CBSE Class 12 Mathematics Question Paper 2019 with Solution PDF-Set 65-4-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 100**

General Instructions

(i) **All** questions are compulsory.

(ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. If A is a square matrix of order 3, with $|A| = 9$, then write the value of $|2 \cdot \operatorname{adj} A|$. [1 mark]

Solution:

Concept:

- For an invertible matrix A of order n, $|\operatorname{adj} A| = |A|^{n-1}$, where $\operatorname{adj} A$ denotes its adjoint.
- Multiplying every entry of an order-n matrix B by k gives $|kB| = k^n|B|$, because each of its n rows is multiplied by k.

Step 1: Find the determinant of the adjoint Here A has order 3 and $|A| = 9 \neq 0$. Therefore

$$|\operatorname{adj} A| = 9^{3-1} = 9^2 = 81.$$

The exponent is one less than the order of the matrix.

Step 2: Apply the scalar multiplier The adjoint also has order 3. Thus the factor 2 multiplies all three rows, so

$$|2 \operatorname{adj} A| = 2^3 |\operatorname{adj} A| = 8 \times 81.$$

Step 3: Conclusion Hence, $\boxed{|2 \operatorname{adj} A| = 648}$.

Quick Tip to Solve: Use $|k \operatorname{adj} A| = k^n |A|^{n-1}$; distinguish the exponents n and $n-1$.

2(a). Find the acute angle between the planes $\vec{r} \cdot (\hat{i} - 2\hat{j} - 2\hat{k}) = 1$ and $\vec{r} \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) = 0$. [1 mark]

Solution:

Concept:

- A plane written as $\vec{r} \cdot \vec{n} = d$ has normal vector $\vec{n}$. The acute angle θ between two planes is the acute angle between their normals.
- Use $\cos\theta = |\vec{n}_1 \cdot \vec{n}_2| / (|\vec{n}_1||\vec{n}_2|)$; the absolute value selects the acute angle.

Step 1: Identify the normal vectors The normals are $\vec{n}_1 = (1, -2, -2)$ and $\vec{n}_2 = (3, -6, 2)$. Their dot product is

$$\vec{n}_1 \cdot \vec{n}_2 = 1(3) + (-2)(-6) + (-2)(2) = 11.$$

The constants on the right of the plane equations do not affect their angle.

Step 2: Calculate the cosine of the angle The lengths are

$$|\vec{n}_1| = \sqrt{1 + 4 + 4} = 3, \qquad |\vec{n}_2| = \sqrt{9 + 36 + 4} = 7.$$

Consequently, $\cos\theta = |11|/(3 \times 7) = 11/21$.

Step 3: Conclusion The acute angle is $\boxed{\theta = \cos^{-1}\left(\frac{11}{21}\right)}$.

Quick Tip to Solve: Read the normals directly from the coefficients of $\hat{i}, \hat{j}, \hat{k}$, and take the absolute dot product.

OR

2(b). Find the length of the intercept, cut off by the plane $2x + y - z = 5$ on the x-axis. [1 mark]

Solution:

Concept:

- Every point on the x-axis has coordinates $(x, 0, 0)$, so set $y = z = 0$ in the plane equation to locate its x-intercept.
- The intercept length is the distance from the origin to this intersection, namely $|x|$, and is non-negative.

Step 1: Find the intersection with the x-axis Substituting $y = 0$ and $z = 0$ into $2x + y - z = 5$ gives

$$2x = 5, \qquad x = \frac{5}{2}.$$

Thus the plane meets the x-axis at $P = (5/2, 0, 0)$.

Step 2: Convert the coordinate into a length The intercepted segment runs from $O = (0, 0, 0)$ to P. Its length is

$$OP = \sqrt{(5/2)^2 + 0^2 + 0^2} = \left|\frac{5}{2}\right|.$$

Here the intersection lies on the positive x-axis.

Step 3: Conclusion The required intercept length is $\boxed{\frac{5}{2} \text{ units}}$.

