

$$\begin{aligned} x^y p \log x - \frac{x}{y} y^x p &= y^x \log y - \frac{y}{x} x^y, \\ p (x^y \log x - x y^{x-1}) &= y^x \log y - y x^{y-1}. \end{aligned}$$

Factor p only after retaining the minus sign preceding the entire derivative of y^x . Dividing by its coefficient gives the derivative at points where that coefficient is nonzero; these are the points where this implicit calculation determines a finite slope.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = \frac{y^x \log y - y x^{y-1}}{x^y \log x - x y^{x-1}}}$$

This formula applies on the positive-base domain wherever the denominator is nonzero.

Quick Tip to Solve: Rewrite each power as an exponential separately, and collect all dy/dx terms before dividing; never take the logarithm of the difference term by term.

15. Find $\int \frac{3x+5}{x^2+3x-18} dx$.

[4 marks]

Solution:

Concept:

- Factor the denominator into linear factors before applying partial fractions. Here $x^2 + 3x - 18 = (x+6)(x-3)$, so the integrand is defined for $x \neq -6, 3$.
- For distinct linear factors, express the proper rational function as $A/(x+6) + B/(x-3)$. The constants are found by equating the numerator after taking a common denominator.
- The standard primitive is $\int (x-c)^{-1} dx = \ln|x-c| + C$. Absolute values give a real logarithm on either side of each excluded point.

Step 1: Factor and set up partial fractions Let the required integral be I . Since the numerator has

Quick Tip to Solve: After factoring into $(x+6)(x-3)$, substitute $x = -6$ and $x = 3$ in the numerator identity to find the two coefficients directly.

lower degree than the denominator, no polynomial division is needed. Write

$$\frac{3x+5}{x^2+3x-18} = \frac{A}{x+6} + \frac{B}{x-3}.$$

Multiplying by $(x+6)(x-3)$ gives the polynomial identity

$$3x+5 = A(x-3) + B(x+6).$$

This identity can be evaluated at the roots of the denominator to isolate each coefficient.

Step 2: Determine the constants Put $x = -6$ in the numerator identity:

$$-13 = -9A \implies A = \frac{13}{9}.$$

Put $x = 3$:

$$14 = 9B \implies B = \frac{14}{9}.$$

As a check, the coefficient of x is $A + B = 3$, and the constant term is $-3A + 6B = 5$. Thus both numerator coefficients agree with the given expression.

Step 3: Integrate the two simple fractions Substituting the constants and integrating term by term yields

$$\begin{aligned} I &= \frac{13}{9} \int \frac{dx}{x+6} + \frac{14}{9} \int \frac{dx}{x-3} \\ &= \frac{13}{9} \ln|x+6| + \frac{14}{9} \ln|x-3| + C. \end{aligned}$$

Differentiating this expression gives $13/[9(x+6)] + 14/[9(x-3)]$, which combines to the original integrand. The constant of integration applies on any interval that avoids -6 and 3 .

Step 4: Conclusion $\int \frac{3x+5}{x^2+3x-18} dx = \frac{13}{9} \ln|x+6| + \frac{14}{9} \ln|x-3| + C$, for $x \neq -6, 3$.

Quick Tip to Solve: After factoring into $(x+6)(x-3)$, substitute $x = -6$ and $x = 3$ in the numerator identity to find the two coefficients directly.

16. Prove that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, **hence evaluate** $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$. [4 marks]

Solution:

Concept:

- In a definite integral, the substitution $t = a - x$ reflects the interval $[0, a]$. Its differential is $dx = -dt$, and its new limits occur in reverse order.
- Adding an integral to its reflected form is useful when the factor multiplying x is unchanged by $x \mapsto a - x$.
- For the remaining integral, use $u = \cos x$, so $du = -\sin x dx$, and $\int du/(1 + u^2) = \tan^{-1} u + C$.

Step 1: Prove the reflection property Suppose f is integrable over the stated interval. In $\int_0^a f(x) dx$,

Quick Tip to Solve: When $\sin x/(1 + \cos^2 x)$ accompanies x on $[0, \pi]$, reflect x to $\pi - x$ and add the two integrals before substituting $\cos x$.

put $t = a - x$. Then $x = a - t$ and $dx = -dt$. When $x = 0$, $t = a$; when $x = a$, $t = 0$. Therefore

$$\begin{aligned}\int_0^a f(x) dx &= \int_a^0 f(a-t)(-dt) \\ &= \int_0^a f(a-t) dt = \int_0^a f(a-x) dx.\end{aligned}$$

In the last equality only the dummy variable is renamed. This proves the required property; the negative differential and reversal of limits cancel.

Step 2: Apply the property to the given integral Let

$$I = \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx.$$

Use the proved property with $a = \pi$. Since $\sin(\pi - x) = \sin x$ and $\cos(\pi - x) = -\cos x$, squaring the latter leaves the denominator unchanged. Consequently,

$$I = \int_0^\pi \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx.$$

Add this identity to the defining expression for I . The numerator factors x and $\pi - x$ sum to π , giving

$$2I = \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} dx.$$

Step 3: Evaluate the remaining definite integral Write $J = \int_0^\pi \sin x / (1 + \cos^2 x) dx$. With $u = \cos x$, the endpoints are $u = 1$ at $x = 0$ and $u = -1$ at $x = \pi$. Hence

$$\begin{aligned}J &= - \int_1^{-1} \frac{du}{1 + u^2} = \int_{-1}^1 \frac{du}{1 + u^2} \\ &= [\tan^{-1} u]_{-1}^1 = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}.\end{aligned}$$

Thus $2I = \pi J = \pi^2/2$. Dividing by 2 completes the evaluation. The denominator is at least 1, so the integrand is continuous throughout the interval and all these definite integrals exist.

Step 4: Conclusion The reflection identity is proved, and $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}$.

Quick Tip to Solve: When $\sin x / (1 + \cos^2 x)$ accompanies x on $[0, \pi]$, reflect x to $\pi - x$ and add the two integrals before substituting $\cos x$.

17. If $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$ respectively are the position vectors of points A , B , C and D , then find the angle between the straight lines AB and CD . Find whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear or not. [4 marks]

Solution:

Concept:

- A displacement vector is obtained by subtracting the initial point's position vector from the final point's position vector: $\overrightarrow{AB} = \vec{b} - \vec{a}$.
- For nonzero direction vectors $\vec{u}, \vec{v}$, the smaller angle θ between their lines satisfies $\cos \theta = |\vec{u} \cdot \vec{v}| / (|\vec{u}||\vec{v}|)$. Vectors are collinear precisely when one is a scalar multiple of the other.

Step 1: Obtain the direction vectors of the two lines The position vectors give $A = (1, 1, 1)$, $B = (2, 5, 0)$, $C = (3, 2, -3)$ and $D = (1, -6, -1)$. Notice that the missing $\hat{k}$ component of B is zero. Subtract coordinates in the stated order:

$$\begin{aligned}\overrightarrow{AB} &= (2 - 1)\hat{i} + (5 - 1)\hat{j} + (0 - 1)\hat{k} = \hat{i} + 4\hat{j} - \hat{k}, \\ \overrightarrow{CD} &= (1 - 3)\hat{i} + (-6 - 2)\hat{j} + (-1 + 3)\hat{k} = -2\hat{i} - 8\hat{j} + 2\hat{k}.\end{aligned}$$

Both vectors are nonzero and therefore furnish valid directions for the respective straight lines.

