

CBSE Class 12 Mathematics Question Paper 2019 with Solution PDF-Set 65-1-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) This question paper contains **29** questions divided into four sections A, B, C and D. Section A comprises of 4 questions of **one mark** each, Section B comprises of 8 questions of **two marks** each, Section C comprises of 11 questions of **four marks** each and Section D comprises of 6 questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is not permitted. You may ask logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. If $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, then find the matrix A .

[1 mark]

Solution:

Concept:

- Matrix addition and scalar multiplication act entry by entry. Matrices of the same order may be added.
- To isolate an unknown matrix in $3A - B = M$, add B to both sides and divide each entry by 3.

Step 1: Isolate the unknown matrix Adding the given matrix B to both sides gives

$$3A = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}.$$

The sign of every entry of B becomes positive when it is moved to the right-hand side.

Step 2: Add corresponding entries and divide Compute each entry independently:

$$3A = \begin{bmatrix} 5+4 & 0+3 \\ 1+2 & 1+5 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}.$$

Multiplication by $1/3$ applies to all four entries. As a check, multiplying the resulting matrix by 3 and subtracting B recovers the first given matrix.

Step 3: Conclusion

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

Quick Tip to Solve: Add B before dividing by 3, and divide every matrix entry, not just the diagonal entries.

2. Write the order and the degree of the following differential equation: $x^3 \left(\frac{d^2y}{dx^2} \right)^2 + x \left(\frac{dy}{dx} \right)^4 = 0$.

[1 mark]

Solution:

Concept:

- The order is the order of the highest derivative present in a differential equation.
- The degree is the power of the highest-order derivative when the equation is polynomial in its derivatives.

Step 1: Identify the highest-order derivative The equation contains dy/dx and d^2y/dx^2 . The sec-

Quick Tip to Solve: Choose the highest derivative first, then read its exponent; the exponent 4 on the first derivative is irrelevant.

ond derivative is the highest-order derivative, irrespective of the powers attached to the different derivatives. Thus the order is determined by d^2y/dx^2 .

Step 2: Inspect its power The given equation is already polynomial in its derivatives; there are no radicals or fractions containing derivatives to remove. The highest-order derivative d^2y/dx^2 is squared. The fourth power of the first derivative does not determine the degree, because degree refers specifically to the highest-order derivative.

Step 3: Conclusion

$$\boxed{\text{Order} = 2, \quad \text{Degree} = 2}.$$

Quick Tip to Solve: Choose the highest derivative first, then read its exponent; the exponent 4 on the first derivative is irrelevant.

3. If $f(x) = x + 1$, find $\frac{d}{dx}(f \circ f)(x)$.

[1 mark]

Solution:

Concept:

- Composition means $(f \circ f)(x) = f(f(x))$, so the output of the inner function becomes the input of the outer function.
- The derivative of $x + c$, with c constant, is 1.

Step 1: Form the composition Using $f(t) = t + 1$ for any input t , insert $t = f(x) = x + 1$. Hence

$$(f \circ f)(x) = f(x + 1) = (x + 1) + 1 = x + 2.$$

Composition here is repeated application, not multiplication of the two expressions.

Step 2: Differentiate the composed expression Differentiating $x + 2$ gives $1 + 0 = 1$. This also agrees with the chain rule: $f'(f(x))f'(x) = 1 \cdot 1$, since the derivative of f is 1 at every input.

Step 3: Conclusion

$$\boxed{\frac{d}{dx}(f \circ f)(x) = 1}.$$

Quick Tip to Solve: For $f(x) = x + 1$, each composition adds another constant, so the slope remains exactly 1.

4(a). If a line makes angles $90^\circ, 135^\circ, 45^\circ$ with the x, y and z axes respectively, find its direction cosines. [1 mark]

Solution:

Concept:

- The direction cosines l, m, n are the cosines of the angles made with the positive x, y, z axes respectively.
- They obey $l^2 + m^2 + n^2 = 1$; an obtuse axis angle has a negative cosine.

Step 1: Evaluate the three cosines In the given axis order,

$$l = \cos 90^\circ = 0, \quad m = \cos 135^\circ = -\frac{1}{\sqrt{2}}, \quad n = \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

The angle with the positive y -axis is obtuse, so its cosine is negative.

Step 2: Check the unit-direction identity The squares add to

$$0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 0 + \frac{1}{2} + \frac{1}{2} = 1.$$

Thus these values describe a valid direction. The zero first cosine means the line is perpendicular to the x -axis.

Step 3: Conclusion

$$(l, m, n) = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Quick Tip to Solve: Keep the x, y, z order and use the negative cosine for the obtuse angle 135 degrees.

OR

4(b). Find the vector equation of the line which passes through the point $(3, 4, 5)$ and is parallel to the vector $2\hat{i} + 2\hat{j} - 3\hat{k}$. [1 mark]

Solution:

Concept:

- A line through position vector $\vec{a}$, parallel to a nonzero vector $\vec{b}$, has equation $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\lambda \in \mathbb{R}$.
- The point coordinates give $\vec{a}$, while the supplied parallel vector provides the direction.

Step 1: Choose the point and direction vectors The point $(3, 4, 5)$ has position vector $\vec{a} = 3\hat{i} + 4\hat{j} + 5\hat{k}$. The stated parallel vector is $\vec{b} = 2\hat{i} + 2\hat{j} - 3\hat{k}$, which is nonzero and can be used directly without normalising it.

Step 2: Construct and check the parametrisation Substituting these vectors in $\vec{r} = \vec{a} + \lambda\vec{b}$ gives components $x = 3 + 2\lambda, y = 4 + 2\lambda, z = 5 - 3\lambda$. At $\lambda = 0$ the required point is obtained, and changing

Quick Tip to Solve: Use the given point as the constant vector and the given parallel vector as the coefficient of the real parameter.

λ produces a displacement parallel to $\vec{b}$.

Step 3: Conclusion

$$\vec{r} = (3\hat{i} + 4\hat{j} + 5\hat{k}) + \lambda(2\hat{i} + 2\hat{j} - 3\hat{k}), \quad \lambda \in \mathbb{R}.$$

Quick Tip to Solve: Use the given point as the constant vector and the given parallel vector as the coefficient of the real parameter.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5. Find $\int \sin x \cdot \log \cos x \, dx$.

[2 marks]

Solution:

Concept:

- Choose $u = \cos x$, so $du = -\sin x \, dx$; the factor $\sin x \, dx$ is therefore exactly $-du$.
- Integration by parts gives $\int \log u \, du = u \log u - u + C$, on $u > 0$.

Step 1: Reduce to a logarithmic integral Let $I = \int \sin x \log(\cos x) \, dx$. The printed real logarithm is defined on intervals where $\cos x > 0$. With $u = \cos x$ and $du = -\sin x \, dx$,

$$I = - \int \log u \, du.$$

The minus sign is essential because the derivative of cosine is negative sine.

Step 2: Apply integration by parts and substitute back Take the logarithm as the first factor and integrate 1 as the second:

$$\int \log u \, du = u \log u - \int u \frac{1}{u} \, du = u \log u - u.$$

Therefore

$$I = -u \log u + u + C = -\cos x \log(\cos x) + \cos x + C.$$

Differentiation verifies the result: the logarithmic term contributes $\sin x \log(\cos x) + \sin x$, while the final cosine contributes $-\sin x$, leaving the required integrand.

Step 3: Conclusion

$$\int \sin x \log(\cos x) \, dx = \cos x - \cos x \log(\cos x) + C.$$

Quick Tip to Solve: Substitute $u = \cos x$ first, preserve the minus sign, then integrate $\log u$ by parts.

6(a). Evaluate $\int_{-\pi}^{\pi} (1 - x^2) \sin x \cos^2 x \, dx$.

[2 marks]

Solution:

Concept:

- An odd function g satisfies $g(-x) = -g(x)$. For an integrable odd function, $\int_{-a}^a g(x) \, dx = 0$.
- The product of an even function and an odd function is odd; squaring cosine produces an even function.

Step 1: Establish the parity of the integrand Let $g(x) = (1 - x^2) \sin x \cos^2 x$. Both $1 - x^2$ and $\cos^2 x$ are even, whereas $\sin x$ is odd. Direct substitution confirms

$$g(-x) = (1 - x^2)(-\sin x) \cos^2 x = -g(x).$$

Thus the entire integrand, rather than just its sine factor, is odd.

Step 2: Use cancellation on the symmetric interval The integrand is continuous throughout $[-\pi, \pi]$, so the odd-function property applies. Explicitly, split at zero and set $t = -x$ in the negative half:

$$\int_{-\pi}^0 g(x) \, dx = \int_0^{\pi} g(-t) \, dt = - \int_0^{\pi} g(t) \, dt.$$

This cancels the integral over $[0, \pi]$, avoiding an unnecessary antiderivative.

Step 3: Conclusion

$$\boxed{\int_{-\pi}^{\pi} (1 - x^2) \sin x \cos^2 x \, dx = 0}.$$

Quick Tip to Solve: On symmetric limits, check the parity of the whole product before attempting integration.

OR

6(b). Evaluate $\int_{-1}^2 \frac{|x|}{x} \, dx$.

[2 marks]

Solution:

Concept:

- For $x < 0$, $|x| = -x$, while for $x > 0$, $|x| = x$. Hence $|x|/x$ equals -1 or 1 respectively.
- Split a modulus integral at the point where its expression changes. Here that point is $x = 0$, inside the integration interval.

Step 1: Write the piecewise integrand The function is

$$\frac{|x|}{x} = \begin{cases} -1, & -1 \leq x < 0, \\ 1, & 0 < x \leq 2. \end{cases}$$

Quick Tip to Solve: Treat $|x|/x$ as the sign function and subtract the length of the negative interval from the positive one.

Although the printed quotient is undefined at 0, the two one-sided integrals exist. Its value at this single point does not affect the integral, so we may evaluate by splitting at zero.

Step 2: Integrate on both sides of zero Using the appropriate constant on each interval gives

$$\begin{aligned} I &= \int_{-1}^0 (-1) dx + \int_0^2 1 dx \\ &= [-x]_{-1}^0 + [x]_0^2 \\ &= -1 + 2 = 1. \end{aligned}$$

Equivalently, the negative interval has length 1 and contributes -1 , while the positive interval has length 2 and contributes 2. These are signed contributions, so adding absolute areas would give an incorrect result.

Step 3: Conclusion

$$\int_{-1}^2 \frac{|x|}{x} dx = 1.$$

Quick Tip to Solve: Treat $|x|/x$ as the sign function and subtract the length of the negative interval from the positive one.

7. Examine whether the operation $*$ defined on $\mathbb{R}$ by $a * b = ab + 1$ is (i) a binary or not. (ii) if a binary operation, is it associative or not? [2 marks]

Solution:

Concept:

- A binary operation on a set assigns to every ordered pair of elements a unique element of the same set. Thus closure on $\mathbb{R}$ must be checked.
- Associativity requires $(a * b) * c = a * (b * c)$ for every $a, b, c \in \mathbb{R}$. One counterexample is enough to disprove it.

Step 1: Check whether the operation is binary For any real numbers a, b , their product ab is real, and adding the real number 1 again gives a real number. Moreover, the formula assigns a unique value to each ordered pair. Hence $a * b = ab + 1$ defines a map from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, so the operation is binary.

