CBSE Class 12 Mathematics Question Paper 2019 with Solution PDF-Set 65-3-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 100**

General Instructions

(i) All questions are compulsory.

(ii) The question paper consists of **29** questions divided into four sections : A, B, C and D. Section A comprises of 4 questions of **one mark** each, Section B comprises of 8 questions of **two marks** each, Section C comprises of 11 questions of **four marks** each and Section D comprises of 6 questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Questions number 1 to 4 carry 1 mark each.

1(a). Find the direction cosines of the line joining the points $P(4, 3, -5)$ and $Q(-2, 1, -8)$. [1 mark]

Solution:

Concept:

- Direction ratios of the line from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ are the coordinate differences.
- Dividing each direction ratio by the length of the direction vector gives direction cosines l, m, n, with $l^2 + m^2 + n^2 = 1$.

Step 1: Find the direction vector For the direction from P to Q, subtract the coordinates of P from those of Q:

$$\overrightarrow{PQ} = (-2 - 4,\ 1 - 3,\ -8 - (-5)) = (-6, -2, -3).$$

All three differences are negative because each coordinate decreases in this direction.

Step 2: Normalize the vector The length of this vector is

$$|\overrightarrow{PQ}| = \sqrt{(-6)^2 + (-2)^2 + (-3)^2} = \sqrt{49} = 7.$$

Hence divide each component by 7. The squared normalized components add to $(36 + 4 + 9)/49 = 1$, as required.

Step 3: Conclusion The direction cosines from P to Q are $\boxed{(-6/7, -2/7, -3/7)}$. For the reverse direction they are $(6/7, 2/7, 3/7)$.

Quick Tip to Solve: Subtract the coordinates in one consistent order, then divide all three differences by the same vector length.

OR

1(b). Find the value of p for which the following lines are perpendicular: $\frac{1-x}{3} = \frac{2y-14}{2p} = \frac{z-3}{2}$; $\frac{1-x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$. [1 mark]

Solution:

Concept:

- Two lines are perpendicular when the dot product of their direction vectors is zero.
- In symmetric form $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$, the denominators a, b, c are direction ratios after the numerators have been normalized.

Step 1: Extract direction ratios carefully Write the first line as

$$\frac{x-1}{-3} = \frac{y-7}{p} = \frac{z-3}{2}.$$

Quick Tip to Solve: Normalize every numerator to a coordinate minus a constant before reading the direction ratios.

Its direction vector is $(-3, p, 2)$. For the second line, write

$$\frac{x-1}{-3p} = \frac{y-5}{1} = \frac{z-6}{-5},$$

giving the direction vector $(-3p, 1, -5)$.

Step 2: Apply perpendicularity The numerator $2y - 14$ has a factor of 2, so its effective direction ratio is p, not $2p$. Now the zero dot-product condition gives

$$(-3)(-3p) + p(1) + 2(-5) = 0, \qquad 10p - 10 = 0.$$

Solving this linear equation determines the required value.

Step 3: Conclusion $\boxed{p = 1}$. The resulting direction vectors $(-3, 1, 2)$ and $(-3, 1, -5)$ have dot product $9 + 1 - 10 = 0$.

Quick Tip to Solve: Normalize every numerator to a coordinate minus a constant before reading the direction ratios.

2. Find the integrating factor of the differential equation $x\dfrac{dy}{dx} - 2y = 2x^2$. [1 mark]

Solution:

Concept:

- For a linear equation $dy/dx + P(x)y = Q(x)$, an integrating factor is $e^{\int P(x)\,dx}$.
- The coefficient of dy/dx must first be made equal to one.

Step 1: Put the equation in standard form On an interval where $x \neq 0$, divide every term of the given equation by x:

$$\frac{dy}{dx} - \frac{2}{x}y = 2x.$$

Thus the function multiplying y in standard linear form is $P(x) = -2/x$.

Step 2: Evaluate the integrating factor Use the coefficient from the normalized equation:

$$\text{I.F.} = e^{\int (-2/x)\,dx} = e^{-2\log|x|} = |x|^{-2} = x^{-2}.$$

A nonzero constant multiplier could also be used; the simplest choice is the expression shown.

Step 3: Conclusion The integrating factor is $\boxed{1/x^2}$, for $x \neq 0$.

Quick Tip to Solve: Divide by the derivative coefficient first; using the unnormalized coefficient of y gives the wrong integrating factor.

3. If A is a square matrix of order 3 with $|A| = 4$, then write the value of $|-2A|$. [1 mark]

Solution:

Concept:

- If A has order n, multiplying the entire matrix by k multiplies its determinant by k^n.
- This follows because each of the n rows contributes one factor k to the determinant.

Step 1: Apply the scaling rule The matrix has three rows and three columns. In $-2A$, every entry in each row is multiplied by -2. Taking this factor out of all three rows yields

$$|-2A| = (-2)^3|A|.$$

The multiplier is cubed because the order is three.

Step 2: Substitute the determinant The supplied determinant is $|A| = 4$, so

$$|-2A| = (-8)(4) = -32.$$

The negative sign remains since an odd number of rows is multiplied by a negative scalar; using only one factor of -2 would ignore two rows.

Step 3: Conclusion $\boxed{|-2A| = -32}$.

Quick Tip to Solve: Raise the scalar multiplier to the matrix order before multiplying by the given determinant.

4. If $y = \sin^{-1} x + \cos^{-1} x$, find $\dfrac{dy}{dx}$. [1 mark]

Solution:

Concept:

- For principal inverse trigonometric functions, $\sin^{-1} x + \cos^{-1} x = \pi/2$ on $[-1, 1]$.
- For $-1 < x < 1$, their derivatives are $1/\sqrt{1-x^2}$ and $-1/\sqrt{1-x^2}$, respectively.

Step 1: Differentiate the two terms At an interior point of the common real domain, apply the standard derivative formulas:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}.$$

The inverse cosine derivative carries a negative sign.

Step 2: Simplify and check the identity The two equal terms cancel exactly. This also follows from the principal-value identity $y = \pi/2$: the sum is constant throughout its domain, so its derivative at every interior point is zero. The individual derivative formulas above are used only for $-1 < x < 1$.

Step 3: Conclusion $\boxed{dy/dx = 0}$, for $-1 < x < 1$.

Quick Tip to Solve: Recognize the complementary inverse-trigonometric identity to see immediately that the given sum is constant.

Section B (16 marks)

Questions number 5 to 12 carry 2 marks each.

5. If $A = \begin{bmatrix} 3 & 9 & 0 \\ 1 & 8 & -2 \\ 7 & 5 & 4 \end{bmatrix}$ **and** $B = \begin{bmatrix} 4 & 0 & 2 \\ 7 & 1 & 4 \\ 2 & 2 & 6 \end{bmatrix}$**, then find the matrix** $B'A'$**.** [2 marks]

Solution:

Concept:

- The transpose reverses multiplication order: $(AB)^{\mathsf{T}} = B^{\mathsf{T}}A^{\mathsf{T}}$. Here the prime denotes transpose.
- The entry in row i, column j of a product is the dot product of row i of the first matrix with column j of the second.

Step 1: Compute the product AB Compute each entry using the supplied rows and columns:

$$AB = \begin{bmatrix} 3(4)+9(7)+0(2) & 3(0)+9(1)+0(2) & 3(2)+9(4)+0(6) \\ 1(4)+8(7)-2(2) & 1(0)+8(1)-2(2) & 1(2)+8(4)-2(6) \\ 7(4)+5(7)+4(2) & 7(0)+5(1)+4(2) & 7(2)+5(4)+4(6) \end{bmatrix}.$$

Simplifying the entries gives

$$AB = \begin{bmatrix} 75 & 9 & 42 \\ 56 & 4 & 22 \\ 71 & 13 & 58 \end{bmatrix}.$$

Step 2: Transpose the product The question asks for $B\prime A\prime$, which equals $(AB)\prime$, so interchange rows and columns in the product just obtained. Its first row becomes the first column, and similarly for the other two rows. As a direct check, the upper-right entry of $B\prime A\prime$ is $(4, 7, 2) \cdot (7, 5, 4) = 71$, agreeing with the transpose.

Step 3: Conclusion

$$\boxed{B\prime A\prime = \begin{bmatrix} 75 & 56 & 71 \\ 9 & 4 & 13 \\ 42 & 22 & 58 \end{bmatrix}}.$$

Quick Tip to Solve: Use B transpose times A transpose equals the transpose of AB; reverse the order when applying the transpose rule.

6. Find: $\int_a^b \frac{\log x}{x}\,dx$**.** [2 marks]

Solution:

Concept:

- The substitution $t = \log x$ gives $dt = dx/x$, converting the integrand into $t\,dt$.

Quick Tip to Solve: Whenever log x is multiplied by 1/x, substitute t = log x and change the two endpoint values immediately.

- For a definite integral, transform both limits along with the variable. The real logarithm requires positive endpoints and an interval in $x > 0$.

Step 1: Substitute the logarithm Let

$$t = \log x, \qquad dt = \frac{dx}{x}.$$

When $x = a$, the new lower limit is $\log a$; when $x = b$, the new upper limit is $\log b$. Thus, for positive a, b,

$$I = \int_{\log a}^{\log b} t\, dt.$$

This substitution uses precisely the factor $1/x$ already present in the integrand.

Step 2: Integrate and apply the limits The antiderivative of t is $t^2/2$. Therefore

$$I = \left[\frac{t^2}{2}\right]_{\log a}^{\log b} = \frac{(\log b)^2}{2} - \frac{(\log a)^2}{2}.$$

Equivalently, differentiating $\frac{1}{2}(\log x)^2$ by the chain rule gives $(\log x)/x$, checking the antiderivative. The expression also gives zero when $a = b$ and changes sign when the endpoints are interchanged.

Step 3: Conclusion

$$\boxed{\int_a^b \frac{\log x}{x}\, dx = \frac{1}{2}\left[(\log b)^2 - (\log a)^2\right]}, \qquad a, b > 0.$$

Quick Tip to Solve: Whenever log x is multiplied by 1/x, substitute t = log x and change the two endpoint values immediately.

7. Form the differential equation representing the family of curves $y^2 = m(a^2 - x^2)$ by eliminating the arbitrary constants m and a. [2 marks]

Solution:

Concept:

- To eliminate two arbitrary constants, differentiate the family twice and remove the constants from the resulting equations.
- The product rule gives $d(yy\prime)/dx = (y\prime)^2 + yy\prime\prime$, where primes denote derivatives with respect to x.

Step 1: Differentiate to remove a The family is $y^2 = m(a^2 - x^2)$, with m, a constant. Differentiating with respect to x gives

$$2yy\prime = -2mx, \qquad yy\prime = -mx.$$

The constant a has disappeared because ma^2 is independent of x. The remaining equation contains only the constant m.

Quick Tip to Solve: Differentiate yy prime using the product rule, then multiply the second derivative equation by x to eliminate m without division.

Step 2: Differentiate again and eliminate m Apply the product rule to the last equation:

$$(y\prime)^2 + yy\prime\prime = -m.$$

Multiply this equation by x to obtain

$$x\left[(y\prime)^2 + yy\prime\prime\right] = -mx.$$

From the first derivative equation, $-mx = yy\prime$. Substitute that value and bring all terms to one side. Multiplying rather than dividing by x avoids introducing an unnecessary restriction at $x = 0$.

Step 3: Conclusion The required differential equation is

$$\boxed{x\left[\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right] - y\frac{dy}{dx} = 0}.$$

Quick Tip to Solve: Differentiate yy prime using the product rule, then multiply the second derivative equation by x to eliminate m without division.