Quick Tip to Solve: For an intercept on one coordinate axis, set the other two coordinates equal to zero before solving.

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3. If $y = \operatorname{cosec}(\cot\sqrt{x})$, then find $\frac{dy}{dx}$. [1 mark]

Solution:

Concept:

- The chain rule differentiates each layer of a composite function: $dy/dx = (dy/dv)(dv/du)(du/dx)$.
- Use $d(\operatorname{cosec} v)/dv = -\operatorname{cosec} v \cot v$, $d(\cot u)/du = -\operatorname{cosec}^2 u$, and $d(\sqrt{x})/dx = 1/(2\sqrt{x})$.

Step 1: Separate the three layers Put $u = \sqrt{x}$ and $v = \cot u$. Then $y = \operatorname{cosec} v$, so

$$\frac{dy}{dv} = -\operatorname{cosec} v \cot v, \qquad \frac{dv}{du} = -\operatorname{cosec}^2 u.$$

These derivatives apply wherever the original trigonometric expressions are defined.

Step 2: Multiply the derivatives For $x > 0$, the inner derivative is $du/dx = 1/(2\sqrt{x})$. Multiplying all

Quick Tip to Solve: Work from the outer cosecant inward; retain the inner cotangent derivative, whose negative sign cancels the first.

three factors cancels the two negative signs and gives

$$\frac{dy}{dx} = \frac{\operatorname{cosec} v \cot v \operatorname{cosec}^2 u}{2\sqrt{x}}.$$

Now substitute the original expressions for u and v.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = \frac{\operatorname{cosec}(\cot\sqrt{x}) \cot(\cot\sqrt{x}) \operatorname{cosec}^2\sqrt{x}}{2\sqrt{x}}}$$

Quick Tip to Solve: Work from the outer cosecant inward; retain the inner cotangent derivative, whose negative sign cancels the first.

4. Write the integrating factor of the differential equation $(\tan^{-1} y - x)\,dy = (1+y^2)\,dx$**.** [1 mark]

Solution:

Concept:

- A linear differential equation in x as a function of y has the form $dx/dy + P(y)x = Q(y)$.
- Its integrating factor is $e^{\int P(y)\,dy}$. Choose the dependent variable from the linear form, rather than automatically using dy/dx.

Step 1: Express the equation in linear form Divide the given equation by $(1+y^2)\,dy$. This gives

$$\frac{dx}{dy} = \frac{\tan^{-1} y - x}{1+y^2}, \qquad \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}.$$

Thus $P(y) = 1/(1+y^2)$; its sign is positive after moving the term in x to the left.

Step 2: Integrate the coefficient of x Since $\int dy/(1+y^2) = \tan^{-1} y$, the exponential formula gives

$$\text{I.F.} = \exp\left(\int \frac{dy}{1+y^2}\right) = e^{\tan^{-1} y}.$$

A nonzero constant multiplier is immaterial for an integrating factor.

Step 3: Conclusion The integrating factor is $\boxed{e^{\tan^{-1} y}}$.

Quick Tip to Solve: Treat x as the dependent variable here; first put the coefficient of dx/dy equal to one.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5. 12 cards numbered 1 to 12 (one number on one card), are placed in a box and mixed up thoroughly. Then a card is drawn at random from the box. If it is known that the number on the drawn card is greater than 5, find the probability that the card bears an odd number. [2 marks]

Solution:

Concept:

- Conditional probability restricts the sample space to the event already known to have occurred. For events A, B, $P(A \mid B) = P(A \cap B)/P(B)$, provided $P(B) > 0$.
- Because every card is equally likely to be drawn, the probability within the restricted set is its favourable count divided by its total count.

Step 1: Identify the condition and the favourable cards Let A denote an odd number and B a number greater than 5. The information given means the possible drawn cards are now

$$B = \{6, 7, 8, 9, 10, 11, 12\}.$$

There are seven such cards, still equally likely after conditioning. The odd numbers within this set are

$$A \cap B = \{7, 9, 11\}.$$

There are three favourable cards; odd numbers $1, 3, 5$ cannot occur under the condition.