Step 2: Test associativity Applying the operation separately to the two possible groupings gives

$$\begin{aligned} (a * b) * c &= (ab + 1)c + 1 = abc + c + 1, \\ a * (b * c) &= a(bc + 1) + 1 = abc + a + 1. \end{aligned}$$

Their difference is $c - a$, which need not vanish. For instance, choose $a = 0, b = 0, c = 1$. Then $(0 * 0) * 1 = 1 * 1 = 2$, whereas $0 * (0 * 1) = 0 * 1 = 1$. Since these values are unequal, the defining identity does not hold for all triples.

Step 3: Conclusion (i) The operation is binary on $\mathbb{R}$. (ii) It is not associative.

Quick Tip to Solve: Compute both groupings symbolically first; their difference $c - a$ immediately supplies a counterexample with $a \neq c$.

8. Find a matrix A such that $2A - 3B + 5C = O$, where $B = \begin{bmatrix} -2 & 2 & 0 \\ 3 & 1 & 4 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 0 & -2 \\ 7 & 1 & 6 \end{bmatrix}$. [2 marks]

Solution:

Concept:

- Matrices of the same order are added and subtracted entry by entry. Multiplication by a scalar multiplies every matrix entry.
- The zero matrix O has the same order as the matrices in the equation. Isolating A gives $A = \frac{1}{2}(3B - 5C)$.

Step 1: Isolate the unknown matrix Both given matrices have two rows and three columns, so the unknown matrix A must also have order 2×3 . Starting from $2A - 3B + 5C = O$, add $3B - 5C$ to both sides to obtain

$$2A = 3B - 5C.$$

Division by the nonzero scalar 2 is performed entry by entry; no matrix inverse is needed.

Step 2: Calculate the entries The scalar multiples and their difference are

$$3B = \begin{bmatrix} -6 & 6 & 0 \\ 9 & 3 & 12 \end{bmatrix}, \quad 5C = \begin{bmatrix} 10 & 0 & -10 \\ 35 & 5 & 30 \end{bmatrix}.$$

Therefore

$$3B - 5C = \begin{bmatrix} -16 & 6 & 10 \\ -26 & -2 & -18 \end{bmatrix}.$$

Multiplying by $\frac{1}{2}$ gives

$$A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}.$$

As a check, twice this matrix is exactly $3B - 5C$, so substituting it makes every entry of $2A - 3B + 5C$ zero.

Step 3: Conclusion $A = \begin{bmatrix} -8 & 3 & 5 \\ -13 & -1 & -9 \end{bmatrix}$.

Quick Tip to Solve: First write $2A = 3B - 5C$; keeping the minus sign on $5C$ prevents entrywise sign errors.

9. A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event “number is even” and B be the event “number is marked red”. Find whether the events A and B are independent or not. [2 marks]

Solution:

Concept:

- For equally likely outcomes, $P(E) = n(E)/n(S)$, where S is the sample space and E is an event.
- Events A and B are independent exactly when $P(A \cap B) = P(A)P(B)$. The intersection consists

Quick Tip to Solve: Count the even red outcomes before multiplying probabilities: only 2 belongs to both events.

of outcomes satisfying both descriptions.

Step 1: List the events and their intersection For the die, the equally likely sample space is $S = \{1, 2, 3, 4, 5, 6\}$. The event that the number is even is $A = \{2, 4, 6\}$, and the event that the number is marked red is $B = \{1, 2, 3\}$. Only the outcome 2 is both even and red, so $A \cap B = \{2\}$. Consequently

$$P(A) = \frac{3}{6} = \frac{1}{2}, \quad P(B) = \frac{3}{6} = \frac{1}{2}, \quad P(A \cap B) = \frac{1}{6}.$$

Step 2: Apply the independence criterion The product of the individual probabilities is

$$P(A)P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}.$$

This differs from $P(A \cap B) = \frac{1}{6}$, so the required equality for independence fails. Equivalently, among the three red outcomes only one is even, giving $P(A | B) = \frac{1}{3}$, whereas $P(A) = \frac{1}{2}$. Knowledge that the result is red therefore changes the probability of an even number.

Step 3: Conclusion The events A and B are not independent.

Quick Tip to Solve: Count the even red outcomes before multiplying probabilities: only 2 belongs to both events.

10. Form the differential equation representing the family of curves $y = e^{2x}(a + bx)$, where a and b are arbitrary constants. [2 marks]

Solution:

Concept:

- To form a differential equation from a family containing two arbitrary constants, differentiate twice and eliminate those constants.
- The product rule is $(uv)' = u'v + uv'$. Here $(e^{2x})' = 2e^{2x}$, and $(a + bx)' = b$.

Step 1: Differentiate the given family Write $y = e^{2x}(a + bx)$, with a and b constant. By the product rule,

$$\begin{aligned} y' &= 2e^{2x}(a + bx) + be^{2x} \\ &= 2y + be^{2x}. \end{aligned}$$

Thus

$$y' - 2y = be^{2x}.$$

The constant a has disappeared from this relation, but b still has to be eliminated.

Step 2: Differentiate again and eliminate the remaining constant Differentiating $y' = 2y + be^{2x}$ once more gives

$$y'' = 2y' + 2be^{2x}.$$

Quick Tip to Solve: After the first differentiation, replace $e^{2x}(a + bx)$ by y ; this makes the second elimination immediate.

Substitute $be^{2x} = y' - 2y$ from the first derivative:

$$\begin{aligned}y'' &= 2y' + 2(y' - 2y) \\ &= 4y' - 4y.\end{aligned}$$

The resulting relation contains neither arbitrary constant. It is of order two, consistent with the two independent constants in the original family, and it is linear in y and its derivatives.

Step 3: Conclusion The required differential equation is $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

Quick Tip to Solve: After the first differentiation, replace $e^{2x}(a + bx)$ by y ; this makes the second elimination immediate.

11(a). A die is thrown 6 times. If “getting an odd number” is a “success”, what is the probability of (i) 5 successes? (ii) at most 5 successes? [2 marks]

Solution:

Concept:

- If X counts successes in n independent trials with success probability p , then $P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$.
- For a fair die, the three odd results 1, 3, 5 give $p = \frac{1}{2}$. Here $n = 6$, and X can take the integer values from 0 through 6.

Step 1: Find the probability of exactly five successes The six throws are independent and each has the same success probability $p = 3/6 = 1/2$, so X follows a binomial distribution with parameters 6 and $1/2$. For five successes there must be five odd results and one even result. The even result can occur in any of six positions. Hence

$$P(X = 5) = \binom{6}{5} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right) = \frac{6}{64} = \frac{3}{32}.$$

Step 2: Use the complement for at most five successes The phrase at most five includes all counts 0, 1, 2, 3, 4, 5. Since there are only six throws, the sole excluded count is 6. All six successes have probability

$$P(X = 6) = \binom{6}{6} \left(\frac{1}{2}\right)^6 = \frac{1}{64}.$$

Therefore

$$P(X \leq 5) = 1 - P(X = 6) = 1 - \frac{1}{64} = \frac{63}{64}.$$

This avoids adding six separate binomial terms and includes the possibility of no success.

Step 3: Conclusion (i) $\frac{3}{32}$. (ii) $\frac{63}{64}$.

Quick Tip to Solve: For at most five successes in six throws, subtract only the probability of six successes from 1.

11(b). The random variable X has a probability distribution $P(X)$ of the following form, where k is some number:

$$P(X = x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the value of k .

[2 marks]

Solution:

Concept:

- For a discrete probability distribution, every probability is nonnegative, and the probabilities over all possible values sum to 1.
- Only $x = 0, 1, 2$ have potentially nonzero probabilities here: their respective probabilities are $k, 2k, 3k$. Every other value contributes zero.

Step 1: Apply the total probability condition The outcomes $X = 0$, $X = 1$, and $X = 2$ are mutually exclusive. The distribution assigns probability zero to all remaining values, so their probabilities account for the entire total. Using the normalization condition gives

$$\begin{aligned} P(X = 0) + P(X = 1) + P(X = 2) &= 1, \\ k + 2k + 3k &= 1, \\ 6k &= 1. \end{aligned}$$

Hence the unique candidate for the constant is $k = \frac{1}{6}$.

Step 2: Verify that the candidate defines valid probabilities Substituting the candidate gives

$$P(X = 0) = \frac{1}{6}, \quad P(X = 1) = \frac{1}{3}, \quad P(X = 2) = \frac{1}{2}.$$

Each number lies between 0 and 1, and

$$\frac{1}{6} + \frac{1}{3} + \frac{1}{2} = \frac{1 + 2 + 3}{6} = 1.$$

All other probabilities remain zero, which is also admissible. Thus both nonnegativity and normalization are satisfied. In particular, the factors 1, 2, 3 are weights whose sum is 6, rather than values to be averaged.

Step 3: Conclusion The required value is $k = \frac{1}{6}$.

Quick Tip to Solve: Add the probability weights $1 + 2 + 3$, then take their reciprocal; check that the resulting probabilities sum to 1.

12(a). If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$. [2 marks]

Solution:

Concept:

- A unit vector has magnitude one. For vectors $\vec{a}$ and $\vec{b}$, the scalar product satisfies $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$, where θ is their included angle.
- The squared magnitude is obtained by taking the scalar product with itself: $|\vec{a} \pm \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 \pm 2\vec{a} \cdot \vec{b}$.

Step 1: Use the condition on the sum Let the given unit vectors be $\vec{a}$ and $\vec{b}$. Then $|\vec{a}| = |\vec{b}| = 1$, and the hypothesis also gives $|\vec{a} + \vec{b}| = 1$. Squaring and expanding the scalar product, we obtain

$$\begin{aligned} 1 &= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) \\ &= 1 + 1 + 2\vec{a} \cdot \vec{b}. \end{aligned}$$

Therefore $2\vec{a} \cdot \vec{b} = -1$, so $\vec{a} \cdot \vec{b} = -\frac{1}{2}$.

Step 2: Find the squared magnitude of the difference Using the same scalar-product identity with the minus sign,

$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b} \\ &= 1 + 1 - 2\left(-\frac{1}{2}\right) = 3. \end{aligned}$$

The negative scalar product is essential: subtracting it increases the squared magnitude. A magnitude is nonnegative, so its value is obtained from the positive square root.

Step 3: Conclusion Hence $|\vec{a} - \vec{b}| = \sqrt{3}$, as required.

Quick Tip to Solve: Add the squared sum and squared difference identities; their cross terms cancel, giving a total of four for two unit vectors.

OR

12(b). If $\vec{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - 2\hat{j} + \hat{k}$ and $\vec{c} = -3\hat{i} + \hat{j} + 2\hat{k}$, find $[\vec{a} \vec{b} \vec{c}]$. [2 marks]

Solution:

Concept:

- The scalar triple product $[\vec{a} \vec{b} \vec{c}]$ means $\vec{a} \cdot (\vec{b} \times \vec{c})$. It equals the determinant whose rows are the components of $\vec{a}, \vec{b}, \vec{c}$, in that order.
- Expansion along the first row uses alternating signs $+, -, +$. The scalar triple product is signed; interchanging two vectors changes its sign.

Step 1: Form the determinant in the stated order The components of the three given vectors are

Quick Tip to Solve: Write the vectors as rows in the given order and retain the minus sign on the middle cofactor; do not take an absolute value.

