

CBSE Class 12 Mathematics Question Paper 2019 with Solution PDF-Set 65-2-2



Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of 8 questions of **two marks** each, Section C comprises of 11 questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. Find $|AB|$, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$.

[1 mark]

Solution:

Concept:

- For square matrices of the same order, the determinant of a product satisfies $|AB| = |A||B|$.
- For a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its determinant is $ad - bc$.

Step 1: Evaluate the two determinants Both given matrices have a zero column or row. Directly,

$$|A| = 0(2) - (-1)(0) = 0, \quad |B| = 3(0) - 5(0) = 0.$$

Thus both matrices are singular.

Step 2: Apply the product rule Using multiplicativity,

$$|AB| = |A||B| = 0 \cdot 0.$$

As a direct check, each entry of AB is zero, because the first column of A and the second row of B are zero. Its determinant therefore vanishes as well.

Step 3: Conclusion $|AB| = 0$.

Quick Tip to Solve: Check for a zero row or column before multiplying two matrices to find a determinant.

2. Differentiate $e^{\sqrt{3x}}$ with respect to x .

[1 mark]

Solution:

Concept:

- The chain rule gives $\frac{d}{dx}e^{u(x)} = e^{u(x)}u'(x)$.
- For $u = \sqrt{3x} = (3x)^{1/2}$, differentiate the outer power and then the inner expression $3x$.

Step 1: Differentiate the exponent For $x > 0$,

$$\frac{du}{dx} = \frac{1}{2}(3x)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{3x}}.$$

The factor 3 is essential because the quantity inside the square root is $3x$.

Step 2: Apply the chain rule Writing $y = e^u$, we have

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx} = e^{\sqrt{3x}} \frac{3}{2\sqrt{3x}}.$$

Quick Tip to Solve: Differentiate the square-root exponent separately so that the inner factor of 3 is retained.

The real-valued function has domain $x \geq 0$; this derivative formula applies for $x > 0$.

Step 3: Conclusion $\frac{d}{dx}e^{\sqrt{3x}} = \frac{3e^{\sqrt{3x}}}{2\sqrt{3x}}$, for $x > 0$.

Quick Tip to Solve: Differentiate the square-root exponent separately so that the inner factor of 3 is retained.

3. Find the order and degree (if defined) of the differential equation:

$$\frac{d^2y}{dx^2} + x \left(\frac{dy}{dx} \right)^2 = 2x^2 \log \left(\frac{d^2y}{dx^2} \right).$$

[1 mark]

Solution:

Concept:

- The order of a differential equation is the order of its highest derivative.
- The degree is the power of the highest-order derivative only when the equation is polynomial in its derivatives after removing radicals and fractions involving derivatives.

Step 1: Identify the highest derivative The equation contains $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Its highest derivative is the second derivative, even though that derivative also occurs inside a logarithm. Hence its order is 2.

Step 2: Check whether degree is defined The term $\log \left(\frac{d^2y}{dx^2} \right)$ is a logarithmic function of a derivative. Thus the equation is not polynomial in the derivatives. Exponentiating would introduce another non-polynomial expression and does not provide a polynomial form. The exponent 2 on the first derivative therefore does not give the degree.

Step 3: Conclusion Order = 2; degree is not defined.

Quick Tip to Solve: Before reading any power as the degree, check whether every derivative occurs polynomially.

4(a). Find the direction cosines of a line which makes equal angles with the coordinate axes. [1 mark]

Solution:

Concept:

- If a directed line makes angles α, β, γ with the positive coordinate axes, its direction cosines are $l = \cos \alpha, m = \cos \beta, n = \cos \gamma$.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$.

Step 1: Use equality of the angles Since the three angles are equal, their cosines are equal: $l = m = n$. Put their common value equal to t . Equality of the angles fixes equal signs as well as equal magnitudes.

Quick Tip to Solve: Equal angles imply equal cosines; use the sum-of-squares identity to find their common magnitude.

Step 2: Use the normalisation identity Substitution gives

$$t^2 + t^2 + t^2 = 1 \implies t = \pm \frac{1}{\sqrt{3}}.$$

Taking the usual direction with equal acute angles to the axes gives the positive sign. Reversing the direction of this same line changes all three signs simultaneously.

Step 3: Conclusion The direction cosines are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$; the reverse direction gives $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$.

Quick Tip to Solve: Equal angles imply equal cosines; use the sum-of-squares identity to find their common magnitude.

OR

4(b). A line passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction of the vector $\hat{i} + \hat{j} - 2\hat{k}$. Find the equation of the line in cartesian form. [1 mark]

Solution:

Concept:

- A line through (x_1, y_1, z_1) with direction ratios (a, b, c) has Cartesian equation $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$.
- The coefficients of a direction vector supply direction ratios; they need not be unit values.

Step 1: Read the point and direction The position vector $2\hat{i} - \hat{j} + 4\hat{k}$ gives the point $(2, -1, 4)$. The direction vector $\hat{i} + \hat{j} - 2\hat{k}$ gives direction ratios $(1, 1, -2)$.

Step 2: Eliminate the line parameter A general point on the line obeys

$$x = 2 + t, \quad y = -1 + t, \quad z = 4 - 2t.$$

Hence $t = x - 2 = y + 1 = (z - 4)/(-2)$. The negative direction ratio must stay in the denominator of the third expression.

Step 3: Conclusion $\frac{x - 2}{1} = \frac{y + 1}{1} = \frac{z - 4}{-2}$.

Quick Tip to Solve: Write the three parametric coordinates first to avoid losing the minus sign in the z direction.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5. Find: $\int \sqrt{3 - 2x - x^2} dx$.

[2 marks]

Solution:

Concept:

- Complete the square to express a quadratic radicand in the form $a^2 - u^2$.
- The standard antiderivative is $\int \sqrt{a^2 - u^2} du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \sin^{-1}(u/a) + C$, where $a > 0$.

Step 1: Complete the square and substitute

$$3 - 2x - x^2 = 4 - (x + 1)^2.$$

Put $u = x + 1$, so $du = dx$. The requested integral becomes

$$I = \int \sqrt{4 - u^2} du.$$

Here $a = 2$. The real radicand is nonnegative when $-3 \leq x \leq 1$.

Step 2: Apply the standard formula Substituting $a = 2$ gives

$$I = \frac{u}{2} \sqrt{4 - u^2} + 2 \sin^{-1} \left(\frac{u}{2} \right) + C.$$

To check the coefficients, differentiating the first term gives $\sqrt{4 - u^2}/2 - u^2/(2\sqrt{4 - u^2})$; differentiating the inverse-sine term gives $2/\sqrt{4 - u^2}$. Their sum is $\sqrt{4 - u^2}$, as required. Finally substitute $u = x + 1$.

Step 3: Conclusion

$$I = \frac{x + 1}{2} \sqrt{3 - 2x - x^2} + 2 \sin^{-1} \left(\frac{x + 1}{2} \right) + C.$$

Quick Tip to Solve: Complete the square first: the radius in the standard formula is 2, so the inverse-sine coefficient is 2.

6. If $A = \begin{bmatrix} p & 2 \\ 2 & p \end{bmatrix}$ and $|A^3| = 125$, then find the values of p .

[2 marks]

Solution:

Concept:

- For a square matrix A , multiplicativity gives $|A^3| = (|A|)^3$.
- The determinant of $\begin{bmatrix} p & 2 \\ 2 & p \end{bmatrix}$ is $p^2 - 4$.

Quick Tip to Solve: Take the determinant before the matrix power; after the cube root, retain both square roots of p squared.

Step 1: Express the given condition using the determinant There is no need to calculate all entries of A^3 . Since $A^3 = AAA$,

$$|A^3| = |A||A||A| = (p^2 - 4)^3.$$

The given value therefore implies

$$(p^2 - 4)^3 = 125 = 5^3.$$

The cube-root step gives a unique real value for the determinant because the function $t \mapsto t^3$ is strictly increasing on the real line. In particular, the negative value -5 would have cube -125 and is excluded.

Step 2: Solve for the real values of p Taking the real cube root gives $p^2 - 4 = 5$, and hence $p^2 = 9$. Factoring,

$$(p - 3)(p + 3) = 0.$$

Both roots must be retained because the determinant depends on p^2 , not on the sign of p . For either root, $|A| = 9 - 4 = 5$, so $|A^3| = 5^3 = 125$, confirming the given condition.

Step 3: Conclusion The required values are $p = 3$ or $p = -3$.

Quick Tip to Solve: Take the determinant before the matrix power; after the cube root, retain both square roots of p squared.

7. Examine whether the operation $*$ defined on $\mathbb{R}$, the set of all real numbers, by $a * b = \sqrt{a^2 + b^2}$ is a binary operation or not, and if it is, find whether it is associative or not. [2 marks]

Solution:

Concept:

- A binary operation on $\mathbb{R}$ must assign one real number to every ordered pair of real numbers.
- Associativity means $(a * b) * c = a * (b * c)$ for every $a, b, c \in \mathbb{R}$, using the given operation at each stage.

Step 1: Check closure and uniqueness For any real a, b , $a^2 + b^2 \geq 0$. Its principal square root is a uniquely defined nonnegative real number. Thus $a * b = \sqrt{a^2 + b^2}$ belongs to $\mathbb{R}$ for every input pair. The fact that outputs are nonnegative does not prevent closure on the larger set $\mathbb{R}$.

Step 2: Compare the two groupings For arbitrary real a, b, c ,

$$\begin{aligned}(a * b) * c &= \sqrt{(\sqrt{a^2 + b^2})^2 + c^2} = \sqrt{a^2 + b^2 + c^2}, \\ a * (b * c) &= \sqrt{a^2 + (\sqrt{b^2 + c^2})^2} = \sqrt{a^2 + b^2 + c^2}.\end{aligned}$$

Both expressions use principal square roots and agree for all real inputs, so equality is universal rather than a check of a few examples.

Step 3: Conclusion The operation is a binary operation on $\mathbb{R}$ and is associative.

Quick Tip to Solve: For associativity, square only the inner square root after substituting the operation into each grouping.

8(a). If $|\vec{a}| = 2$, $|\vec{b}| = 7$ and $\vec{a} \times \vec{b} = 3\hat{i} + 2\hat{j} + 6\hat{k}$, find the angle between $\vec{a}$ and $\vec{b}$.

[2 marks]

Solution:

Concept:

- For nonzero vectors, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$, where $0 \leq \theta \leq \pi$ is their angle.
- The magnitude of $u\hat{i} + v\hat{j} + w\hat{k}$ is $\sqrt{u^2 + v^2 + w^2}$. Sine has the same value for supplementary angles.

Step 1: Calculate the magnitude of the cross product The given cross product has mutually perpendicular Cartesian components. Hence

$$|\vec{a} \times \vec{b}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7.$$

This is a magnitude and is nonnegative; it is not the sum of the three components.