Step 2: Apply conditional probability In the original twelve-card sample space, $P(B) = 7/12$ and $P(A \cap B) = 3/12$. Therefore

$$P(A \mid B) = \frac{3/12}{7/12} = \frac{3}{7}.$$

Equivalently, count three odd cards among the seven admissible cards. Using all twelve cards in the final denominator would ignore the known condition.

Step 3: Conclusion The required probability is $\boxed{\frac{3}{7}}$.

Quick Tip to Solve: Write the numbers satisfying the condition first, then count odd numbers only inside that reduced set.

6(a). Out of 8 outstanding students of a school, in which there are 3 boys and 5 girls, a team of 4 students is to be selected for a quiz competition. Find the probability that 2 boys and 2 girls are selected. [2 marks]

Solution:

Concept:

- If every team of four is equally likely, probability equals the number of favourable teams divided by the number of all possible teams.
- For selection without regard to order, use $\binom{n}{r} = n!/[r!(n-r)!]$. Independent choices from the boys and girls are counted by multiplication.

Step 1: Count all possible teams There are $3 + 5 = 8$ distinct students. Selecting any four gives

$$N = \binom{8}{4} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2 \times 1} = 70.$$

The order of selection does not create a new team, so combinations, rather than permutations, are appropriate. Each of these seventy subsets has the same chance in a random selection.

Step 2: Count teams with the required composition To obtain exactly two boys and two girls, choose two of the three boys and two of the five girls. Thus

$$N_f = \binom{3}{2}\binom{5}{2} = 3 \times 10 = 30.$$

Each boy-pair can be combined with any girl-pair, explaining the product. Hence the favourable-to-total ratio is

$$P = \frac{30}{70} = \frac{3}{7}.$$

The team has four members in every favourable case, so no additional choice is required.

Step 3: Conclusion The probability of selecting two boys and two girls is $\boxed{\frac{3}{7}}$.

Quick Tip to Solve: For a prescribed team composition, multiply the separate group combinations and divide by the overall team combination.

OR

6(b). In a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing? [2 marks]

Solution:

Concept:

- Under independent random guessing, each question has probability $p = 1/3$ of a correct answer and $q = 2/3$ of an incorrect answer.
- The number X of correct answers in five questions has a binomial distribution: $P(X = r) =$

Quick Tip to Solve: Translate four or more into the two disjoint cases $X = 4$ and $X = 5$, and remember the incorrect-answer probability $2/3$.

$\binom{5}{r}p^r q^{5-r}$, where r is an integer from 0 to 5.

Step 1: Calculate the probability of exactly four correct answers Four or more correct answers means either four or five correct answers. For exactly four, choose which one of the five answers is incorrect. Equivalently, choose four correct positions in $\binom{5}{4} = 5$ ways. Independence gives

$$P(X = 4) = \binom{5}{4}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right) = \frac{10}{243}.$$

The factor $2/3$ accounts for either of the two incorrect options on the remaining question.

Step 2: Include the all-correct outcome There is only one choice of five correct positions. Thus

$$P(X = 5) = \binom{5}{5}\left(\frac{1}{3}\right)^5 = \frac{1}{243}.$$

The events $X = 4$ and $X = 5$ are mutually exclusive, so their probabilities add:

$$P(X \geq 4) = \frac{10}{243} + \frac{1}{243} = \frac{11}{243}.$$

Including the all-correct case is necessary because the question says four or more.

Step 3: Conclusion The required probability is $\boxed{\frac{11}{243}}$.

Quick Tip to Solve: Translate four or more into the two disjoint cases $X = 4$ and $X = 5$, and remember the incorrect-answer probability $2/3$.