$(2, 3, 1)$, $(1, -2, 1)$ and $(-3, 1, 2)$, respectively. Thus

$$[\vec{a} \ \vec{b} \ \vec{c}] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix}.$$

Keeping the vector order unchanged avoids introducing an unwanted negative sign. In particular, the first component of $\vec{c}$ is -3 , and the middle component of $\vec{b}$ is -2 .

Step 2: Expand along the first row The three minors are $(-2)(2) - (1)(1) = -5$, $(1)(2) - (1)(-3) = 5$, and $(1)(1) - (-2)(-3) = -5$. Therefore

$$\begin{aligned} [\vec{a} \ \vec{b} \ \vec{c}] &= 2(-5) - 3(5) + 1(-5) \\ &= -10 - 15 - 5. \end{aligned}$$

Equivalently, $\vec{b} \times \vec{c} = -5\hat{i} - 5\hat{j} - 5\hat{k}$, whose scalar product with $2\hat{i} + 3\hat{j} + \hat{k}$ gives the same signed value.

Step 3: Conclusion The required scalar triple product is $[\vec{a} \ \vec{b} \ \vec{c}] = -30$.

Quick Tip to Solve: Write the vectors as rows in the given order and retain the minus sign on the middle cofactor; do not take an absolute value.

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13. Using properties of determinants, prove the following: $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$ [4 marks]

Solution:

Concept:

- Adding one row of a determinant to another leaves its value unchanged. A common factor from a row can be taken outside the determinant.
- For $S = a + b + c$, the identity $S(a^2 + b^2 + c^2 - ab - bc - ca) = a^3 + b^3 + c^3 - 3abc$ follows by multiplication and cancellation of the mixed terms.

Step 1: Create a row with equal entries Let the determinant on the left be D . Apply the determinant-preserving operation $R_3 \rightarrow R_3 + R_1$. Each entry of the new third row is $a + b + c$. Taking this

Quick Tip to Solve: Adding the first row to the third produces the common sum in every column; factor it out before expanding the determinant.

common factor out of that row gives

$$D = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ 1 & 1 & 1 \end{vmatrix}.$$

No division by $a+b+c$ has been used, so this step remains valid even when that sum is zero.

Step 2: Evaluate the remaining determinant Expand the smaller determinant along the first row with cofactor signs $+, -, +$:

$$D = (a+b+c) \{ a[(b-c) - (c-a)] \\ - b[(a-b) - (c-a)] \\ + c[(a-b) - (b-c)] \}.$$

The three bracketed differences are $a+b-2c$, $2a-b-c$, and $a-2b+c$, respectively. Hence

$$D = (a+b+c) \{ a(a+b-2c) \\ - b(2a-b-c) + c(a-2b+c) \}$$

$$= (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca).$$

For example, the coefficient of ab is $1 - 2 = -1$; similarly the coefficients of bc and ca are each -1 .

Step 3: Recognise the cubic factorisation Multiplying the two factors gives the pure cubic terms $a^3 + b^3 + c^3$. Each mixed square term, such as a^2b , occurs once with positive sign and once with negative sign, and therefore cancels. The product terms $-a(bc)$, $-b(ca)$, and $-c(ab)$ together give $-3abc$. Thus the expression obtained by row operations is exactly the right-hand side of the proposed identity.

Step 4: Conclusion Therefore $D = a^3 + b^3 + c^3 - 3abc$, proving the required identity using properties of determinants.

Quick Tip to Solve: Adding the first row to the third produces the common sum in every column; factor it out before expanding the determinant.

14. Solve: $\tan^{-1} 4x + \tan^{-1} 6x = \frac{\pi}{4}$.

[4 marks]

Solution:

Concept:

- The principal value of $\tan^{-1} t$ lies in $(-\pi/2, \pi/2)$ and has the same sign as t .
- The tangent addition formula is $\tan(A+B) = (\tan A + \tan B)/(1 - \tan A \tan B)$, when the denominator is nonzero. Taking tangents loses information about the angle, so every candidate must satisfy the original principal-value equation.

Quick Tip to Solve: Check the sign of the principal inverse tangents before taking tangents; this immediately eliminates the negative quadratic root.

Step 1: Restrict the sign of the unknown Put $A = \tan^{-1}(4x)$ and $B = \tan^{-1}(6x)$. If $x < 0$, both principal angles are negative and their sum cannot equal the positive angle $\pi/4$. If $x = 0$, their sum is zero. Thus every solution must satisfy $x > 0$, for which $A, B \in (0, \pi/2)$. This sign restriction will remove any extraneous negative root introduced by taking tangents.

Step 2: Use the addition formula and solve the quadratic Since the required sum is $\pi/4$, its tangent is 1. Its cosine is nonzero, so the addition formula has a nonzero denominator. Hence

$$\frac{4x + 6x}{1 - 24x^2} = 1, \quad 24x^2 + 10x - 1 = 0.$$

Applying the quadratic formula gives

$$x = \frac{-10 \pm \sqrt{100 + 96}}{48} = \frac{-10 \pm 14}{48}.$$

The algebraic candidates are therefore $x = 1/12$ and $x = -1/2$. The latter is incompatible with the necessary condition $x > 0$.

Step 3: Verify the remaining candidate with principal angles At $x = 1/12$, the two arguments are $4x = 1/3$ and $6x = 1/2$. Both are between zero and one, so each inverse tangent lies in $(0, \pi/4)$; their sum lies in $(0, \pi/2)$. Also,

$$\tan\left(\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{2}\right) = \frac{1/3 + 1/2}{1 - 1/6} = 1.$$

In $(0, \pi/2)$, the unique angle whose tangent is one is $\pi/4$. Thus the positive candidate satisfies the original equation. The rejected negative candidate has two negative principal angles, so its sum cannot be the required angle even though its tangent is one.

Step 4: Conclusion The unique solution is $x = \frac{1}{12}$.

Quick Tip to Solve: Check the sign of the principal inverse tangents before taking tangents; this immediately eliminates the negative quadratic root.

15(a). Show that the relation R on $\mathbb{R}$ defined as $R = \{(a, b) : a \leq b\}$, is reflexive, and transitive but not symmetric. [4 marks]

Solution:

Concept:

- A relation R on a set S is reflexive if $(a, a) \in R$ for every $a \in S$.
- It is symmetric if $(a, b) \in R$ implies $(b, a) \in R$ for every $a, b \in S$. One counterexample disproves symmetry.
- It is transitive if $(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$ for every $a, b, c \in S$.

Step 1: Prove reflexivity for every real number Here $R = \{(a, b) : a \leq b\}$ is a relation on $\mathbb{R}$. Let a be an arbitrary real number. The order relation on real numbers always satisfies $a \leq a$, because

Quick Tip to Solve: For an inequality relation, use arbitrary elements to prove reflexivity and transitivity, then choose two distinct ordered numbers to disprove symmetry.

equality is included in the symbol $\leq$. By the defining condition of R , it follows that $(a, a) \in R$. Since no restriction was placed on a , every element of $\mathbb{R}$ is related to itself. This establishes reflexivity on the whole given set.

Step 2: Prove transitivity for arbitrary related pairs Let $a, b, c \in \mathbb{R}$ and suppose $(a, b) \in R$ and $(b, c) \in R$. Translating these hypotheses using the definition of the relation gives

$$a \leq b \quad \text{and} \quad b \leq c.$$

The transitive property of the usual order of real numbers gives $a \leq c$. Thus the ordered pair (a, c) satisfies precisely the defining condition for membership in R , so $(a, c) \in R$. This reasoning works whether either inequality is strict or an equality, and for every triple satisfying the hypotheses. Hence it proves the required transitivity.

Step 3: Disprove symmetry using a counterexample Choose the real numbers $a = 1$ and $b = 2$. Since $1 \leq 2$, we have $(1, 2) \in R$. However, reversing the ordered pair would require $2 \leq 1$, which is false; therefore $(2, 1) \notin R$. Symmetry requires the reverse pair for every pair in the relation, whereas this one pair has no reverse pair in R . Consequently the symmetry condition fails. Although pairs of the form (a, a) do reverse to themselves, their presence does not make the entire relation symmetric.

Step 4: Conclusion Therefore R is reflexive and transitive, but not symmetric.

Quick Tip to Solve: For an inequality relation, use arbitrary elements to prove reflexivity and transitivity, then choose two distinct ordered numbers to disprove symmetry.

OR

15(b). Prove that the function $f : \mathbb{N} \rightarrow \mathbb{N}$, defined by $f(x) = x^2 + x + 1$ is one-one but not onto. Find inverse of $f : \mathbb{N} \rightarrow S$, where S is range of f . [4 marks]

Solution:

Concept:

- Injectivity means that $f(a) = f(b)$ implies $a = b$ for any two domain elements.
- Surjectivity means that every codomain element has a preimage. Here $\mathbb{N} = \{1, 2, 3, \dots\}$.
- A one-one function becomes bijective when its codomain is restricted to its range; solve $y = f(x)$ for the admissible input to find its inverse.

Step 1: Prove the one-one property Take arbitrary $a, b \in \mathbb{N}$ and suppose that their images are equal. Then

$$\begin{aligned} a^2 + a + 1 &= b^2 + b + 1, \\ a^2 - b^2 + a - b &= 0, \\ (a - b)(a + b + 1) &= 0. \end{aligned}$$

Since a, b are positive natural numbers, $a + b + 1 > 0$. Consequently the second factor is nonzero, and the first factor must vanish. Thus $a = b$. The implication holds for every pair of domain elements, so distinct inputs cannot share an image, proving injectivity.

Quick Tip to Solve: Factor the difference of equal images for injectivity, and reject the negative quadratic root when finding the inverse on the range.

Step 2: Exhibit an element with no preimage For every natural number x , we have $x \geq 1$, whence

$$f(x) = x^2 + x + 1 \geq 1 + 1 + 1 = 3.$$

The number 1 belongs to the codomain $\mathbb{N}$, yet no natural number maps to it. Therefore the function is not onto $\mathbb{N}$. It is sufficient to identify one codomain element without a preimage; there is no need to list all the omitted numbers. The actual range is $S = \{n^2 + n + 1 : n \in \mathbb{N}\}$, beginning with 3, 7, 13, 21, ...

Step 3: Derive the inverse on the range For $f : \mathbb{N} \rightarrow S$, every element of S has a preimage by the definition of range, and injectivity ensures that this preimage is unique. This restricted function is therefore bijective. Let $y = x^2 + x + 1 \in S$. Rearrange and use the quadratic formula:

$$x^2 + x + (1 - y) = 0 \implies x = \frac{-1 \pm \sqrt{4y - 3}}{2}.$$

The minus sign gives a negative value, which cannot belong to the domain $\mathbb{N}$; only the plus sign is admissible. To verify that this expression really maps the range into natural numbers, write $y = n^2 + n + 1$ with $n \in \mathbb{N}$. Then

$$\frac{\sqrt{4y - 3} - 1}{2} = \frac{\sqrt{(2n + 1)^2 - 1} - 1}{2} = \frac{2n + 1 - 1}{2} = n.$$

Thus it returns precisely the original input. Conversely, substituting this recovered input into f returns y , verifying the inverse relation in both directions.

Step 4: Conclusion The function is one-one but not onto $\mathbb{N}$. Its inverse after restricting its codomain to S is

$$f^{-1} : S \rightarrow \mathbb{N}, \quad f^{-1}(y) = \frac{\sqrt{4y - 3} - 1}{2}.$$

Quick Tip to Solve: Factor the difference of equal images for injectivity, and reject the negative quadratic root when finding the inverse on the range.