Step 2: Apply the cross-product formula and retain both angles Substitute the given vector lengths into the magnitude formula:

$$7 = 2 \cdot 7 \sin \theta \implies \sin \theta = \frac{1}{2}.$$

Within the angle range $0 \leq \theta \leq \pi$, this equation has the two solutions $\theta = \pi/6$ and $\theta = 5\pi/6$. The cross product alone does not specify the sign of $\vec{a} \cdot \vec{b}$, so the information does not distinguish an acute angle from its supplementary obtuse angle.

Step 3: Conclusion The possible angles are 30° or 150° . If the acute angle is intended, it is 30° ; no acute-angle restriction is stated.

Quick Tip to Solve: After finding sine from a cross product, check for the supplementary angle unless the dot product or an acute-angle condition is supplied.

OR

8(b). Find the volume of a cuboid whose edges are given by $-3\hat{i} + 7\hat{j} + 5\hat{k}$, $-5\hat{i} + 7\hat{j} - 3\hat{k}$ and $7\hat{i} - 5\hat{j} - 3\hat{k}$.

[2 marks]

Solution:

Concept:

- The volume determined by three edge vectors $\vec{a}, \vec{b}, \vec{c}$ is $V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$.
- The scalar triple product equals the determinant of the three component rows; its absolute value gives a nonnegative volume.

Step 1: Set up the scalar triple product Take the vectors in their printed order as $\vec{a} = (-3, 7, 5)$, $\vec{b} = (-5, 7, -3)$ and $\vec{c} = (7, -5, -3)$. The volume formula uses all three directions, rather than multiplying their lengths. Thus

$$V = \left| \begin{vmatrix} -3 & 7 & 5 \\ -5 & 7 & -3 \\ 7 & -5 & -3 \end{vmatrix} \right|.$$

Quick Tip to Solve: Expand using the first row with signs plus, minus, plus, and take the absolute value only after evaluating the determinant.

Changing the order could change the determinant sign but not its absolute value.

Step 2: Expand the determinant along its first row Using the alternating cofactor signs +, −, +, the signed scalar triple product is

$$\begin{aligned} D &= -3[7(-3) - (-3)(-5)] \\ &\quad - 7[(-5)(-3) - (-3)(7)] \\ &\quad + 5[(-5)(-5) - 7(7)] \\ &= -3(-36) - 7(36) + 5(-24) \\ &= 108 - 252 - 120 = -264. \end{aligned}$$

The negative sign expresses the orientation of the three vectors, so the physical volume is $|D|$.

Step 3: Conclusion The required volume is 264 cubic units.

Quick Tip to Solve: Expand using the first row with signs plus, minus, plus, and take the absolute value only after evaluating the determinant.

9(a). A coin is tossed 5 times. What is the probability of getting (i) 3 heads, (ii) at most 3 heads?

[2 marks]

Solution:

Concept:

- For five independent tosses of a fair coin, the number of heads X has binomial probabilities $P(X = r) = \binom{5}{r}(1/2)^r(1/2)^{5-r}$.
- Exactly three heads means $X = 3$; at most three heads means $X \in \{0, 1, 2, 3\}$.

Step 1: Find the probability of exactly three heads Each sequence of five heads or tails has probability $(1/2)^5 = 1/32$. Choose the three positions occupied by heads in $\binom{5}{3} = 10$ ways; the remaining two positions are tails. Therefore

$$P(X = 3) = \binom{5}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = \frac{10}{32} = \frac{5}{16}.$$

This counts all arrangements, rather than a single sequence such as HHHTT.

Step 2: Find the probability of at most three heads The outcomes with zero, one, two or three heads are disjoint. Adding their binomial probabilities gives

$$\begin{aligned} P(X \leq 3) &= \frac{\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3}}{32} \\ &= \frac{1 + 5 + 10 + 10}{32} = \frac{26}{32} = \frac{13}{16}. \end{aligned}$$

Equivalently, exclude four and five heads: $1 - (5 + 1)/32 = 13/16$, providing a check on the sum.

Step 3: Conclusion (i) 5/16. (ii) 13/16.

Quick Tip to Solve: For at most three heads in five tosses, subtract the probabilities of four and five heads from one to shorten the calculation.

9(b). Find the probability distribution of X , the number of heads in a simultaneous toss of two coins. [2 marks]

Solution:

Concept:

- For two independent fair coins, the ordered outcomes HH, HT, TH, TT are equally likely, each with probability $1/4$.
- A probability distribution lists every possible value of the random variable X with its probability; these probabilities sum to one.

Step 1: Assign the number of heads to each outcome Distinguish the two coins, even though they are tossed simultaneously. The sample space is $S = \{HH, HT, TH, TT\}$. For the random variable counting heads,

$$X(TT) = 0, \quad X(HT) = X(TH) = 1, \quad X(HH) = 2.$$

Thus the possible values of X are 0, 1, 2. In particular, HT and TH are distinct equally likely outcomes, although both give the same value of X .

Step 2: Calculate and check the distribution Only TT gives no heads, two outcomes give one head, and only HH gives two heads. Dividing the corresponding counts by four yields

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

All probabilities are nonnegative, and $1/4 + 1/2 + 1/4 = 1$, so the table is a complete probability distribution. The middle probability is twice either end probability because it combines HT and TH.

Step 3: Conclusion $P(X = 0) = \frac{1}{4}, \quad P(X = 1) = \frac{1}{2}, \quad P(X = 2) = \frac{1}{4}.$

Quick Tip to Solve: List the four ordered coin outcomes first: treating HT and TH separately prevents the common error of assigning one-third to each head count.

10(a). Find: $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx.$

[2 marks]

Solution:

Concept:

- Split a fraction with a sum in the numerator into separate terms before choosing an integration method.
- The derivatives $\frac{d}{dx} \sec x = \sec x \tan x$ and $\frac{d}{dx} \csc x = -\csc x \cot x$ identify the two antiderivatives.

Step 1: Rewrite the integrand in standard trigonometric form On every interval where $\sin x \cos x \neq 0$,

Quick Tip to Solve: Split the numerator before integrating, and remember that a positive cosecant-cotangent term integrates to negative cosecant.

cancel the powers in each numerator term separately:

$$\begin{aligned}\frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} &= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x} \\ &= \sec x \tan x + \csc x \cot x.\end{aligned}$$

The first term follows from $\sec x \tan x = (1/\cos x)(\sin x/\cos x)$; the second follows in the same way from cosecant and cotangent.

Step 2: Integrate the two terms and verify their signs By linearity,

$$\begin{aligned}I &= \int \sec x \tan x \, dx + \int \csc x \cot x \, dx \\ &= \sec x - \csc x + C.\end{aligned}$$

The minus sign is essential because the derivative of cosecant is negative. Differentiating $\sec x - \csc x$ gives $\sec x \tan x + \csc x \cot x$, which equals the original integrand after the preceding simplification.

Step 3: Conclusion $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} \, dx = \sec x - \csc x + C$, on intervals where the integrand is defined.

Quick Tip to Solve: Split the numerator before integrating, and remember that a positive cosecant-cotangent term integrates to negative cosecant.

OR

10(b). Find: $\int \frac{x-3}{(x-1)^3} e^x \, dx$.

[2 marks]

Solution:

Concept:

- The product rule gives $\frac{d}{dx}[e^x f(x)] = e^x[f(x) + f'(x)]$.
- For an exponential times a rational expression, look for f whose value plus derivative produces the given rational factor.

Step 1: Rewrite the rational factor to reveal a product derivative For $x \neq 1$, write $x - 3 = (x - 1) - 2$. Then

$$\frac{x-3}{(x-1)^3} = \frac{1}{(x-1)^2} - \frac{2}{(x-1)^3}.$$

Choose $f(x) = (x-1)^{-2}$. The power rule gives $f'(x) = -2(x-1)^{-3}$, so the two terms on the right are exactly $f(x) + f'(x)$. This avoids a lengthy integration-by-parts sequence.

Quick Tip to Solve: Rewrite x minus 3 as $(x$ minus 1) minus 2, then recognize the product-rule derivative of the exponential divided by $(x$ minus 1) squared.

Step 2: Use the product rule and check the antiderivative The product derivative is

$$\begin{aligned}\frac{d}{dx} \left[\frac{e^x}{(x-1)^2} \right] &= \frac{e^x}{(x-1)^2} - \frac{2e^x}{(x-1)^3} \\ &= \frac{e^x[(x-1) - 2]}{(x-1)^3} \\ &= \frac{(x-3)e^x}{(x-1)^3}.\end{aligned}$$

It reproduces both the numerator and the denominator of the printed integrand. Therefore this product supplies an antiderivative, to which an arbitrary constant is added on each interval avoiding $x = 1$.

Step 3: Conclusion $\int \frac{x-3}{(x-1)^3} e^x dx = \frac{e^x}{(x-1)^2} + C$, for $x \neq 1$.

Quick Tip to Solve: Rewrite x minus 3 as $(x$ minus 1) minus 2, then recognize the product-rule derivative of the exponential divided by $(x$ minus 1) squared.

11. If $P(\text{not } A) = 0.7$, $P(B) = 0.7$ and $P(B/A) = 0.5$, then find $P(A/B)$.

[2 marks]

Solution:

Concept:

- The complement rule gives $P(A) = 1 - P(\text{not } A)$, because an event and its complement together exhaust the sample space.
- Conditional probability is $P(B/A) = P(A \cap B)/P(A)$ and $P(A/B) = P(A \cap B)/P(B)$, when the conditioning event has positive probability.

Step 1: Find the probability of A The given complement probability is $P(\text{not } A) = 0.7$. Hence

$$P(A) = 1 - 0.7 = 0.3.$$

The given notation B/A means B conditional on A; it is not the reverse conditional probability.

Step 2: Calculate the intersection and reverse the conditioning Using $P(B/A) = 0.5$, the probability that both events occur is

$$P(A \cap B) = P(B/A)P(A) = 0.5 \times 0.3 = 0.15.$$

Now the conditioning event is B, whose probability is 0.7. Therefore

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.7} = \frac{3}{14}.$$

Both denominators are nonzero, so the conditional probabilities used here are defined.

Step 3: Conclusion Thus, $P(A/B) = \frac{3}{14}$.

Quick Tip to Solve: Find $P(A)$ from its complement first; then use the same intersection probability in both conditional-probability formulas.

12. Find the general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$.

[2 marks]

Solution:

Concept:

- A separable differential equation can be solved by collecting factors involving y with dy and factors involving x with dx .
- The exponential identity $e^{x+y} = e^x e^y$ gives the required separation. Also, $\int e^{-y} dy = -e^{-y}$.