Step 2: Calculate the angle using the scalar product Put $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{CD}$. Their scalar product and lengths are

$$\begin{aligned}\vec{u} \cdot \vec{v} &= 1(-2) + 4(-8) + (-1)(2) = -36, \\ |\vec{u}| &= \sqrt{1 + 16 + 1} = \sqrt{18}, \\ |\vec{v}| &= \sqrt{4 + 64 + 4} = \sqrt{72} = 2\sqrt{18}.\end{aligned}$$

Thus the smaller angle between the straight lines satisfies

$$\cos \theta = \frac{|-36|}{\sqrt{18}\sqrt{72}} = \frac{36}{36} = 1.$$

The absolute value matters here: reversing a line's direction vector does not change the line. If instead we compute the angle ϕ between the directed vectors themselves, then $\cos \phi = -1$, so that angle is 180° .

Step 3: Test collinearity of the displacement vectors Compare the corresponding components:

$$\frac{-2}{1} = \frac{-8}{4} = \frac{2}{-1} = -2.$$

Equivalently, $\overrightarrow{CD} = -2\overrightarrow{AB}$. A negative scalar multiple means that the vectors have opposite senses along a common direction. This satisfies the criterion for collinear vectors; collinearity of vectors does not require the four position points to lie on one common line.

Step 4: Conclusion The angle between the straight lines AB and CD is 0° . The vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear, since $\overrightarrow{CD} = -2\overrightarrow{AB}$; their directed-vector angle is 180° .

Quick Tip to Solve: Check component ratios first: a common ratio immediately gives collinearity. A negative ratio gives opposite vectors but a zero smaller angle between their lines.

18. Using properties of determinants, prove the following:
$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a).$$
 [4 marks]

Solution:

Concept:

- Adding one row to another row leaves a determinant unchanged. The same property holds for columns.
- A common factor in a row can be taken outside the determinant. Expansion along a column containing two zeros reduces a third-order determinant to a second-order determinant.

Step 1: Add rows to expose the first linear factor Denote the given determinant by Δ , and write $s = a + b + c$ for brevity. Apply $R_1 \leftarrow R_1 + R_2$, which preserves its value:

$$\begin{aligned} \Delta &= \begin{vmatrix} s & -c & -b \\ -c & s & -a \\ -b & -a & s \end{vmatrix} \\ &= \begin{vmatrix} a+b & a+b & -(a+b) \\ -c & s & -a \\ -b & -a & s \end{vmatrix}. \end{aligned}$$

In the new first row, the entries follow from $s - c = a + b$ and $-b - a = -(a + b)$. Taking the factor $a + b$ outside gives

$$\Delta = (a+b) \begin{vmatrix} 1 & 1 & -1 \\ -c & s & -a \\ -b & -a & s \end{vmatrix}.$$

Step 2: Expose the second factor by another row addition Now apply $R_2 \leftarrow R_2 + R_3$ to the remaining determinant. Its first row stays fixed, and the second row becomes

$$(-c - b, s - a, -a + s) = (-(b+c), b+c, b+c).$$

Therefore, factoring $b + c$ out of the second row,

$$\begin{aligned} \Delta &= (a+b) \begin{vmatrix} 1 & 1 & -1 \\ -(b+c) & b+c & b+c \\ -b & -a & s \end{vmatrix} \\ &= (a+b)(b+c) \begin{vmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ -b & -a & s \end{vmatrix}. \end{aligned}$$

These factors are taken out by the multilinearity property; no division by $a + b$ or $b + c$ is involved. Consequently the argument also applies when either factor vanishes.

Step 3: Create two zeros and expand the determinant Apply the column operation $C_1 \leftarrow C_1 + C_3$.

Quick Tip to Solve: Add the first two rows and then the last two rows: the factors $a+b$ and $b+c$ appear immediately. Adding column 3 to column 1 then creates two zeros.

The first two entries of the resulting first column are $1 - 1 = 0$ and $-1 + 1 = 0$, while the third is $-b + s = a + c$. Thus

$$\Delta = (a+b)(b+c) \begin{vmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ a+c & -a & s \end{vmatrix}.$$

Expand along the first column. The only nonzero contribution is from row 3, column 1; its cofactor sign is $(-1)^{3+1} = +1$. Hence

$$\begin{aligned} \Delta &= (a+b)(b+c)(a+c) \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} \\ &= (a+b)(b+c)(a+c)(1 \cdot 1 - (-1) \cdot 1). \end{aligned}$$

The remaining second-order determinant has value 2. Every operation used either preserves the determinant or extracts an explicit row factor.

Step 4: Conclusion Therefore $\Delta = 2(a+b)(b+c)(c+a)$, proving the required identity by properties of determinants.

Quick Tip to Solve: Add the first two rows and then the last two rows: the factors $a+b$ and $b+c$ appear immediately. Adding column 3 to column 1 then creates two zeros.

19. If $x = \cos t + \log \tan\left(\frac{t}{2}\right)$, $y = \sin t$, then find the values of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$. [4 marks]

Solution:

Concept:

- For parametric functions $x = x(t)$, $y = y(t)$, the first derivative is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided $dx/dt \neq 0$.
- The second derivative with respect to x is $\frac{d^2y}{dx^2} = \frac{d}{dt}(dy/dx) \div \frac{dx}{dt}$. It is different from the second derivative with respect to t .
- Apply the chain rule to $\log \tan(t/2)$; its derivative is $1/\sin t$ wherever the given expression is defined.

Step 1: Differentiate y with respect to t Since $y = \sin t$, differentiation gives

$$\frac{dy}{dt} = \cos t, \quad \frac{d^2y}{dt^2} = -\sin t.$$

Thus the first requested derivative can be evaluated directly at the specified parameter. For the derivative with respect to x , however, we must also differentiate $x(t)$; simply dividing the second derivatives would be incorrect.

Quick Tip to Solve: Simplify dy/dx to $\tan t$ before differentiating again, and divide by dx/dt once more.

Step 2: Differentiate x and form dy/dx Using the derivative of the logarithm and the chain rule,

$$\begin{aligned}\frac{d}{dt} \log \tan(t/2) &= \frac{\frac{1}{2} \sec^2(t/2)}{\tan(t/2)} \\ &= \frac{1}{2 \sin(t/2) \cos(t/2)} = \frac{1}{\sin t}.\end{aligned}$$

Consequently,

$$\frac{dx}{dt} = -\sin t + \frac{1}{\sin t} = \frac{\cos^2 t}{\sin t}, \quad \frac{dy}{dx} = \frac{\cos t}{\cos^2 t / \sin t} = \tan t.$$

At $t = \pi/4$, both sine and cosine are nonzero, and $dx/dt = 1/\sqrt{2}$, so all divisions used here are valid in a neighbourhood of the required parameter.

Step 3: Differentiate the first parametric derivative and substitute Apply the parametric second-derivative rule to $dy/dx = \tan t$:

$$\frac{d^2 y}{dx^2} = \frac{\sec^2 t}{\cos^2 t / \sin t} = \frac{\sin t}{\cos^4 t}.$$

The exact trigonometric values are $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$. Hence the evaluations are

$$-\sin(\pi/4) = -\frac{1}{\sqrt{2}}, \quad \frac{\sin(\pi/4)}{\cos^4(\pi/4)} = \frac{1/\sqrt{2}}{1/4} = 2\sqrt{2}.$$

Step 4: Conclusion At $t = \pi/4$, $\boxed{\frac{d^2 y}{dx^2} = -\frac{1}{\sqrt{2}}}$ and $\boxed{\frac{d^2 y}{dx^2} = 2\sqrt{2}}$.

Quick Tip to Solve: Simplify dy/dx to $\tan t$ before differentiating again, and divide by dx/dt once more.

20(a). Show that the relation R on $\mathbb{R}$ defined as $R = \{(a, b) : a \leq b\}$, is reflexive, and transitive but not symmetric. [4 marks]

Solution:

Concept:

- A relation R on a set A is reflexive if $(a, a) \in R$ for every $a \in A$.
- Symmetry requires $(a, b) \in R \Rightarrow (b, a) \in R$ for every pair. A single counterexample disproves symmetry.
- Transitivity requires that $(a, b) \in R$ and $(b, c) \in R$ together imply $(a, c) \in R$, for arbitrary $a, b, c \in A$.