16. Find the equation of tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$. Also, write the equation of normal to the curve at the point of contact. [4 marks]

Solution:

Concept:

- At a differentiable point of a curve, the derivative dy/dx equals the slope of the tangent.
- Parallel lines have equal slopes. A line through (x_0, y_0) with slope m has equation $y - y_0 = m(x - x_0)$.
- The normal is perpendicular to the tangent. When the tangent slope m is finite and nonzero, the normal slope is $-1/m$.

Quick Tip to Solve: Equate the derivative to the given line's slope, locate the contact point, and use the negative reciprocal slope for the normal.

Step 1: Find the slope required for the tangent Express the given line in slope-intercept form:

$$4x - 2y + 5 = 0 \implies y = 2x + \frac{5}{2}.$$

Its slope is 2. Therefore the tangent sought must also have slope 2. The intercept of the given line does not determine the tangent's intercept; parallelism specifies the slope only. The contact point must be obtained from the derivative of the curve.

Step 2: Determine the point of contact For $y = \sqrt{3x - 2}$, the real domain is $x \geq 2/3$, and $y \geq 0$. At an interior point $x > 2/3$, differentiation gives

$$\frac{dy}{dx} = \frac{3}{2\sqrt{3x - 2}}.$$

Equating this slope to 2 gives

$$\begin{aligned}\frac{3}{2\sqrt{3x - 2}} &= 2, \\ \sqrt{3x - 2} &= \frac{3}{4}, \\ 3x - 2 &= \frac{9}{16}, \\ 3x &= \frac{41}{16}, \quad x = \frac{41}{48}.\end{aligned}$$

Thus the ordinate is $y = 3/4$, and the point of contact is $P(41/48, 3/4)$. This point is in the interior of the real domain and satisfies the original square-root equation. Since the positive square root was retained before squaring, no extraneous sign choice has been introduced.

Step 3: Write the tangent through the contact point Using slope 2 and point P , the tangent equation is

$$y - \frac{3}{4} = 2 \left(x - \frac{41}{48} \right).$$

Simplifying gives $y = 2x - 23/24$. Multiplying throughout by 24, it can also be written as $48x - 24y - 23 = 0$. Its slope is indeed 2, as required, and substituting the coordinates of P verifies that it passes through the contact point.

Step 4: Write the perpendicular normal The normal slope is the negative reciprocal of the tangent slope:

$$m_n = -\frac{1}{2}.$$

Through the same point P , its equation is

$$y - \frac{3}{4} = -\frac{1}{2} \left(x - \frac{41}{48} \right).$$

Hence $y = -x/2 + 113/96$, or $48x + 96y - 113 = 0$. The product of the tangent and normal slopes is $2(-1/2) = -1$, confirming perpendicularity; both lines pass through the same point of the curve.

Quick Tip to Solve: Equate the derivative to the given line's slope, locate the contact point, and use the negative reciprocal slope for the normal.

Step 5: Conclusion The contact point is $(\frac{41}{48}, \frac{3}{4})$. The required equations are

$$\text{Tangent: } 48x - 24y - 23 = 0,$$

$$\text{Normal: } 48x + 96y - 113 = 0.$$

Quick Tip to Solve: Equate the derivative to the given line's slope, locate the contact point, and use the negative reciprocal slope for the normal.

17(a). If $\log(x^2 + y^2) = 2 \tan^{-1} \left(\frac{y}{x} \right)$, show that $\frac{dy}{dx} = \frac{x+y}{x-y}$.

[4 marks]

Solution:

Concept:

- Regard y as a differentiable function of x and write $p = dy/dx$. The chain rule gives $d(y^2)/dx = 2yp$.
- For $u > 0$, $d(\log u)/dx = u'/u$; also $d(\tan^{-1} v)/dx = v'/(1 + v^2)$.
- The quotient rule gives $d(y/x)/dx = (xp - y)/x^2$. The given expression requires $x \neq 0$.

Step 1: Differentiate the logarithmic side Put $p = dy/dx$. Differentiating the inside of the logarithm first gives $2x + 2yp$. Therefore

$$\frac{d}{dx} \log(x^2 + y^2) = \frac{2x + 2yp}{x^2 + y^2}.$$

We retain the factor p because y depends on x ; treating y as constant would lose the essential implicit derivative.

Step 2: Differentiate the inverse tangent side Apply the chain rule and then the quotient rule to obtain

$$\begin{aligned} \frac{d}{dx} \left[2 \tan^{-1} \left(\frac{y}{x} \right) \right] &= \frac{2}{1 + y^2/x^2} \frac{xp - y}{x^2} \\ &= \frac{2(xp - y)}{x^2 + y^2}. \end{aligned}$$

The two sides now have the same denominator. This simplification avoids carrying separate fractions when collecting the derivative terms.

Step 3: Equate derivatives and isolate the derivative Differentiation of the given equality yields

$$\frac{2(x + yp)}{x^2 + y^2} = \frac{2(xp - y)}{x^2 + y^2}.$$

Since $x \neq 0$, the common denominator is positive. Cancel it and the factor 2:

$$x + yp = xp - y, \quad x + y = (x - y)p.$$

Thus we divide by $x - y$ wherever a finite derivative exists. If $x = y$, the last identity would force $2x = 0$ for finite p , contradicting $x \neq 0$; the requested formula accordingly applies away from such points.

Quick Tip to Solve: Simplify the inverse tangent derivative to denominator $x^2 + y^2$ before collecting all terms containing dy/dx .

Step 4: Conclusion Hence, wherever the finite derivative exists,

$$\boxed{\frac{dy}{dx} = \frac{x+y}{x-y}}.$$

Quick Tip to Solve: Simplify the inverse tangent derivative to denominator $x^2 + y^2$ before collecting all terms containing dy/dx .

OR

17(b). If $x^y - y^x = a^b$, find $\frac{dy}{dx}$.

[4 marks]

Solution:

Concept:

- When both the base and the exponent vary, write $u^v = e^{v \log u}$, for $u > 0$, before differentiating.
- The chain and product rules give $\frac{d}{dx} u^v = u^v \left(v' \log u + v \frac{u'}{u} \right)$. Here $y = y(x)$.
- The parameters a, b are constants, so $d(a^b)/dx = 0$. Real logarithmic differentiation is taken on $x > 0, y > 0$.

Step 1: Differentiate the first variable power Let $p = dy/dx$. The exponent in $x^y = e^{y \log x}$ is a product of two varying quantities. Its derivative is $p \log x + y/x$, so

$$\frac{d}{dx}(x^y) = x^y \left(p \log x + \frac{y}{x} \right).$$

Both terms must be retained: one comes from the varying exponent y , and the other from the varying base x .

Step 2: Differentiate the second variable power Similarly, write $y^x = e^{x \log y}$. The derivative of $x \log y$ is $\log y + xp/y$, because the derivative of $\log y$ is p/y . Hence

$$\frac{d}{dx}(y^x) = y^x \left(\log y + \frac{x}{y} p \right).$$

Although the base is y , differentiation is still with respect to x ; this is why its logarithmic derivative includes p .

Step 3: Differentiate the relation and collect terms Since a^b is constant, differentiation of the given equation gives

$$x^y \left(p \log x + \frac{y}{x} \right) - y^x \left(\log y + \frac{x}{y} p \right) = 0.$$

Expand and move the terms without p to the other side:

$$\left(x^y \log x - \frac{x}{y} y^x \right) p = y^x \log y - \frac{y}{x} x^y.$$

Quick Tip to Solve: Differentiate each variable power separately using its exponential form; never take the logarithm of the difference $x^y - y^x$ as if it were a quotient.

Division by the coefficient of p is valid at points where it is nonzero. The constants a, b select the curve, but disappear from its differentiated relation.

Step 4: Conclusion Therefore, at regular points with nonzero denominator,

$$\frac{dy}{dx} = \frac{y^x \log y - yx^{y-1}}{x^y \log x - xy^{x-1}}.$$

Quick Tip to Solve: Differentiate each variable power separately using its exponential form; never take the logarithm of the difference $x^y - y^x$ as if it were a quotient.

18. If $y = (\sin^{-1} x)^2$, prove that $(1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - 2 = 0$.

[4 marks]

Solution:

Concept:

- For $-1 < x < 1$, $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$. The notation $\sin^{-1} x$ means the principal inverse sine.
- The chain rule gives $(u^2)' = 2uu'$; the product rule gives $(uv)' = u'v + uv'$. A prime denotes differentiation with respect to x .

Step 1: Obtain a convenient first-derivative identity For the given function $y = (\sin^{-1} x)^2$, differentiate the outer square by the chain rule and then differentiate the inverse sine:

$$\frac{dy}{dx} = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}}.$$

Multiply both sides by $\sqrt{1-x^2}$ to obtain

$$\sqrt{1-x^2} \frac{dy}{dx} = 2 \sin^{-1} x.$$

This rearrangement is useful because differentiating a product produces the precise coefficient of the first derivative appearing in the required identity. We work on $(-1, 1)$, where the inverse sine is differentiable and the square-root denominator is nonzero.

Step 2: Differentiate again using the product rule First compute the derivative of the coefficient:

$$\frac{d}{dx} \sqrt{1-x^2} = \frac{1}{2} (1-x^2)^{-1/2} (-2x) = -\frac{x}{\sqrt{1-x^2}}.$$

Differentiating the first-derivative identity therefore gives

$$\sqrt{1-x^2} \frac{d^2 y}{dx^2} - \frac{x}{\sqrt{1-x^2}} \frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}.$$

The first term comes from differentiating dy/dx , whereas the second comes from differentiating its

Quick Tip to Solve: Multiply the first derivative by $\sqrt{1-x^2}$ before differentiating again; the product rule produces the desired coefficients directly.

coefficient. Multiplying every term by $\sqrt{1-x^2}$ clears the denominator and gives

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2.$$

The negative coefficient of the first derivative follows from the derivative of $1-x^2$, so no sign changes are introduced in this simplification.

Step 3: Conclusion Transposing the constant proves

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0}, \quad -1 < x < 1.$$

Quick Tip to Solve: Multiply the first derivative by $\sqrt{1-x^2}$ before differentiating again; the product rule produces the desired coefficients directly.

19. Prove that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, **hence evaluate** $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx$. [4 marks]

Solution:

Concept:

- Under the substitution $t = a - x$, the differential is $dx = -dt$. The bounds must be transformed along with the variable.
- Definite integration is linear. Adding two equal expressions for an integral can eliminate a factor such as x when the remaining function is symmetric about the midpoint.
- The standard antiderivative $\int du/(1+u^2) = \tan^{-1} u + C$ uses the principal inverse tangent.

Step 1: Prove the reflection property Assume the indicated definite integrals exist. In $\int_0^a f(x) dx$, set $t = a - x$. Then $dx = -dt$, and the limits $x = 0$ and $x = a$ become $t = a$ and $t = 0$, respectively. Consequently,

$$\int_0^a f(x) dx = - \int_a^0 f(a-t) dt = \int_0^a f(a-t) dt.$$

Replacing the dummy integration variable t by x proves

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx.$$

Reversing the limits accounts for the negative sign; the dummy-variable renaming does not change the value.