Step 1: Separate the variables The given equation is

$$\frac{dy}{dx} = e^{x+y} = e^x e^y.$$

Since $e^y > 0$ for every real y , division by e^y loses no real solution. Multiplying by $e^{-y} dx$ gives

$$e^{-y} dy = e^x dx.$$

Step 2: Integrate and verify Integrating both sides yields

$$-e^{-y} = e^x + C_1.$$

Rearranging and writing $C = -C_1$, we obtain

$$e^{-y} + e^x = C.$$

Differentiating this relation gives

$$-e^{-y} \frac{dy}{dx} + e^x = 0,$$

which recovers $dy/dx = e^{x+y}$. Equivalently, $y = -\ln(C - e^x)$, on any interval where $C - e^x > 0$, so the real solution is well defined.

Step 3: Conclusion The general solution is $e^{-y} + e^x = C$, where C is an arbitrary constant compatible with the real domain.

Quick Tip to Solve: When integrating e to the power minus y , keep the minus sign: the derivative of minus e to the power minus y is positive e to the power minus y .

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13. Find the value of $\sin\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right)$.

[4 marks]

Solution:

Concept:

- The principal range of $\cos^{-1} t$ is $[0, \pi]$, and that of $\tan^{-1} t$ is $(-\pi/2, \pi/2)$. Positive arguments here put both angles in the first quadrant.
- The addition identity is $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$. The identity $\sin^2 \alpha + \cos^2 \alpha = 1$ determines the remaining ratio with its correct sign.

Step 1: Determine the ratios of the first angle Let

$$\alpha = \cos^{-1} \frac{4}{5}, \quad \beta = \tan^{-1} \frac{2}{3}.$$

Then $\cos \alpha = 4/5$. Since $0 < 4/5 < 1$ and α is a principal inverse cosine value, $0 < \alpha < \pi/2$. In particular, its sine is positive. Thus

$$\sin \alpha = \sqrt{1 - \cos^2 \alpha} = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}.$$

Choosing the negative square root would contradict the first-quadrant range of α .

Step 2: Determine the ratios of the second angle We have $\tan \beta = 2/3$. Because the principal inverse tangent value of a positive number lies in the first quadrant, both $\sin \beta$ and $\cos \beta$ are positive. The opposite-to-adjacent ratio is 2 : 3, so the corresponding hypotenuse ratio is $\sqrt{2^2 + 3^2} = \sqrt{13}$. Consequently,

$$\sin \beta = \frac{2}{\sqrt{13}}, \quad \cos \beta = \frac{3}{\sqrt{13}}.$$

These ratios also satisfy $\sin^2 \beta + \cos^2 \beta = (4 + 9)/13 = 1$.

Step 3: Apply the sine addition formula Substitute the four ratios into the addition identity:

$$\begin{aligned} \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ &= \frac{3}{5} \cdot \frac{3}{\sqrt{13}} + \frac{4}{5} \cdot \frac{2}{\sqrt{13}} \\ &= \frac{9 + 8}{5\sqrt{13}} = \frac{17}{5\sqrt{13}}. \end{aligned}$$

Rationalising the denominator multiplies numerator and denominator by $\sqrt{13}$; the denominator becomes $5 \times 13 = 65$.

Step 4: Conclusion Hence, $\sin\left(\cos^{-1}\frac{4}{5} + \tan^{-1}\frac{2}{3}\right) = \frac{17}{5\sqrt{13}} = \frac{17\sqrt{13}}{65}$.

Quick Tip to Solve: Check the principal quadrants before taking square roots, then use the 3-4-5 ratio and the 2-3-square-root-13 ratio directly in the addition formula.

14. Using properties of determinants, show that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca).$$

[4 marks]

Solution:

Concept:

- Adding other columns to a column preserves the determinant. Thus $C_1 \rightarrow C_1 + C_2 + C_3$ is allowed, while the other columns remain unchanged.
- A common factor in a column can be taken outside the determinant. Subtracting one row from another also preserves its value.
- Expansion along a column containing two zeros reduces the determinant to a second-order determinant. The latter equals $ps - qr$ for rows (p, q) and (r, s) .

Step 1: Add columns and extract the common factor Denote the given determinant by D and put $S = a + b + c$. Under $C_1 \rightarrow C_1 + C_2 + C_3$, every entry of the first column becomes S . For instance, its top entry is $3a + (-a + b) + (-a + c) = S$. Thus

$$D = \begin{vmatrix} S & b-a & c-a \\ S & 3b & c-b \\ S & b-c & 3c \end{vmatrix} = S \begin{vmatrix} 1 & b-a & c-a \\ 1 & 3b & c-b \\ 1 & b-c & 3c \end{vmatrix}.$$

We have factored out S , without dividing by it, so this argument remains valid even when $a + b + c = 0$.

Step 2: Create zeros and expand along the first column Apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ to the last determinant. For example, its second-row entries become 0 , $3b - (b - a) = a + 2b$, $(c - b) - (c - a) = a - b$. Similarly the last row becomes $(0, a - c, a + 2c)$. Hence

$$D = S \begin{vmatrix} 1 & b-a & c-a \\ 0 & a+2b & a-b \\ 0 & a-c & a+2c \end{vmatrix}.$$

Expansion along the first column has only one nonzero term, whose cofactor sign is positive. Therefore

$$D = S[(a + 2b)(a + 2c) - (a - b)(a - c)].$$

Step 3: Simplify the remaining determinant Expand each product separately, retaining the subtraction before the second:

$$\begin{aligned} (a + 2b)(a + 2c) &= a^2 + 2ab + 2ac + 4bc, \\ (a - b)(a - c) &= a^2 - ab - ac + bc. \end{aligned}$$

Their difference is $3ab + 3ac + 3bc = 3(ab + bc + ca)$. Substitute this simplification into the previous expression and use $S = a + b + c$. This establishes the claimed identity directly by determinant properties, without any restrictions on a, b, c .

Step 4: Conclusion Thus the given determinant equals $\boxed{3(a + b + c)(ab + bc + ca)}$, as required.

Quick Tip to Solve: All three row sums equal $a + b + c$: add columns into the first column, then subtract the first row from each remaining row.

15(a). Check whether the relation R defined on the set $A = \{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive. [4 marks]

Solution:

Concept:

- A relation on A is reflexive if $(a, a) \in R$ for every $a \in A$.
- It is symmetric if $(a, b) \in R$ always implies $(b, a) \in R$.
- It is transitive if $(a, b), (b, c) \in R$ always imply $(a, c) \in R$. A single counterexample disproves a universal property.

Step 1: List the pairs allowed by the defining rule Both coordinates must belong to $A = \{1, 2, 3, 4, 5, 6\}$, and the second must be exactly one greater than the first. Thus

$$R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}.$$

For $a = 6$, the required second coordinate would be 7, which is outside A , so $(6, 7)$ is excluded. This explicit list lets us test the three properties using their definitions, rather than assuming that the rule itself guarantees any of them.

Step 2: Check reflexivity A diagonal pair (a, a) would have to satisfy $a = a + 1$, which is impossible. In particular, $1 \in A$ but $(1, 1) \notin R$. Reflexivity requires every diagonal pair, so even this one missing pair proves that R is not reflexive. In fact no diagonal pair belongs to the relation.

Step 3: Check symmetry The pair $(1, 2)$ belongs to R because $2 = 1 + 1$. Its reverse $(2, 1)$ does not belong to R , since the defining condition for that pair would be $1 = 2 + 1$, which is false. This violates the implication required for symmetry, so R is not symmetric. The rule points from a number to its successor and does not include the reverse direction.

Step 4: Check transitivity We have $(1, 2) \in R$ and $(2, 3) \in R$. Transitivity would therefore require $(1, 3) \in R$. But $3 \neq 1 + 1$, so that pair is absent, and R is not transitive. More generally, chaining two successor pairs gives $c = b + 1 = a + 2$, whereas a direct pair (a, c) in this relation requires $c = a + 1$. The explicit chain above lies entirely in A and supplies the needed counterexample.

Step 5: Conclusion The relation R is neither reflexive, nor symmetric, nor transitive.

Quick Tip to Solve: For this successor relation, use $(1, 1)$ for reflexivity, reverse $(1, 2)$ for symmetry, and chain $(1, 2), (2, 3)$ for transitivity.

15(b). Let $f : \mathbb{N} \rightarrow Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in \mathbb{N} : y = 4x + 3, \text{ for some } x \in \mathbb{N}\}$. Show that f is invertible. Find its inverse. [4 marks]

Solution:

Concept:

- A function is invertible precisely when it is bijective, that is, both one-one and onto its stated codomain.
- One-one means equal outputs imply equal inputs. Onto means every element of the codomain has a preimage in the domain.
- The inverse reverses the correspondence: $f(x) = y$ implies $f^{-1}(y) = x$. Its domain is the codomain of f .

Step 1: Prove that the function is one-one Let $a, b \in \mathbb{N}$ and suppose that $f(a) = f(b)$. Then

$$4a + 3 = 4b + 3 \implies 4a = 4b \implies a = b.$$

Thus two distinct natural numbers cannot have the same image. Since this reasoning applies to every pair of inputs, f is one-one on its entire domain.

Step 2: Prove that the function is onto Take any $y \in Y$. By the given definition of Y , there exists $n \in \mathbb{N}$ such that $y = 4n + 3$. Consequently $f(n) = y$, so every element of Y is attained by an input in $\mathbb{N}$. Therefore $f : \mathbb{N} \rightarrow Y$ is onto. The specified codomain is essential: we need preimages for all elements of Y , rather than for all natural numbers.

Step 3: Obtain and verify the inverse The function is both one-one and onto, hence bijective and invertible. Solving $y = 4x + 3$ for the input yields $x = (y - 3)/4$. Define $g : Y \rightarrow \mathbb{N}$ by $g(y) = (y - 3)/4$. This is well-defined because each $y \in Y$ equals $4n + 3$ for some natural number n , and then $g(y) = n$. To verify that this rule really reverses f , calculate both compositions:

$$g(f(x)) = \frac{(4x + 3) - 3}{4} = x \quad (x \in \mathbb{N}),$$

$$f(g(y)) = 4 \left(\frac{y - 3}{4} \right) + 3 = y \quad (y \in Y).$$

They are the identity maps on the respective sets, so $g = f^{-1}$.

Step 4: Conclusion Hence f is invertible and $f^{-1} : Y \rightarrow \mathbb{N}, \quad f^{-1}(y) = \frac{y - 3}{4}.$

Quick Tip to Solve: Use the definition of the codomain to prove onto; then reverse the linear rule and verify both compositions.

16(a). If $x\sqrt{1+y} + y\sqrt{1+x} = 0$ and $x \neq y$, prove that $\frac{dy}{dx} = -\frac{1}{(x+1)^2}$.

[4 marks]

Solution:

Concept:

- Principal square roots are nonnegative. For real square roots here, $1+x \geq 0$ and $1+y \geq 0$.
- A substitution $a = \sqrt{1+x}$, $b = \sqrt{1+y}$ converts the radical relation into a factorisable algebraic equation.
- In implicit differentiation, y is a function of x . The product rule gives $\frac{d}{dx}[(1+x)(1+y)] = (1+y) + (1+x)\frac{dy}{dx}$.