Step 1: Prove reflexivity The relation is defined on all real numbers by

$$(a, b) \in R \iff a \leq b.$$

Let a be an arbitrary real number. The order relation satisfies $a \leq a$, since equality is included in the symbol $\leq$. Therefore $(a, a) \in R$. This argument holds for every real number, including negative

Quick Tip to Solve: Use arbitrary elements to prove reflexivity and transitivity, but choose unequal numbers such as 1 and 2 to disprove symmetry.

numbers and zero, so every diagonal pair is present in R . Thus R is reflexive.

Step 2: Prove transitivity Choose arbitrary real numbers a, b, c such that $(a, b) \in R$ and $(b, c) \in R$. By the definition of the relation these assumptions mean

$$a \leq b \quad \text{and} \quad b \leq c.$$

To see the resulting inequality explicitly, $b - a \geq 0$ and $c - b \geq 0$, so

$$c - a = (c - b) + (b - a) \geq 0.$$

Hence $a \leq c$, which means $(a, c) \in R$. This covers all triples for which both assumed pairs belong to R , including the cases where one or both inequalities are equalities. Thus R is transitive.

Step 3: Disprove symmetry by a counterexample Take the real numbers $a = 1$ and $b = 2$. Since $1 \leq 2$, the pair $(1, 2)$ belongs to R . But $2 \leq 1$ is false, so the reversed pair $(2, 1)$ does not belong to R .

$$(1, 2) \in R, \quad (2, 1) \notin R.$$

Symmetry is a universal requirement: every pair in the relation must have its reverse in the relation. The displayed pair violates that requirement and is therefore sufficient to disprove symmetry. Although pairs with equal entries do have their reverse in R , that does not make the entire relation symmetric.

Step 4: Conclusion Therefore, R is reflexive and transitive, but not symmetric.

Quick Tip to Solve: Use arbitrary elements to prove reflexivity and transitivity, but choose unequal numbers such as 1 and 2 to disprove symmetry.

OR

20(b). Prove that the function $f : \mathbb{N} \rightarrow \mathbb{N}$, defined by $f(x) = x^2 + x + 1$ is one-one but not onto. Find inverse of $f : \mathbb{N} \rightarrow S$, where S is range of f . [4 marks]

Solution:

Concept:

- A function is one-one if $f(a) = f(b)$ implies $a = b$ for any two elements of its domain.
- A function is onto if every element of its codomain has a preimage. Restricting the codomain to its range makes a function onto.
- A one-one and onto function has an inverse; solve $y = f(x)$ for the admissible domain value x .

Step 1: Prove that the function is one-one Let $a, b \in \mathbb{N}$ and suppose $f(a) = f(b)$. Then

$$\begin{aligned} a^2 + a + 1 &= b^2 + b + 1, \\ a^2 - b^2 + a - b &= 0, \\ (a - b)(a + b + 1) &= 0. \end{aligned}$$

Since a, b are natural numbers, $a + b + 1 > 0$. Therefore the second factor cannot be zero, so $a - b = 0$,

Quick Tip to Solve: Factor $f(a) - f(b)$ to prove injectivity, and use the missing codomain value 2 to disprove surjectivity.

giving $a = b$. This proves injectivity on the entire domain, rather than merely checking a few values.

Step 2: Show that the original codomain is not covered The natural number 2 belongs to the codomain $\mathbb{N}$. If it had a preimage, some $x \in \mathbb{N}$ would satisfy

$$x^2 + x + 1 = 2 \implies x(x + 1) = 1.$$

The product of two consecutive integers is even, and so cannot equal 1. Thus 2 has no preimage and $f : \mathbb{N} \rightarrow \mathbb{N}$ is not onto. This argument also works if the convention for natural numbers includes zero.

Step 3: Find the inverse after restricting the codomain Put $S = \{n^2 + n + 1 : n \in \mathbb{N}\}$. The function $f : \mathbb{N} \rightarrow S$ is onto by the definition of S , and is one-one by the preceding proof. Hence its inverse exists. For $y \in S$, solve

$$x^2 + x + 1 = y \implies x = \frac{-1 \pm \sqrt{4y - 3}}{2}.$$

The negative sign gives a negative number and is inadmissible in $\mathbb{N}$. If $y = n^2 + n + 1 \in S$, then $4y - 3 = (2n + 1)^2$; the positive sign therefore gives exactly n . This verifies that the expression maps every element of S back to its unique natural-number preimage.

Step 4: Conclusion Thus $f : \mathbb{N} \rightarrow \mathbb{N}$ is one-one but not onto. With codomain S , its inverse is

$$f^{-1} : S \rightarrow \mathbb{N}, \quad f^{-1}(y) = \frac{\sqrt{4y - 3} - 1}{2}.$$

Quick Tip to Solve: Factor $f(a) - f(b)$ to prove injectivity, and use the missing codomain value 2 to disprove surjectivity.

21. Find the equation of tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$. Also, write the equation of normal to the curve at the point of contact. [4 marks]

Solution:

Concept:

- The derivative dy/dx at a point on a curve is the slope of its tangent there. Parallel lines have equal slopes.
- A line through (x_1, y_1) with slope m has equation $y - y_1 = m(x - x_1)$.
- The normal is perpendicular to the tangent, so its slope is $-1/m$ when the tangent slope m is nonzero.

Step 1: Obtain the required tangent slope Rearrange the given line into slope-intercept form:

$$4x - 2y + 5 = 0 \implies y = 2x + \frac{5}{2}.$$

Its slope is 2. Hence the tangent parallel to this line must also have slope 2. For the curve $y = \sqrt{3x - 2}$,

Quick Tip to Solve: Set the derivative equal to the given line's slope first, then use the same contact point for both tangent and normal.

its real domain is $x \geq 2/3$, with $y \geq 0$. Differentiation gives

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}.$$

Step 2: Determine the point of contact Equate the derivative to the required slope:

$$\begin{aligned} \frac{3}{2\sqrt{3x-2}} &= 2, \\ \sqrt{3x-2} &= \frac{3}{4} \implies 3x-2 = \frac{9}{16} \implies x = \frac{41}{48}. \end{aligned}$$

The corresponding ordinate is $y = 3/4$, so the point of contact is $P(41/48, 3/4)$. Its abscissa is greater than $2/3$, ensuring the derivative exists.

Step 3: Write the tangent through the contact point Use slope 2 and the point P :

$$y - \frac{3}{4} = 2 \left(x - \frac{41}{48} \right) \implies y = 2x - \frac{23}{24}.$$

Multiplying by 24 gives the tangent in general form, $48x - 24y - 23 = 0$.

Step 4: Write the normal at the same point Since the tangent slope is 2, the normal slope is $-1/2$. Therefore

$$y - \frac{3}{4} = -\frac{1}{2} \left(x - \frac{41}{48} \right) \implies y = -\frac{x}{2} + \frac{113}{96}.$$

Multiplication by 96 yields $48x + 96y - 113 = 0$.

Step 5: Conclusion At $P(41/48, 3/4)$, the required lines are

$$\text{Tangent: } 48x - 24y - 23 = 0,$$

$$\text{Normal: } 48x + 96y - 113 = 0.$$

Quick Tip to Solve: Set the derivative equal to the given line's slope first, then use the same contact point for both tangent and normal.

22(a). Solve the differential equation: $x dy - y dx = \sqrt{x^2 + y^2} dx$, given that $y = 0$ when $x = 1$.

[4 marks]

Solution:

Concept:

- A homogeneous differential equation can be reduced by putting $y = vx$, where $v = y/x$, so that $dy/dx = v + x dv/dx$.
- On the interval $x > 0$ containing the initial point $x = 1$, $\sqrt{x^2 + y^2} = x\sqrt{1 + v^2}$. The standard integral is $\int dv/\sqrt{1 + v^2} = \log(v + \sqrt{1 + v^2})$.