8(a). Find: $\displaystyle\int \frac{\sin(x-a)}{\sin(x+a)}\,dx$. [2 marks]

Solution:

Concept:

- Treat a as a constant. Rewrite the numerator in terms of the angle $x + a$ using $\sin(u - v) = \sin u \cos v - \cos u \sin v$.
- The standard integral is $\int \cot u\,du = \log|\sin u| + C$. Work on intervals where $\sin(x + a) \neq 0$.

Step 1: Rewrite the integrand Since $x - a = (x + a) - 2a$, the subtraction identity gives

$$\sin(x-a) = \sin(x+a)\cos 2a - \cos(x+a)\sin 2a.$$

Dividing by the nonzero denominator,

$$\frac{\sin(x-a)}{\sin(x+a)} = \cos 2a - \sin 2a \cot(x+a).$$

This form isolates the constant term and a standard trigonometric integral.

Step 2: Integrate and check As a is constant and $d(x + a)/dx = 1$,

$$\begin{aligned} I &= \cos 2a \int dx - \sin 2a \int \cot(x+a)\,dx \\ &= x\cos 2a - \sin 2a \log|\sin(x+a)| + C. \end{aligned}$$

Differentiating this expression gives $\cos 2a - \sin 2a \cot(x + a)$, exactly the rewritten integrand. The absolute value makes the logarithm valid on either sign interval of the sine.

Quick Tip to Solve: Write x-a as (x+a)-2a first, so the denominator cancels one term immediately.

Step 3: Conclusion

$$\boxed{\int \frac{\sin(x-a)}{\sin(x+a)}\,dx = x\cos 2a - \sin 2a \log|\sin(x+a)| + C.}$$

Quick Tip to Solve: Write x-a as (x+a)-2a first, so the denominator cancels one term immediately.

OR

8(b). Find: $\int (\log x)^2\,dx$. [2 marks]

Solution:

Concept:

- For $x > 0$, $\log x$ denotes the natural logarithm and $d(\log x)/dx = 1/x$.
- Integration by parts: $\int u\,dv = uv - \int v\,du$. Choosing a logarithmic expression as u lowers its power after differentiation.

Step 1: Apply integration by parts to the square Put $u = (\log x)^2$ and $dv = dx$. Then $du = 2\log x\,dx/x$ and $v = x$. Consequently,

$$I = x(\log x)^2 - 2\int \log x\,dx.$$

The factor x from integrating dx cancels the denominator in du, leaving a simpler integral.

Step 2: Evaluate the remaining integral Apply integration by parts again with $u = \log x$ and $dv = dx$. This gives

$$\int \log x\,dx = x\log x - \int 1\,dx = x\log x - x.$$

Therefore

$$I = x(\log x)^2 - 2x\log x + 2x + C.$$

For verification, its derivative is $(\log x)^2 + 2\log x - 2\log x - 2 + 2 = (\log x)^2$, so all extra terms cancel. Only one arbitrary constant is needed in the final antiderivative.

Step 3: Conclusion

$$\boxed{\int (\log x)^2\,dx = x\big[(\log x)^2 - 2\log x + 2\big] + C, \quad x > 0.}$$

Quick Tip to Solve: Choose the logarithmic factor for differentiation in each integration by parts; its power decreases.

9(a). Find a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$, where $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$. [2 marks]

Solution:

Concept:

- The cross product $\vec{a} \times \vec{b}$ is perpendicular to both input vectors. If it is nonzero, divide it by its magnitude to obtain a unit vector.
- For $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$, $|\vec{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$. Both opposite unit directions satisfy perpendicularity.

Step 1: Calculate the cross product Using the given components,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} \\ &= \hat{i}(-14 + 14) - \hat{j}(2 - 21) + \hat{k}(-2 + 21) \\ &= 19\hat{j} + 19\hat{k}.\end{aligned}$$

The result is nonzero, so the two given vectors determine a perpendicular direction.

Step 2: Normalize and verify the direction The magnitude is

$$|\vec{a} \times \vec{b}| = \sqrt{19^2 + 19^2} = 19\sqrt{2}.$$

Thus choose $\hat{n} = (\hat{j} + \hat{k})/\sqrt{2}$. Its magnitude is $\sqrt{1/2 + 1/2} = 1$. Also,

$$\hat{n} \cdot \vec{a} = \frac{-7 + 7}{\sqrt{2}} = 0, \qquad \hat{n} \cdot \vec{b} = \frac{-2 + 2}{\sqrt{2}} = 0.$$

These checks verify both the unit length and the two perpendicularity conditions.

Step 3: Conclusion One required unit vector is

$$\boxed{\hat{n} = \frac{\hat{j} + \hat{k}}{\sqrt{2}}.}$$

Its negative is also a valid answer.

Quick Tip to Solve: Notice that each given vector has opposite j and k components; their sum direction is perpendicular.

OR

9(b). Show that the vectors $\hat{i}-2\hat{j}+3\hat{k},\ -2\hat{i}+3\hat{j}-4\hat{k}$ and $\hat{i}-3\hat{j}+5\hat{k}$ are coplanar. [2 marks]

Solution:

Concept:

- Three vectors are coplanar when their scalar triple product is zero. Equivalently, one vector can be written as a linear combination of the other two.
- If $\vec{u},\vec{v}$ are independent, all vectors $s\vec{u}+t\vec{v}$ lie in the plane spanned by them.

Step 1: Find a linear relation Let $\vec{u}=\hat{i}-2\hat{j}+3\hat{k}$, $\vec{v}=-2\hat{i}+3\hat{j}-4\hat{k}$, and $\vec{w}=\hat{i}-3\hat{j}+5\hat{k}$. Calculate

$$\begin{aligned}3\vec{u}+\vec{v}&=3(\hat{i}-2\hat{j}+3\hat{k})+(-2\hat{i}+3\hat{j}-4\hat{k})\\&=(3-2)\hat{i}+(-6+3)\hat{j}+(9-4)\hat{k}\\&=\hat{i}-3\hat{j}+5\hat{k}=\vec{w}.\end{aligned}$$

Thus the third vector belongs to the span of the first two.

Step 2: Confirm by the scalar triple product The determinant of their components is

$$\begin{aligned}[\vec{u}\ \vec{v}\ \vec{w}]&=\begin{vmatrix}1&-2&3\\-2&3&-4\\1&-3&5\end{vmatrix}\\&=1(15-12)+2(-10+4)+3(6-3)\\&=3-12+9=0.\end{aligned}$$

A zero scalar triple product means that the parallelepiped formed by the vectors has zero volume; hence all three vectors lie in a common plane.

Step 3: Conclusion $\boxed{\vec{w}=3\vec{u}+\vec{v}}$, and the scalar triple product is zero. Therefore the given vectors are coplanar.

Quick Tip to Solve: Compare components to find a small integer linear combination before expanding a full determinant.

10. Mother, father and son line up at random for a family photo. If A and B are two events given by A = Son on one end, B = Father in the middle, find $P(B/A)$. [2 marks]

Solution:

Concept:

- Conditional probability is $P(B\mid A)=P(A\cap B)/P(A)$, provided $P(A)>0$. Here conditioning restricts attention to arrangements with the son at an end.
- The three distinct family members can be arranged in $3!=6$ equally likely orders. Write M,F,S for mother, father and son respectively.

Quick Tip to Solve: After imposing that the son occupies an end, count favourable orders only among the four remaining orders.

Step 1: Count the arrangements satisfying A The six left-to-right orders are

$$MFS,\ MSF,\ FMS,\ FSM,\ SMF,\ SFM.$$

The son is at an end in MFS, FMS, SMF, SFM, giving four outcomes. Equivalently, choose one of two ends for the son, then arrange the other two people in 2! ways. Hence $P(A) = 4/6 = 2/3$. The remaining two arrangements put the son in the middle and are excluded after conditioning.

Step 2: Count the intersection and apply the formula The father is in the middle in exactly MFS and SFM. In both these outcomes the son is necessarily at an end, so both belong to $A \cap B$. Thus

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3}, \qquad P(B \mid A) = \frac{1/3}{2/3} = \frac{2}{4}.$$

The conditional denominator is the four arrangements allowed by A, rather than all six original arrangements.

Step 3: Conclusion The required conditional probability is

$$\boxed{P(B/A) = P(B \mid A) = \frac{1}{2}.}$$

Quick Tip to Solve: After imposing that the son occupies an end, count favourable orders only among the four remaining orders.

11(a). Let X be a random variable which assumes values x_1, x_2, x_3, x_4 such that $2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4)$. Find the probability distribution of X. [2 marks]

Solution:

Concept:

- A probability distribution assigns a nonnegative probability to every possible value, with their sum equal to 1.
- Set the common value of the given weighted probabilities equal to k; divide by each stated multiplier to recover the individual probabilities.

Step 1: Express all probabilities using one constant Let

$$2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4) = k.$$

Then

$$P(X = x_1) = \frac{k}{2}, \quad P(X = x_2) = \frac{k}{3}, \quad P(X = x_3) = k, \quad P(X = x_4) = \frac{k}{5}.$$

Notice that the probabilities are inversely proportional to the printed multipliers, so the coefficient 5 produces the smallest probability.

Quick Tip to Solve: Turn equal weighted probabilities into the ratio 15:10:30:6, then divide each entry by their sum 61.

Step 2: Normalize the probabilities Since these are all the possible values of X,

$$\frac{k}{2}+\frac{k}{3}+k+\frac{k}{5}=1,$$
$$\frac{15k+10k+30k+6k}{30}=1,$$
$$k=\frac{30}{61}.$$

Substitution gives probabilities $15/61, 10/61, 30/61, 6/61$. They are all positive and add to $(15+10+30+6)/61 = 1$. Furthermore, multiplying them by $2, 3, 1, 5$, respectively, gives $30/61$ every time, verifying the original relation.

Step 3: Conclusion The probability distribution is

X	x_1	x_2	x_3	x_4
$P(X)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

.

Quick Tip to Solve: Turn equal weighted probabilities into the ratio 15:10:30:6, then divide each entry by their sum 61.

OR

11(b). A coin is tossed 5 times. Find the probability of getting (i) at least 4 heads, and (ii) at most 4 heads. [2 marks]

Solution:

Concept:

- For five independent tosses of a fair coin, the number of heads X follows the binomial distribution with $n = 5$ and $p = \frac{1}{2}$.
- The probability of exactly r heads is $P(X = r) = \binom{5}{r}(\frac{1}{2})^r(\frac{1}{2})^{5-r} = \binom{5}{r}/32$. Each of the $2^5 = 32$ ordered outcomes is equally likely.

Step 1: (i) Identify and add the cases for at least four heads At least four heads means either exactly four heads or exactly five heads. These events are disjoint, so their probabilities add. There are $\binom{5}{4} = 5$ ways to choose the four head positions, and $\binom{5}{5} = 1$ way to obtain five heads. Thus,

$$P(X \geq 4) = P(X = 4) + P(X = 5) = \frac{5}{32} + \frac{1}{32} = \frac{6}{32}.$$

Step 2: (ii) Use the complement for at most four heads At most four heads includes zero, one, two, three and four heads. The only excluded case is five heads. Since all possibilities together have probability one,

$$P(X \leq 4) = 1 - P(X = 5) = 1 - \left(\frac{1}{2}\right)^5 = 1 - \frac{1}{32} = \frac{31}{32}.$$

Quick Tip to Solve: For at most four heads in five tosses, subtract only the all-heads outcome from one; for at least four, count the five single-tail outcomes and the all-heads outcome.

Notice that exactly four heads belongs to both events in the question; the two requested events are therefore not complements of each other.