7(a). Find: $\int x \cdot \tan^{-1} x \, dx$. [2 marks]

Solution:

Concept:

- Integration by parts: $\int u\,dv = uv - \int v\,du$. Choose the inverse trigonometric factor as u, since its derivative is a rational function.
- Use $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$ and $\int x\,dx = \frac{x^2}{2}$. The arbitrary integration constant is denoted by C.

Step 1: Apply integration by parts Let $I = \int x \tan^{-1} x \, dx$. Take $u = \tan^{-1} x$ and $dv = x\,dx$, so that $du = \frac{dx}{1+x^2}$ and $v = \frac{x^2}{2}$. Thus

$$I = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx.$$

Step 2: Simplify the remaining rational integral The numerator can be written as $x^2 = (1+x^2) - 1$.

Quick Tip to Solve: After integration by parts, use $x^2/(1+x^2) = 1 - 1/(1+x^2)$ to avoid a second substitution.

Therefore

$$\int \frac{x^2}{1+x^2}\,dx = \int \left(1 - \frac{1}{1+x^2}\right) dx$$
$$= x - \tan^{-1} x.$$

Substitution gives $I = \frac{x^2}{2}\tan^{-1} x - \frac{1}{2}(x - \tan^{-1} x) + C$. Combining the two terms containing $\tan^{-1} x$ yields the required antiderivative.

Step 3: Conclusion

$$\boxed{\int x \tan^{-1} x\,dx = \frac{x^2+1}{2}\tan^{-1} x - \frac{x}{2} + C.}$$

Quick Tip to Solve: After integration by parts, use $x^2/(1+x^2) = 1 - 1/(1+x^2)$ to avoid a second substitution.

OR

7(b). Find: $\int \frac{dx}{\sqrt{5 - 4x - 2x^2}}$. [2 marks]

Solution:

Concept:

- Complete the square in a quadratic under a square root to obtain the standard form $a^2 - u^2$.
- The standard antiderivative is $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}(u/a) + C$, for $a > 0$ and $|u| < a$.

Step 1: Complete the square The expression under the radical is

$$5 - 4x - 2x^2 = 5 - 2(x^2 + 2x)$$
$$= 5 - 2\big((x+1)^2 - 1\big)$$
$$= 7 - 2(x+1)^2.$$

This form shows that a linear substitution in $x + 1$ will reduce the integral to the inverse sine formula.

Step 2: Reduce to the standard inverse sine integral Put $u = \sqrt{2}(x+1)$. Then $du = \sqrt{2}\,dx$, and

$$\int \frac{dx}{\sqrt{5 - 4x - 2x^2}} = \frac{1}{\sqrt{2}} \int \frac{du}{\sqrt{7 - u^2}}$$
$$= \frac{1}{\sqrt{2}} \sin^{-1}\left(\frac{u}{\sqrt{7}}\right) + C.$$

Here the real integrand is defined for $|x+1| < \sqrt{7/2}$. Substituting $u = \sqrt{2}(x+1)$ gives the result on this interval.

Step 3: Conclusion

$$\boxed{\frac{1}{\sqrt{2}} \sin^{-1}\left(\sqrt{\frac{2}{7}}(x+1)\right) + C.}$$

Quick Tip to Solve: Complete the square first: the shift is $x + 1$, and the remaining constant is 7, not 5.

8. Find: $\displaystyle\int_{-\pi/4}^{0} \frac{1+\tan x}{1-\tan x}\,dx$. [2 marks]

Solution:

Concept:

- The tangent addition identity gives $\tan(x+\pi/4) = \dfrac{1+\tan x}{1-\tan x}$.
- For $\cos u > 0$, $\int \tan u\,du = -\log(\cos u) + C$. A definite integral is obtained by evaluating the antiderivative at the upper and lower limits.