Step 1: Rewrite and factor the given equation Put $a = \sqrt{1+x}$ and $b = \sqrt{1+y}$. Then $a, b \geq 0$, $x = a^2 - 1$ and $y = b^2 - 1$. Substituting these expressions in the given equation yields

$$\begin{aligned}(a^2 - 1)b + (b^2 - 1)a &= 0, \\ a^2b + ab^2 - a - b &= 0, \\ (a+b)(ab-1) &= 0.\end{aligned}$$

This factorisation uses the original equation directly, without squaring it and possibly introducing an unwanted branch.

Step 2: Use the condition to select the admissible factor If $a+b=0$, nonnegativity forces $a=b=0$. It would follow that $x=y=-1$, contrary to the hypothesis $x \neq y$. Therefore $a+b \neq 0$, and the factored equation gives $ab=1$. In particular both square roots are strictly positive. Squaring this positive equality now gives

$$(1+x)(1+y) = 1.$$

Hence $1+x > 0$ and $1+y = 1/(1+x)$; all divisions used below are legitimate on the given branch.

Step 3: Differentiate the resulting relation implicitly Differentiate $(1+x)(1+y) = 1$ with respect to x , using the product rule and treating y as dependent on x :

$$(1+y) + (1+x)\frac{dy}{dx} = 0.$$

Isolating the derivative and substituting the relation found above gives

$$\frac{dy}{dx} = -\frac{1+y}{1+x} = -\frac{1/(1+x)}{1+x}.$$

Equivalently, the same branch is explicitly $y = 1/(1+x) - 1$, whose differentiation confirms this result.

Step 4: Conclusion Therefore $\boxed{\frac{dy}{dx} = -\frac{1}{(x+1)^2}}$, as required.

Quick Tip to Solve: Replace the two square roots by nonnegative variables and factor; use the given inequality before dividing out a factor.

16(b). If $(\cos x)^y = (\sin y)^x$, find $\frac{dy}{dx}$.

[4 marks]

Solution:

Concept:

- Logarithmic differentiation converts variable powers into products: $\log(u^v) = v \log u$. Work locally where the real logarithms are defined, with $\cos x > 0$ and $\sin y > 0$.
- Since y depends on x , use the product rule and chain rule: $\frac{d}{dx} \log(\cos x) = -\tan x$ and $\frac{d}{dx} \log(\sin y) = \cot y \frac{dy}{dx}$.

Step 1: Take logarithms Both exponents contain variables, so the ordinary power rule alone is insufficient. Taking natural logarithms of the given equality gives

$$y \log(\cos x) = x \log(\sin y).$$

This is an implicit equation; both factors in each product depend on the independent variable and must be differentiated.

Step 2: Apply the product and chain rules Writing $y' = \frac{dy}{dx}$, differentiation with respect to x yields

$$y' \log(\cos x) - y \tan x = \log(\sin y) + x \cot y y'.$$

The negative sign comes from the derivative of cosine. The factor y' in the last term is necessary because the argument of the sine function is $y(x)$, not the independent variable x .

Step 3: Isolate the derivative Bring all terms containing y' to the left and the remaining terms to the right:

$$y' [\log(\cos x) - x \cot y] = \log(\sin y) + y \tan x.$$

Divide by the coefficient of y' wherever it is nonzero. At a point where this coefficient vanishes, division is invalid and this expression cannot be used to assert a finite derivative without separately examining the implicit relation.

Step 4: Conclusion Hence, where the denominator is nonzero,

$$\frac{dy}{dx} = \frac{\log(\sin y) + y \tan x}{\log(\cos x) - x \cot y}.$$

Quick Tip to Solve: Take logarithms before differentiating variable powers; retain dy/dx whenever differentiating a function of y .

17. Prove that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ and hence evaluate $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$. [4 marks]

Solution:

Concept:

- A change of variable in a definite integral changes both the differential and the limits. For $t = a - x$, we have $dx = -dt$.
- If the denominator is invariant under $x \mapsto a - x$, adding the original and transformed integrals replaces the numerator x by the constant a .
- Use $\sin x + \cos x = \sqrt{2} \cos(x - \pi/4)$ and $\int \sec u du = \log |\sec u + \tan u| + C$.

Step 1: Prove the reflection property Suppose f is integrable on $[0, a]$. In $\int_0^a f(a-x) dx$, put $t = a - x$. Then $dx = -dt$; the lower limit $x = 0$ becomes $t = a$, and the upper limit $x = a$ becomes $t = 0$. Therefore

$$\int_0^a f(a-x) dx = - \int_a^0 f(t) dt = \int_0^a f(t) dt = \int_0^a f(x) dx.$$

The final equality simply renames the dummy variable, proving the required identity.

Step 2: Apply the property and add the two forms Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$. The denominator is positive throughout this interval, so the integral exists. The proved identity with $a = \pi/2$ gives

$$I = \int_0^{\pi/2} \frac{\pi/2 - x}{\cos x + \sin x} dx.$$

Adding this to the original expression eliminates the variable numerator:

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}, \quad I = \frac{\pi}{4} J,$$

where J denotes the remaining integral.

Step 3: Evaluate the remaining integral Set $u = x - \pi/4$. Using the trigonometric identity in the concept block,

$$\begin{aligned} J &= \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} \sec u du \\ &= \frac{1}{\sqrt{2}} [\log |\sec u + \tan u|]_{-\pi/4}^{\pi/4} \\ &= \frac{1}{\sqrt{2}} \log \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right). \end{aligned}$$

Both logarithm arguments are positive at the endpoints. Since $(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$, the ratio equals $(\sqrt{2} + 1)^2$, so $J = \sqrt{2} \log(1 + \sqrt{2})$.

Step 4: Conclusion The reflection property is proved, and the required integral is

$$I = \frac{\pi}{2\sqrt{2}} \log(1 + \sqrt{2}).$$

Quick Tip to Solve: Reflect x about $\pi/4$ first and add the two integrals; then only the reciprocal of $\sin x$ plus $\cos x$ remains.

18. Find: $\int \frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} dx$.

[4 marks]

Solution:

Concept:

- A rational function with a linear factor and an irreducible quadratic factor is decomposed as $A/(x + 2) + (Bx + C)/(x^2 + 1)$, where A, B, C are constants.
- Use $\int dx/(x + a) = \ln|x + a|$, $\int x dx/(x^2 + 1) = \frac{1}{2} \ln(x^2 + 1)$, and $\int dx/(x^2 + 1) = \tan^{-1} x$.

Step 1: Set up the partial fractions Write

$$\frac{x^2 + x + 1}{(x + 2)(x^2 + 1)} = \frac{A}{x + 2} + \frac{Bx + C}{x^2 + 1}.$$

Multiplying by the denominator gives the polynomial identity

$$\begin{aligned} x^2 + x + 1 &= A(x^2 + 1) + (Bx + C)(x + 2) \\ &= (A + B)x^2 + (2B + C)x + (A + 2C). \end{aligned}$$

Comparing coefficients of equal powers of x , we obtain $A + B = 1$, $2B + C = 1$, and $A + 2C = 1$. This comparison is valid because the identity must hold for every value of x .

Step 2: Determine the constants From the first two equations, $B = 1 - A$ and $C = 1 - 2B = 2A - 1$. Substituting into the third equation gives $A + 2(2A - 1) = 1$, so $5A = 3$. Thus

$$A = \frac{3}{5}, \quad B = \frac{2}{5}, \quad C = \frac{1}{5}.$$

Hence the integrand is $\frac{3}{5(x+2)} + \frac{2x+1}{5(x^2+1)}$. The numerator of the quadratic part is separated into a multiple of the denominator's derivative and a constant.

Step 3: Integrate and verify Therefore, writing I for the required integral,

$$\begin{aligned} I &= \frac{3}{5} \int \frac{dx}{x + 2} + \frac{1}{5} \int \frac{2x dx}{x^2 + 1} + \frac{1}{5} \int \frac{dx}{x^2 + 1} \\ &= \frac{3}{5} \ln|x + 2| + \frac{1}{5} \ln(x^2 + 1) + \frac{1}{5} \tan^{-1} x + C_0. \end{aligned}$$

Differentiation returns $3/[5(x + 2)] + (2x + 1)/[5(x^2 + 1)]$, whose combined numerator is $x^2 + x + 1$. Thus the result is valid on each interval excluding $x = -2$; the quadratic denominator is positive for all real x .

Step 4: Conclusion

$$I = \frac{3}{5} \ln|x + 2| + \frac{1}{5} \ln(x^2 + 1) + \frac{1}{5} \tan^{-1} x + C_0.$$

Quick Tip to Solve: Use a linear numerator over the quadratic factor, then split it into the derivative term $2x$ and a constant before integrating.

19(a). Solve the differential equation: $x \frac{dy}{dx} = y - x \tan\left(\frac{y}{x}\right)$.

[4 marks]

Solution:

Concept:

- A homogeneous first-order equation has the form $dy/dx = F(y/x)$. Substituting $v = y/x$, so that $y = vx$, gives $dy/dx = v + x dv/dx$.
- Separation of variables reduces the equation to integrals in v and x . Here $\int \cot v dv = \ln |\sin v|$, and $x \neq 0$ throughout.

Step 1: Transform the homogeneous equation Dividing the given equation by x , we have

$$\frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right).$$

The right-hand side depends only on the ratio y/x ; therefore the equation is homogeneous. Put $v = y/x$, or $y = vx$. The product rule then gives $dy/dx = v + x dv/dx$. Substituting this expression into the equation yields

$$v + x \frac{dv}{dx} = v - \tan v, \quad x \frac{dv}{dx} = -\tan v.$$

This substitution cancels the repeated v term and leaves a separable equation.

Step 2: Separate and integrate On an interval where $\sin v \neq 0$, divide by $\tan v$ and rearrange:

$$\cot v dv = -\frac{dx}{x}.$$

Integration gives

$$\ln |\sin v| = -\ln |x| + C_1, \quad \ln |x \sin v| = C_1.$$

Absorbing the nonzero magnitude and its sign into an arbitrary constant C , this becomes $x \sin v = C$. Substituting $v = y/x$ restores the original variables. If $\tan v = 0$, the separated division is unavailable, but the transformed equation directly gives the constant-ratio solutions $v = n\pi$, or $y = n\pi x$. These are also included by $C = 0$.

Step 3: Check the implicit solution and its domain Differentiate $x \sin(y/x) = C$ by the product and chain rules:

$$\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right) \left(\frac{dy}{dx} - \frac{y}{x}\right) = 0.$$

Where the original tangent is defined, $\cos(y/x) \neq 0$. Dividing by this cosine gives $dy/dx - y/x = -\tan(y/x)$. Multiplication by x recovers exactly the printed differential equation. Thus each differentiable branch of the implicit family is a solution on intervals where $x \neq 0$ and $\cos(y/x) \neq 0$.