Step 3: Conclusion The required probabilities are (i) $\boxed{\frac{3}{16}}$ for at least four heads and (ii) $\boxed{\frac{31}{32}}$ for at most four heads.

Quick Tip to Solve: For at most four heads in five tosses, subtract only the all-heads outcome from one; for at least four, count the five single-tail outcomes and the all-heads outcome.

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12. If $*$ is defined on the set $\mathbb{R}$ of all real numbers by $*$: $a * b = \sqrt{a^2 + b^2}$, find the identity element, if it exists in $\mathbb{R}$ with respect to $*$. [2 marks]

Solution:

Concept:

- An identity element e for a binary operation on $\mathbb{R}$ must satisfy $a * e = e * a = a$ for every real number a, including negative numbers.
- The square root symbol denotes the non-negative principal square root. In particular, $\sqrt{a^2} = |a|$, which equals a only when $a \geq 0$.

Step 1: Find the only possible candidate Suppose an identity $e \in \mathbb{R}$ exists. Applying the identity condition to $a = 0$ gives

$$0 * e = \sqrt{0^2 + e^2} = |e| = 0.$$

Therefore $e = 0$ is the only possible identity candidate. It must still be tested for every real input; finding a candidate alone does not establish an identity.

Step 2: Test the candidate on negative real numbers For $e = 0$, the operation yields

$$a * 0 = 0 * a = \sqrt{a^2} = |a|.$$

If $a = -1$, this gives $(-1) * 0 = 1$, whereas the identity condition requires the result to be -1. Thus the candidate fails on an element of the given set. More generally, every output $\sqrt{a^2 + e^2}$ is non-negative, so it cannot equal a negative input, whatever e is chosen.

Step 3: Conclusion Hence, $\boxed{\text{no identity element exists in } \mathbb{R} \text{ with respect to } *}$.

Quick Tip to Solve: After obtaining the candidate zero, test a negative input: the principal square root returns the absolute value, so zero cannot be an identity on all real numbers.

Section C (44 marks)

Questions number 13 to 23 carry 4 marks each.

13. Find the value of x, if $\tan\left[\sec^{-1}\left(\frac{1}{x}\right)\right] = \sin(\tan^{-1} 2),\ x > 0$. [4 marks]

Solution:

Concept:

- Principal values of $\tan^{-1} t$ lie in $(-\pi/2, \pi/2)$. If $t > 0$, the angle is in the first quadrant, where its sine and cosine are positive.
- The real domain of $\sec^{-1} u$ is $|u| \geq 1$, with principal values in $[0, \pi] \setminus \{\pi/2\}$. Since $x > 0$, the expression $\sec^{-1}(1/x)$ requires $0 < x \leq 1$.
- Use $1 + \tan^2\theta = \sec^2\theta$, and $\cos(\sec^{-1}(1/x)) = x$, to translate the inverse trigonometric equation into algebra while preserving the principal-value restriction.

Step 1: Evaluate the right-hand side Let $\alpha = \tan^{-1} 2$. Then $\tan\alpha = 2$ and $0 < \alpha < \pi/2$. By the identity $\sec^2\alpha = 1 + \tan^2\alpha$, we get $\sec^2\alpha = 5$. As $\cos\alpha > 0$,

$$\cos\alpha = \frac{1}{\sqrt{5}}, \qquad \sin\alpha = \tan\alpha\cos\alpha = \frac{2}{\sqrt{5}}.$$

Thus the given equation becomes $\tan[\sec^{-1}(1/x)] = 2/\sqrt{5}$.

Step 2: Apply the secant identity on the allowed branch Put $\theta = \sec^{-1}(1/x)$. The restriction $0 < x \leq 1$ means $1/x \geq 1$, so $\theta \in [0, \pi/2)$ and $\sec\theta = 1/x$. From the equation, $\tan\theta = 2/\sqrt{5}$. Therefore,

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{4}{5} = \frac{9}{5},$$
$$\frac{1}{x^2} = \frac{9}{5},$$
$$x^2 = \frac{5}{9}.$$

The condition $x > 0$ selects the positive square root. The negative root cannot be used in the printed question.

Step 3: Verify the candidate in the original equation For the positive candidate $x = \sqrt{5}/3$, we have $0 < x < 1$, so the inverse secant is defined. Its secant is $3/\sqrt{5} > 1$, and its principal angle lies in the first quadrant. Hence,

$$\tan[\sec^{-1}(3/\sqrt{5})] = \sqrt{\left(\frac{3}{\sqrt{5}}\right)^2 - 1} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}.$$

This agrees with $\sin(\tan^{-1} 2)$, establishing that the candidate satisfies the original equation and its branch conditions.

Step 4: Conclusion The required value is $\boxed{x = \frac{\sqrt{5}}{3}}$.

Quick Tip to Solve: Evaluate the sine on the right first, then use one plus tangent squared equals secant squared; select the positive root only after checking the inverse-secant domain.

14(a). **If $e^y(x+1)=1$, then show that $\dfrac{d^2y}{dx^2}=\left(\dfrac{dy}{dx}\right)^2$.** [4 marks]

Solution:

Concept:

- In implicit differentiation, y is a function of x. Thus the chain rule gives $\dfrac{d}{dx}e^y=e^y\dfrac{dy}{dx}$.
- The product rule is $\dfrac{d}{dx}(uv)=u'v+uv'$. Since $e^y>0$ and $e^y(x+1)=1$, the real domain is $x>-1$.

Step 1: Differentiate the given relation once Write $y'=\dfrac{dy}{dx}$. Differentiating both sides of the equation $e^y(x+1)=1$ with respect to x, and applying the product rule to its left side, gives

$$e^y y'(x+1)+e^y=0.$$

The derivative of the constant on the right is zero. As e^y is never zero, division by e^y is valid, so

$$(x+1)y'+1=0, \qquad y'=-\frac{1}{x+1}.$$

The denominator is nonzero throughout the domain already obtained from the original equation.

Step 2: Differentiate the first-derivative relation Differentiate $(x+1)y'+1=0$ once more. The factor y' also depends on x, so another application of the product rule gives

$$(x+1)y''+y'=0.$$

Substituting the value of the first derivative into this relation yields

$$y''=-\frac{y'}{x+1}=-\frac{-1/(x+1)}{x+1}=\frac{1}{(x+1)^2}.$$

Independently, squaring the expression for y' gives

$$(y')^2=\left(-\frac{1}{x+1}\right)^2=\frac{1}{(x+1)^2}.$$

Both calculations hold at every point of the real domain. One may also check the differentiation by taking logarithms of the original relation: $y=-\ln(x+1)$, whose first two derivatives agree with those found above.

Step 3: Conclusion Hence, for $x>-1$,

$$\boxed{\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2}.$$

Quick Tip to Solve: After the first differentiation, cancel the nonzero factor e^y before differentiating again; this makes the second product rule much shorter.

OR

14(b). Find $\frac{dy}{dx}$, if $y = \sin^{-1}\left[\frac{2^{x+1}}{1+4^x}\right]$. [4 marks]

Solution:

Concept:

- For $-1 < u < 1$, $\frac{d}{dx}\sin^{-1} u = \frac{u'}{\sqrt{1-u^2}}$, where the square root is nonnegative.
- Put $t = 2^x > 0$; then $\frac{dt}{dx} = t \ln 2$. The identity $\sqrt{a^2} = |a|$ is essential when simplifying the inverse sine derivative.

Step 1: Rewrite and differentiate the inner function Since $4^x = (2^x)^2 = t^2$, the given function is

$$y = \sin^{-1} u, \qquad u = \frac{2t}{1+t^2}.$$

For all $t > 0$, $0 < u \leq 1$, because $1 + t^2 - 2t = (t-1)^2 \geq 0$. Equality $u = 1$ occurs only when $t = 1$, that is, $x = 0$. Using the quotient rule and the derivative of t,

$$\frac{du}{dx} = \frac{2t\ln 2(1+t^2) - 2t(2t^2\ln 2)}{(1+t^2)^2} = \frac{2t\ln 2(1-t^2)}{(1+t^2)^2}.$$

Step 2: Apply the inverse sine derivative away from zero For $x \neq 0$, the inner function is strictly less than 1, so the chain rule applies. Its denominator simplifies as

$$\sqrt{1-u^2} = \sqrt{\frac{(1+t^2)^2 - 4t^2}{(1+t^2)^2}} = \frac{|1-t^2|}{1+t^2}.$$

Therefore

$$\frac{dy}{dx} = \frac{2t\ln 2}{1+t^2}\frac{1-t^2}{|1-t^2|}.$$

When $x < 0$, $t < 1$, and the sign factor is $+1$. When $x > 0$, $t > 1$, and it is -1. The absolute value must be retained until these two cases are separated.

Step 3: Check differentiability at zero Set $\theta = \tan^{-1} t$, so $0 < \theta < \pi/2$ and $u = \sin(2\theta)$. The principal inverse sine value is 2θ for $t \leq 1$, and $\pi - 2\theta$ for $t \geq 1$. Hence the left-hand derivative at $x = 0$ is $\ln 2$, whereas the right-hand derivative is $-\ln 2$. These differ, so the derivative does not exist at zero, even though $y(0) = \pi/2$ is defined.

Step 4: Conclusion

$$\boxed{\frac{dy}{dx} = \begin{cases} \dfrac{2^{x+1}\ln 2}{1+4^x}, & x < 0, \\ -\dfrac{2^{x+1}\ln 2}{1+4^x}, & x > 0. \end{cases}}$$

The function is not differentiable at $x = 0$.

Quick Tip to Solve: Simplify the square root to $|1-4^x|/(1+4^x)$, then split at $x = 0$; cancelling without the absolute value loses the negative branch.

15. Find the intervals in which the function f given by $f(x) = 4x^3 - 6x^2 - 72x + 30$ is (a) strictly increasing, (b) strictly decreasing. [4 marks]

Solution:

Concept:

- For a differentiable function, a positive first derivative on an interval implies strict increase, and a negative first derivative implies strict decrease there.
- A polynomial is differentiable for every real number. The zeros of its derivative divide the real line into intervals on which its derivative has a constant sign.

Step 1: Differentiate and locate the critical numbers For the given function $f(x) = 4x^3 - 6x^2 - 72x + 30$, differentiate each term with respect to x:

$$\begin{aligned} f'(x) &= 12x^2 - 12x - 72 \\ &= 12(x^2 - x - 6) \\ &= 12(x - 3)(x + 2). \end{aligned}$$

Thus $f'(x) = 0$ at $x = -2$ and $x = 3$. Since the derivative is defined everywhere, these are the only critical numbers. We examine the three intervals separated by them.

Step 2: Determine the sign on each interval If $x < -2$, both $x - 3$ and $x + 2$ are negative, so their product is positive and $f'(x) > 0$.

If $-2 < x < 3$, the factor $x - 3$ is negative while $x + 2$ is positive, giving $f'(x) < 0$.

If $x > 3$, both factors are positive, so $f'(x) > 0$. The positive constant factor 12 never affects the sign. Hence the signs, in order from left to right, are positive, negative and positive.

Step 3: Apply the monotonicity criterion The derivative is strictly positive throughout each of $(-\infty, -2)$ and $(3, \infty)$, so the function strictly increases on each interval separately. It is strictly negative throughout $(-2, 3)$, so the function strictly decreases there. The derivative vanishes at the two boundary points only; this does not introduce a further interval of constancy. These intervals cover all noncritical real numbers.

Step 4: Conclusion (a) f is strictly increasing on $\boxed{(-\infty, -2) \text{ and } (3, \infty)}$.