Step 2: Apply the property and add the two expressions Write

$$I = \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} dx.$$

Apply the proved property with $a = \pi$. Since $\sin(\pi - x) = \sin x$ and $\cos^2(\pi - x) = \cos^2 x$, it follows

Quick Tip to Solve: When the factor multiplying x is unchanged by $x \mapsto \pi - x$, add the original and reflected integrals to replace x by $\pi/2$.

that

$$I = \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx.$$

Add this expression to the original one. The numerator factors x and $\pi - x$ add to the constant π , giving

$$2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx.$$

The denominator is always positive, so the integral is continuous throughout these limits.

Step 3: Evaluate the remaining standard integral Set $u = \cos x$, so $du = -\sin x dx$. At $x = 0$, $u = 1$; at $x = \pi$, $u = -1$. Thus

$$\begin{aligned} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx &= - \int_1^{-1} \frac{du}{1 + u^2} \\ &= \int_{-1}^1 \frac{du}{1 + u^2} \\ &= [\tan^{-1} u]_{-1}^1 \\ &= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}. \end{aligned}$$

Hence the previous equation becomes $2I = \pi(\pi/2) = \pi^2/2$.

Step 4: Conclusion The reflection property is proved, and the required value is

$$\boxed{\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx = \frac{\pi^2}{4}}.$$

Quick Tip to Solve: When the factor multiplying x is unchanged by $x \mapsto \pi - x$, add the original and reflected integrals to replace x by $\pi/2$.

20. Find $\int \frac{\cos x}{(1 + \sin x)(2 + \sin x)} dx$.

[4 marks]

Solution:

Concept:

- Substitution converts a composite integrand into a rational function: if $t = \sin x$, then $dt = \cos x dx$.
- Partial fractions separate distinct linear factors. The standard antiderivative is $\int dt/(t + a) = \ln|t + a| + C$, where C is an arbitrary constant.

Step 1: Substitute for the sine term Let the required integral be I . Since the numerator is the derivative of $\sin x$, take $t = \sin x$, so that $dt = \cos x dx$. Therefore

$$I = \int \frac{dt}{(1 + t)(2 + t)}.$$

Both factors in the denominator are linear in the new variable, so a partial-fraction decomposition is appropriate. The original integrand is defined on intervals where $1 + \sin x \neq 0$; the other factor

Quick Tip to Solve: Substitute $t = \sin x$ first; the two factors differ by one, so their reciprocal difference immediately gives the partial fractions.

$2 + \sin x$ is always positive for real x .

Step 2: Determine the partial fractions Write

$$\frac{1}{(1+t)(2+t)} = \frac{A}{1+t} + \frac{B}{2+t}.$$

Multiplying by the common denominator gives the polynomial identity

$$1 = A(2+t) + B(1+t) = (A+B)t + (2A+B).$$

Comparison of coefficients yields $A+B=0$ and $2A+B=1$. Thus $A=1$ and $B=-1$. The minus sign is essential: the numerator of the difference is $(2+t) - (1+t) = 1$.

Step 3: Integrate and return to the original variable Using the decomposition,

$$\begin{aligned} I &= \int \left(\frac{1}{1+t} - \frac{1}{2+t} \right) dt \\ &= \ln|1+t| - \ln|2+t| + C \\ &= \ln \left| \frac{1+\sin x}{2+\sin x} \right| + C. \end{aligned}$$

As a check, differentiating the logarithmic expression gives

$$\frac{\cos x}{1+\sin x} - \frac{\cos x}{2+\sin x} = \frac{\cos x}{(1+\sin x)(2+\sin x)},$$

which recovers the printed integrand on every interval of its domain.

Step 4: Conclusion

$$I = \ln \left| \frac{1+\sin x}{2+\sin x} \right| + C.$$

Quick Tip to Solve: Substitute $t = \sin x$ first; the two factors differ by one, so their reciprocal difference immediately gives the partial fractions.

21(a). Solve the differential equation: $\frac{dy}{dx} - \frac{2x}{1+x^2} y = x^2 + 2$.

[4 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $dy/dx + P(x)y = Q(x)$, where P and Q are functions of the independent variable x .
- Its integrating factor is $\text{I.F.} = e^{\int P(x) dx}$. Multiplication by this factor turns the left-hand side into the derivative of $y \text{ I.F.}$.

Step 1: Identify the linear equation and its integrating factor Comparing the given equation with

Quick Tip to Solve: Keep the negative sign in P : it makes the integrating factor $1/(1+x^2)$, after which divide $x^2 + 2$ as $(x^2 + 1) + 1$.

$dy/dx + Py = Q$, we have

$$P(x) = -\frac{2x}{1+x^2}, \quad Q(x) = x^2 + 2.$$

The negative sign belongs to P . Since $1+x^2 > 0$ for every real x ,

$$\int P(x) dx = -\int \frac{2x}{1+x^2} dx = -\ln(1+x^2),$$
$$\text{I.F.} = e^{-\ln(1+x^2)} = \frac{1}{1+x^2}.$$

Step 2: Multiply through and integrate Multiplying the differential equation by $1/(1+x^2)$ gives

$$\frac{1}{1+x^2} \frac{dy}{dx} - \frac{2x}{(1+x^2)^2} y = \frac{x^2+2}{1+x^2}.$$

By the product rule, the left-hand side is $d[y/(1+x^2)]/dx$. Also, $x^2+2 = (1+x^2) + 1$. Hence

$$\frac{d}{dx} \left(\frac{y}{1+x^2} \right) = 1 + \frac{1}{1+x^2},$$
$$\frac{y}{1+x^2} = \int \left(1 + \frac{1}{1+x^2} \right) dx$$
$$= x + \tan^{-1} x + C.$$

No initial condition is supplied, so C remains arbitrary.

Step 3: Verify the obtained family Write $F(x) = x + \tan^{-1} x + C$, so $y = (1+x^2)F(x)$. The product rule gives $dy/dx = 2xF(x) + x^2 + 2$. Also, $2xy/(1+x^2) = 2xF(x)$. Subtracting these expressions recovers $dy/dx - 2xy/(1+x^2) = x^2 + 2$, the given equation, for every real x .

Step 4: Conclusion The general solution is

$$y = (1+x^2)(x + \tan^{-1} x + C), \quad C \in \mathbb{R}.$$

Quick Tip to Solve: Keep the negative sign in P : it makes the integrating factor $1/(1+x^2)$, after which divide x^2+2 as $(x^2+1)+1$.

OR

21(b). Solve the differential equation: $(x+1)\frac{dy}{dx} = 2e^{-y} - 1$; $y(0) = 0$.

[4 marks]

Solution:

Concept:

- Separate the variables by placing expressions involving y with dy and expressions involving x with dx .
- After multiplying by e^y , use $u = e^y$, since $du = e^y dy$. The initial condition determines the

Quick Tip to Solve: Multiply by e^y before separating so that $e^y dy$ becomes exactly du under the substitution $u = e^y$.

integration constant.

Step 1: Separate the differential equation Starting from $(x+1)y' = 2e^{-y} - 1$, multiply by e^y to obtain

$$(x+1)e^y \frac{dy}{dx} = 2 - e^y, \quad \frac{e^y}{2 - e^y} dy = \frac{dx}{x+1}.$$

At $(x, y) = (0, 0)$, both denominators are nonzero. Thus separation is valid near the initial point. The constant solution $e^y = 2$ excluded by division cannot satisfy $y(0) = 0$.

Step 2: Integrate and determine the constant With $u = e^y$ and $du = e^y dy$, integrate both sides:

$$\int \frac{du}{2 - u} = \int \frac{dx}{x+1},$$
$$-\log |2 - e^y| = \log |x+1| + C.$$

The minus sign on the left arises because the derivative of $2 - u$ is -1 . Combining logarithms gives $\log |(2 - e^y)(x+1)| = -C$. On an interval containing the initial point the signs are fixed, so write

$$(2 - e^y)(x+1) = K.$$

Substituting $x = 0$ and $y = 0$ yields $K = (2 - 1)(1) = 1$. Hence

$$e^y = 2 - \frac{1}{x+1} = \frac{2x+1}{x+1}.$$

Step 3: Verify the equation and real domain To take a real logarithm we require $(2x+1)/(x+1) > 0$. The maximal interval containing 0 is $x > -\frac{1}{2}$. Differentiating the logarithmic expression gives

$$y' = \frac{2}{2x+1} - \frac{1}{x+1} = \frac{1}{(2x+1)(x+1)}.$$

Therefore $(x+1)y' = 1/(2x+1)$. Also,

$$2e^{-y} - 1 = \frac{2(x+1)}{2x+1} - 1 = \frac{1}{2x+1}.$$

The two sides coincide. At $x = 0$, the logarithmic expression equals $\log 1 = 0$, so the initial condition is satisfied.

Step 4: Conclusion The required solution is $y = \log \left(\frac{2x+1}{x+1} \right)$, on the interval $x > -\frac{1}{2}$ containing the initial point.

Quick Tip to Solve: Multiply by e^y before separating so that $e^y dy$ becomes exactly du under the substitution $u = e^y$.

22. If $\hat{i} + \hat{j} + \hat{k}$, $2\hat{i} + 5\hat{j}$, $3\hat{i} + 2\hat{j} - 3\hat{k}$ and $\hat{i} - 6\hat{j} - \hat{k}$ respectively are the position vectors of points A , B , C and D , then find the angle between the straight lines AB and CD . Find whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear or not. [4 marks]

Solution:

Concept:

- For position vectors $\vec{a}, \vec{b}$, the direction vector from A to B is $\overrightarrow{AB} = \vec{b} - \vec{a}$.
- For nonzero direction vectors $\vec{u}, \vec{v}$, the angle between the straight lines is the smaller angle θ , with $\cos \theta = |\vec{u} \cdot \vec{v}| / (|\vec{u}||\vec{v}|)$.
- Two nonzero vectors are collinear when one is a scalar multiple of the other. A negative scalar makes their directions opposite.

Step 1: Obtain the direction vectors The position vectors are $\vec{a} = (1, 1, 1)$, $\vec{b} = (2, 5, 0)$, $\vec{c} = (3, 2, -3)$ and $\vec{d} = (1, -6, -1)$. Subtract the initial point's vector from the final point's vector:

$$\begin{aligned}\overrightarrow{AB} &= (2 - 1)\hat{i} + (5 - 1)\hat{j} + (0 - 1)\hat{k} \\ &= \hat{i} + 4\hat{j} - \hat{k}, \\ \overrightarrow{CD} &= (1 - 3)\hat{i} + (-6 - 2)\hat{j} + (-1 + 3)\hat{k} \\ &= -2\hat{i} - 8\hat{j} + 2\hat{k}.\end{aligned}$$

In particular, $\overrightarrow{CD} = -2\overrightarrow{AB}$. This already shows the two lines have parallel directions.

Step 2: Calculate the angle between the straight lines Compute the scalar product and magnitudes:

$$\begin{aligned}\overrightarrow{AB} \cdot \overrightarrow{CD} &= (1)(-2) + (4)(-8) + (-1)(2) = -36, \\ |\overrightarrow{AB}| &= \sqrt{1 + 16 + 1} = \sqrt{18}, \\ |\overrightarrow{CD}| &= \sqrt{4 + 64 + 4} = \sqrt{72}.\end{aligned}$$

A straight line has no preferred orientation, so its smaller angle with another line uses the absolute value of the dot product:

$$\cos \theta = \frac{|-36|}{\sqrt{18}\sqrt{72}} = \frac{36}{36} = 1.$$

If instead we retain the specified directions of the two vectors, their cosine is $-36/36 = -1$; their directed-vector angle is 180° . This differs from the smaller angle between the straight lines.