Step 4: Conclusion The general implicit solution is

$$x \sin\left(\frac{y}{x}\right) = C,$$

with arbitrary constant C , on intervals where $x \neq 0$ and $\cos(y/x) \neq 0$. It includes $y = n\pi x$ when $C = 0$.

Quick Tip to Solve: After putting $y = vx$, cancel v before separating; check $\tan v = 0$ separately so no constant-ratio solution is lost.

19(b). Solve the differential equation: $\frac{dy}{dx} = -\left[\frac{x + y \cos x}{1 + \sin x}\right]$.

[4 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $y' + P(x)y = Q(x)$. Its integrating factor is $I(x) = e^{\int P(x) dx}$, so that $(Iy)' = IQ$.
- The product rule gives $\frac{d}{dx}[y(1 + \sin x)] = (1 + \sin x)y' + y \cos x$. Work on any interval where $1 + \sin x \neq 0$, as required by the given equation.

Step 1: Put the equation in linear form Move the term containing y to the left to obtain

$$\frac{dy}{dx} + \frac{\cos x}{1 + \sin x}y = -\frac{x}{1 + \sin x}.$$

Thus $P(x) = \cos x/(1 + \sin x)$ and $Q(x) = -x/(1 + \sin x)$. The denominator must be nonzero throughout the interval on which the solution is considered.

Step 2: Find the integrating factor Substitute $u = 1 + \sin x$, so $du = \cos x dx$. Then

$$\int \frac{\cos x}{1 + \sin x} dx = \log(1 + \sin x).$$

On an allowed interval, $1 + \sin x > 0$, and hence the integrating factor can be chosen as $I(x) = 1 + \sin x$. An arbitrary nonzero constant multiplier in an integrating factor has no effect on the solution.

Step 3: Multiply and integrate Multiplication by the integrating factor yields

$$(1 + \sin x) \frac{dy}{dx} + y \cos x = -x, \quad \frac{d}{dx}[y(1 + \sin x)] = -x.$$

Integrating both sides with respect to x , we obtain

$$y(1 + \sin x) = -\frac{x^2}{2} + C,$$

where C is an arbitrary real constant.

Step 4: Check by differentiation Differentiate the last identity: $(1 + \sin x)y' + y \cos x = -x$. Dividing by the nonzero factor $1 + \sin x$ recovers $y' = -(x + y \cos x)/(1 + \sin x)$, exactly the given differential equation. Thus the integration introduces no extra restriction apart from its original domain.

Step 5: Conclusion The general solution is $y = \frac{C - x^2/2}{1 + \sin x}$, on any interval where $1 + \sin x \neq 0$.

Quick Tip to Solve: Multiply by $1 + \sin x$ immediately: the two terms on the left are the derivative of $y(1 + \sin x)$.

20. The scalar product of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\vec{b} + \vec{c}$. [4 marks]

Solution:

Concept:

- For a nonzero vector $\vec{v}$, the unit vector along its direction is $\hat{v} = \vec{v}/|\vec{v}|$, with the positive magnitude in the denominator.
- In an orthonormal basis, $(a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = a_1b_1 + a_2b_2 + a_3b_3$. A candidate obtained by squaring must satisfy the original equation and its sign condition.

Step 1: Find the sum and its direction Adding corresponding components gives

$$\vec{b} + \vec{c} = (\lambda + 2)\hat{i} + 6\hat{j} - 2\hat{k}.$$

This vector is nonzero for every real λ , since its $\hat{j}$ component is 6. Its magnitude is

$$|\vec{b} + \vec{c}| = \sqrt{(\lambda + 2)^2 + 36 + 4} = \sqrt{(\lambda + 2)^2 + 40}.$$

Therefore divide the sum by this positive magnitude to obtain its unit vector.

Step 2: Apply the scalar product condition Since $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, the condition in the question becomes

$$\frac{(\lambda + 2) + 6 - 2}{\sqrt{(\lambda + 2)^2 + 40}} = 1, \quad \lambda + 6 = \sqrt{(\lambda + 2)^2 + 40}.$$

The right-hand side is positive, so any valid value must satisfy $\lambda + 6 > 0$. Squaring the equation gives

$$\begin{aligned} (\lambda + 6)^2 &= (\lambda + 2)^2 + 40, \\ \lambda^2 + 12\lambda + 36 &= \lambda^2 + 4\lambda + 44, \\ 8\lambda &= 8. \end{aligned}$$

The resulting candidate is $\lambda = 1$; it satisfies the required positivity condition.

Step 3: Compute and verify the unit vector Substitution yields $\vec{b} + \vec{c} = 3\hat{i} + 6\hat{j} - 2\hat{k}$, whose magnitude is $\sqrt{9 + 36 + 4} = 7$. Hence the desired direction is represented by

$$\hat{u} = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{7}.$$

Its norm squared is $(9 + 36 + 4)/49 = 1$, confirming that it is a unit vector. Also $\vec{a} \cdot \hat{u} = (3 + 6 - 2)/7 = 1$, which checks the original unsquared condition. The negative of this unit vector would point opposite to the sum and would have scalar product -1 , so it is not the requested direction.

Step 4: Conclusion $\lambda = 1$, and the unit vector along $\vec{b} + \vec{c}$ is $\boxed{\frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}}$.

Quick Tip to Solve: Keep $\lambda + 6 > 0$ before squaring, then check the resulting unit vector by both its norm and its dot product.

21. If $(a + bx)e^{y/x} = x$, then prove that $x^3 \frac{d^2y}{dx^2} = \left(x \frac{dy}{dx} - y\right)^2$.

[4 marks]

Solution:

Concept:

- In implicit differentiation, y is a differentiable function of x , while a and b are constants. The product rule and chain rule must both be applied to $(a + bx)e^{y/x}$.
- By the quotient rule, $\frac{d}{dx}(y/x) = \frac{xy' - y}{x^2}$, where $y' = dy/dx$. Also, $\frac{d}{dx}(xy' - y) = xy''$, with $y'' = d^2y/dx^2$.

Step 1: Differentiate the given relation once The given relation is defined for $x \neq 0$. Moreover, $a + bx \neq 0$ on any interval satisfying it, because the exponential is positive and the right-hand side is nonzero. Differentiating both sides gives

$$be^{y/x} + (a + bx)e^{y/x} \frac{xy' - y}{x^2} = 1.$$

Use the original equation to substitute $(a + bx)e^{y/x} = x$ and $e^{y/x} = x/(a + bx)$. Therefore,

$$\frac{bx}{a + bx} + \frac{xy' - y}{x} = 1.$$

Hence, after subtracting the first term and multiplying by x ,

$$xy' - y = x \left(1 - \frac{bx}{a + bx}\right) = \frac{ax}{a + bx}.$$

Step 2: Differentiate again and compare the two expressions Differentiating $xy' - y = ax/(a + bx)$, the two first-derivative terms on the left cancel. Applying the quotient rule on the right gives

$$\begin{aligned} y' + xy'' - y' &= \frac{a(a + bx) - abx}{(a + bx)^2}, \\ xy'' &= \frac{a^2}{(a + bx)^2}. \end{aligned}$$

Multiplying by x^2 , we obtain

$$x^3 y'' = \frac{a^2 x^2}{(a + bx)^2}.$$

On the other hand, squaring the expression found in the first step yields

$$(xy' - y)^2 = \left(\frac{ax}{a + bx}\right)^2 = \frac{a^2 x^2}{(a + bx)^2}.$$

Both sides agree. The proof also covers $a = 0$, since no division by a is used.

Step 3: Conclusion Hence, $x^3 \frac{d^2y}{dx^2} = \left(x \frac{dy}{dx} - y\right)^2$, as required.

Quick Tip to Solve: Replace the exponential using the original equation immediately after differentiating; then differentiate the combination $xy' - y$, whose derivative is simply xy'' .

22. The volume of a cube is increasing at the rate of $8 \text{ cm}^3/\text{s}$. How fast is the surface area increasing when the length of its edge is 12 cm? [4 marks]

Solution:

Concept:

- For a cube of edge length x , its volume is $V = x^3$ and its total surface area is $S = 6x^2$. The factor 6 accounts for all six square faces.
- Since the edge length changes with time t , the chain rule gives $dV/dt = 3x^2(dx/dt)$ and $dS/dt = 12x(dx/dt)$. Here dx/dt is the rate of increase of the edge length.

Step 1: Find the rate at which the edge length increases Measure x in centimetres and t in seconds. The given volume rate is

$$\frac{dV}{dt} = 8 \text{ cm}^3/\text{s}.$$

Differentiate the volume formula $V = x^3$ with respect to time, treating x as a function of t :

$$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}.$$

At the instant when $x = 12 \text{ cm}$, substitute the specified edge length and volume rate:

$$8 = 3(12)^2 \frac{dx}{dt} = 432 \frac{dx}{dt}.$$

Therefore,

$$\frac{dx}{dt} = \frac{8}{432} = \frac{1}{54} \text{ cm/s}.$$

This intermediate rate is positive, consistent with the increasing volume. It is the edge-length rate at the specified instant, which will now be used in the area formula.

Step 2: Relate the surface-area rate to the edge-length rate The total surface area is $S = 6x^2$, so differentiation with respect to time gives

$$\frac{dS}{dt} = 6 \cdot 2x \frac{dx}{dt} = 12x \frac{dx}{dt}.$$

At the same instant, $x = 12$ and $dx/dt = 1/54$. Thus,

$$\frac{dS}{dt} = 12(12) \left(\frac{1}{54} \right) = \frac{144}{54} \text{ cm}^2/\text{s}.$$

As a direct check, eliminating dx/dt between the two rate equations yields $dS/dt = (4/x)(dV/dt)$. Substitution gives $(4/12) \cdot 8$, agreeing with the value above. The area rate has units of square centimetres per second because $x(dx/dt)$ has those units.

Step 3: Conclusion The surface area of the cube is increasing at $\boxed{\frac{8}{3} \text{ cm}^2/\text{s}}$ when its edge length is 12 cm.

Quick Tip to Solve: Divide the differentiated area formula by the differentiated volume formula to obtain $dS/dt = (4/x)(dV/dt)$, then insert the edge length at the specified instant.

23. Find the cartesian and vector equations of the plane passing through the points $A(2, 5, -3)$, $B(-2, -3, 5)$ and $C(5, 3, -3)$. [4 marks]

Solution:

Concept:

- Two non-parallel vectors lying in a plane determine its direction. Their cross product gives a vector normal to the plane.
- If a plane passes through a point with position vector $\vec{a}$ and has a normal vector $\vec{n}$, its vector equation is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Step 1: Obtain two vectors lying in the plane Taking the given point A as the common initial point, subtract its coordinates from those of B and C :

$$\begin{aligned}\vec{AB} &= (-2 - 2)\hat{i} + (-3 - 5)\hat{j} + (5 + 3)\hat{k} \\ &= -4\hat{i} - 8\hat{j} + 8\hat{k}, \\ \vec{AC} &= (5 - 2)\hat{i} + (3 - 5)\hat{j} + (-3 + 3)\hat{k} \\ &= 3\hat{i} - 2\hat{j}.\end{aligned}$$

These vectors are not scalar multiples, since one has a nonzero third component and the other has a zero third component. Hence the three given points are non-collinear and determine a unique plane.