(b) f is strictly decreasing on $\boxed{(-2, 3)}$.

Quick Tip to Solve: Factor the derivative as $12(x - 3)(x + 2)$; its sign is positive outside the two roots and negative between them.

16(a). Show that the relation R on the set $\mathbb{Z}$ of all integers, given by $R = \{(a, b) : 2 \text{ divides } (a - b)\}$ is an equivalence relation. [4 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation when it is reflexive, symmetric and transitive.
- The notation $2 \mid (a - b)$ means that $a - b = 2k$ for some integer k. We use this definition to verify

Quick Tip to Solve: Express each divisible difference as twice an integer; negate it for symmetry and add two such differences for transitivity.

each property for arbitrary integers.

Step 1: Verify reflexivity Let a be any integer. Its difference from itself is

$$a - a = 0 = 2 \cdot 0.$$

Since 0 is an integer, $2 \mid (a - a)$. Therefore $(a, a) \in R$ for every $a \in \mathbb{Z}$. This is exactly the condition for R to be reflexive on $\mathbb{Z}$.

Step 2: Verify symmetry Take any $a, b \in \mathbb{Z}$ and suppose that $(a, b) \in R$. By the definition of R, there exists an integer k such that

$$a - b = 2k.$$

Reversing the order of the difference gives

$$b - a = -(a - b) = -2k = 2(-k).$$

As $-k$ is also an integer, $2 \mid (b - a)$, and consequently $(b, a) \in R$. Thus membership of (a, b) implies membership of its reversed pair, proving symmetry.

Step 3: Verify transitivity Let $a, b, c \in \mathbb{Z}$ and assume that $(a, b) \in R$ and $(b, c) \in R$. There are integers m, n for which

$$a - b = 2m, \qquad b - c = 2n.$$

Adding the two equalities cancels the intermediate integer b:

$$\begin{aligned} a - c &= (a - b) + (b - c) \\ &= 2m + 2n = 2(m + n). \end{aligned}$$

The sum $m + n$ is an integer. Hence $2 \mid (a - c)$, so $(a, c) \in R$. This establishes transitivity for every such choice of a, b, c.

Step 4: Conclusion The relation R is reflexive, symmetric and transitive on $\mathbb{Z}$. Therefore $\boxed{R \text{ is an equivalence relation on } \mathbb{Z}}$.

Quick Tip to Solve: Express each divisible difference as twice an integer; negate it for symmetry and add two such differences for transitivity.

OR

16(b). If $f(x) = \dfrac{4x + 3}{6x - 4}$, $x \neq \dfrac{2}{3}$, show that $fof(x) = x$ for all $x \neq \dfrac{2}{3}$. Also, find the inverse of f.

[4 marks]

Solution:

Concept:

- A composite function is evaluated by substituting the output of the inner function into the outer function: $(f \circ f)(x) = f(f(x))$.

Quick Tip to Solve: Before cancelling in a rational composite, check that the inner output avoids the excluded input; here the difference from $2/3$ has constant numerator 17.

- The inverse reverses the mapping. If a function maps its domain to itself and $f(f(x)) = x$, it is bijective and its inverse is the same function.

Step 1: Check the domain of the composite Let $D = \mathbb{R} \setminus \{2/3\}$ and take any $x \in D$. The denominator $6x - 4$ is nonzero. We must also ensure that $f(x) \neq 2/3$, so that the second application of f is defined. Compute

$$f(x) - \frac{2}{3} = \frac{3(4x+3) - 2(6x-4)}{3(6x-4)} = \frac{17}{3(6x-4)} \neq 0.$$

Hence f maps D into D, and the composite is defined for every permitted x.

Step 2: Substitute and simplify the composite Substituting the given expression for $f(x)$, we obtain

$$\begin{aligned}(f \circ f)(x) &= \frac{4\left(\frac{4x+3}{6x-4}\right) + 3}{6\left(\frac{4x+3}{6x-4}\right) - 4} \\ &= \frac{\frac{16x+12+18x-12}{6x-4}}{\frac{24x+18-24x+16}{6x-4}} \\ &= \frac{34x/(6x-4)}{34/(6x-4)} = x.\end{aligned}$$

Cancellation is valid because $6x - 4 \neq 0$, and the denominator of the outer fraction is also nonzero.

Step 3: Solve for the inverse and justify its existence Put $y = f(x)$. Then

$$\begin{aligned}y(6x-4) &= 4x+3, \\ (6y-4)x &= 4y+3, \\ x &= \frac{4y+3}{6y-4}, \qquad y \neq \frac{2}{3}.\end{aligned}$$

The domain check above guarantees that every output satisfies this restriction. The composite identity proves injectivity: if $f(u) = f(v)$, applying f gives $u = v$. It also proves surjectivity onto D: for every $y \in D$, choosing $x = f(y) \in D$ gives $f(x) = y$. Thus the inverse exists on all of D.

Step 4: Conclusion For every $x \neq 2/3$, $(f \circ f)(x) = x$, and $f^{-1}(x) = \dfrac{4x+3}{6x-4} = f(x)$, with domain $\mathbb{R} \setminus \{2/3\}$.

Quick Tip to Solve: Before cancelling in a rational composite, check that the inner output avoids the excluded input; here the difference from $2/3$ has constant numerator 17.

17. Using properties of determinants, prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$**.** [4 marks]

Solution:

Concept:

- Adding a multiple of one row to another row does not change the determinant. Thus a row can be replaced by itself minus two other rows through two successive valid row operations.
- In expansion along the first row, the cofactor signs are $+, -, +$. A 2×2 determinant is the product of its main diagonal entries minus the product of its other diagonal entries.

Step 1: Create a zero by row operations Denote the given determinant by

$$\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}.$$

First apply $R_1 \to R_1 - R_2$, and then $R_1 \to R_1 - R_3$. The second and third rows remain fixed throughout. These operations preserve the determinant. The resulting first row is $(0, -2c, -2b)$: subtracting the first entries gives $b + c - b - c = 0$, the second entries give $a - (c + a) - c = -2c$, and the third entries give $a - b - (a + b) = -2b$. Therefore

$$\Delta = \begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}.$$

Step 2: Expand along the simplified first row The zero first entry removes one minor. Using the first-row cofactor signs carefully gives

$$\begin{aligned} \Delta &= 2c \begin{vmatrix} b & b \\ c & a+b \end{vmatrix} - 2b \begin{vmatrix} b & c+a \\ c & c \end{vmatrix} \\ &= 2c\{b(a+b) - bc\} - 2b\{bc - c(c+a)\}. \end{aligned}$$

The first contribution is positive because the negative second entry multiplies a negative cofactor sign; the third entry retains its negative sign.

Step 3: Collect the common factors Factoring the two terms and combining their brackets yields

$$\begin{aligned} \Delta &= 2bc(a+b-c) - 2bc(b-c-a) \\ &= 2bc\{a+b-c-b+c+a\} \\ &= 2bc(2a) = 4abc. \end{aligned}$$

The b and c terms inside the bracket cancel exactly. No division by a, b, or c has been used, so the argument also holds if any of these numbers is zero.

Step 4: Conclusion Hence the given determinant equals $4abc$, as required.

Quick Tip to Solve: Use $R_1 - R_2 - R_3$: the first entry becomes zero and the remaining entries become $-2c$ and $-2b$, leaving only two small minors to evaluate.

18. If $y = (\sec^{-1} x)^2,\ x > 0$, show that $x^2(x^2-1)\dfrac{d^2y}{dx^2} + (2x^3 - x)\dfrac{dy}{dx} - 2 = 0$. [4 marks]

Solution:

Concept:

- For the positive branch, $\dfrac{d}{dx}(\sec^{-1} x) = \dfrac{1}{x\sqrt{x^2-1}}$. The real-valued function in the question has $x \geq 1$; its derivatives are evaluated for $x > 1$.
- The chain rule gives $\dfrac{d}{dx}(u^2) = 2u\dfrac{du}{dx}$. Differentiating a suitable first-derivative identity with the product rule avoids a lengthy quotient calculation.

Step 1: Calculate the first derivative Put $u = \sec^{-1} x$, so that $y = u^2$. Since $x > 0$, the absolute value in the usual inverse-secant derivative equals x. Thus, on the differentiable domain $x > 1$,

$$\frac{dy}{dx} = 2u\frac{du}{dx} = \frac{2\sec^{-1} x}{x\sqrt{x^2-1}}.$$

Multiplying by the denominator produces the simpler identity

$$x\sqrt{x^2-1}\,\frac{dy}{dx} = 2\sec^{-1} x.$$

Both the inverse function and the square-root factor must be differentiated in the next step; the inverse-secant factor is not a constant.

Step 2: Differentiate the identity by the product rule First compute the derivative of the coefficient:

$$\begin{aligned}\frac{d}{dx}\left(x\sqrt{x^2-1}\right) &= \sqrt{x^2-1} + \frac{x^2}{\sqrt{x^2-1}} \\ &= \frac{2x^2-1}{\sqrt{x^2-1}}.\end{aligned}$$

Therefore, differentiating the identity obtained above gives

$$x\sqrt{x^2-1}\,\frac{d^2y}{dx^2} + \frac{2x^2-1}{\sqrt{x^2-1}}\,\frac{dy}{dx} = \frac{2}{x\sqrt{x^2-1}}.$$

Here the first term comes from differentiating dy/dx, and the second comes from differentiating its coefficient.

Step 3: Clear the denominators Multiply every term by $x\sqrt{x^2-1}$, which is nonzero for $x > 1$:

$$x^2(x^2-1)\frac{d^2y}{dx^2} + x(2x^2-1)\frac{dy}{dx} = 2.$$

Expanding only the second coefficient gives $x(2x^2-1) = 2x^3 - x$. Moving the constant on the right to the left puts the expression in exactly the form requested.

Quick Tip to Solve: Multiply the first derivative by $x\sqrt{x^2-1}$ before differentiating again; the product rule then produces the required coefficients directly.

Step 4: Conclusion Hence, wherever the real derivatives exist,

$$\boxed{x^2(x^2-1)\frac{d^2y}{dx^2}+(2x^3-x)\frac{dy}{dx}-2=0}.$$

Quick Tip to Solve: Multiply the first derivative by $x\sqrt{x^2-1}$ before differentiating again; the product rule then produces the required coefficients directly.

19. Prove that $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$ and hence evaluate $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}$. [4 marks]

Solution:

Concept:

- In a definite integral, the substitution $t=a+b-x$ reflects the variable about the midpoint of $[a,b]$. The differential and both limits must be changed together.
- Complementary angles satisfy $\tan(\pi/2-x)=\cot x$. For positive $\tan x$, $\sqrt{\cot x}=1/\sqrt{\tan x}$.
- Adding two equivalent forms of an integral is useful when their integrands sum to a constant.

Step 1: Prove the reflection property Assume f is integrable on $[a,b]$, and write

$$J=\int_a^b f(x)\,dx.$$

Make the substitution $t=a+b-x$. Then $dx=-dt$, while $x=a$ gives $t=b$ and $x=b$ gives $t=a$. Consequently,

$$\begin{aligned}J&=-\int_b^a f(a+b-t)\,dt\\&=\int_a^b f(a+b-t)\,dt.\end{aligned}$$

The letter used for the integration variable is a dummy variable, so replacing t by x proves

$$\boxed{\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx}.$$

Reversing the limits cancels the minus sign from the differential; omitting either operation would give the wrong sign.

Step 2: Apply the property to the given integral Let

$$I=\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}}.$$

Quick Tip to Solve: When the limits add to $\pi/2$, reflect x to $\pi/2-x$; the resulting reciprocal square root makes the two integrands add to 1.