Step 1: Use the tangent addition identity Let $I = \int_{-\pi/4}^{0} \dfrac{1+\tan x}{1-\tan x}\,dx$. The identity converts this into $I = \int_{-\pi/4}^{0} \tan(x+\pi/4)\,dx$. Set $u = x+\pi/4$, so $du = dx$. When $x = -\pi/4$, $u = 0$; when $x = 0$, $u = \pi/4$. Thus

$$I = \int_{0}^{\pi/4} \tan u\,du.$$

Step 2: Evaluate at the transformed limits On $[0, \pi/4]$, $\cos u$ is positive, so

$$\begin{aligned} I &= [-\log(\cos u)]_0^{\pi/4} \\ &= -\log\left(\frac{1}{\sqrt{2}}\right) + \log(1) \\ &= \frac{1}{2}\log 2. \end{aligned}$$

The answer is positive, consistent with the nonnegative integrand on the original interval. The lower limit contributes zero because $\cos 0 = 1$.

Step 3: Conclusion

$$\boxed{\int_{-\pi/4}^{0} \frac{1+\tan x}{1-\tan x}\,dx = \frac{1}{2}\log 2.}$$

Quick Tip to Solve: Recognize $(1+\tan x)/(1-\tan x)$ as $\tan(x+\pi/4)$, then shift both integration limits.

9(a). X and Y are two points with position vectors $3\vec{a}+\vec{b}$ and $\vec{a}-3\vec{b}$ respectively. Write the position vector of a point Z which divides the line segment XY in the ratio $2:1$ externally. [2 marks]

Solution:

Concept:

- If position vectors $\vec{x}$ and $\vec{y}$ represent endpoints X, Y, the point dividing XY externally in the ratio $m:n$ has position vector $\vec{z} = \dfrac{m\vec{y} - n\vec{x}}{m-n}$, where $m \neq n$.
- External division uses a difference in both numerator and denominator. For $m > n$, the point lies on the extension beyond Y.

Step 1: Insert the endpoint vectors in the external section formula Here $\vec{x} = 3\vec{a}+\vec{b}$, $\vec{y} = \vec{a}-3\vec{b}$, and

Quick Tip to Solve: For external division in the ratio $2:1$, use $2\vec{y} - \vec{x}$; the denominator equals one.

$m : n = 2 : 1$. Therefore

$$\vec{z} = \frac{2(\vec{a} - 3\vec{b}) - (3\vec{a} + \vec{b})}{2 - 1}.$$

The coefficient 2 multiplies the position vector of Y, since the ratio is $XZ : ZY = 2 : 1$.

Step 2: Simplify and verify the division ratio Expanding the numerator gives

$$\vec{z} = 2\vec{a} - 6\vec{b} - 3\vec{a} - \vec{b} = -\vec{a} - 7\vec{b}.$$

To check the external division, compute $\vec{z} - \vec{x} = -4\vec{a} - 8\vec{b}$ and $\vec{z} - \vec{y} = -2\vec{a} - 4\vec{b}$. Hence $\vec{z} - \vec{x} = 2(\vec{z} - \vec{y})$, so the distances are in the required ratio and the point is on the extension beyond Y.

Step 3: Conclusion The position vector of Z is $\boxed{-\vec{a} - 7\vec{b}}$.

Quick Tip to Solve: For external division in the ratio $2 : 1$, use $2\vec{y} - \vec{x}$; the denominator equals one.

OR

9(b). Let $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ be two vectors. Show that the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular to each other. [2 marks]

Solution:

Concept:

- Two nonzero vectors are perpendicular if their scalar product is zero: $\vec{u} \cdot \vec{v} = |\vec{u}|\,|\vec{v}| \cos\theta$.
- For Cartesian vectors, take the products of corresponding components and add. The identity $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$ provides another check.

Step 1: Calculate the sum and difference Using the given components,

$$\begin{aligned} \vec{a} + \vec{b} &= (1 + 3)\hat{i} + (2 - 1)\hat{j} + (-3 + 2)\hat{k} \\ &= 4\hat{i} + \hat{j} - \hat{k}, \\ \vec{a} - \vec{b} &= (1 - 3)\hat{i} + (2 + 1)\hat{j} + (-3 - 2)\hat{k} \\ &= -2\hat{i} + 3\hat{j} - 5\hat{k}. \end{aligned}$$

Both vectors are nonzero.