Step 2: Calculate a normal vector A vector perpendicular to both vectors, and therefore perpendicular to the required plane, is

$$\begin{aligned}\vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} \\ &= 16\hat{i} + 24\hat{j} + 32\hat{k} \\ &= 8(2\hat{i} + 3\hat{j} + 4\hat{k}).\end{aligned}$$

Multiplying a normal vector by a nonzero scalar does not change its direction, so we may use the simpler normal vector $\vec{n} = 2\hat{i} + 3\hat{j} + 4\hat{k}$.

Step 3: Use the point-normal equation and verify it The position vector of A is $\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$. Substituting this and the normal into the point-normal equation gives

$$(\vec{r} - 2\hat{i} - 5\hat{j} + 3\hat{k}) \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 0.$$

Expanding in Cartesian coordinates,

$$2(x - 2) + 3(y - 5) + 4(z + 3) = 0.$$

The constant on the right after rearrangement is $4 + 15 - 12 = 7$. As a check, the expression $2x + 3y + 4z$ takes the values $4 + 15 - 12 = 7$, $-4 - 9 + 20 = 7$, and $10 + 9 - 12 = 7$ at A, B, C , respectively. Thus all three specified points satisfy the equation.

Step 4: Conclusion The Cartesian equation is $2x + 3y + 4z = 7$. The vector equation is $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 7$.

Quick Tip to Solve: Simplify the cross-product normal by its common factor before substituting a point; then check the resulting constant at all three given points.

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24. Find the point on the curve $y^2 = 4x$, which is nearest to the point $(2, -8)$.

[6 marks]

Solution:

Concept:

- To find a nearest point on a curve, express the squared distance from the fixed point as a function of one coordinate and minimise that function.
- The distance formula is $d^2 = (x - x_0)^2 + (y - y_0)^2$. Since the square-root function is increasing on nonnegative numbers, the distance and its square have the same minimiser.
- For a differentiable function on the real line, a change of derivative from negative to positive gives a minimum. A sign change valid on the whole domain establishes the absolute minimum.

Step 1: Represent every point of the parabola with one variable Let $P(x, y)$ be a point on $y^2 = 4x$, and let the fixed point be $Q(2, -8)$. Write $y = t$, so the curve equation gives $x = t^2/4$. Thus

$$P\left(\frac{t^2}{4}, t\right), \quad t \in \mathbb{R}.$$

Every real value of t yields a point on the parabola, and every point on the parabola is obtained this way. In particular, this representation includes both the upper and lower branches; restricting y to the positive square root would miss possible nearest points.

Step 2: Form and simplify the squared distance By the distance formula, the squared distance from P to Q is

$$\begin{aligned} S(t) = PQ^2 &= \left(\frac{t^2}{4} - 2\right)^2 + (t + 8)^2 \\ &= \left(\frac{t^4}{16} - t^2 + 4\right) + (t^2 + 16t + 64) \\ &= \frac{t^4}{16} + 16t + 68. \end{aligned}$$

The quadratic terms cancel exactly. It is enough to minimise $S(t)$, avoiding differentiation of a square root and preserving the point at which the shortest distance occurs.

Step 3: Find the stationary point Differentiate the simplified expression with respect to t :

$$S'(t) = \frac{t^3}{4} + 16.$$

At a stationary point, $S'(t) = 0$, giving

$$\frac{t^3}{4} + 16 = 0 \implies t^3 = -64 \implies t = -4.$$

The real cube function is strictly increasing, so this equation has exactly one real solution.

Step 4: Establish that the minimum is absolute Factor the derivative to determine its sign throughout

Quick Tip to Solve: Set y equal to a real parameter and minimise the squared distance; here the quadratic terms cancel, leaving a simple cubic stationary-point equation.

the domain:

$$S'(t) = \frac{t^3 + 64}{4} = \frac{(t+4)(t^2 - 4t + 16)}{4}.$$

The quadratic factor is always positive because

$$t^2 - 4t + 16 = (t-2)^2 + 12 > 0.$$

Therefore $S'(t) < 0$ when $t < -4$, and $S'(t) > 0$ when $t > -4$. The squared distance decreases up to $t = -4$ and increases afterwards on the entire real line. This establishes a unique absolute minimum. As a local check, $S''(t) = 3t^2/4$ and $S''(-4) = 12 > 0$.

Step 5: Recover the coordinates and check the distance For $t = -4$, we obtain $y = -4$ and

$$x = \frac{(-4)^2}{4} = 4.$$

This point satisfies the curve equation, since $(-4)^2 = 16 = 4(4)$. Its squared distance from the fixed point is

$$(4-2)^2 + (-4+8)^2 = 4 + 16 = 20,$$

agreeing with $S(-4) = 256/16 - 64 + 68 = 20$. Hence the corresponding minimum distance is $\sqrt{20} = 2\sqrt{5}$.

Step 6: Conclusion The point on $y^2 = 4x$ nearest to $(2, -8)$ is $(4, -4)$. The minimum distance is $2\sqrt{5}$.

Quick Tip to Solve: Set y equal to a real parameter and minimise the squared distance; here the quadratic terms cancel, leaving a simple cubic stationary-point equation.

25(a). Find $\int_1^3 (x^2 + 2 + e^{2x}) dx$ as the limit of sums.

[6 marks]

Solution:

Concept:

- For a continuous function f on $[a, b]$, divide the interval into n equal parts of width $h = (b - a)/n$. The definite integral is the limit of the right-endpoint sum $h \sum_{r=1}^n f(a + rh)$.
- Use $\sum_{r=1}^n r = n(n+1)/2$, $\sum_{r=1}^n r^2 = n(n+1)(2n+1)/6$, and the finite geometric-series formula to evaluate the separate sums.
- The exponential limit $\lim_{t \rightarrow 0} (e^t - 1)/t = 1$ evaluates the denominator arising from the geometric sum.

Step 1: Set up the limit of sums Here $a = 1$, $b = 3$, so $h = 2/n$. The right endpoint of the r -th subinterval is $x_r = 1 + 2r/n$, for $r = 1, \dots, n$. Thus the required integral is

$$I = \lim_{n \rightarrow \infty} \frac{2}{n} \sum_{r=1}^n \left[\left(1 + \frac{2r}{n} \right)^2 + 2 + e^{2+4r/n} \right].$$

Quick Tip to Solve: Separate the polynomial and exponential sums first; the exponential ratio is $e^{4/n}$, because the interval width is $2/n$ and the exponent is $2x$.

All three summands are continuous on the interval, so we can evaluate their sums separately and add their limits.

Step 2: Evaluate the polynomial contribution Expanding the square gives $(1 + 2r/n)^2 + 2 = 3 + 4r/n + 4r^2/n^2$. Hence

$$P_n = \frac{2}{n} \left[3n + \frac{4}{n} \frac{n(n+1)}{2} + \frac{4}{n^2} \frac{n(n+1)(2n+1)}{6} \right] \\ = 6 + 4 \frac{n+1}{n} + \frac{4}{3} \frac{(n+1)(2n+1)}{n^2}.$$

Since $(n+1)/n \rightarrow 1$ and $(n+1)(2n+1)/n^2 \rightarrow 2$, this contribution tends to $6 + 4 + 8/3 = 38/3$.

Step 3: Evaluate the exponential contribution as a geometric sum Let $q = e^{4/n}$. The exponential terms form a geometric progression with first term $e^2 q$ and ratio q . Since $q^n = e^4$, their contribution is

$$E_n = \frac{2e^2}{n} \sum_{r=1}^n q^r = \frac{2e^2}{n} \frac{q(q^n - 1)}{q - 1} = \frac{2e^2 e^{4/n} (e^4 - 1)}{n(e^{4/n} - 1)}.$$

Put $t = 4/n$. Then $n(e^{4/n} - 1) = 4(e^t - 1)/t \rightarrow 4$, while $e^{4/n} \rightarrow 1$. Therefore the exponential contribution tends to $e^2(e^4 - 1)/2 = (e^6 - e^2)/2$.

Step 4: Combine the two limits Both component limits exist and are finite. The polynomial and constant terms contribute $38/3$, and the exponential terms contribute $(e^6 - e^2)/2$. Adding these values evaluates the original right-endpoint sum, without replacing the requested limit-of-sums method by an antiderivative calculation.

Step 5: Conclusion

$$\int_1^3 (x^2 + 2 + e^{2x}) dx = \frac{38}{3} + \frac{e^6 - e^2}{2}.$$

Quick Tip to Solve: Separate the polynomial and exponential sums first; the exponential ratio is $e^{4/n}$, because the interval width is $2/n$ and the exponent is $2x$.

OR

25(b). Using integration, find the area of the triangular region whose sides have the equations $y = 2x + 1$, $y = 3x + 1$ and $x = 4$. [6 marks]

Solution:

Concept:

- The area between two curves $y = u(x)$ and $y = l(x)$, with $u(x) \geq l(x)$ for $a \leq x \leq b$, is $A = \int_a^b [u(x) - l(x)] dx$.
- First find the intersections of the boundary lines. They determine the vertices and the correct limits of integration; the vertical line $x = 4$ provides the right boundary.
- A vertical strip has height equal to the upper ordinate minus the lower ordinate. Integrating its

Quick Tip to Solve: Equal intercepts make the oblique lines meet at $x = 0$; subtract the lower line from the upper before integrating, so the intercepts cancel.

area over the interval gives the area of the bounded triangular region in square units.

Step 1: Find the intersection of the two oblique sides The lines $y = 2x + 1$ and $y = 3x + 1$ meet where their ordinates are equal. Therefore

$$2x + 1 = 3x + 1 \implies x = 0.$$

Substituting into either line gives $y = 1$. Their common vertex is thus $P = (0, 1)$. This is the leftmost point of the bounded triangle. It also shows why the integration begins at $x = 0$, even though both lines have a nonzero intercept on the y -axis.

Step 2: Locate the vertices on the vertical boundary On $x = 4$, the line $y = 2x + 1$ gives $y = 9$, while $y = 3x + 1$ gives $y = 13$. The other vertices are $Q = (4, 9)$ and $R = (4, 13)$. The region bounded by PQ , PR and QR therefore lies entirely within $0 \leq x \leq 4$. Every vertical line through this interval meets the triangle in one segment between the two oblique sides, so a single integral suffices.