Here the sum of the limits is $\pi/6 + \pi/3 = \pi/2$. The property just proved therefore gives

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan(\pi/2 - x)}} = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}}.$$

Throughout this interval $\tan x > 0$, so the positive square roots obey $\sqrt{\cot x} = 1/\sqrt{\tan x}$. Multiplying numerator and denominator by $\sqrt{\tan x}$ yields

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\, dx.$$

Step 3: Add the two equal expressions Adding the original and reflected forms over the same limits gives

$$\begin{aligned} 2I &= \int_{\pi/6}^{\pi/3} \left(\frac{1}{1+\sqrt{\tan x}} + \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}} \right) dx \\ &= \int_{\pi/6}^{\pi/3} 1\, dx \\ &= \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}. \end{aligned}$$

The numerator of the sum equals its denominator, so the square-root expression disappears without finding an antiderivative. Divide the resulting equality by 2 to obtain the value of the original integral.

Step 4: Conclusion The reflection identity is proved, and the required value is

$$\boxed{\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{12}}.$$

Quick Tip to Solve: When the limits add to $\pi/2$, reflect x to $\pi/2 - x$; the resulting reciprocal square root makes the two integrands add to 1.

20. Find: $\int \frac{\sin 2x}{(\sin^2 x + 1)(\sin^2 x + 3)}\, dx$. [4 marks]

Solution:

Concept:

- Use substitution when the numerator is the derivative of an expression in the denominator. Here $\frac{d}{dx}(\sin^2 x) = 2\sin x \cos x = \sin 2x$.
- A rational function with distinct linear factors can be decomposed into partial fractions, and $\int \frac{du}{u+a} = \ln|u+a| + C$.

Step 1: Substitute for the squared sine Denote the required integral by I. Put $u = \sin^2 x$, so that

Quick Tip to Solve: Set $u = \sin^2 x$ immediately, then use $\frac{1}{(u+1)(u+3)} = \frac{1}{2}[\frac{1}{u+1} - \frac{1}{u+3}]$ to avoid unnecessary trigonometric expansion.

$du = \sin 2x\,dx$. The numerator together with dx is exactly du, with no extra factor. Therefore

$$I = \int \frac{du}{(u+1)(u+3)}.$$

Both factors are positive for real x, since $u = \sin^2 x \geq 0$. Thus the original integrand has no real singularity.

Step 2: Resolve the rational function into partial fractions Write

$$\frac{1}{(u+1)(u+3)} = \frac{A}{u+1} + \frac{B}{u+3}.$$

Multiplying by $(u+1)(u+3)$ gives $1 = A(u+3) + B(u+1)$. Comparing coefficients yields $A + B = 0$ and $3A + B = 1$, hence $A = \frac{1}{2}$ and $B = -\frac{1}{2}$. Consequently,

$$I = \frac{1}{2}\int \left(\frac{1}{u+1} - \frac{1}{u+3}\right) du = \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u+3| + C.$$

Step 3: Restore the variable and verify the antiderivative Substituting $u = \sin^2 x$ combines the two logarithms into one. Absolute-value signs may be omitted here because both logarithm arguments are positive. To check the sign and coefficient, differentiate the expression obtained:

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{2}\ln\frac{\sin^2 x+1}{\sin^2 x+3}\right] &= \frac{\sin 2x}{2}\left(\frac{1}{\sin^2 x+1} - \frac{1}{\sin^2 x+3}\right)\\ &= \frac{\sin 2x}{(\sin^2 x+1)(\sin^2 x+3)}.\end{aligned}$$

This recovers the given integrand exactly; C denotes an arbitrary real constant.

Step 4: Conclusion

$$\boxed{\int \frac{\sin 2x}{(\sin^2 x+1)(\sin^2 x+3)}\,dx = \frac{1}{2}\ln\left(\frac{\sin^2 x+1}{\sin^2 x+3}\right) + C.}$$

Quick Tip to Solve: Set $u = \sin^2 x$ immediately, then use $\frac{1}{(u+1)(u+3)} = \frac{1}{2}[\frac{1}{u+1} - \frac{1}{u+3}]$ to avoid unnecessary trigonometric expansion.

21. Find the value of λ for which the following lines are perpendicular to each other: $\frac{x-5}{5\lambda+2} = \frac{2-y}{5} = \frac{1-z}{-1}$; $\frac{x}{1} = \frac{y+\frac{1}{2}}{2\lambda} = \frac{z-1}{3}$. Hence, find whether the lines intersect or not. [4 marks]

Solution:

Concept:

- Direction ratios are read from the standard symmetric form $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$. Reversing a numerator changes the sign of its direction ratio.
- Two direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are perpendicular when $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$. Intersection additionally requires a common point satisfying all three coordinate equations.

Step 1: Read the direction ratios with the correct signs For the first line, $2 - y = -(y - 2)$ and $1 - z = -(z - 1)$. Thus its standard form is

$$\frac{x-5}{5\lambda+2} = \frac{y-2}{-5} = \frac{z-1}{1}.$$

It passes through $(5, 2, 1)$ with direction ratios $(5\lambda+2, -5, 1)$. The second line passes through $(0, -\frac{1}{2}, 1)$ with direction ratios $(1, 2\lambda, 3)$. In particular, the direction ratio for z in the first line is positive, despite the printed denominator being -1.

Step 2: Apply the perpendicularity condition The dot product of the direction vectors must vanish:

$$\begin{aligned}(5\lambda+2)(1) + (-5)(2\lambda) + (1)(3) &= 0,\\ 5\lambda + 2 - 10\lambda + 3 &= 0,\\ -5\lambda + 5 &= 0.\end{aligned}$$

Solving this linear equation gives $\lambda = 1$. At this value the direction vectors are $(7, -5, 1)$ and $(1, 2, 3)$; their dot product is $7 - 10 + 3 = 0$, confirming the condition.

Step 3: Test for a common point using independent parameters With $\lambda = 1$, use parameters t and s for the first and second lines respectively:

$$\begin{aligned}L_1: &\quad x = 5 + 7t, \quad y = 2 - 5t, \quad z = 1 + t,\\ L_2: &\quad x = s, \quad y = -\tfrac{1}{2} + 2s, \quad z = 1 + 3s.\end{aligned}$$

At an intersection, equality of the z-coordinates would give $t = 3s$. Equality of the x-coordinates would then give $5 + 21s = s$, so $s = -\frac{1}{4}$ and $t = -\frac{3}{4}$. These uniquely determined parameters must also satisfy the remaining coordinate equation. But the two y-coordinates are

$$y_{L_1} = 2 - 5(-\tfrac{3}{4}) = \tfrac{23}{4}, \qquad y_{L_2} = -\tfrac{1}{2} + 2(-\tfrac{1}{4}) = -1.$$

They differ, so no common point exists. Since the direction vectors are not parallel, the two lines are skew, although their directions are perpendicular.

Step 4: Conclusion $\boxed{\lambda = 1}$. For this value, the lines do not intersect; they are skew lines with perpendicular direction vectors.

Quick Tip to Solve: First rewrite $2 - y$ and $1 - z$ in standard form to get the signs right. After the dot-product test, check all three coordinates before claiming intersection.

22. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}| = 1,\ |\vec{b}| = 2$ and $|\vec{c}| = 3$. If the projection of $\vec{b}$ along $\vec{a}$ is equal to the projection of $\vec{c}$ along $\vec{a}$; and $\vec{b}, \vec{c}$ are perpendicular to each other, then find $|3\vec{a} - 2\vec{b} + 2\vec{c}|$.

[4 marks]

Solution:

Concept:

- The scalar projection of a vector $\vec{u}$ along a nonzero vector $\vec{a}$ is $\dfrac{\vec{u} \cdot \vec{a}}{|\vec{a}|}$.
- Perpendicular vectors have zero dot product. Also, $|\vec{u}|^2 = \vec{u} \cdot \vec{u}$, so the magnitude of a linear combination can be found by expanding its dot product with itself.

Step 1: Translate the projection condition The two projections are along the same vector $\vec{a}$. Therefore their denominators are identical and nonzero:

$$\frac{\vec{b} \cdot \vec{a}}{|\vec{a}|} = \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|} \quad \Longrightarrow \quad \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}.$$

Equivalently, $\vec{a} \cdot (\vec{c} - \vec{b}) = 0$. Thus the difference $\vec{c} - \vec{b}$ is perpendicular to $\vec{a}$, even though neither of the two individual vectors need be perpendicular to $\vec{a}$.

Step 2: Use the perpendicularity of the other two vectors Since $\vec{b} \perp \vec{c}$, we have $\vec{b} \cdot \vec{c} = 0$. The given magnitudes give

$$|\vec{c} - \vec{b}|^2 = |\vec{c}|^2 + |\vec{b}|^2 - 2\vec{b} \cdot \vec{c} = 9 + 4 - 0 = 13.$$

We can write the requested vector as $3\vec{a} + 2(\vec{c} - \vec{b})$, whose two terms are perpendicular by the projection condition.

Step 3: Expand the square of the required magnitude Applying the dot-product identity and then substituting the values,

$$\begin{aligned} |3\vec{a} - 2\vec{b} + 2\vec{c}|^2 &= 9|\vec{a}|^2 + 4|\vec{b}|^2 + 4|\vec{c}|^2 \\ &\quad - 12\vec{a} \cdot \vec{b} + 12\vec{a} \cdot \vec{c} - 8\vec{b} \cdot \vec{c} \\ &= 9(1)^2 + 4(2)^2 + 4(3)^2 + 0 \\ &= 61. \end{aligned}$$

The two mixed products involving $\vec{a}$ cancel because the projections are equal, and the remaining mixed product vanishes because $\vec{b}$ and $\vec{c}$ are perpendicular. Magnitudes are nonnegative, so we take the positive square root.

Step 4: Conclusion The required magnitude is $\boxed{|3\vec{a} - 2\vec{b} + 2\vec{c}| = \sqrt{61}}$.

Quick Tip to Solve: Rewrite the vector as $3\vec{a} + 2(\vec{c} - \vec{b})$; equal projections make these terms perpendicular and allow a direct Pythagorean calculation.

23(a). Solve the differential equation: $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$. [4 marks]

Solution:

Concept:

- A differential equation $\dfrac{dy}{dx} = F(y/x)$ is homogeneous. Put $y = vx$, so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$, on an interval where $x \neq 0$.
- Separate the variables and integrate. Here the denominator $1 + v^2$ gives the standard integral $\displaystyle\int \frac{dv}{1+v^2} = \tan^{-1} v$.

Step 1: Reduce the homogeneous equation Both numerator and denominator of the given right-hand side are homogeneous expressions of degree one. Dividing by x,

$$\frac{dy}{dx} = \frac{1 + y/x}{1 - y/x}.$$

Put $y = vx$. By the product rule, the differential equation becomes

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}, \qquad x\frac{dv}{dx} = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}.$$

The original equation requires $x - y \neq 0$, corresponding to $v \neq 1$ in this substitution.

Step 2: Separate and integrate Since $1 + v^2 > 0$ for real v, separation does not discard any real constant-ratio solution:

$$\frac{1-v}{1+v^2}\,dv = \frac{dx}{x}.$$

Integrating both sides gives

$$\int \frac{dv}{1+v^2} - \int \frac{v\,dv}{1+v^2} = \int \frac{dx}{x},$$
$$\tan^{-1} v - \frac{1}{2}\ln(1+v^2) = \ln|x| + C.$$

In the second integral, use $u = 1 + v^2$, with $du = 2v\,dv$. The logarithm on the right must have an absolute value because x may be negative.