Step 3: Test collinearity of the vectors The coefficients satisfy $(-2)/1 = (-8)/4 = 2/(-1) = -2$. Thus every component has the same scalar ratio, rather than merely one matching component. Equivalently, the exact vector relation $\overrightarrow{CD} = -2\overrightarrow{AB}$ proves collinearity. The negative factor means the vectors are antiparallel. This statement concerns the direction vectors; it does not require all four points to lie on one common line.

Step 4: Conclusion The angle between the straight lines AB and CD is 0° . The vectors are collinear, since $\overrightarrow{CD} = -2\overrightarrow{AB}$; their own angle is 180° .

Quick Tip to Solve: Check proportional direction vectors first: a negative ratio gives opposite vector directions but a zero smaller angle between their lines.

23. Find the value of λ , so that the lines $\frac{1-x}{3} = \frac{7y-14}{\lambda} = \frac{z-3}{2}$ and $\frac{7-7x}{3\lambda} = \frac{y-5}{1} = \frac{6-z}{5}$ are at right angles. Also, find whether the lines are intersecting or not. [4 marks]

Solution:

Concept:

- The direction ratios of a line are the coefficients of its parameter in its coordinate equations. Numerator signs and numerical factors must be included when reading these ratios.
- Lines with direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ have perpendicular directions when $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$. Intersection additionally requires a common point satisfying all three coordinate equations.

Step 1: Read the directions from parameter equations Set the common ratio in the first line equal to t . Then

$$x = 1 - 3t, \quad y = 2 + \frac{\lambda}{7}t, \quad z = 3 + 2t.$$

Thus its direction ratios are $(-3, \lambda/7, 2)$. Similarly, setting the common ratio in the second line equal to s gives

$$x = 1 - \frac{3\lambda}{7}s, \quad y = 5 + s, \quad z = 6 - 5s,$$

so its direction ratios are $(-3\lambda/7, 1, -5)$. In particular, the numerator $6 - z$ contributes a negative direction ratio, while the factor 7 in $7 - 7x$ divides the corresponding denominator.

Step 2: Apply the condition for perpendicular directions The scalar product must vanish:

$$\begin{aligned} (-3) \left(-\frac{3\lambda}{7} \right) + \frac{\lambda}{7}(1) + 2(-5) &= 0, \\ \frac{10\lambda}{7} - 10 &= 0, \\ \lambda &= 7. \end{aligned}$$

For this value, the directions become $(-3, 1, 2)$ and $(-3, 1, -5)$; their dot product is $9 + 1 - 10 = 0$, verifying the angle condition directly.

Step 3: Test whether a common point exists Substitute the obtained value into both parameter equations. A common point would require

$$1 - 3t = 1 - 3s, \quad 2 + t = 5 + s, \quad 3 + 2t = 6 - 5s.$$

The first equation gives $t = s$, whereas the second gives $t - s = 3$. These conditions contradict each other. Hence no pair of parameter values can satisfy all three coordinates; the third equation cannot remove the contradiction. The two direction vectors are also not proportional, so the lines are not parallel. They are therefore skew, despite having perpendicular directions.

Step 4: Conclusion The required value is $\lambda = 7$. The lines do not intersect; they are skew lines with perpendicular direction vectors.

Quick Tip to Solve: Convert each line to parameter form before taking the dot product: this prevents losing the negative signs in $1 - x$ and $6 - z$, and the factor 7 in the numerators.

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3 . If building of tank costs ₹70 per square metre for the base and ₹45 per square metre for the sides, what is the cost of least expensive tank? [6 marks]

Solution:

Concept:

- For a rectangular tank, volume equals length times breadth times depth. The fixed volume and depth determine the area of its base.
- Total construction cost is the sum of the area of each surface multiplied by its own rate. An open tank has one base and four side walls, with no top surface.
- A differentiable cost function has a stationary value where its first derivative vanishes. A positive second derivative throughout the feasible domain establishes a strict minimum.

Step 1: Express the dimensions using the volume constraint Let the length and breadth of the rectangular base be x m and y m, where $x > 0$ and $y > 0$. The depth is fixed at 2 m. Therefore,

$$2xy = 8 \implies xy = 4, \quad y = \frac{4}{x}.$$

In particular, the base area is always 4 m^2 . Its cost does not depend on the choice of the base dimensions. The sides, however, change their total area when the length and breadth change.

Step 2: Form the total cost function The cost of the base is $70xy = 70 \times 4 = 280$ rupees. Two opposite side walls each have area $2x \text{ m}^2$, and the other two each have area $2y \text{ m}^2$. Hence their combined area is

$$2(2x) + 2(2y) = 4x + 4y \text{ square metres.}$$

At 45 rupees per square metre, the sides cost $180(x + y)$ rupees. Since the tank is open at the top, no top area is included. Thus the total cost, in rupees, is

$$C(x) = 280 + 180 \left(x + \frac{4}{x} \right), \quad x > 0.$$

Step 3: Find and classify the stationary point Differentiate with respect to the length:

$$C'(x) = 180 \left(1 - \frac{4}{x^2} \right).$$

Setting $C'(x) = 0$ gives $x^2 = 4$. Only the positive root is feasible for a length, so $x = 2$ m. The corresponding breadth is $y = 4/2 = 2$ m. To verify the minimum, differentiate again:

$$C''(x) = \frac{1440}{x^3} > 0 \quad \text{for every } x > 0.$$

Thus the cost function is strictly convex on the entire feasible domain, and this stationary point is its

Quick Tip to Solve: Use the volume to fix $xy = 4$ first: the base cost is constant, so minimise only $x + 4/x$, remembering that the open top contributes no cost.

unique global minimum. Equivalently, $C'(x) < 0$ for $0 < x < 2$ and $C'(x) > 0$ for $x > 2$, so the cost decreases up to this dimension and increases afterwards.

Step 4: Evaluate the minimum construction cost For the optimal tank, length, breadth and depth are all 2 m. Its volume is $2 \times 2 \times 2 = 8 \text{ m}^3$, as required. The base area is 4 m^2 , and the combined area of the four sides is 16 m^2 . Consequently,

$$\begin{aligned} C_{\min} &= 70(4) + 45(16) \\ &= 280 + 720 \\ &= 1000 \text{ rupees.} \end{aligned}$$

This also agrees with substituting $x = 2$ directly into the cost function. The equal length and breadth minimise the perimeter of the base while its area remains fixed, which explains why they minimise the side-wall cost.

Step 5: Conclusion The least expensive tank has a square base of side 2 m and depth 2 m. Its minimum construction cost is ₹1000.

Quick Tip to Solve: Use the volume to fix $xy = 4$ first: the base cost is constant, so minimise only $x + 4/x$, remembering that the open top contributes no cost.

25(a). If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, find A^{-1} . Hence, solve the system of equations $x + y + z = 6$, $x + 2z = 7$, $3x + y + z = 12$. [6 marks]

Solution:

Concept:

- For a square matrix A , the cofactor C_{ij} is $(-1)^{i+j}$ times the minor obtained by deleting row i and column j . The adjoint is the transpose of the cofactor matrix.
- If $|A| \neq 0$, then $A^{-1} = \frac{\text{adj } A}{|A|}$. A system $AX = B$, where X is the column of unknowns and B the column of constants, has the unique solution $X = A^{-1}B$.

Step 1: Calculate the determinant Expanding along the first row gives

$$\begin{aligned} |A| &= 1(0 \cdot 1 - 2 \cdot 1) - 1(1 \cdot 1 - 2 \cdot 3) + 1(1 \cdot 1 - 0 \cdot 3) \\ &= -2 - (-5) + 1 = 4. \end{aligned}$$

Since the determinant is nonzero, the inverse exists and the given equations have a unique solution.

Step 2: Find the cofactors and form the inverse The nine cofactors, including their alternating signs, are

$$\begin{aligned} C_{11} &= -2, & C_{12} &= 5, & C_{13} &= 1, \\ C_{21} &= -(1 - 1) = 0, & C_{22} &= 1 - 3 = -2, & C_{23} &= -(1 - 3) = 2, \\ C_{31} &= 2 - 0 = 2, & C_{32} &= -(2 - 1) = -1, & C_{33} &= 0 - 1 = -1. \end{aligned}$$

Quick Tip to Solve: After finding the inverse, multiply its rows by the constants column $(6, 7, 12)^T$; retain the common factor $1/4$ until all three dot products are complete.

Transposing the cofactor matrix, and then dividing by the determinant, gives

$$\text{adj } A = \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}, \quad A^{-1} = \frac{1}{4} \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}.$$

The transpose is essential: the matrix of cofactors itself is not the adjoint.

Step 3: Write the equations in matrix form and multiply The coefficients of x, y, z , in their printed order, form precisely the matrix A ; the absent y -term in the second equation has coefficient zero. Thus

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}.$$

Premultiplying both sides by A^{-1} , we obtain

$$\begin{aligned} \begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{4} \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} -12 + 24 \\ 30 - 14 - 12 \\ 6 + 14 - 12 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix}. \end{aligned}$$

Substitution checks every equation: $3 + 1 + 2 = 6$, $3 + 2(2) = 7$, and $3(3) + 1 + 2 = 12$.

Step 4: Conclusion Therefore,

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}, \quad \boxed{x = 3, y = 1, z = 2}.$$

Quick Tip to Solve: After finding the inverse, multiply its rows by the constants column $(6, 7, 12)^T$; retain the common factor $1/4$ until all three dot products are complete.

25(b). Find the inverse of the following matrix using elementary operations: $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$. [6 marks]

Solution:

Concept:

- To find A^{-1} by elementary row operations, augment A with the identity matrix I . Apply every row operation to the entire augmented matrix $[A | I]$.
- If the left block becomes I , the right block becomes A^{-1} . A row replacement $R_i \rightarrow R_i + kR_j$ preserves equivalence and is reversible by subtracting kR_j .

Step 1: Augment the matrix and eliminate the first-column entry Start with

$$[A | I] = \left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right].$$

Use $R_2 \rightarrow R_2 + R_1$. The first entry of the second row becomes zero, and the operation must also be performed on its identity-block entries:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 5 & -2 & 1 & 1 & 0 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right].$$

Step 2: Obtain a unit pivot in the second column The third-column entries -2 and 1 allow elimination without introducing fractions. Apply $R_2 \rightarrow R_2 + 2R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & -2 & 1 & 0 & 0 & 1 \end{array} \right].$$

Here the new second-column entry is $5 + 2(-2) = 1$, while the third-column entry is $-2 + 2(1) = 0$. Now use this updated second row in $R_3 \rightarrow R_3 + 2R_2$, giving

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 5 \end{array} \right].$$

The third-row right block is $(0, 0, 1) + 2(1, 1, 2) = (2, 2, 5)$.

Step 3: Reduce the first row and read the inverse Clear the second entry of the first row using $R_1 \rightarrow R_1 - 2R_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & -2 & -4 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 5 \end{array} \right].$$

Quick Tip to Solve: After $R_2 \rightarrow R_2 + R_1$, use $R_2 \rightarrow R_2 + 2R_3$ to create the pivot $5 - 4 = 1$; this avoids fractions throughout the reduction.

Finally apply $R_1 \rightarrow R_1 + 2R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 2 & 6 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 2 & 5 \end{array} \right] = [I \mid A^{-1}].$$

Thus the left block is exactly the identity, so the right block is the required inverse. Each step used an elementary operation on both blocks, as stipulated in the question.