Step 3: Choose the upper and lower boundaries Throughout $0 \leq x \leq 4$, the difference of the ordinates is

$$(3x + 1) - (2x + 1) = x \geq 0.$$

Thus $y = 3x + 1$ is the upper boundary and $y = 2x + 1$ is the lower boundary. A thin vertical strip at abscissa x has height x and width dx , giving the elemental area $dA = x dx$. At the common vertex the height is zero, and at the right boundary it is four, consistent with the two endpoint ordinates.

Step 4: Integrate the strip area Adding all the vertical strips gives

$$\begin{aligned} A &= \int_0^4 [(3x + 1) - (2x + 1)] dx \\ &= \int_0^4 x dx = \left[\frac{x^2}{2} \right]_0^4 = \frac{16}{2} - 0. \end{aligned}$$

The integrand is nonnegative on the entire interval, so this definite integral is the geometric area directly; no splitting of the interval or change of sign is needed. As a check, the vertical side has length $13 - 9 = 4$, and its perpendicular distance from P is also four.

Step 5: Conclusion The area of the triangular region, evaluated using integration, is 8 square units.

Quick Tip to Solve: Equal intercepts make the oblique lines meet at $x = 0$; subtract the lower line from the upper before integrating, so the intercepts cancel.

26(a). If $A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 1 & 1 \end{pmatrix}$, find A^{-1} . Hence solve the system of equations $x + 3y + 4z = 8$, $2x + y + 2z = 5$ and $5x + y + z = 7$. [6 marks]

Solution:

Concept:

- A square matrix is invertible when its determinant is non-zero. Its inverse is $A^{-1} = \frac{\text{adj } A}{|A|}$, where the adjugate is the transpose of the cofactor matrix.
- A system with coefficient matrix A , unknown column X and constant column B is $AX = B$. Multiplication on the left by A^{-1} gives the unique solution $X = A^{-1}B$.

Step 1: Calculate the determinant Expanding along the first row, with alternating signs, gives

$$\begin{aligned} |A| &= 1(1 - 2) - 3(2 - 10) + 4(2 - 5) \\ &= -1 + 24 - 12 = 11. \end{aligned}$$

Since $|A| \neq 0$, the inverse exists and the given system has a unique solution.

Step 2: Form the cofactors and adjugate Each cofactor is $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant obtained by deleting row i and column j . For example, $C_{12} = -(2 - 10) = 8$ and $C_{23} = -(1 - 15) = 14$. Computing all nine entries gives

$$C = \begin{pmatrix} -1 & 8 & -3 \\ 1 & -19 & 14 \\ 2 & 6 & -5 \end{pmatrix}, \quad \text{adj } A = C^T = \begin{pmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{pmatrix}.$$

Thus the inverse is obtained by dividing each entry of this adjugate by 11. Transposition is essential: the cofactor matrix itself is not the adjugate.

Step 3: Apply the inverse to the equations Write the system as

$$AX = B, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 8 \\ 5 \\ 7 \end{pmatrix}.$$

Multiplying by the inverse on the left gives

$$\begin{aligned} X &= \frac{1}{11} \begin{pmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{pmatrix} \begin{pmatrix} 8 \\ 5 \\ 7 \end{pmatrix} \\ &= \frac{1}{11} \begin{pmatrix} -8 + 5 + 14 \\ 64 - 95 + 42 \\ -24 + 70 - 35 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \end{aligned}$$

Substitution checks all three equations: $1 + 3 + 4 = 8$, $2 + 1 + 2 = 5$, and $5 + 1 + 1 = 7$, respectively.

Quick Tip to Solve: After transposing the cofactors, multiply each row of the adjugate by the constant column; check the resulting values in all three original equations.

Step 4: Conclusion

$$A^{-1} = \frac{1}{11} \begin{pmatrix} -1 & 1 & 2 \\ 8 & -19 & 6 \\ -3 & 14 & -5 \end{pmatrix}, \quad \boxed{x = y = z = 1}.$$

Quick Tip to Solve: After transposing the cofactors, multiply each row of the adjugate by the constant column; check the resulting values in all three original equations.

OR

26(b). Find the inverse of the following matrix, using elementary transformations: $A = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$.
[6 marks]

Solution:**Concept:**

- To find an inverse by elementary transformations, reduce the augmented matrix $[A \mid I]$ to $[I \mid A^{-1}]$. Every row operation must act on both halves of the augmented matrix.
- Interchanging two rows, adding a multiple of one row to another, and multiplying a row by a non-zero constant are reversible operations. Reducing the left half to the identity establishes the inverse on the right.

Step 1: Set up the augmented matrix and obtain a unit pivot Start with

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array} \right].$$

Apply $R_2 \leftarrow R_2 - 2R_1$. This produces the first entry $5 - 2(2) = 1$ without fractions. Then interchange R_1 and R_2 , obtaining

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -2 & 1 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \end{array} \right].$$

Step 2: Clear the first column and create the remaining pivots Apply $R_2 \leftarrow R_2 - 2R_1$, followed by $R_2 \leftrightarrow R_3$. The augmented matrix becomes

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & -2 & -5 & 5 & -2 & 0 \end{array} \right].$$

Now apply $R_3 \leftarrow R_3 + 2R_2$. Its second entry becomes zero and its third entry becomes $-5 + 2(3) = 1$,

Quick Tip to Solve: First use the entry 5 minus twice the entry 2 to obtain a unit pivot; this keeps every row transformation free of fractions.

giving

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & -2 & 1 & 0 \\ 0 & 1 & 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array} \right].$$

Step 3: Reduce the left half to the identity Use the third-row pivot to remove the entries above it: apply $R_2 \leftarrow R_2 - 3R_3$ and $R_1 \leftarrow R_1 - 2R_3$. This gives

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -12 & 5 & -4 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array} \right].$$

Finally, $R_1 \leftarrow R_1 - R_2$ produces

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -1 & 1 \\ 0 & 1 & 0 & -15 & 6 & -5 \\ 0 & 0 & 1 & 5 & -2 & 2 \end{array} \right].$$

The left half is now I , so the right half is the required inverse. As a direct check, multiplication of the original matrix by this right half gives rows $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. For example, the first row is twice the first inverse row minus its third row, namely $(6, -2, 2) - (5, -2, 2) = (1, 0, 0)$.

Step 4: Conclusion By elementary row transformations,

$$A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}.$$

Quick Tip to Solve: First use the entry 5 minus twice the entry 2 to obtain a unit pivot; this keeps every row transformation free of fractions.

27. A company produces two types of goods, A and B, that require gold and silver. Each unit of type A requires 3 g of silver and 1 g of gold while that of type B requires 1 g of silver and 2 g of gold. The company can use at the most 9 g of silver and 8 g of gold. If each unit of type A brings a profit of ₹ 40 and that of type B ₹ 50, find the number of units of each type that the company should produce to maximize profit. Formulate the above problem as an LPP and solve it graphically. Also find the maximum profit. [6 marks]

Solution:

Concept:

- Let x and y denote the numbers of units of goods A and B. Resource limits give linear inequalities, and production quantities must satisfy $x, y \geq 0$.
- The objective function is total profit $Z = 40x + 50y$, measured in rupees. On a bounded feasible polygon, its maximum occurs at a vertex.

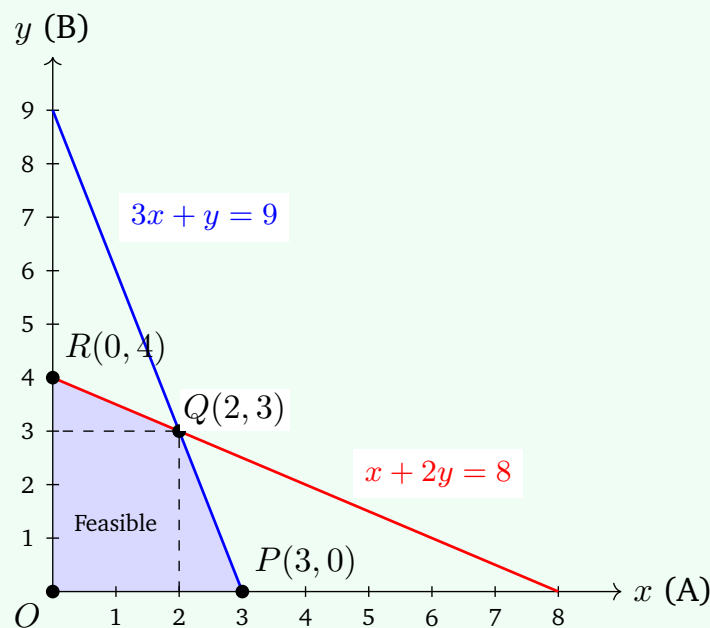
Quick Tip to Solve: Plot each resource line using its two intercepts, then compare profit at every vertex of the common first-quadrant region.

Step 1: Formulate the linear programming problem Each unit of A consumes 3 g of silver and 1 g of gold, whereas each unit of B consumes 1 g of silver and 2 g of gold. Therefore the silver and gold restrictions respectively are

$$3x + y \leq 9, \quad x + 2y \leq 8.$$

The required LPP is to maximize $Z = 40x + 50y$, subject to these two inequalities and $x \geq 0, y \geq 0$. The inequalities are non-strict because all the available material may be used; unused material is also permitted.

Step 2: Plot the boundary lines and identify the feasible region The line $3x + y = 9$ passes through $(3, 0)$ and $(0, 9)$. The line $x + 2y = 8$ passes through $(8, 0)$ and $(0, 4)$. For both resource inequalities, the origin satisfies the inequality, so take the half-plane containing the origin. Restricting to the first quadrant gives the shaded common region in the graph. Its vertices are $O = (0, 0)$, $P = (3, 0)$, the intersection Q of the two lines, and $R = (0, 4)$.



Feasible region OPQR for the two resource constraints

Step 3: Calculate the intersection vertex Solve the two active resource constraints simultaneously:

$$\begin{aligned} 3x + y &= 9, \\ x + 2y &= 8. \end{aligned}$$

Substituting $y = 9 - 3x$ into the second equation gives $x + 2(9 - 3x) = 8$, hence $-5x = -10$ and $x = 2$. Then $y = 9 - 6 = 3$, so $Q = (2, 3)$. This point lies in the first quadrant and satisfies both restrictions with equality.

Step 4: Compare profits at all vertices Since the feasible region is nonempty and bounded, evaluate

Quick Tip to Solve: Plot each resource line using its two intercepts, then compare profit at every vertex of the common first-quadrant region.

the objective at every corner:

Vertex (x, y)	$Z = 40x + 50y$ (rupees)
$(0, 0)$	0
$(3, 0)$	120
$(2, 3)$	230
$(0, 4)$	200

The largest value in this comparison is 230, attained at Q . As a resource check, this production uses $3(2) + 3 = 9$ g of silver and $2 + 2(3) = 8$ g of gold. Both quantities are whole numbers, so this solution is also valid when units of goods are indivisible.

Step 5: Conclusion The company should produce 2 units of A and 3 units of B. The maximum profit is ₹ 230.

Quick Tip to Solve: Plot each resource line using its two intercepts, then compare profit at every vertex of the common first-quadrant region.