Step 3: Return to the original variables and verify Substituting $v = y/x$ and combining the logarithms yields

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\ln(x^2+y^2) + C.$$

To check, differentiate this implicit relation on an interval where $x \neq 0$:

$$\frac{xy' - y}{x^2+y^2} = \frac{x + yy'}{x^2+y^2} \quad \Longrightarrow \quad (x-y)y' = x + y.$$

Dividing by $x - y \neq 0$ recovers the printed equation exactly. The inverse tangent is understood on a continuous local branch, with any branch change absorbed into the arbitrary constant.

Step 4: Conclusion The general solution is $\boxed{\tan^{-1}\left(\dfrac{y}{x}\right) = \dfrac{1}{2}\ln(x^2+y^2) + C}$, locally for $x \neq 0$ and $x - y \neq 0$.

Quick Tip to Solve: After putting $y = vx$, split $(1-v)/(1+v^2)$ into two integrals; they give an inverse tangent and a logarithm immediately.

OR

23(b). Solve the differential equation: $(1+x^2)\,dy + 2xy\,dx = \cot x\,dx$. [4 marks]

Solution:

Concept:

- A linear differential equation has the form $y' + P(x)y = Q(x)$. Its integrating factor is $I(x) = e^{\int P(x)\,dx}$, and multiplication by it makes the left side $\dfrac{d}{dx}[I(x)y]$.
- The product rule gives $\dfrac{d}{dx}[(1+x^2)y] = (1+x^2)y' + 2xy$. Also, $\int \cot x\,dx = \ln|\sin x| + C$ on any interval where $\sin x \neq 0$.

Step 1: Put the equation in linear form Divide the given equation by dx, and then by $1+x^2$. This divisor is positive for every real x, so this operation does not lose any solutions:

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cot x}{1+x^2}.$$

Thus $P(x) = \dfrac{2x}{1+x^2}$ and $Q(x) = \dfrac{\cot x}{1+x^2}$. The original equation is defined only away from integral multiples of π; solve it on any interval between consecutive such points.

Step 2: Determine the integrating factor Use the substitution $u = 1+x^2$, so $du = 2x\,dx$. It follows that

$$I(x) = e^{\int \frac{2x}{1+x^2}\,dx} = e^{\ln(1+x^2)} = 1+x^2.$$

Multiplying the linear equation by this factor gives

$$(1+x^2)y' + 2xy = \cot x.$$

Step 3: Integrate the resulting product derivative By the product rule, the equation now reads

$$\frac{d}{dx}[(1+x^2)y] = \cot x.$$

Integrate both sides. Since $\cot x = \dfrac{\cos x}{\sin x}$, substitution of $v = \sin x$ yields

$$(1+x^2)y = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C.$$

Differentiating this relation gives $(1+x^2)y' + 2xy = \cot x$, exactly the original equation after division by dx.

Step 4: Conclusion The general solution is

$$\boxed{y = \frac{\ln|\sin x| + C}{1+x^2}},$$

where C is arbitrary and the solution is taken on an interval with $\sin x \neq 0$.

Quick Tip to Solve: Spot the product derivative directly: the coefficient $2x$ of y is the derivative of $1+x^2$, so the left side is $d[(1+x^2)y]$.

Section D (36 marks)

Questions number 24 to 29 carry 6 marks each.

24(a). Find the area of the region $\{(x, y) : 0 \le y \le x^2,\ 0 \le y \le x + 2,\ -1 \le x \le 3\}$**.** [6 marks]

Solution:

Concept:

- For a region between $y = 0$ and an upper boundary $y = f(x) \ge 0$, the area is $\int_a^b f(x)\,dx$. Here a and b are the limiting abscissae.
- Two simultaneous upper bounds $y \le f(x)$ and $y \le g(x)$ mean that the height of a vertical strip is $\min\{f(x), g(x)\}$, not their difference. Split the integral where the smaller function changes.
- The power rule $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C$, for $n \ne -1$, evaluates the polynomial areas.

Step 1: Identify the upper boundary of the region The inequalities specify $-1 \le x \le 3$, with y above the x-axis and below both $y = x^2$ and $y = x + 2$. On this interval, $x^2 \ge 0$ and $x + 2 \ge 1$, so each vertical cross-section starts at $y = 0$. Its upper endpoint is the lower of the two curves. Thus the required region is under their common lower envelope, rather than the region between the curves.

Step 2: Locate the points where the boundary changes Equate the two upper bounds:

$$x^2 = x + 2 \quad \Longrightarrow \quad x^2 - x - 2 = (x - 2)(x + 1) = 0.$$

Their intersection abscissae are $x = -1$ and $x = 2$, both within the prescribed range. For $-1 \le x \le 2$, the product $(x - 2)(x + 1) \le 0$; hence $x^2 \le x + 2$, and the parabola is the upper boundary of the region. For $2 \le x \le 3$, that product is nonnegative, so $x + 2 \le x^2$, and the line becomes the upper boundary.

Step 3: Integrate the area under the parabola For the first part the vertical-strip height is x^2. Therefore

$$\begin{aligned} A_1 &= \int_{-1}^{2} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{2} \\ &= \frac{8}{3} - \left(-\frac{1}{3}\right) = 3. \end{aligned}$$

The integrand stays nonnegative throughout, so no signed-area correction is needed.

Step 4: Integrate the area under the line and combine For the remaining interval the strip height is $x + 2$. Thus

$$\begin{aligned} A_2 &= \int_{2}^{3} (x + 2)\,dx = \left[\frac{x^2}{2} + 2x\right]_{2}^{3} \\ &= \left(\frac{9}{2} + 6\right) - (2 + 4) = \frac{9}{2}. \end{aligned}$$

The two intervals share only their endpoint $x = 2$, so their areas add without overlap. As a check, the second region is a trapezium with parallel vertical sides of lengths 4 and 5, separated by distance 1; its area is $\frac{1}{2}(4 + 5) \cdot 1 = \frac{9}{2}$, consistent with the integral.

Quick Tip to Solve: With two conditions of the form $0 \le y \le \cdots$, integrate the smaller upper bound. Here the switch occurs at $x = 2$: use x^2 first and $x + 2$ afterwards.

Step 5: Conclusion The required area is

$$\boxed{A = A_1 + A_2 = 3 + \frac{9}{2} = \frac{15}{2} \text{ square units}}.$$

Quick Tip to Solve: With two conditions of the form $0 \le y \le \cdots$, integrate the smaller upper bound. Here the switch occurs at $x = 2$: use x^2 first and $x + 2$ afterwards.

OR

24(b). Evaluate $\int_1^4 (1 + x + e^{2x})\,dx$ as limit of sums. [6 marks]

Solution:

Concept:

- For a continuous function f on $[a, b]$, divide the interval into n equal pieces of width $h = (b - a)/n$. The definite integral is the limit of the right-endpoint sums: $\int_a^b f(x)\,dx = \lim_{n\to\infty} h \sum_{r=1}^{n} f(a + rh)$.
- Use $\sum_{r=1}^{n} r = n(n + 1)/2$ for the linear term and $\sum_{r=1}^{n} t^r = t(t^n - 1)/(t - 1)$, for $t \neq 1$, for the exponential term. The standard exponential limit is $\lim_{u\to 0}(e^u - 1)/u = 1$.

Step 1: Form the sum on equal subintervals Here $a = 1$, $b = 4$, and $h = 3/n$. The right endpoint of the r-th subinterval is $x_r = 1 + 3r/n$. As $1 + x + e^{2x}$ is continuous throughout $[1, 4]$, its Riemann sums converge to the required integral. Thus, writing the desired integral as I,

$$I = \lim_{n\to\infty} \frac{3}{n} \sum_{r=1}^{n} \left(2 + \frac{3r}{n} + e^2 e^{6r/n}\right).$$

Each of the three terms can now be summed separately before taking the limit. This uses the definition of a definite integral as a limit of sums directly.

Step 2: Sum the constant and linear terms There are n copies of the constant 2. Using the sum of the first n natural numbers gives

$$\frac{3}{n} \sum_{r=1}^{n} \left(2 + \frac{3r}{n}\right) = \frac{3}{n} \left(2n + \frac{3}{n} \frac{n(n+1)}{2}\right)$$
$$= 6 + \frac{9}{2} \frac{n+1}{n}.$$

As $n \to \infty$, $(n + 1)/n \to 1$, so this portion tends to $6 + 9/2 = 21/2$. Keeping the width $3/n$ outside both sums is essential because the interval has length 3.

Step 3: Sum and take the limit of the exponential terms The exponential values form a geometric

Quick Tip to Solve: For the exponential sum, factor the denominator as $(e^{6/n} - 1)/(6/n)$ to apply the standard exponential limit immediately.

progression with common ratio $t = e^{6/n}$. Its first term is $e^{6/n}$, and $t^n = e^6$. Therefore

$$\frac{3e^2}{n}\sum_{r=1}^{n} e^{6r/n} = \frac{3e^2}{n}\frac{e^{6/n}(e^6-1)}{e^{6/n}-1}$$
$$= \frac{e^2(e^6-1)}{2}\, e^{6/n}\frac{6/n}{e^{6/n}-1}.$$

Put $u = 6/n$, so that $u \to 0$ as $n \to \infty$. Then $e^{6/n} \to 1$ and $u/(e^u - 1) \to 1$. Consequently this contribution tends to $e^2(e^6-1)/2$. All the limits used exist, so the limits of the two contributions may be added.

Step 4: Conclusion Hence, as a limit of sums,

$$\boxed{\int_1^4 (1 + x + e^{2x})\, dx = \frac{21 + e^8 - e^2}{2}}.$$

Quick Tip to Solve: For the exponential sum, factor the denominator as $(e^{6/n}-1)/(6/n)$ to apply the standard exponential limit immediately.

25. Find the mean and variance of the random variable X which denotes the number of doublets in four throws of a pair of dice. [6 marks]

Solution:

Concept:

- A doublet means that both dice show the same number. For a pair of fair dice the 36 ordered outcomes are equally likely, and 6 of them are doublets.
- The count X of successes in n independent trials, each with success probability p, has binomial distribution $P(X = k) = \binom{n}{k}p^k(1-p)^{n-k}$. Its mean is np and its variance is $np(1-p)$.

Step 1: Identify the probability of a doublet In one throw, the sample space consists of ordered pairs (i, j), where $i, j \in \{1, 2, 3, 4, 5, 6\}$. There are $6 \times 6 = 36$ equally likely outcomes. The favourable outcomes are

$$(1,1),\ (2,2),\ (3,3),\ (4,4),\ (5,5),\ (6,6).$$

Thus the success probability is $p = 6/36 = 1/6$, and the probability of a non-doublet is $q = 1 - p = 5/6$. A success is a doublet on a single throw; it is not restricted to any particular face value.

Step 2: Construct the distribution of the count The four throws are independent, and the probability $p = 1/6$ remains the same on every throw. Each throw gives either a doublet or a non-doublet. Hence X is a binomial random variable with $n = 4$, and its possible values are $0, 1, 2, 3, 4$. For exactly k doublets, choose their positions in $\binom{4}{k}$ ways; each such sequence has probability $(1/6)^k(5/6)^{4-k}$. Therefore

$$P(X = k) = \binom{4}{k}\left(\frac{1}{6}\right)^k\left(\frac{5}{6}\right)^{4-k}, \qquad k = 0, 1, 2, 3, 4.$$

Quick Tip to Solve: Count the six matching ordered pairs first, then use $n = 4$, $p = 1/6$, and $q = 5/6$ directly in the binomial formulas.