Step 1: Reduce the equation using a homogeneous substitution Dividing the given equation by dx

Quick Tip to Solve: Put $y = vx$ so that the terms containing v cancel, and use the initial condition before squaring the radical equation.

gives

$$x \frac{dy}{dx} - y = \sqrt{x^2 + y^2}.$$

The initial point has positive x , so work first on $x > 0$. Put $y = vx$ and substitute $dy/dx = v + x dv/dx$:

$$x \left(v + x \frac{dv}{dx} \right) - vx = x \sqrt{1 + v^2}.$$

Cancelling the terms xv , and then dividing by x , yields

$$x \frac{dv}{dx} = \sqrt{1 + v^2}, \quad \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}.$$

Step 2: Integrate and determine the constant Integrating both sides,

$$\log \left(v + \sqrt{1 + v^2} \right) = \log x + C.$$

The logarithm on the left is defined since $\sqrt{1 + v^2} > |v|$. At $x = 1$, the condition $y = 0$ gives $v = 0$, hence $C = 0$. Exponentiating therefore gives

$$v + \sqrt{1 + v^2} = x.$$

Thus $\sqrt{1 + v^2} = x - v$. Squaring and simplifying gives

$$1 + v^2 = x^2 - 2xv + v^2, \quad 2xv = x^2 - 1.$$

Since $xv = y$, the dependent variable is now determined explicitly.

Step 3: Check the initial condition and the original radical For the expression obtained, $dy/dx = x$. Consequently,

$$x \frac{dy}{dx} - y = x^2 - \frac{x^2 - 1}{2} = \frac{x^2 + 1}{2}.$$

Also,

$$\sqrt{x^2 + y^2} = \sqrt{x^2 + \frac{(x^2 - 1)^2}{4}} = \sqrt{\frac{(x^2 + 1)^2}{4}} = \frac{x^2 + 1}{2}.$$

Both sides agree, and substituting $x = 1$ gives $y = 0$. This check also ensures that squaring has introduced no extraneous solution.

Step 4: Conclusion The required solution is $y = \frac{x^2 - 1}{2}$. It satisfies the original equation and the given initial condition; although derived on $x > 0$, this expression satisfies the equation for every real x .

Quick Tip to Solve: Put $y = vx$ so that the terms containing v cancel, and use the initial condition before squaring the radical equation.

22(b). Solve the differential equation: $(1 + x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$, **subject to the initial condition** $y(0) = 0$. [4 marks]

Solution:

Concept:

- A first-order linear equation has the form $dy/dx + P(x)y = Q(x)$. Multiplying by its integrating factor $I = e^{\int P(x) dx}$ turns its left side into $d(Iy)/dx$.
- The product rule gives $d[(1 + x^2)y]/dx = (1 + x^2) dy/dx + 2xy$. The initial condition fixes the otherwise arbitrary integration constant.

Step 1: Put the equation in linear form Transpose the term $-4x^2$ and divide by $1 + x^2$, which is positive for every real x . The equation becomes

$$\frac{dy}{dx} + \frac{2x}{1 + x^2}y = \frac{4x^2}{1 + x^2}.$$

Thus the coefficient of y is $P(x) = 2x/(1 + x^2)$, and the right-hand function is $Q(x) = 4x^2/(1 + x^2)$. No value of real x , including the initial point $x = 0$, is excluded by this division.

Step 2: Obtain the integrating factor and integrate Using the substitution $u = 1 + x^2$, $du = 2x dx$, the integrating factor is

$$I = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1 + x^2.$$

Multiplication of the linear equation by this factor gives

$$(1 + x^2)\frac{dy}{dx} + 2xy = 4x^2.$$

By the product rule, the entire left-hand side is one derivative:

$$\frac{d}{dx}[(1 + x^2)y] = 4x^2.$$

Integration with respect to x therefore yields

$$(1 + x^2)y = \int 4x^2 dx = \frac{4x^3}{3} + C.$$

Step 3: Apply the initial condition and verify At $x = 0$, the given value is $y(0) = 0$. Substitution in the integrated equation gives $0 = C$. Hence the expression for y has numerator $4x^3/3$ and denominator $1 + x^2$. For a direct check, differentiating it gives

$$\frac{dy}{dx} = \frac{4}{3} \frac{3x^2 + x^4}{(1 + x^2)^2}.$$

Quick Tip to Solve: Recognise $(1 + x^2)y' + 2xy$ immediately as the derivative of $(1 + x^2)y$; this gives the integrating factor without extra algebra.

Substituting into the left side of the printed equation,

$$\begin{aligned}(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 &= \frac{4x^2 + \frac{4}{3}x^4 + \frac{8}{3}x^4}{1+x^2} - 4x^2 \\ &= \frac{4x^2(1+x^2)}{1+x^2} - 4x^2 = 0.\end{aligned}$$

The expression also gives zero at $x = 0$, as required.

Step 4: Conclusion The solution satisfying $y(0) = 0$ is $y = \frac{4x^3}{3(1+x^2)}$, valid for all real x .

Quick Tip to Solve: Recognise $(1+x^2)y' + 2xy$ immediately as the derivative of $(1+x^2)y$; this gives the integrating factor without extra algebra.

23. Find the value of λ , so that the lines $\frac{1-x}{3} = \frac{7y-14}{\lambda} = \frac{z-3}{2}$ and $\frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not. [4 marks]

Solution:

Concept:

- Direction ratios must be read after putting each numerator in the form $x - x_0$, $y - y_0$, or $z - z_0$. A negative sign in a numerator changes the corresponding direction ratio.
- Two lines have perpendicular directions when the scalar product of their direction vectors is zero. To decide whether they intersect, one must separately check whether their parametric equations have a common point.

Step 1: Write the two lines in parametric form Let the common ratio for the first line be t , and for the second line be s . Solving the three component equations gives

$$\begin{aligned}L_1 : \quad x &= 1 - 3t, \quad y = 2 + \frac{\lambda}{7}t, \quad z = 3 + 2t, \\ L_2 : \quad x &= 1 - \frac{3\lambda}{7}s, \quad y = 5 + s, \quad z = 6 - 5s.\end{aligned}$$

Thus direction vectors for the lines are

$$\vec{d}_1 = \left(-3, \frac{\lambda}{7}, 2\right), \quad \vec{d}_2 = \left(-\frac{3\lambda}{7}, 1, -5\right).$$

In particular, the coefficients of x and z in the printed numerators must be retained: the first line has negative x -direction, and the second has negative z -direction.

Step 2: Apply the perpendicularity condition For the lines to be at right angles, their direction vectors

Quick Tip to Solve: Clear the numerator coefficients before taking the dot product, then test intersection independently: perpendicular directions need not give intersecting lines in space.

must satisfy $\vec{d}_1 \cdot \vec{d}_2 = 0$. Therefore

$$\begin{aligned}(-3) \left(-\frac{3\lambda}{7} \right) + \frac{\lambda}{7}(1) + 2(-5) &= 0, \\ \frac{10\lambda}{7} - 10 &= 0, \\ \lambda &= 7.\end{aligned}$$

As a check, the resulting vectors $(-3, 1, 2)$ and $(-3, 1, -5)$ have scalar product $9 + 1 - 10 = 0$.

Step 3: Test for a common point At $\lambda = 7$, the two lines become

$$L_1 : (x, y, z) = (1 - 3t, 2 + t, 3 + 2t), \quad L_2 : (x, y, z) = (1 - 3s, 5 + s, 6 - 5s).$$

Equality of the x -coordinates would require $t = s$. But equality of the y -coordinates would require $2 + t = 5 + s$, or $t - s = 3$. These two requirements contradict each other, so no pair of parameters can give a common point. It is unnecessary to solve the remaining coordinate equation once this contradiction is obtained. The vectors are also not proportional, since their equal first and second components would force the proportionality factor to be one, while their third components differ.