Step 2: Compute their scalar product The dot product is

$$\begin{aligned} (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) &= 4(-2) + 1(3) + (-1)(-5) \\ &= -8 + 3 + 5 = 0. \end{aligned}$$

As an independent check, $|\vec{a}|^2 = 1^2 + 2^2 + (-3)^2 = 14$ and $|\vec{b}|^2 = 3^2 + (-1)^2 + 2^2 = 14$, whose difference is zero. Since the two vectors are nonzero, their zero scalar product implies that the angle between them is $90°$.

Step 3: Conclusion Thus $\boxed{(\vec{a} + \vec{b}) \perp (\vec{a} - \vec{b})}$, as required.

Quick Tip to Solve: Check that $|\vec{a}|^2 = |\vec{b}|^2$; equal lengths make the sum and difference perpendicular.

10. If A and B are symmetric matrices, such that AB and BA are both defined, then prove that $AB - BA$ is a skew symmetric matrix. [2 marks]

Solution:

Concept:

- A symmetric matrix satisfies $A^T = A$, while a skew symmetric matrix C satisfies $C^T = -C$.
- Transposition reverses the order in a product: $(AB)^T = B^T A^T$. It also preserves subtraction: $(P - Q)^T = P^T - Q^T$.

Step 1: Transpose the given difference Let $C = AB - BA$. Since symmetric matrices are square and both products are defined, A and B have the same order, so C is a square matrix. Using the transpose properties,

$$\begin{aligned} C^T &= (AB - BA)^T \\ &= (AB)^T - (BA)^T \\ &= B^T A^T - A^T B^T. \end{aligned}$$

The order of factors is reversed in each term; matrix multiplication need not be commutative.

Step 2: Use the symmetry of both matrices Given that A and B are symmetric, $A^T = A$ and $B^T = B$. Substituting these identities gives

$$\begin{aligned} C^T &= BA - AB \\ &= -(AB - BA) \\ &= -C. \end{aligned}$$

This is exactly the defining condition for a skew symmetric matrix. The proof does not require $AB = BA$; if the products happen to be equal, then C is the zero matrix and the same condition still holds.

Step 3: Conclusion As $(AB - BA)^T = -(AB - BA)$, the matrix $\boxed{AB - BA}$ is skew symmetric.

Quick Tip to Solve: When transposing AB, reverse the factors first, then replace A^T, B^T by A, B.

11. Let $* : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be an operation defined as $a * b = a + ab,\ \forall a, b \in \mathbb{N}$. Check if $*$ is a binary operation. If yes, find if it is associative too. [2 marks]

Solution:

Concept:

- A binary operation on $\mathbb{N}$ assigns to every ordered pair $(a, b) \in \mathbb{N} \times \mathbb{N}$ a unique element of $\mathbb{N}$; thus closure must hold.
- Associativity requires $(a * b) * c = a * (b * c)$ for every $a, b, c \in \mathbb{N}$. A single counterexample disproves it.

Step 1: Check closure in the natural numbers For any natural numbers a and b, both ab and $a + ab$ are natural numbers. The formula assigns exactly one value to each ordered pair. Therefore the stated rule defines a binary operation on $\mathbb{N}$.

Quick Tip to Solve: For $a * b = a(1 + b)$, keep the order of the inputs fixed while expanding each grouping; the extra ac term immediately disproves associativity.

Step 2: Compare the two groupings Apply the same rule to the outer operation in each expression:

$$\begin{aligned}(a*b)*c &= (a+ab)+(a+ab)c\\ &= a+ab+ac+abc,\\ a*(b*c) &= a+a(b+bc)\\ &= a+ab+abc.\end{aligned}$$

These expressions differ by ac, which is positive for positive natural numbers. In particular, taking $a=b=c=1$, we get $(1*1)*1=2*1=4$, whereas $1*(1*1)=1*2=3$. Hence the two groupings are not always equal.