Step 4: Verify by matrix multiplication An independent multiplication check gives

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 3+2-4 & 2+2-4 & 6+4-10 \\ -3+3 & -2+3 & -6+6 \\ -2+2 & -2+2 & -4+5 \end{bmatrix} = I.$$

This confirms that all signs and entries carried through the row operations are consistent.

Step 5: Conclusion The inverse obtained using elementary row operations is

$$A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}.$$

Quick Tip to Solve: After $R_2 \rightarrow R_2 + R_1$, use $R_2 \rightarrow R_2 + 2R_3$ to create the pivot $5 - 4 = 1$; this avoids fractions throughout the reduction.

26(a). Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by sides $x = 0$, $x = 4$, $y = 4$ and $y = 0$ into three equal parts. [6 marks]

Solution:

Concept:

- For vertical strips, the area between an upper curve $y = f(x)$ and a lower curve $y = g(x)$ is $\int_a^b [f(x) - g(x)] dx$, provided $f(x) \geq g(x)$ throughout the interval.
- Inside the given square, $x, y \geq 0$. Thus $y^2 = 4x$ has the branch $y = 2\sqrt{x}$, while $x^2 = 4y$ is $y = x^2/4$. The square has side 4 and total area 16 square units.

Step 1: Locate the intersections within the square Substituting $y = x^2/4$ into $y^2 = 4x$ gives

$$\frac{x^4}{16} = 4x \implies x(x^3 - 64) = 0.$$

Hence $x = 0$ or $x = 4$, and the corresponding values of y are 0 and 4. Both curves therefore join $(0, 0)$ to $(4, 4)$ inside the square. For $0 < x < 4$,

$$\frac{x^2}{4} < 2\sqrt{x},$$

Quick Tip to Solve: Express both parabolas as functions of x on the nonnegative square, establish their order, and use lower, middle and upper strip heights to cover the square exactly.

since dividing by $\sqrt{x} > 0$ reduces this to $x^{3/2} < 8$. Thus the curves enclose a middle region and leave one region below the lower curve and another above the upper curve.

Step 2: Integrate the region below the lower parabola The lower region is bounded by $y = 0$, $y = x^2/4$, and the right side of the square. Its vertical strips extend from $x = 0$ to $x = 4$, so its area is

$$A_1 = \int_0^4 \frac{x^2}{4} dx = \left[\frac{x^3}{12} \right]_0^4 = \frac{64}{12} = \frac{16}{3}.$$

The lower boundary is the horizontal side of the square, so no portion outside the square enters this integral.

Step 3: Integrate the region between the two parabolas For each vertical strip, subtract the lower ordinate $x^2/4$ from the upper ordinate $2\sqrt{x}$. The middle area is

$$\begin{aligned} A_2 &= \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx \\ &= \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12} \right]_0^4 \\ &= \frac{32}{3} - \frac{16}{3} = \frac{16}{3}. \end{aligned}$$

The verified ordering of the curves ensures that this integral is an area without a sign change.

Step 4: Integrate the region above the upper parabola The upper region lies between $y = 2\sqrt{x}$ and the top side $y = 4$. Its area is

$$\begin{aligned} A_3 &= \int_0^4 (4 - 2\sqrt{x}) dx \\ &= \left[4x - \frac{4}{3}x^{3/2} \right]_0^4 \\ &= 16 - \frac{32}{3} = \frac{16}{3}. \end{aligned}$$

As a check, $A_1 + A_2 + A_3 = 16$, which accounts for the entire area of the square.

Step 5: Conclusion Each of the three regions has area $\boxed{16/3}$ square units. Therefore the two parabolas divide the given square into three equal parts.

Quick Tip to Solve: Express both parabolas as functions of x on the nonnegative square, establish their order, and use lower, middle and upper strip heights to cover the square exactly.

26(b). Using integration, find the area of the triangle whose vertices are $(2, 3)$, $(3, 5)$ and $(4, 4)$.

[6 marks]

Solution:

Concept:

- The equation of a nonvertical line through (x_1, y_1) and (x_2, y_2) is $y - y_1 = m(x - x_1)$, where $m = (y_2 - y_1)/(x_2 - x_1)$.
- A region with upper boundary $y = f(x)$ and lower boundary $y = g(x)$ has area $\int_a^b [f(x) - g(x)] dx$. Split the integral wherever the boundary changes from one side of the triangle to another.

Step 1: Obtain equations of the three sides Denote the vertices by $A(2, 3)$, $B(3, 5)$, and $C(4, 4)$. For side AB , the slope is $(5 - 3)/(3 - 2) = 2$. Therefore

$$AB : y - 3 = 2(x - 2), \quad y = 2x - 1.$$

For side BC , the slope is $(4 - 5)/(4 - 3) = -1$, giving

$$BC : y - 5 = -(x - 3), \quad y = 8 - x.$$

Finally, the slope of AC is $(4 - 3)/(4 - 2) = 1/2$. Hence

$$AC : y - 3 = \frac{1}{2}(x - 2), \quad y = \frac{x}{2} + 2.$$

Each equation gives the stated coordinates at its two endpoints, confirming the boundary lines.

Step 2: Identify the strip heights on the two intervals The triangle extends horizontally from $x = 2$ to $x = 4$, with the upper vertex at $x = 3$. Throughout this range, side AC is the lower boundary. On $2 \leq x \leq 3$, side AB is the upper boundary, so the vertical height is

$$(2x - 1) - \left(\frac{x}{2} + 2\right) = \frac{3}{2}x - 3.$$

On $3 \leq x \leq 4$, side BC is the upper boundary, and the height becomes

$$(8 - x) - \left(\frac{x}{2} + 2\right) = 6 - \frac{3}{2}x.$$

Both heights are nonnegative on their respective intervals. They are zero at the outer vertices and agree at $x = 3$, where each equals $3/2$.

Quick Tip to Solve: Split at the middle vertex's x-coordinate, $x=3$: the upper boundary changes there, while the line joining the outer vertices stays the lower boundary.

Step 3: Integrate the left portion of the triangle The area between AB and AC for $2 \leq x \leq 3$ is

$$\begin{aligned} A_1 &= \int_2^3 \left(\frac{3}{2}x - 3 \right) dx \\ &= \left[\frac{3}{4}x^2 - 3x \right]_2^3 \\ &= \left(\frac{27}{4} - 9 \right) - (3 - 6) = \frac{3}{4}. \end{aligned}$$

Using the difference of the ordinates ensures that the integral includes only the triangular region, not the area beneath its lower side.

Step 4: Integrate the right portion and combine For $3 \leq x \leq 4$, the area between BC and AC is

$$\begin{aligned} A_2 &= \int_3^4 \left(6 - \frac{3}{2}x \right) dx \\ &= \left[6x - \frac{3}{4}x^2 \right]_3^4 \\ &= (24 - 12) - \left(18 - \frac{27}{4} \right) = \frac{3}{4}. \end{aligned}$$

These two portions cover the triangle and overlap only on the line $x = 3$, which has no area. Their combined area is therefore $A_1 + A_2 = 3/4 + 3/4 = 3/2$ square units.

Step 5: Conclusion Using integration, the area of the triangle is $\boxed{\frac{3}{2}}$ square units.

Quick Tip to Solve: Split at the middle vertex's x-coordinate, $x=3$: the upper boundary changes there, while the line joining the outer vertices stays the lower boundary.

27. A manufacturer has employed 5 skilled men and 10 semi-skilled men and makes two models A and B of an article. The making of one item of model A requires 2 hours work by a skilled man and 2 hours work by a semi-skilled man. One item of model B requires 1 hour by a skilled man and 3 hours by a semi-skilled man. No man is expected to work more than 8 hours per day. The manufacturer's profit on an item of model A is ₹15 and on an item of model B is ₹10. How many of items of each model should be made per day in order to maximize daily profit? Formulate the above LPP and solve it graphically and find the maximum profit. [6 marks]

Solution:

Concept:

- Let x and y denote the numbers of models A and B , respectively, produced per day. Production quantities satisfy $x \geq 0$ and $y \geq 0$.
- Each category of labour gives a separate resource constraint: total hours required cannot exceed the number of workers multiplied by their daily working hours.
- For a bounded feasible polygon, a linear objective function attains its maximum at a corner point. Evaluate the profit at every vertex of the common feasible region.

Step 1: Formulate the labour constraints and objective The five skilled workers provide $5 \times 8 = 40$ skilled hours per day, while the ten semi-skilled workers provide $10 \times 8 = 80$ semi-skilled hours per day. An item of A uses two hours from each category; an item of B uses one skilled hour and three semi-skilled hours. Thus the LPP is

$$\begin{aligned} &\text{Maximise } Z = 15x + 10y, \\ &\text{subject to } 2x + y \leq 40, \\ &\quad \quad \quad 2x + 3y \leq 80, \\ &\quad \quad \quad x \geq 0, \quad y \geq 0. \end{aligned}$$

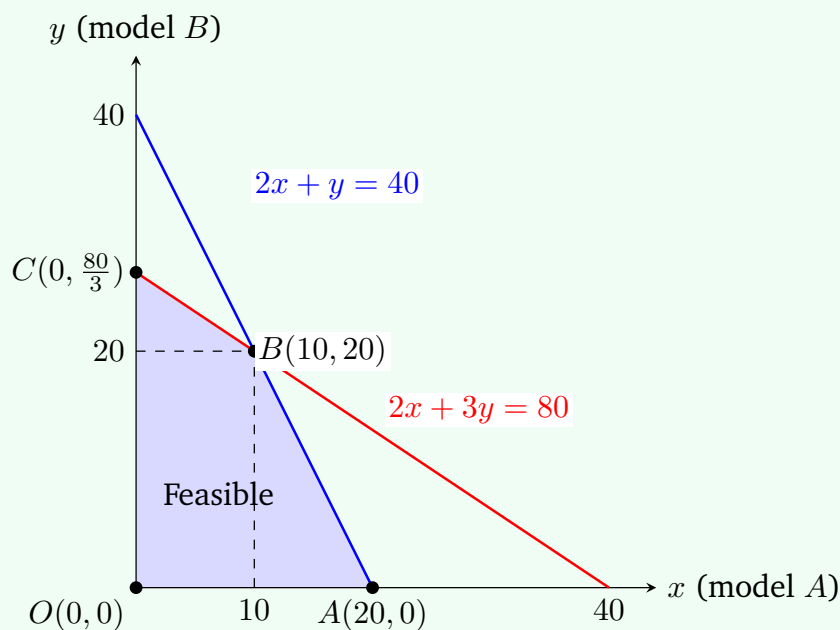
Here Z is the daily profit in rupees. Hours of the two worker categories cannot be combined into a single constraint because the models require different proportions of skilled and semi-skilled work.

Step 2: Draw the boundary lines and locate the feasible region The line $2x + y = 40$ passes through $(20, 0)$ and $(0, 40)$. The line $2x + 3y = 80$ passes through $(40, 0)$ and $(0, 80/3)$. For each inequality, the origin satisfies the constraint, so shade the side containing the origin. Restricting this intersection to the first quadrant gives the bounded quadrilateral $OABC$ shown below. To obtain the intersection of the two oblique boundaries, subtract their equations:

$$(2x + 3y) - (2x + y) = 80 - 40 \implies y = 20.$$

Substituting into $2x + y = 40$ gives $x = 10$. Hence the vertices are $O(0, 0)$, $A(20, 0)$, $B(10, 20)$ and $C(0, 80/3)$.