28. There are three coins. One is a two-headed coin, another is a biased coin that comes up heads 75% of the time and the third is an unbiased coin. One of the three coins is chosen at random and tossed. If it shows heads, what is the probability that it is the two-headed coin? [6 marks]

Solution:

Concept:

- Bayes' theorem updates the probability of a cause after an outcome is observed. For mutually exclusive and exhaustive events C_1, C_2, C_3 and an event H with positive probability, $P(C_i | H) = \frac{P(C_i)P(H | C_i)}{P(H)}$.
- The law of total probability gives $P(H) = \sum_{i=1}^3 P(C_i)P(H | C_i)$, where C_i specifies which coin was selected and H is the event of getting a head.

Step 1: Define the coin-selection events and their prior probabilities Let C_1 be the event that the two-headed coin is selected, C_2 the event that the biased coin is selected, and C_3 the event that the unbiased coin is selected. Exactly one of these three events occurs. The choice is random among the three coins, so

$$P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}.$$

Let H denote the event that the selected coin shows heads on the toss. The probability requested is $P(C_1 | H)$, since the observed head is now known.

Step 2: Write the probability of heads for each coin A two-headed coin always shows heads; the biased coin shows heads in 75% of tosses; and an unbiased coin has equal probabilities of heads and tails. Consequently,

$$P(H | C_1) = 1, \quad P(H | C_2) = \frac{3}{4}, \quad P(H | C_3) = \frac{1}{2}.$$

Quick Tip to Solve: Equal prior coin probabilities cancel in Bayes' ratio: divide the two-headed coin's head likelihood by the sum $1 + 3/4 + 1/2$.

These are conditional probabilities for the outcome given the coin, rather than the desired conditional probability for the coin given the outcome.

Step 3: Find the total probability of observing a head The head may come from any of the three selected coins. Since the selection events partition the sample space, the law of total probability yields

$$\begin{aligned} P(H) &= P(C_1)P(H | C_1) + P(C_2)P(H | C_2) + P(C_3)P(H | C_3) \\ &= \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{3}{4} + \frac{1}{3} \cdot \frac{1}{2} \\ &= \frac{1}{3} + \frac{1}{4} + \frac{1}{6} = \frac{3}{4}. \end{aligned}$$

The probability that the chosen coin is two-headed and a head occurs is $P(C_1 \cap H) = \frac{1}{3} \cdot 1 = \frac{1}{3}$.

Step 4: Apply Bayes' theorem to the observed outcome Conditioning on the observed head gives

$$P(C_1 | H) = \frac{P(C_1 \cap H)}{P(H)} = \frac{\frac{1}{3}}{\frac{3}{4}} = \frac{4}{9}.$$

For a check, the other two posterior probabilities are $P(C_2 | H) = (1/4)/(3/4) = 1/3$ and $P(C_3 | H) = (1/6)/(3/4) = 2/9$. They sum with $4/9$ to 1. Also, $4/9 > 1/3$, as expected: observing a head makes selection of the always-heads coin more likely than it was before the toss.

Step 5: Conclusion Given that the toss shows heads, the probability that the chosen coin is the two-headed coin is $\boxed{\frac{4}{9}}$.

Quick Tip to Solve: Equal prior coin probabilities cancel in Bayes' ratio: divide the two-headed coin's head likelihood by the sum $1 + 3/4 + 1/2$.

29(a). Find the vector and cartesian equations of the plane passing through the points having position vectors $\hat{i} + \hat{j} - 2\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$. Write the equation of a plane passing through a point $(2, 3, 7)$ and parallel to the plane obtained above. Hence, find the distance between the two parallel planes. [6 marks]

Solution:

Concept:

- A plane through a point with position vector $\vec{a}$ and with nonzero normal vector $\vec{n}$ has vector equation $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ represents an arbitrary point on it.
- For three noncollinear points A, B, C , the cross product $\vec{AB} \times \vec{AC}$ is perpendicular to both directions in their plane and supplies its normal.
- Parallel planes have proportional normal vectors. For equations $ax + by + cz = d_1$ and $ax + by + cz = d_2$, their perpendicular separation is $|d_2 - d_1|/\sqrt{a^2 + b^2 + c^2}$.

Step 1: Obtain two directions in the required plane Write the three points as $A = (1, 1, -2)$,

Quick Tip to Solve: Keep the same normal coefficients for both parallel planes; then subtract only their constants and divide by the normal's length.

$B = (2, -1, 1)$ and $C = (1, 2, 1)$. Subtract the coordinates of A from those of the other points to obtain

$$\overrightarrow{AB} = (1, -2, 3), \quad \overrightarrow{AC} = (0, 1, 3).$$

Both vectors lie in the plane. Their cross product is

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 0 & 1 & 3 \end{vmatrix} \\ &= -9\hat{i} - 3\hat{j} + \hat{k}. \end{aligned}$$

It is nonzero, so the points are noncollinear and determine a unique plane. We may use the opposite vector $\vec{n} = 9\hat{i} + 3\hat{j} - \hat{k}$, which gives simpler positive leading coefficients.

Step 2: Write the vector and Cartesian equations The plane passes through A , whose position vector is $\hat{i} + \hat{j} - 2\hat{k}$. Therefore its point-normal vector equation is

$$(\vec{r} - \hat{i} - \hat{j} + 2\hat{k}) \cdot (9\hat{i} + 3\hat{j} - \hat{k}) = 0.$$

Since $(\hat{i} + \hat{j} - 2\hat{k}) \cdot \vec{n} = 9 + 3 + 2 = 14$, the equivalent vector equation is $\vec{r} \cdot (9\hat{i} + 3\hat{j} - \hat{k}) = 14$. Expanding in coordinates gives

$$9(x - 1) + 3(y - 1) - (z + 2) = 0, \quad 9x + 3y - z = 14.$$

As a check, the left side equals $18 - 3 - 1 = 14$ at B and $9 + 6 - 1 = 14$ at C , so all three supplied points lie on this plane.

Step 3: Find the parallel plane through the supplied point A parallel plane can use exactly the same normal vector, so write it as $9x + 3y - z = d$. Substituting $P = (2, 3, 7)$ determines the constant:

$$d = 9(2) + 3(3) - 7 = 20.$$

Thus its equation is $9x + 3y - z = 20$, or in vector form $\vec{r} \cdot (9\hat{i} + 3\hat{j} - \hat{k}) = 20$. The unequal constants show that these parallel planes are distinct.

Step 4: Calculate their perpendicular separation The normal length is $\sqrt{9^2 + 3^2 + (-1)^2} = \sqrt{91}$. Because the two equations use identical normal coefficients, their separation is

$$D = \frac{|20 - 14|}{\sqrt{91}} = \frac{6}{\sqrt{91}}.$$

Equivalently, the point-to-plane distance from P to the original plane has numerator $|18 + 9 - 7 - 14| = 6$, confirming the same result.

Step 5: Conclusion The original plane is $\boxed{\vec{r} \cdot (9\hat{i} + 3\hat{j} - \hat{k}) = 14}$, or $\boxed{9x + 3y - z = 14}$. The parallel plane is $\boxed{9x + 3y - z = 20}$, and the distance between them is $\boxed{6/\sqrt{91}}$ units.

Quick Tip to Solve: Keep the same normal coefficients for both parallel planes; then subtract only their constants and divide by the normal's length.

29(b). Find the equation of the line passing through $(2, -1, 2)$ and $(5, 3, 4)$, and of the plane passing through $(2, 0, 3)$, $(1, 1, 5)$ and $(3, 2, 4)$. Also find their point of intersection. [6 marks]

Solution:

Concept:

- A line through points A and B has vector equation $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$, where t is a real parameter. Its direction ratios are the coordinate differences of the two points.
- A plane through three noncollinear points P, Q, R has normal $\vec{PQ} \times \vec{PR}$ and equation $(\vec{r} - \vec{p}) \cdot \vec{n} = 0$.
- To find a line-plane intersection, substitute the parametric coordinates of the line into the plane equation. A unique value of the parameter gives the unique intersection point.

Step 1: Find the equation of the line For $A = (2, -1, 2)$ and $B = (5, 3, 4)$, the direction vector is

$$\vec{AB} = (5 - 2, 3 - (-1), 4 - 2) = (3, 4, 2).$$

Therefore the vector equation of the line is

$$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + t(3\hat{i} + 4\hat{j} + 2\hat{k}), \quad t \in \mathbb{R}.$$

Its parametric equations are $x = 2 + 3t$, $y = -1 + 4t$, $z = 2 + 2t$. Since all three direction ratios are nonzero, we can also write the symmetric Cartesian form

$$\frac{x - 2}{3} = \frac{y + 1}{4} = \frac{z - 2}{2}.$$

The parameters $t = 0$ and $t = 1$ reproduce A and B , respectively, verifying the required line.

Step 2: Calculate a normal to the plane Denote the three plane points by $P = (2, 0, 3)$, $Q = (1, 1, 5)$, $R = (3, 2, 4)$. The two directions from P are

$$\vec{PQ} = (-1, 1, 2), \quad \vec{PR} = (1, 2, 1).$$

Their cross product is

$$\begin{aligned} \vec{PQ} \times \vec{PR} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} \\ &= -3\hat{i} + 3\hat{j} - 3\hat{k}. \end{aligned}$$

The product is nonzero, proving the points are noncollinear. Dividing by -3 gives the convenient normal vector $\vec{n} = \hat{i} - \hat{j} + \hat{k}$. Its scalar product with each direction vector is zero, as required.

Step 3: Write and check the plane equation Using point P and the normal $(1, -1, 1)$, the point-normal equation is

$$(x - 2) - (y - 0) + (z - 3) = 0.$$

It simplifies to $x - y + z = 5$; in vector notation this is $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$. Check the other supplied points: at Q , $1 - 1 + 5 = 5$, and at R , $3 - 2 + 4 = 5$. Thus all three points satisfy the equation.

Quick Tip to Solve: Check whether either given line point already satisfies the plane equation; here the first point does, and the nonzero direction-normal product proves uniqueness.

Step 4: Determine the common point of the line and plane Substitute the line coordinates into $x - y + z = 5$:

$$(2 + 3t) - (-1 + 4t) + (2 + 2t) = 5,$$

$$5 + t = 5,$$

$$t = 0.$$

The coefficient of t is $3 - 4 + 2 = 1$, which is nonzero, so the line intersects the plane once rather than lying in it or being parallel to it. At $t = 0$, the coordinates are $(2, -1, 2)$. Substitution into the plane gives $2 - (-1) + 2 = 5$, confirming that this original line point also belongs to the plane.

Step 5: Conclusion The line is $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$, the plane is $x - y + z = 5$, and their point of intersection is $(2, -1, 2)$.

Quick Tip to Solve: Check whether either given line point already satisfies the plane equation; here the first point does, and the nonzero direction-normal product proves uniqueness.