Step 3: Conclusion The operation $*$ is a binary operation on $\mathbb{N}$, but it is not associative.

Quick Tip to Solve: For $a*b=a(1+b)$, keep the order of the inputs fixed while expanding each grouping; the extra ac term immediately disproves associativity.

12. Solve the following differential equation: $\dfrac{dx}{dy}+x=(\tan y+\sec^2 y)$. [2 marks]

Solution:

Concept:

- Treat x as a function of y. A linear differential equation $\dfrac{dx}{dy}+P(y)x=Q(y)$ has integrating factor $e^{\int P(y)\,dy}$.
- Multiplication by the integrating factor converts the left side into a product derivative. Use $\dfrac{d}{dy}(\tan y)=\sec^2 y$ to recognize the integral on the right.

Step 1: Find and apply the integrating factor Here $P(y)=1$ and $Q(y)=\tan y+\sec^2 y$. Consequently the integrating factor is $e^{\int 1\,dy}=e^y$. Multiplying the given equation throughout by e^y, we obtain

$$e^y\frac{dx}{dy}+e^yx=e^y(\tan y+\sec^2 y).$$

By the product rule, the left side is $\dfrac{d}{dy}(xe^y)$.

Step 2: Integrate the product derivative The right side is also a product derivative, since

$$\frac{d}{dy}(e^y\tan y)=e^y\tan y+e^y\sec^2 y.$$

Thus integrating both sides with respect to y gives

$$xe^y=e^y\tan y+C,$$

where C is an arbitrary real constant. This holds on any interval where $\cos y\neq 0$, so the coefficients of the original equation are defined. Dividing by e^y, which never vanishes, isolates x.

Quick Tip to Solve: After multiplying by e^y, recognize $e^y(\tan y+\sec^2 y)$ as the derivative of $e^y\tan y$; no integration by parts is needed.

Step 3: Conclusion The general solution is $\boxed{x = \tan y + Ce^{-y}}$, on any interval avoiding the zeros of $\cos y$.

Quick Tip to Solve: After multiplying by e^y, recognize $e^y(\tan y + \sec^2 y)$ as the derivative of $e^y \tan y$; no integration by parts is needed.

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13. If $y = (x)^{\cos x} + (\cos x)^{\sin x}$, find $\frac{dy}{dx}$. [4 marks]

Solution:

Concept:

- For a positive differentiable base $f(x)$, define $u = f(x)^{g(x)}$. Taking logarithms gives $\ln u = g(x) \ln f(x)$, so $\frac{u'}{u} = g'(x) \ln f(x) + g(x)\frac{f'(x)}{f(x)}$.
- Differentiate the two summands separately and add their derivatives. The product rule applies to the logarithm of each summand, and the chain rule gives $\frac{d}{dx} \ln(\cos x) = -\tan x$.

Step 1: Separate the two variable powers Put $u = x^{\cos x}$ and $v = (\cos x)^{\sin x}$, so $y = u + v$ and $y' = u' + v'$. Use logarithmic differentiation on each power individually; taking the logarithm of the sum would not separate its terms. Work on an interval where $x > 0$ and $\cos x > 0$, so both real logarithms exist and the powers are differentiable.