Step 4: Conclusion The required value is $\lambda = 7$. The lines do not intersect; they are skew lines whose directions are perpendicular.

Quick Tip to Solve: Clear the numerator coefficients before taking the dot product, then test intersection independently: perpendicular directions need not give intersecting lines in space.

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find the maximum volume of cone. [6 marks]

Solution:

Concept:

- The volume of a right circular cone of base radius a and altitude h is $V = \frac{1}{3}\pi a^2 h$. For an inscribed cone, the sphere's radius relates a and h , allowing the volume to be expressed in one variable.
- An interior maximum of a differentiable function occurs at a stationary point. The second derivative test verifies a local maximum, and comparison with the endpoints verifies the absolute maximum on the admissible interval.

Step 1: Relate the base radius and altitude Let O be the centre of the sphere of radius $r > 0$, A the cone's vertex on the sphere, and C the centre of its circular base. Let $AC = h$ and let the base radius be a . For a right circular cone, AC is perpendicular to the base plane. The centre of the circular section cut out by that plane lies on the perpendicular through O , so the cone's axis passes through O . Taking the coordinate of A along this axis to be r , the base plane has coordinate $r - h$. A point

Quick Tip to Solve: Use altitude as the variable: the sphere immediately gives $a^2 = 2rh - h^2$, making the volume a cubic whose derivative factors easily.

B on the base circumference satisfies $OB = r$ and $CB = a$. By the right triangle formed by the perpendicular distance from O to the base plane and the base radius,

$$a^2 + (r - h)^2 = r^2.$$

Hence

$$a^2 = r^2 - (r - h)^2 = 2rh - h^2.$$

This relation remains valid whether the base plane is above or below the sphere's centre, because the axial distance is squared. A non-degenerate inscribed cone has $0 < h < 2r$.

Step 2: Express and differentiate the volume Substituting the geometric relation into the volume formula gives

$$V(h) = \frac{\pi}{3}(2rh - h^2)h = \frac{\pi}{3}(2rh^2 - h^3).$$

Differentiating with respect to h , with the sphere's radius held fixed, gives

$$V'(h) = \frac{\pi}{3}(4rh - 3h^2) = \frac{\pi h}{3}(4r - 3h),$$

$$V'(h) = 0 \implies h = 0 \quad \text{or} \quad h = \frac{4r}{3}.$$

The value $h = 0$ is a degenerate endpoint, whereas $4r/3$ lies strictly between 0 and $2r$ and is the only interior stationary altitude.

Step 3: Verify that the stationary altitude gives the maximum The second derivative is

$$V''(h) = \frac{\pi}{3}(4r - 6h).$$

At $h = 4r/3$,

$$V''\left(\frac{4r}{3}\right) = -\frac{4\pi r}{3} < 0,$$

so the stationary point is a strict local maximum. It is also the absolute maximum: V is continuous on $[0, 2r]$, its endpoint values are $V(0) = V(2r) = 0$, and its volume is positive for every non-degenerate cone. Alternatively, $V'(h) > 0$ for $0 < h < 4r/3$, while $V'(h) < 0$ for $4r/3 < h < 2r$, so the volume increases up to this altitude and decreases thereafter.

Step 4: Evaluate the maximum volume For the maximizing altitude, the square of the base radius is

$$a^2 = 2r\left(\frac{4r}{3}\right) - \left(\frac{4r}{3}\right)^2 = \frac{8r^2}{9}.$$

Therefore the maximum volume is

$$V_{\max} = \frac{1}{3}\pi\left(\frac{8r^2}{9}\right)\left(\frac{4r}{3}\right) = \frac{32\pi r^3}{81}.$$

The factor r^3 is consistent with the dimensions of a volume.

Step 5: Conclusion The cone of maximum volume has altitude $\boxed{h = \frac{4r}{3}}$, and its maximum volume is

Quick Tip to Solve: Use altitude as the variable: the sphere immediately gives $a^2 = 2rh - h^2$, making the volume a cubic whose derivative factors easily.

$$\frac{32\pi r^3}{81}.$$

Quick Tip to Solve: Use altitude as the variable: the sphere immediately gives $a^2 = 2rh - h^2$, making the volume a cubic whose derivative factors easily.

25(a). If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find A^{-1} . Hence solve the following system of equations: $2x - 3y + 5z = 11$, $3x + 2y - 4z = -5$, $x + y - 2z = -3$. [6 marks]

Solution:

Concept:

- For a square matrix A , the inverse exists exactly when $|A| \neq 0$, and $A^{-1} = \frac{\text{adj } A}{|A|}$.
- The cofactor C_{ij} is $(-1)^{i+j}$ times the minor obtained by deleting row i and column j . The adjugate is the transpose of the cofactor matrix.
- With the variables ordered as $X = (x, y, z)^T$, the given equations become $AX = B$. Multiplication on the left by A^{-1} gives $X = A^{-1}B$.

Step 1: Evaluate the determinant Expanding along the first row, keeping the alternating signs, gives

$$\begin{aligned} |A| &= 2\{2(-2) - (-4)(1)\} + 3\{3(-2) - (-4)(1)\} \\ &\quad + 5\{3(1) - 2(1)\} \\ &= 2(0) + 3(-2) + 5(1) = -1. \end{aligned}$$

Thus the matrix is nonsingular and both its inverse and the unique solution of the system exist.

Step 2: Compute the cofactors and the inverse The nine cofactors, with their signs included, are

$$\begin{aligned} C_{11} &= -4 + 4 = 0, & C_{12} &= -(-6 + 4) = 2, \\ C_{13} &= 3 - 2 = 1, & C_{21} &= -\{6 - 5\} = -1, \\ C_{22} &= -4 - 5 = -9, & C_{23} &= -\{2 + 3\} = -5, \\ C_{31} &= 12 - 10 = 2, & C_{32} &= -\{-8 - 15\} = 23, \\ C_{33} &= 4 + 9 = 13. \end{aligned}$$

Transposing the cofactor matrix and dividing by the determinant yields

$$\text{adj } A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}.$$

Step 3: Use the inverse to solve the equations The coefficient matrix is exactly the given A , and the

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; then check the computed variables in all three original equations.

constants form $B = (11, -5, -3)^T$. Hence

$$\begin{aligned} X &= A^{-1}B \\ &= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} -5 + 6 \\ -22 - 45 + 69 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}. \end{aligned}$$

Check in the original equations: $2 - 6 + 15 = 11$, $3 + 4 - 12 = -5$, and $1 + 2 - 6 = -3$. All three constants are reproduced, verifying the matrix computation.

Step 4: Conclusion $A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}$, and the unique solution is $x = 1, y = 2, z = 3$.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; then check the computed variables in all three original equations.

OR

25(b). Obtain the inverse of the following matrix using elementary operations: $A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$. [6 marks]

Solution:

Concept:

- To find an inverse by elementary row operations, augment A with the identity matrix I and reduce $[A \mid I]$ to $[I \mid A^{-1}]$.
- Each row operation must be applied to the entire augmented row. Swapping rows, adding a multiple of another row and multiplying a row by a nonzero number preserve equivalence.
- Obtaining three nonzero pivots permits reduction to the identity and establishes that the inverse exists. The final right-hand block can be checked by multiplication with the original matrix.

Step 1: Form the augmented matrix and obtain the first pivot Write the matrix beside the identity and interchange R_1 and R_2 to obtain a first pivot equal to 1:

$$\left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 & 1 \end{array} \right].$$

Quick Tip to Solve: Swap the first two rows to start with a unit pivot, and combine the lower rows to create a third-column pivot of 1 before introducing fractions.