Quick Tip to Solve: Keep skilled and semi-skilled hours in separate inequalities, then subtract the two boundary equations to find the intersection quickly.



The shaded polygon is the common feasible region in the first quadrant.

Step 3: Compare the profit at all corner points Evaluate the same objective function at each feasible vertex:

Vertex (x, y)	$Z = 15x + 10y$ (rupees)
$(0, 0)$	0
$(20, 0)$	300
$(10, 20)$	$150 + 200 = 350$
$(0, 80/3)$	$800/3$

The largest value occurs at B . This vertex gives whole numbers of articles, so the graphical optimum is also a practicable production schedule. It uses $2(10) + 20 = 40$ skilled hours and $2(10) + 3(20) = 80$ semi-skilled hours; both available resources are fully utilised. The two incident edges have different profit values at their other endpoints, so there is no edge of alternative optimal schedules.

Step 4: Conclusion The manufacturer should produce 10 items of model A and 20 items of model B per day. The maximum daily profit is ₹350.

Quick Tip to Solve: Keep skilled and semi-skilled hours in separate inequalities, then subtract the two boundary equations to find the intersection quickly.

28(a). Find the vector and Cartesian equations of the plane passing through the points $(2, 2, -1)$, $(3, 4, 2)$ and $(7, 0, 6)$. Also find the vector equation of a plane passing through $(4, 3, 1)$ and parallel to the plane obtained above. [6 marks]

Solution:

Concept:

- A plane through a point with position vector $\vec{a}$, perpendicular to a nonzero normal vector $\vec{n}$, has equation $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.
- If A, B, C are three noncollinear points in a plane, the cross product $\overrightarrow{AB} \times \overrightarrow{AC}$ supplies a normal vector. A nonzero cross product also verifies that the three points determine a unique plane.
- Parallel planes have proportional normal vectors. To obtain a parallel plane through a new point, retain the normal and recompute the constant by taking its dot product with that point's position vector.

Step 1: Obtain two independent vectors within the plane Denote the three given points by $A(2, 2, -1)$, $B(3, 4, 2)$ and $C(7, 0, 6)$. Subtract the coordinates of A from those of the other points:

$$\begin{aligned}\overrightarrow{AB} &= (3 - 2)\hat{i} + (4 - 2)\hat{j} + (2 + 1)\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}, \\ \overrightarrow{AC} &= (7 - 2)\hat{i} + (0 - 2)\hat{j} + (6 + 1)\hat{k} = 5\hat{i} - 2\hat{j} + 7\hat{k}.\end{aligned}$$

These vectors lie in the desired plane and are not proportional, so their cross product can be used to find its normal.

Step 2: Calculate a normal and write the first plane The cross product is

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} \\ &= (14 + 6)\hat{i} - (7 - 15)\hat{j} + (-2 - 10)\hat{k} \\ &= 20\hat{i} + 8\hat{j} - 12\hat{k}.\end{aligned}$$

Dividing by four gives the simpler normal $\vec{n} = 5\hat{i} + 2\hat{j} - 3\hat{k}$. Its nonzero value establishes noncollinearity. Using A , the vector equation of the plane is

$$[\vec{r} - (2\hat{i} + 2\hat{j} - \hat{k})] \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 0,$$

or equivalently $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17$. With $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, the Cartesian equation becomes

$$5(x - 2) + 2(y - 2) - 3(z + 1) = 0, \quad \text{that is} \quad 5x + 2y - 3z = 17.$$

As a direct check, the left side takes values $10 + 4 + 3 = 17$, $15 + 8 - 6 = 17$ and $35 + 0 - 18 = 17$ at A, B, C , respectively. Thus every given point lies on this plane.

Step 3: Find the parallel plane through the new point Let $D(4, 3, 1)$. A plane parallel to the one just obtained has the same normal $5\hat{i} + 2\hat{j} - 3\hat{k}$. Therefore its required vector equation is

$$[\vec{r} - (4\hat{i} + 3\hat{j} + \hat{k})] \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 0.$$

Quick Tip to Solve: Simplify the cross-product normal before forming the equation; for the parallel plane only the dot-product constant changes.

Taking the dot product of $\overrightarrow{OD}$ with the normal gives $5(4) + 2(3) - 3(1) = 23$, so this can also be written as $\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 23$. The differing constants show that the two planes are distinct, and their common normal verifies parallelism.

Step 4: Conclusion The plane through the three given points has vector and Cartesian equations

$$\boxed{\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 17}, \quad \boxed{5x + 2y - 3z = 17}.$$

The required parallel plane through $(4, 3, 1)$ is

$$\boxed{\vec{r} \cdot (5\hat{i} + 2\hat{j} - 3\hat{k}) = 23}.$$

Quick Tip to Solve: Simplify the cross-product normal before forming the equation; for the parallel plane only the dot-product constant changes.

OR

28(b). Find the vector equation of the plane that contains the lines $\vec{r} = (\hat{i} + \hat{j}) + \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and the point $(-1, 3, -4)$. Also, find the length of the perpendicular drawn from the point $(2, 1, 4)$ to the plane, thus obtained. [6 marks]

Solution:

Concept:

- A plane containing two independent direction vectors has a normal vector perpendicular to both; their cross product gives such a normal.
- If a plane passes through a point with position vector $\vec{a}$ and has normal $\vec{n}$, its vector equation is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$.
- The perpendicular distance of (x_0, y_0, z_0) from $Ax + By + Cz + D = 0$ is $\frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$.

Step 1: Identify two directions in the required plane The given line passes through $A(1, 1, 0)$, whose position vector is $\vec{a} = \hat{i} + \hat{j}$, and has direction vector $\vec{d} = \hat{i} + 2\hat{j} - \hat{k}$. The plane must also pass through $B(-1, 3, -4)$. Consequently, the vector joining these two points lies in the plane:

$$\overrightarrow{AB} = (-1 - 1)\hat{i} + (3 - 1)\hat{j} + (-4 - 0)\hat{k} = -2\hat{i} + 2\hat{j} - 4\hat{k}.$$

These two directions are not proportional, so they determine a unique plane through A .

Step 2: Obtain a normal and the vector equation A vector perpendicular to both directions is

$$\begin{aligned} \vec{d} \times \overrightarrow{AB} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 2 & -4 \end{vmatrix} \\ &= -6\hat{i} + 6\hat{j} + 6\hat{k}. \end{aligned}$$

Quick Tip to Solve: Cross the line direction with the vector from a point on the line to the given point; simplify the normal before applying the distance formula.

Dividing by the nonzero factor 6, choose $\vec{n} = -\hat{i} + \hat{j} + \hat{k}$. Substituting the point A into the point-normal form gives

$$(\vec{r} - \hat{i} - \hat{j}) \cdot (-\hat{i} + \hat{j} + \hat{k}) = 0.$$

Since $(\hat{i} + \hat{j}) \cdot (-\hat{i} + \hat{j} + \hat{k}) = -1 + 1 = 0$, this simplifies to $\vec{r} \cdot (-\hat{i} + \hat{j} + \hat{k}) = 0$. Its Cartesian form is $-x + y + z = 0$. As a check, the line has coordinates $(1 + \lambda, 1 + 2\lambda, -\lambda)$, which satisfy this equation for every λ ; the point B also gives $1 + 3 - 4 = 0$.

Step 3: Calculate the perpendicular length For $P(2, 1, 4)$, use the coefficients $A = -1$, $B = 1$, $C = 1$, $D = 0$ of the plane:

$$d = \frac{|-2 + 1 + 4|}{\sqrt{(-1)^2 + 1^2 + 1^2}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

The absolute value ensures a nonnegative length, while division by the normal's magnitude converts the scalar plane expression into a distance.

Step 4: Conclusion The required vector equation is $\vec{r} \cdot (-\hat{i} + \hat{j} + \hat{k}) = 0$, and the perpendicular length from $(2, 1, 4)$ is $\sqrt{3}$ units.

Quick Tip to Solve: Cross the line direction with the vector from a point on the line to the given point; simplify the normal before applying the distance formula.

29. Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean and variance of the number of kings. [6 marks]

Solution:

Concept:

- For a discrete random variable X , the mean is $E(X) = \sum_x xP(X = x)$, and the variance is $\text{Var}(X) = E(X^2) - [E(X)]^2$.
- Drawing two cards without replacement makes all unordered pairs equally likely. There are $\binom{52}{2}$ such pairs, with four kings and 48 other cards available.
- To obtain exactly x kings, choose x of the four kings and $2 - x$ of the 48 non-kings; multiply these independent counting choices.

Step 1: Define the random variable and count the outcomes Let X denote the number of kings among the two cards. Its possible values are 0, 1, 2. The total number of unordered two-card selections is

$$\binom{52}{2} = \frac{52 \cdot 51}{2} = 1326.$$

Simultaneous drawing and successive drawing without replacement give the same distribution: each unordered pair corresponds to exactly two equally likely ordered draws. Thus we may use unordered selections in both cases. The four kings are distinct cards, as are the 48 non-kings; choosing one group does not replace any card in the other group.

Step 2: Construct the probability distribution For no king, both cards come from the 48 non-kings;

Quick Tip to Solve: Count unordered pairs without replacement, check that all three probabilities sum to one, and compute the second moment before subtracting the squared mean.

for one king, one card comes from each group; for two kings, both come from the four kings. Hence

$$\begin{aligned}P(X = 0) &= \frac{\binom{48}{2}}{1326} = \frac{1128}{1326} = \frac{188}{221}, \\P(X = 1) &= \frac{\binom{4}{1}\binom{48}{1}}{1326} = \frac{192}{1326} = \frac{32}{221}, \\P(X = 2) &= \frac{\binom{4}{2}}{1326} = \frac{6}{1326} = \frac{1}{221}.\end{aligned}$$

These possibilities are mutually exclusive and cover all draws. Their probabilities sum to $(188 + 32 + 1)/221 = 1$, providing a check on the distribution.

Step 3: Evaluate the mean and the second moment Weight each value by its probability to obtain

$$E(X) = 0 \cdot \frac{188}{221} + 1 \cdot \frac{32}{221} + 2 \cdot \frac{1}{221} = \frac{34}{221} = \frac{2}{13}.$$

For the second moment, square the value of X , not its probability:

$$E(X^2) = 0^2 \cdot \frac{188}{221} + 1^2 \cdot \frac{32}{221} + 2^2 \cdot \frac{1}{221} = \frac{36}{221}.$$

The small mean is consistent with only four of the 52 cards being kings.

Step 4: Compute the variance Subtract the square of the mean from the second moment:

$$\begin{aligned}\text{Var}(X) &= \frac{36}{221} - \left(\frac{2}{13}\right)^2 \\&= \frac{36}{13 \cdot 17} - \frac{4}{13^2} \\&= \frac{468 - 68}{13^2 \cdot 17} = \frac{400}{2873}.\end{aligned}$$

The positive result describes the spread of the king count around its expected value. No independence assumption between the two draws has been used: removal of the first card was included in the counting of pairs.

Step 5: Conclusion The mean number of kings is $\boxed{\frac{2}{13}}$, and the variance is $\boxed{\frac{400}{2873}}$.

Quick Tip to Solve: Count unordered pairs without replacement, check that all three probabilities sum to one, and compute the second moment before subtracting the squared mean.