The resulting probabilities are

k	0	1	2	3	4
$P(X=k)$	$625/1296$	$500/1296$	$150/1296$	$20/1296$	$1/1296$

Their numerators add to 1296, confirming that the total probability is 1.

Step 3: Compute the mean and variance Applying the binomial formulas with $n=4$, $p=1/6$, and $q=5/6$ gives

$$E(X)=np=4\cdot\frac{1}{6}=\frac{2}{3},\qquad \mathrm{Var}(X)=npq=4\cdot\frac{1}{6}\cdot\frac{5}{6}=\frac{5}{9}.$$

These values can also be checked directly from the distribution. The first and second moments are

$$E(X)=\frac{500+2(150)+3(20)+4}{1296}=\frac{864}{1296}=\frac{2}{3},$$
$$E(X^2)=\frac{500+4(150)+9(20)+16}{1296}=\frac{1296}{1296}=1.$$

Therefore $E(X^2)-[E(X)]^2=1-4/9=5/9$, in agreement with the binomial variance. The mean need not be an integer: it represents the average doublet count over many groups of four throws.

Step 4: Conclusion The number of doublets in four throws has

$$\boxed{\text{Mean}=\frac{2}{3},\qquad \text{Variance}=\frac{5}{9}}.$$

Quick Tip to Solve: Count the six matching ordered pairs first, then use $n=4$, $p=1/6$, and $q=5/6$ directly in the binomial formulas.

26(a). Find the vector and cartesian equations of the plane passing through the points $(2,5,-3)$, $(-2,-3,5)$ and $(5,3,-3)$. Also, find the point of intersection of this plane with the line passing through points $(3,1,5)$ and $(-1,-3,-1)$. [6 marks]

Solution:

Concept:

- For three non-collinear points A,B,C, a normal to their plane is $\overrightarrow{AB}\times\overrightarrow{AC}$. A nonzero scalar multiple gives the same plane.
- If $\vec{n}$ is a normal and $\vec{a}$ is the position vector of a point on the plane, its vector equation is $(\vec{r}-\vec{a})\cdot\vec{n}=0$.
- The line through points with position vectors $\vec{p},\vec{q}$ has equation $\vec{r}=\vec{p}+t(\vec{q}-\vec{p})$, where t is real. Substitution in the plane equation locates their intersection.

Step 1: Obtain two directions in the plane Denote the given points by $A(2,5,-3)$, $B(-2,-3,5)$ and $C(5,3,-3)$. Then

$$\overrightarrow{AB}=-4\hat{i}-8\hat{j}+8\hat{k},\qquad \overrightarrow{AC}=3\hat{i}-2\hat{j}.$$

Quick Tip to Solve: Simplify the cross-product normal before forming the plane equation; then use one line parameter in all three coordinates to find the intersection.

Both vectors lie in the required plane. Hence their cross product is perpendicular to that plane:

$$\begin{aligned}\overrightarrow{AB} \times \overrightarrow{AC} &= ((-8)(0) - 8(-2))\hat{i} + (8(3) - (-4)(0))\hat{j} \\ &\quad + ((-4)(-2) - (-8)(3))\hat{k} \\ &= 16\hat{i} + 24\hat{j} + 32\hat{k}.\end{aligned}$$

This vector is nonzero, so the points are non-collinear and determine a unique plane. Dividing by 8, choose the normal $\vec{n} = 2\hat{i} + 3\hat{j} + 4\hat{k}$.

Step 2: Form the vector and cartesian equations The plane passes through A, whose position vector is $\vec{a} = 2\hat{i} + 5\hat{j} - 3\hat{k}$. Thus

$$(\vec{r} - (2\hat{i} + 5\hat{j} - 3\hat{k})) \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 0.$$

Since $\vec{a} \cdot \vec{n} = 4 + 15 - 12 = 7$, this becomes $\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 7$. Writing $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, we obtain

$$2(x - 2) + 3(y - 5) + 4(z + 3) = 0, \quad \text{or} \quad 2x + 3y + 4z = 7.$$

As a check, the left side at B is $-4 - 9 + 20 = 7$, and at C it is $10 + 9 - 12 = 7$, so all three given points satisfy the equation.

Step 3: Find the intersection with the given line Let $P(3, 1, 5)$ and $Q(-1, -3, -1)$ be the points defining the line. Its direction vector is $\overrightarrow{PQ} = (-4, -4, -6)$. Therefore a general point is

$$(x, y, z) = (3 - 4t,\ 1 - 4t,\ 5 - 6t).$$

For this point to lie on the plane,

$$\begin{aligned}2(3 - 4t) + 3(1 - 4t) + 4(5 - 6t) &= 7, \\ 29 - 44t &= 7, \\ t &= \frac{1}{2}.\end{aligned}$$

The coefficient of t is nonzero, so there is exactly one intersection. Substituting gives $x = 1$, $y = -1$ and $z = 2$. The check $2(1) + 3(-1) + 4(2) = 7$ confirms that this point lies on the plane.

Step 4: Conclusion The vector equation is $\boxed{\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 7}$, the cartesian equation is $\boxed{2x + 3y + 4z = 7}$, and the intersection point is $\boxed{(1, -1, 2)}$.

Quick Tip to Solve: Simplify the cross-product normal before forming the plane equation; then use one line parameter in all three coordinates to find the intersection.

OR

26(b). Find the equation of the plane passing through the intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to x-axis. Hence, find the distance of the plane from x-axis. [6 marks]

Solution:

Concept:

- A plane through the line of intersection of two planes $P_1 = 0$ and $P_2 = 0$ can be written as $P_1 + \lambda P_2 = 0$, where λ is a real parameter chosen to satisfy the additional condition.
- A plane is parallel to a line when its normal is perpendicular to the line's direction. The x-axis has direction $\hat{i}$, so a plane parallel to it has zero coefficient of x.
- The perpendicular distance of (x_0, y_0, z_0) from $ax + by + cz + d = 0$ is $\dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$. For a parallel line, this distance is the same for every point on the line.

Step 1: Express the family of planes through the intersection Write $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. The two given planes are

$$P_1 = x + y + z - 1 = 0, \qquad P_2 = 2x + 3y - z + 4 = 0.$$

Their normals $(1, 1, 1)$ and $(2, 3, -1)$ are not proportional, so the planes intersect in a line. Every point on this line satisfies both equations and hence also satisfies

$$(x + y + z - 1) + \lambda(2x + 3y - z + 4) = 0.$$

On collecting coefficients, this family is

$$(1 + 2\lambda)x + (1 + 3\lambda)y + (1 - \lambda)z - 1 + 4\lambda = 0.$$

Neither given plane itself is parallel to the x-axis, so the required plane is obtained by choosing a parameter in this family.

Step 2: Apply the parallelism condition A normal to a member of the family is

$$\vec{n} = (1 + 2\lambda)\hat{i} + (1 + 3\lambda)\hat{j} + (1 - \lambda)\hat{k}.$$

Parallelism with the x-axis means $\vec{n} \cdot \hat{i} = 0$. Consequently,

$$1 + 2\lambda = 0 \quad \Rightarrow \quad \lambda = -\frac{1}{2}.$$

Substituting in the family gives

$$-\frac{1}{2}y + \frac{3}{2}z - 3 = 0.$$

Multiplying by -2, the required plane is $y - 3z + 6 = 0$. Its vector form is $\vec{r} \cdot (\hat{j} - 3\hat{k}) = -6$. The equation is also obtained directly by taking $P_2 - 2P_1 = 0$, which verifies that it contains the given intersection line.

Step 3: Calculate the distance from the x-axis A general point on the x-axis is $T(t, 0, 0)$, for any real

Quick Tip to Solve: Eliminate the x-term between the two plane equations to impose parallelism with the x-axis, then calculate the distance using the origin.

t. The plane's normal $(0, 1, -3)$ is perpendicular to $(1, 0, 0)$; moreover, substituting any point of the x-axis in its equation gives 6, never zero. Thus the x-axis is parallel to and disjoint from the plane. Applying the point-to-plane distance formula to T,

$$d = \frac{|0(t) + 1(0) - 3(0) + 6|}{\sqrt{0^2 + 1^2 + (-3)^2}} = \frac{6}{\sqrt{10}}.$$

This expression does not depend on t, so it is the distance between the entire x-axis and the plane. Equivalently, one may use the origin, which is a point on the x-axis. To check the perpendicular explicitly, the normal from the origin meets the plane at $H = (0, -3/5, 9/5)$; its length is $\sqrt{9/25 + 81/25} = 6/\sqrt{10}$, as above.

Step 4: Conclusion The required plane is $\boxed{y - 3z + 6 = 0}$, or $\boxed{\vec{r} \cdot (\hat{j} - 3\hat{k}) = -6}$. Its distance from the x-axis is $\boxed{\frac{6}{\sqrt{10}} \text{ units}}$.

Quick Tip to Solve: Eliminate the x-term between the two plane equations to impose parallelism with the x-axis, then calculate the distance using the origin.

27. Show that the height of a cylinder, which is open at the top, having a given surface area and greatest volume, is equal to the radius of its base. [6 marks]

Solution:

Concept:

- For a cylinder open at the top, the surface area consists of its circular base and curved surface: $S = \pi r^2 + 2\pi rh$, where $r > 0$ is the base radius and $h > 0$ is the height.
- Its volume is $V = \pi r^2 h$. A fixed surface area allows us to express the height, and hence the volume, in terms of the radius alone.
- At an interior maximum of a differentiable function, its first derivative vanishes. A negative second derivative at that point establishes a local maximum; the derivative signs establish the greatest volume on the admissible interval.

Step 1: Express the height using the fixed area Let the given surface area be the positive constant S. Since the cylinder has no top, there is only one circular face. Thus

$$S = \pi r^2 + 2\pi rh, \qquad h = \frac{S - \pi r^2}{2\pi r}.$$

Both dimensions must be positive. Consequently $S - \pi r^2 > 0$, so the admissible radii satisfy $0 < r < \sqrt{S/\pi}$. Throughout this interval the formula gives a positive height.

Step 2: Form the volume as a function of radius Substituting the height into the volume formula gives

$$V(r) = \pi r^2 \left(\frac{S - \pi r^2}{2\pi r}\right) = \frac{Sr - \pi r^3}{2}.$$

Quick Tip to Solve: Count only one circular base for an open-top cylinder, then eliminate the height before differentiating the volume.

Here S remains constant as the dimensions vary. Differentiating with respect to r,

$$V'(r) = \frac{S - 3\pi r^2}{2}, \qquad V''(r) = -3\pi r.$$

Step 3: Identify and verify the greatest-volume dimensions Setting $V'(r) = 0$ gives $r^2 = S/(3\pi)$. The only admissible critical radius is therefore $r_0 = \sqrt{S/(3\pi)}$. At this radius $V''(r_0) = -3\pi r_0 < 0$. Moreover, $V'(r) > 0$ for $0 < r < r_0$ and $V'(r) < 0$ for $r_0 < r < \sqrt{S/\pi}$. The volume therefore increases up to this point and decreases thereafter, so this is the greatest volume on the entire admissible interval. The volume approaches zero at both ends, consistent with this conclusion.

Step 4: Compare height with radius at the maximum At the maximizing radius, $S = 3\pi r_0^2$. Returning to the fixed-area equation gives

$$\begin{aligned} h_0 &= \frac{S - \pi r_0^2}{2\pi r_0} \\ &= \frac{3\pi r_0^2 - \pi r_0^2}{2\pi r_0} = r_0. \end{aligned}$$

Thus the height and radius at the greatest-volume configuration are obtained from the same positive quantity $\sqrt{S/(3\pi)}$.