Now use $R_2 \rightarrow R_2 + R_1$ and $R_3 \rightarrow R_3 - 3R_1$, leaving the pivot row unchanged:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & -5 & -8 & 0 & -3 & 1 \end{array} \right].$$

Step 2: Obtain a unit pivot in the third column First multiply R_3 by 3, and then add $5R_2$ to it. In the left block, the second entry becomes $-15 + 15 = 0$, and the third entry becomes $-24 + 25 = 1$. The same operations in the right block give $(5, -4, 3)$. Thus

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 1 & 5 & -4 & 3 \end{array} \right].$$

Apply $R_2 \rightarrow R_2 - 5R_3$, followed by $R_2 \rightarrow \frac{1}{3}R_2$. The second row becomes $(0, 1, 0 \mid -8, 7, -5)$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -8 & 7 & -5 \\ 0 & 0 & 1 & 5 & -4 & 3 \end{array} \right].$$

Step 3: Clear the first row and verify the resulting block Use $R_1 \rightarrow R_1 - 3R_3$; this gives $(1, 2, 0 \mid -15, 13, -9)$. Then use $R_1 \rightarrow R_1 - 2R_2$. The resulting augmented matrix is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -8 & 7 & -5 \\ 0 & 0 & 1 & 5 & -4 & 3 \end{array} \right].$$

The left block is the identity, so the right block is the desired inverse. For an independent check, multiply it by the original matrix:

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

For example, the first product row is $(-1 - 8 + 10, 1 + 7 - 8, -1 - 5 + 6) = (1, 0, 0)$; the other rows likewise reproduce the identity.

Step 4: Conclusion By elementary row operations,

$$A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}.$$

Quick Tip to Solve: Swap the first two rows to start with a unit pivot, and combine the lower rows to create a third-column pivot of 1 before introducing fractions.

26. A manufacturer has three machine operators A , B and C . The first operator A produces 1% of defective items, whereas the other two operators B and C produces 5% and 7% defective items respectively. A is on the job for 50% of the time, B on the job 30% of the time and C on the job for 20% of the time. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by A ? [6 marks]

Solution:

Concept:

- Let A , B , C denote the events that the selected item was made by the corresponding operator, and let D denote the event that the item is defective. The three source events are mutually exclusive and exhaustive.
- The production shares give the prior probabilities $P(A) = 0.50$, $P(B) = 0.30$ and $P(C) = 0.20$. The defect rates are conditional probabilities: $P(D | A) = 0.01$, $P(D | B) = 0.05$ and $P(D | C) = 0.07$.
- The total probability theorem gives $P(D) = \sum_{X=A,B,C} P(X)P(D | X)$. Bayes' theorem reverses the conditioning: $P(A | D) = P(A)P(D | A)/P(D)$.

Step 1: Identify the prior probabilities and conditional defect rates An item is selected from the combined stockpile. The proportions of production associated with the three operators are the stated fractions of time on the job. Thus the probabilities that an item originated from A , B or C are respectively $1/2$, $3/10$ and $1/5$. These add to one, as every item must have one of these three sources. The information that the selected item is defective is new evidence, so the required probability is $P(A | D)$, rather than the prior probability $P(A)$ or the defect rate $P(D | A)$.

Step 2: Calculate the defective contribution from each operator Using the multiplication rule for conditional probability,

$$\begin{aligned} P(A \cap D) &= \frac{1}{2} \cdot \frac{1}{100} = \frac{5}{1000}, \\ P(B \cap D) &= \frac{3}{10} \cdot \frac{5}{100} = \frac{15}{1000}, \\ P(C \cap D) &= \frac{1}{5} \cdot \frac{7}{100} = \frac{14}{1000}. \end{aligned}$$

Each product measures the share of the entire stockpile that is both made by the specified operator and defective. The three contributions cannot overlap because an item has only one source.

Step 3: Apply total probability and then Bayes' theorem Since the defective item must come from one of the three operators,

$$P(D) = \frac{5}{1000} + \frac{15}{1000} + \frac{14}{1000} = \frac{34}{1000}.$$

Conditioning on the observed defect restricts the sample space to defective items. The fraction of this restricted sample space supplied by A is

$$P(A | D) = \frac{P(A \cap D)}{P(D)} = \frac{5/1000}{34/1000} = \frac{5}{34}.$$

As a check, the corresponding conditional contributions from B and C would be $15/34$ and $14/34$; together with the contribution from A , they sum to one.

Quick Tip to Solve: Multiply each production share by its defect rate first; divide A 's contribution by the sum of all three contributions, not by A 's production share.

Step 4: Conclusion The probability that the defective item was produced by operator A is $\frac{5}{34}$, approximately 0.1471.

Quick Tip to Solve: Multiply each production share by its defect rate first; divide A's contribution by the sum of all three contributions, not by A's production share.

27(a). Find the vector and Cartesian equations of the plane passing through the points $(2, 2, -1)$, $(3, 4, 2)$ and $(7, 0, 6)$. Also find the vector equation of a plane passing through $(4, 3, 1)$ and parallel to the plane obtained above. [6 marks]

Solution:

Concept:

- Two non-parallel vectors lying in a plane determine its direction. Their cross product gives a normal vector, perpendicular to every direction in that plane.
- A plane with normal $\vec{n}$ through a point with position vector $\vec{a}$ has equation $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$. For $\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$, its Cartesian form is $ax + by + cz = \vec{a} \cdot \vec{n}$.
- Parallel planes have proportional normal vectors. Thus, the same normal can be used when changing the point through which the plane passes.

Step 1: Form two directions in the required plane Let $A = (2, 2, -1)$, $B = (3, 4, 2)$ and $C = (7, 0, 6)$. Subtract the coordinates of A from those of the other two points:

$$\vec{AB} = \hat{i} + 2\hat{j} + 3\hat{k}, \quad \vec{AC} = 5\hat{i} - 2\hat{j} + 7\hat{k}.$$

Both vectors lie in the required plane because their endpoints are points of the plane.

Step 2: Obtain a normal vector Taking their cross product gives

$$\begin{aligned} \vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} \\ &= (14 + 6)\hat{i} - (7 - 15)\hat{j} + (-2 - 10)\hat{k} \\ &= 20\hat{i} + 8\hat{j} - 12\hat{k}. \end{aligned}$$

This is nonzero, so the three points are non-collinear and determine a unique plane. Dividing the normal by 4, choose $\vec{n} = 5\hat{i} + 2\hat{j} - 3\hat{k}$.

Step 3: Use the first point to determine the plane constant With $\vec{a} = 2\hat{i} + 2\hat{j} - \hat{k}$, the point-normal equation is

$$(\vec{r} - 2\hat{i} - 2\hat{j} + \hat{k}) \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 0.$$

Here $\vec{a} \cdot \vec{n} = 10 + 4 + 3 = 17$. For $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, expansion therefore gives $5x + 2y - 3z = 17$. Checking the remaining points gives $15 + 8 - 6 = 17$ and $35 + 0 - 18 = 17$, respectively, confirming that both also satisfy the equation.

Quick Tip to Solve: After simplifying the cross-product normal, substitute each given point to check the plane constant; for a parallel plane, only this constant changes.

Step 4: Determine the parallel plane through the specified point A plane parallel to the one obtained above has the same normal $5\hat{i} + 2\hat{j} - 3\hat{k}$. The given point has position vector $\vec{p} = 4\hat{i} + 3\hat{j} + \hat{k}$. Its plane constant is

$$\vec{p} \cdot \vec{n} = 5(4) + 2(3) - 3(1) = 23.$$

Thus its point-normal form is $(\vec{r} - \vec{p}) \cdot \vec{n} = 0$. The different constants, 17 and 23, show that these are distinct parallel planes.

Step 5: Conclusion The first plane has vector equation $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$ and Cartesian equation $5x + 2y - 3z = 17$. The required parallel plane has vector equation $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 23$.

Quick Tip to Solve: After simplifying the cross-product normal, substitute each given point to check the plane constant; for a parallel plane, only this constant changes.

OR

27(b). Find the vector equation of the plane that contains the lines $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and the point $(-1, 3, -4)$. Also, find the length of the perpendicular drawn from the point $(2, 1, 4)$ to the plane, thus obtained. [6 marks]

Solution:

Concept:

- A line $\vec{r} = \vec{a} + \lambda\vec{u}$ supplies a point with position vector $\vec{a}$ and a direction vector $\vec{u}$. A plane containing this line must contain that point and be parallel to that direction.
- If another point B lies in the plane, then $\vec{AB}$ is a second direction in it. A normal is $\vec{n} = \vec{u} \times \vec{AB}$, provided this cross product is nonzero.
- The perpendicular distance of (x_0, y_0, z_0) from $ax + by + cz + d = 0$ is $\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$. The absolute value ensures a non-negative length.

Step 1: Extract two directions in the plane The given line passes through $A = (1, 1, 0)$ and has direction vector $\vec{u} = \hat{i} + 2\hat{j} - \hat{k}$. The additional point is $B = (-1, 3, -4)$, so

$$\vec{AB} = (-1 - 1)\hat{i} + (3 - 1)\hat{j} + (-4 - 0)\hat{k} = -2\hat{i} + 2\hat{j} - 4\hat{k}.$$

Since both A and B belong to the required plane, this connecting vector also lies in it. It is not proportional to $\vec{u}$, so the line and the additional point determine a unique plane.

Step 2: Compute a normal to the plane A vector perpendicular to both directions is

$$\begin{aligned}\vec{u} \times \vec{AB} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 2 & -4 \end{vmatrix} \\ &= (-8 + 2)\hat{i} - (-4 - 2)\hat{j} + (2 + 4)\hat{k} \\ &= -6\hat{i} + 6\hat{j} + 6\hat{k}.\end{aligned}$$

Quick Tip to Solve: Use the line's base point and direction separately; verify the final plane by substituting the general line point before applying the distance formula.

Dividing by 6, take $\vec{n} = -\hat{i} + \hat{j} + \hat{k}$. Its scalar products with the two plane directions are $-1 + 2 - 1 = 0$ and $2 + 2 - 4 = 0$, confirming perpendicularity.

Step 3: Determine and verify the plane equation Through A , the point-normal equation is

$$(\vec{r} - \hat{i} - \hat{j}) \cdot (-\hat{i} + \hat{j} + \hat{k}) = 0.$$

Since $(\hat{i} + \hat{j}) \cdot \vec{n} = -1 + 1 = 0$, this simplifies to $\vec{r} \cdot (-\hat{i} + \hat{j} + \hat{k}) = 0$, whose Cartesian form is $-x + y + z = 0$. A general point of the given line is $(1 + \lambda, 1 + 2\lambda, -\lambda)$. Substitution yields

$$-(1 + \lambda) + (1 + 2\lambda) - \lambda = 0.$$

Hence the entire line lies in the plane. The point B satisfies it too, since $-(-1) + 3 - 4 = 0$.

Step 4: Calculate the perpendicular distance For $P = (2, 1, 4)$, use the Cartesian equation with coefficients $a = -1$, $b = 1$, $c = 1$ and $d = 0$. The length of the perpendicular is

$$D = \frac{|-2 + 1 + 4|}{\sqrt{(-1)^2 + 1^2 + 1^2}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

The denominator is the magnitude of the plane's normal, so this divides the scalar displacement along that normal by its length and gives the shortest distance to the plane.

Step 5: Conclusion The required vector equation is $\boxed{\vec{r} \cdot (-\hat{i} + \hat{j} + \hat{k}) = 0}$. The length of the perpendicular from $(2, 1, 4)$ to this plane is $\boxed{\sqrt{3} \text{ units}}$.

Quick Tip to Solve: Use the line's base point and direction separately; verify the final plane by substituting the general line point before applying the distance formula.

28(a). Using integration, find the area of triangle ABC , whose vertices are $A(2, 5)$, $B(4, 7)$ and $C(6, 2)$.

[6 marks]

Solution:

Concept:

- The area between two curves is $\int_a^b (y_{\text{upper}} - y_{\text{lower}}) dx$, where the limits are the horizontal endpoints of the region.
- The line through (x_1, y_1) and (x_2, y_2) has equation $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$. Split the integral wherever the upper boundary changes.

Step 1: Obtain the equations of the three sides For AB , the slope is $(7 - 5)/(4 - 2) = 1$, so $y - 5 = x - 2$, or $y = x + 3$. For AC , the slope is $(2 - 5)/(6 - 2) = -3/4$, so $y - 5 = -\frac{3}{4}(x - 2)$, or $y = \frac{13}{2} - \frac{3}{4}x$. For BC , the slope is $(2 - 7)/(6 - 4) = -5/2$, so $y - 7 = -\frac{5}{2}(x - 4)$, or $y = 17 - \frac{5}{2}x$. Each equation gives the stated coordinates at both endpoints, so these lines describe the three sides of the triangle.

Step 2: Identify the upper and lower boundaries The horizontal extent of the triangle is from $x = 2$ to $x = 6$. Side AC forms the lower boundary throughout this interval. The upper boundary is AB

Quick Tip to Solve: Split at the middle vertex's horizontal coordinate $x = 4$; subtract side AC from the appropriate upper side on each interval.

from $x = 2$ to $x = 4$, and BC from $x = 4$ to $x = 6$. Therefore, using integration,

$$S = \int_2^4 \left[(x+3) - \left(\frac{13}{2} - \frac{3x}{4} \right) \right] dx + \int_4^6 \left[\left(17 - \frac{5x}{2} \right) - \left(\frac{13}{2} - \frac{3x}{4} \right) \right] dx.$$

Step 3: Evaluate both integrals The first height is $\frac{7}{4}(x-2)$, and the second is $\frac{7}{4}(6-x)$. Both are nonnegative on their respective intervals, as required for an area calculation. Thus

$$S_1 = \frac{7}{4} \int_2^4 (x-2) dx = \frac{7}{8} [(x-2)^2]_2^4 = \frac{7}{2},$$

$$S_2 = \frac{7}{4} \int_4^6 (6-x) dx = \frac{7}{4} \left[6x - \frac{x^2}{2} \right]_4^6 = \frac{7}{2}.$$

Adding the areas of the two vertical-strip regions gives $S = S_1 + S_2 = \frac{7}{2} + \frac{7}{2}$.

Step 4: Conclusion The area of triangle ABC , found by integration, is 7 square units.

Quick Tip to Solve: Split at the middle vertex's horizontal coordinate $x = 4$; subtract side AC from the appropriate upper side on each interval.

OR

28(b). Find the area of the region lying about x -axis and included between the circle $x^2 + y^2 = 8x$ and inside of the parabola $y^2 = 4x$. [6 marks]

Solution:

Concept:

- For horizontal strips, the area between a right boundary $x_R(y)$ and a left boundary $x_L(y)$ is $\int_c^d [x_R(y) - x_L(y)] dy$.
- Completing the square gives the circle's centre and radius. The interior of the right-opening parabola $y^2 = 4x$ is the region $x \geq y^2/4$.

Step 1: Find the intersections and the upper-half limits The circle is $(x-4)^2 + y^2 = 16$, with centre $(4, 0)$ and radius 4. Substitute $y^2 = 4x$ into its equation:

$$x^2 + 4x = 8x \implies x(x-4) = 0.$$

Thus the intersections are $(0, 0)$ and $(4, \pm 4)$. For the region above the x -axis, the vertical limits are $y = 0$ and $y = 4$.