Step 5: Conclusion Hence, for a cylinder open at the top with given surface area and greatest volume, $\boxed{h = r}$.

Quick Tip to Solve: Count only one circular base for an open-top cylinder, then eliminate the height before differentiating the volume.

28(a). If $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$**, find** A^{-1}**. Hence, solve the following system of equations:** $x + y + z = 6$, $y + 3z = 11$ **and** $x - 2y + z = 0$**.** [6 marks]

Solution:

Concept:

- A square matrix A has an inverse precisely when $|A| \neq 0$. In that case $A^{-1} = \operatorname{adj} A/|A|$.
- The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor obtained by deleting row i and column j. The adjugate is the transpose of the cofactor matrix.
- Writing a system as $AX = B$, with the variables in the same order as the coefficient columns, gives the unique solution $X = A^{-1}B$ when A is invertible.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant, and preserve the variable order when forming the column vectors.

Step 1: Calculate the determinant For the given matrix, expand along the first row:

$$\begin{aligned}|A| &= 1\begin{vmatrix}1 & 3\\ -2 & 1\end{vmatrix} - 1\begin{vmatrix}0 & 3\\ 1 & 1\end{vmatrix} + 1\begin{vmatrix}0 & 1\\ 1 & -2\end{vmatrix}\\ &= 1(1+6) - 1(0-3) + 1(0-1)\\ &= 7 + 3 - 1 = 9.\end{aligned}$$

Since this determinant is nonzero, the inverse exists and the corresponding system has a unique solution.

Step 2: Find the cofactors and adjugate Applying the alternating cofactor signs gives

$$\begin{aligned}&C_{11} = 1+6 = 7, && C_{12} = -(0-3) = 3, && C_{13} = 0-1 = -1,\\ &C_{21} = -(1+2) = -3, && C_{22} = 1-1 = 0, && C_{23} = -(-2-1) = 3,\\ &C_{31} = 3-1 = 2, && C_{32} = -(3-0) = -3, && C_{33} = 1-0 = 1.\end{aligned}$$

Therefore the cofactor matrix and its transpose are

$$C = \begin{bmatrix}7 & 3 & -1\\ -3 & 0 & 3\\ 2 & -3 & 1\end{bmatrix}, \qquad \operatorname{adj} A = C^T = \begin{bmatrix}7 & -3 & 2\\ 3 & 0 & -3\\ -1 & 3 & 1\end{bmatrix}.$$

Hence

$$A^{-1} = \frac{1}{9}\begin{bmatrix}7 & -3 & 2\\ 3 & 0 & -3\\ -1 & 3 & 1\end{bmatrix}.$$

Step 3: Write the equations in matrix form and use the inverse Keeping the order of variables as x, y, z, the coefficient matrix is exactly the given matrix. Thus

$$\begin{bmatrix}1 & 1 & 1\\ 0 & 1 & 3\\ 1 & -2 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}6\\ 11\\ 0\end{bmatrix}.$$

Multiplying on the left by A^{-1}, we obtain

$$\begin{aligned}\begin{bmatrix}x\\ y\\ z\end{bmatrix} &= \frac{1}{9}\begin{bmatrix}7 & -3 & 2\\ 3 & 0 & -3\\ -1 & 3 & 1\end{bmatrix}\begin{bmatrix}6\\ 11\\ 0\end{bmatrix}\\ &= \frac{1}{9}\begin{bmatrix}42-33+0\\ 18+0+0\\ -6+33+0\end{bmatrix} = \begin{bmatrix}1\\ 2\\ 3\end{bmatrix}.\end{aligned}$$

Step 4: Check the computed values Substituting the obtained values into the three original left-hand sides gives $1+2+3 = 6$, $2+3(3) = 11$, and $1-2(2)+3 = 0$, respectively. Each equals its stated right-hand side, confirming that the inverse multiplication has solved the required system.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant, and preserve the variable order when forming the column vectors.

Step 5: Conclusion $\boxed{A^{-1} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}}$, and the unique solution is $\boxed{x = 1,\ y = 2,\ z = 3}$.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant, and preserve the variable order when forming the column vectors.

OR

28(b). Find the inverse of the following matrix, using elementary transformations: $A = \begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix}$.

[6 marks]

Solution:

Concept:

- For an invertible square matrix, elementary row operations taking A to I take I to A^{-1}. Thus we reduce the augmented matrix $[A \mid I]$.
- Each row operation must be applied to the entire augmented row. Swapping rows, adding a multiple of another row, and multiplying a row by a nonzero number preserve the equivalent matrix equation.

Step 1: Set up the augmented matrix Attach the identity matrix to the right of the given matrix:

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 3 & 7 & 2 & 0 & 0 & 1 \end{array}\right].$$

Our target is an identity matrix on the left. To obtain a convenient pivot without introducing fractions, first subtract the first row from the second.

Step 2: Create a unit pivot in the second column Apply $R_2 \to R_2 - R_1$. This gives

$$\left[\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 7 & 2 & 0 & 0 & 1 \end{array}\right].$$

Now eliminate the second-column entries in the other rows with $R_1 \to R_1 - 3R_2$ and $R_3 \to R_3 - 7R_2$:

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 1 & 4 & -3 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 3 & 0 & 2 & 7 & -7 & 1 \end{array}\right].$$

Step 3: Obtain and use a pivot in the first column Apply $R_3 \to R_3 - R_1$, followed by interchange

Quick Tip to Solve: Start with $R_2 - R_1$: it creates the row $(0, 1, 0)$, allowing the whole second column to be cleared immediately.

$R_1 \leftrightarrow R_3$. The augmented matrix becomes

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -4 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 2 & 0 & 1 & 4 & -3 & 0 \end{array}\right].$$

Next use $R_3 \to R_3 - 2R_1$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 3 & -4 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 5 & -2 \end{array}\right].$$

Step 4: Complete the reduction and verify Multiply the third row by -1, then apply $R_1 \to R_1 - R_3$. We obtain

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -5 & 2 \end{array}\right].$$

The left block is now I. As a check, multiply the original matrix by the right block:

$$\begin{bmatrix} 2 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 7 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This confirms both the arithmetic and the order of multiplication.

Step 5: Conclusion $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 0 \\ 2 & -5 & 2 \end{bmatrix}$.

Quick Tip to Solve: Start with $R_2 - R_1$: it creates the row $(0, 1, 0)$, allowing the whole second column to be cleared immediately.

29. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours and 20 minutes available for cutting and 4 hours available for assembling. The profit is Rs. 50 each for type A and Rs. 60 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximize profit? Formulate the above LPP and solve it graphically and also find the maximum profit. [6 marks]

Solution:

Concept:

- Let x and y denote the numbers of type A and type B souvenirs. Resource availability imposes upper-bound inequalities, with $x, y \geq 0$.
- The profit function is $Z = 50x + 60y$ rupees. On a bounded feasible polygon, a linear objective reaches its maximum at a corner point.

Quick Tip to Solve: Convert hours to minutes first; then compare all four corner profits, including both axis vertices.

Step 1: Formulate the LPP Cutting time available is $3 \times 60 + 20 = 200$ minutes and assembling time available is $4 \times 60 = 240$ minutes. Each type A souvenir uses 5 cutting minutes and 10 assembling minutes; each type B uses 8 minutes for each activity. Therefore the LPP is

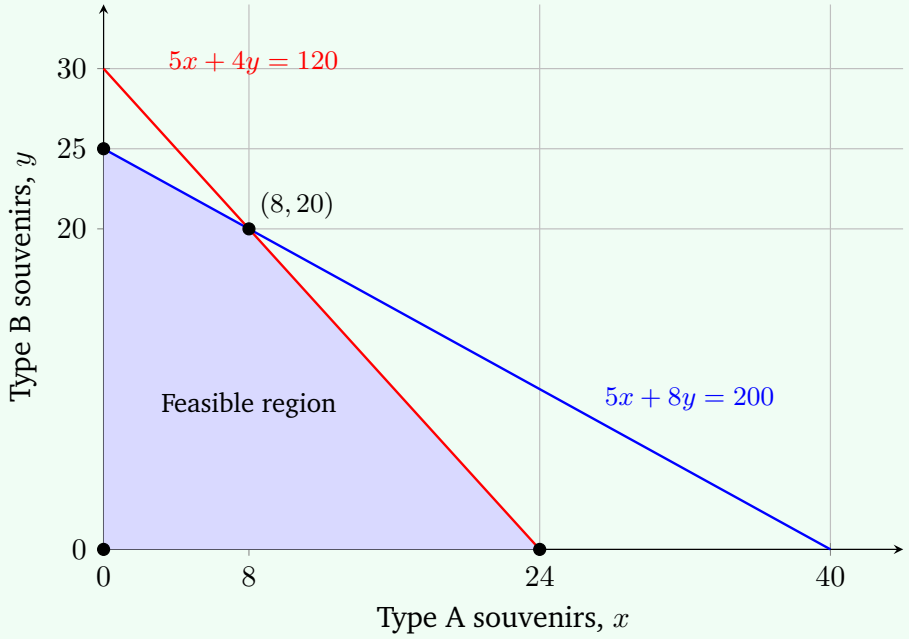
$$\text{Maximize } Z = 50x + 60y$$

subject to

$$5x + 8y \le 200, \qquad 10x + 8y \le 240, \qquad x \ge 0, \quad y \ge 0.$$

Equivalently, the assembling constraint is $5x + 4y \le 120$. Both counts must be nonnegative because negative production has no meaning.

Step 2: Plot the boundary lines and feasible region The cutting boundary $5x + 8y = 200$ meets the axes at $(40, 0)$ and $(0, 25)$. The assembling boundary $5x + 4y = 120$ meets them at $(24, 0)$ and $(0, 30)$. Draw both lines using these intercepts. The origin satisfies each resource inequality, so take the side of each line containing the origin and restrict to the first quadrant. Their common region is the shaded polygon shown below. On the horizontal axis assembling is the tighter constraint, whereas on the vertical axis cutting is tighter.



Feasible region and boundary lines for souvenir production

Step 3: Find the intersection of the two boundaries At the intersection, both available resources are exhausted:

$$5x + 8y = 200, \qquad 5x + 4y = 120.$$

Subtracting the second equation from the first gives $4y = 80$, so $y = 20$. Substitution gives $5x + 80 = 120$, hence $x = 8$. Thus the fourth corner is $(8, 20)$, and the feasible polygon has vertices $(0, 0), (24, 0), (8, 20), (0, 25)$. The plotted intersection is consistent with both resource boundaries.

Step 4: Evaluate profit at every corner The feasible region is closed and bounded, so the corner-point

Quick Tip to Solve: Convert hours to minutes first; then compare all four corner profits, including both axis vertices.

method applies. The objective values, in rupees, are

Corner (x, y)	$Z = 50x + 60y$
$(0, 0)$	0
$(24, 0)$	1200
$(8, 20)$	1600
$(0, 25)$	1500

The largest value is 1600 at $(8, 20)$. Check the times for this production plan: cutting uses $5(8)+8(20) = 200$ minutes and assembling uses $10(8) + 8(20) = 240$ minutes. Both availability limits are satisfied exactly. Also, 8 and 20 are whole numbers, so the graphical optimum is a valid production plan for individual souvenirs.

Step 5: Conclusion The company should manufacture 8 type A souvenirs and 20 type B souvenirs. The maximum profit is $\boxed{\text{Rs. } 1600}$.

Quick Tip to Solve: Convert hours to minutes first; then compare all four corner profits, including both axis vertices.