Step 2: Identify the boundaries of a horizontal strip At a height $0 \leq y \leq 4$, the circle has branches $x = 4 \pm \sqrt{16 - y^2}$, while the parabola has $x = y^2/4$. The region inside both curves extends from the parabola to the right branch of the circle. To verify the left boundary, put $t = \sqrt{16 - y^2}$, so $0 \leq t \leq 4$.

Quick Tip to Solve: Integrate with respect to y : horizontal strips run directly from $x = y^2/4$ to $x = 4 + \sqrt{16 - y^2}$, avoiding a split into two integrals in x .

Then

$$\frac{y^2}{4} - \left(4 - \sqrt{16 - y^2}\right) = t - \frac{t^2}{4} \geq 0.$$

Hence the parabola is to the right of the circle's left branch throughout the interval. The strip width is consequently $4 + \sqrt{16 - y^2} - y^2/4$.

Step 3: Integrate the strip widths The required area is

$$S = \int_0^4 \left(4 + \sqrt{16 - y^2} - \frac{y^2}{4}\right) dy.$$

To evaluate the radical term, use $y = 4 \sin \theta$, with $0 \leq \theta \leq \pi/2$. Then $dy = 4 \cos \theta d\theta$ and $\sqrt{16 - y^2} = 4 \cos \theta$, so

$$\begin{aligned} \int_0^4 \sqrt{16 - y^2} dy &= 16 \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= 8 \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = 4\pi. \end{aligned}$$

The remaining terms give $\int_0^4 4 dy = 16$ and $\int_0^4 y^2/4 dy = [y^3/12]_0^4 = 16/3$. Therefore $S = 16 + 4\pi - 16/3$.

Step 4: Conclusion The required area is $\boxed{\left(\frac{32}{3} + 4\pi\right)} \text{ square units}.$

Quick Tip to Solve: Integrate with respect to y : horizontal strips run directly from $x = y^2/4$ to $x = 4 + \sqrt{16 - y^2}$, avoiding a split into two integrals in x .

29. A manufacturer has employed 5 skilled men and 10 semi-skilled men and makes two models A and B of an article. The making of one item of model A requires 2 hours work by a skilled man and 2 hours work by a semi-skilled man. One item of model B requires 1 hour by a skilled man and 3 hours by a semi-skilled man. No man is expected to work more than 8 hours per day. The manufacturer's profit on an item of model A is ₹ 15 and on an item of model B is ₹ 10. How many of items of each model should be made per day in order to maximize daily profit? Formulate the above LPP and solve it graphically and find the maximum profit. [6 marks]

Solution:

Concept:

- A linear programming problem maximises a linear objective subject to linear resource constraints and non-negativity restrictions.
- Available labour is measured in man-hours. If x and y denote daily quantities of models A and B , respectively, the resource usage is obtained by multiplying each quantity by its labour requirement.
- For a nonempty bounded feasible polygon, a linear objective attains its maximum at a corner point. Thus, evaluate the profit at every vertex of the feasible region.

Step 1: Define variables and calculate daily resources Let x be the number of items of model A and y the number of items of model B made per day. Both quantities must be non-negative. Each of the

Quick Tip to Solve: Convert worker counts to man-hours first, then test every vertex; the intersection here uses both labour supplies fully and has integer coordinates.

five skilled workers can work at most eight hours, giving $5 \times 8 = 40$ skilled man-hours. Similarly, ten semi-skilled workers provide $10 \times 8 = 80$ semi-skilled man-hours each day. These are separate resources; unused time of one category cannot replace time of the other.

Step 2: Formulate the linear programming problem Model A uses two skilled hours and model B uses one, so the skilled labour restriction is $2x + y \leq 40$. Their semi-skilled requirements are two and three hours, respectively, giving $2x + 3y \leq 80$. The profit per item is Rs. 15 for A and Rs. 10 for B. Hence the LPP is

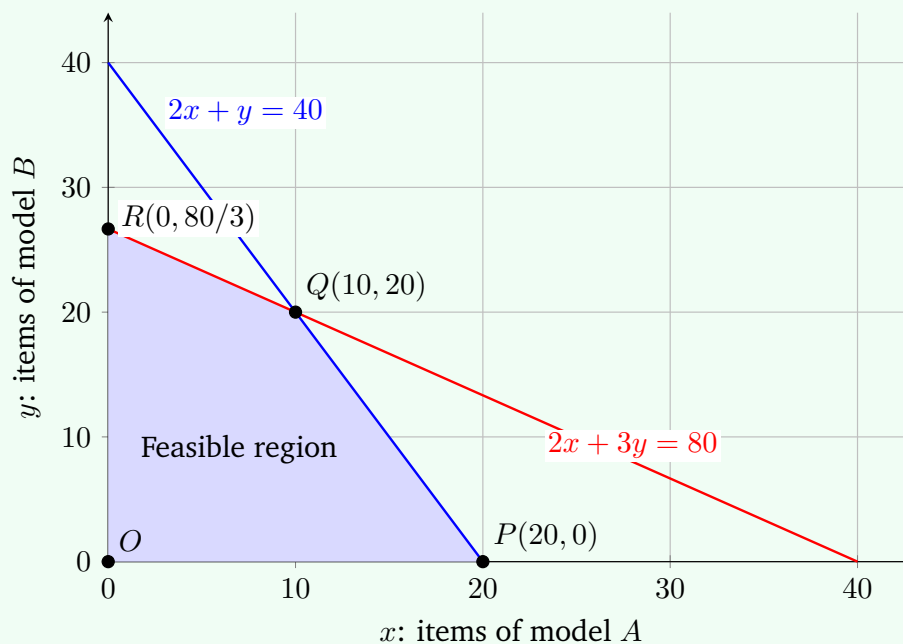
$$\begin{aligned} \text{Maximise } Z &= 15x + 10y, \\ \text{subject to } 2x + y &\leq 40, \\ 2x + 3y &\leq 80, \\ x &\geq 0, \quad y \geq 0. \end{aligned}$$

Here Z is the daily profit in rupees.

Step 3: Draw the constraint lines and identify the feasible region The line $2x + y = 40$ passes through $(20, 0)$ and $(0, 40)$. The line $2x + 3y = 80$ passes through $(40, 0)$ and $(0, 80/3)$. For each inequality the origin satisfies the restriction, so select the half-plane containing the origin. Intersect these half-planes with the first quadrant. The shaded polygon in the graph is the feasible region. To find the intersection of the two lines, subtract their equations:

$$(2x + 3y) - (2x + y) = 80 - 40 \implies 2y = 40.$$

Thus $y = 20$, and substituting into $2x + y = 40$ gives $x = 10$. The four vertices are therefore $O(0, 0)$, $P(20, 0)$, $Q(10, 20)$, and $R(0, 80/3)$.



The shaded polygon satisfies both labour constraints in the first quadrant.

Step 4: Evaluate the objective at all corner points Since the feasible region is bounded, the corner-

Quick Tip to Solve: Convert worker counts to man-hours first, then test every vertex; the intersection here uses both labour supplies fully and has integer coordinates.

point method applies. Calculate

Vertex (x, y)	$Z = 15x + 10y$ (rupees)
$(0, 0)$	0
$(20, 0)$	300
$(10, 20)$	$150 + 200 = 350$
$(0, 80/3)$	$800/3$

The largest table entry occurs at Q . At this point the skilled time used is $2(10) + 20 = 40$ hours and the semi-skilled time used is $2(10) + 3(20) = 80$ hours. Both daily resource limits are exactly satisfied. Moreover, the coordinates are whole numbers, so this feasible production plan can be implemented with complete items.

Step 5: Conclusion The manufacturer should produce items of model A and items of model B per day. The maximum daily profit is .

Quick Tip to Solve: Convert worker counts to man-hours first, then test every vertex; the intersection here uses both labour supplies fully and has integer coordinates.