Step 2: Differentiate the first summand Taking natural logarithms of $u = x^{\cos x}$, and then differentiating with respect to x, gives

$$\ln u = \cos x \ln x,$$
$$\frac{u'}{u} = -\sin x \ln x + \frac{\cos x}{x}.$$

The first term comes from differentiating $\cos x$; the second comes from differentiating $\ln x$. Multiplying by $u = x^{\cos x}$ yields

$$u' = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right).$$

Step 3: Differentiate the second summand Similarly, taking logarithms of $v = (\cos x)^{\sin x}$, we have

$$\ln v = \sin x \ln(\cos x),$$
$$\frac{v'}{v} = \cos x \ln(\cos x) + \sin x \frac{-\sin x}{\cos x}$$
$$= \cos x \ln(\cos x) - \sin x \tan x.$$

The negative sign in the second term is due to $\frac{d}{dx}(\cos x) = -\sin x$. Multiplying by v gives

$$v' = (\cos x)^{\sin x}\left[\cos x \ln(\cos x) - \sin x \tan x\right].$$

Quick Tip to Solve: Logarithmically differentiate each variable power separately. For the second power, check the negative sign from differentiating the base $\cos x$.

Step 4: Conclusion Adding the derivatives of the two summands,

$$\frac{dy}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \sin x \ln x\right) + (\cos x)^{\sin x}\left[\cos x \ln(\cos x) - \sin x \tan x\right],$$

on the real differentiable domain $x > 0$, $\cos x > 0$.

Quick Tip to Solve: Logarithmically differentiate each variable power separately. For the second power, check the negative sign from differentiating the base $\cos x$.

14. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, and hence evaluate $\int_0^1 x^2(1-x)^n\,dx$. [4 marks]

Solution:

Concept:

- A change of variable must change both the differential and the limits. For $t = a - x$, we have $dx = -dt$, while $x = 0, a$ correspond to $t = a, 0$.
- The reflection property $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ replaces a polynomial factor in x by one in $a - x$.
- The power integral is $\int_0^1 x^r\,dx = 1/(r+1)$ for $r > -1$. Here the required integral converges for $n > -1$, including all non-negative integer values of n.

Step 1: Prove the reflection property Let $t = a - x$. Then $x = a - t$ and $dx = -dt$. Replacing the limits as well as the integrand gives

$$\begin{aligned}\int_0^a f(x)\,dx &= \int_a^0 f(a-t)(-dt)\\ &= \int_0^a f(a-t)\,dt\\ &= \int_0^a f(a-x)\,dx.\end{aligned}$$

The last line merely renames the dummy variable of integration. Thus the property holds whenever the indicated integrals exist; the reversal of limits cancels the minus sign in the differential.

Step 2: Apply the property to the given integral Write $I = \int_0^1 x^2(1-x)^n\,dx$. Using the proved property with $a = 1$ and $f(x) = x^2(1-x)^n$, we obtain

$$I = \int_0^1 (1-x)^2 x^n\,dx.$$

This substitution is useful because the factor $(1-x)^2$ expands into just three terms, irrespective of the value of n. Therefore

$$I = \int_0^1 \left(x^n - 2x^{n+1} + x^{n+2}\right)dx.$$

Quick Tip to Solve: Reflect the variable before expanding: expanding $(1-x)^2$ gives only three power integrals, whereas expanding $(1-x)^n$ can be lengthy.

Step 3: Integrate the three powers and simplify For $n > -1$, each power has a finite integral at the lower endpoint. Consequently,

$$\begin{aligned} I &= \left[\frac{x^{n+1}}{n+1} - \frac{2x^{n+2}}{n+2} + \frac{x^{n+3}}{n+3}\right]_0^1 \\ &= \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3} \\ &= \frac{(n+2)(n+3) - 2(n+1)(n+3) + (n+1)(n+2)}{(n+1)(n+2)(n+3)}. \end{aligned}$$

Expanding the numerator gives $n^2 + 5n + 6 - 2n^2 - 8n - 6 + n^2 + 3n + 2 = 2$.

Step 4: Conclusion The reflection property is proved, and

$$\boxed{\int_0^1 x^2(1-x)^n\,dx = \frac{2}{(n+1)(n+2)(n+3)}} \qquad (n > -1).$$

Quick Tip to Solve: Reflect the variable before expanding: expanding $(1-x)^2$ gives only three power integrals, whereas expanding $(1-x)^n$ can be lengthy.

15. Find the value of x, for which the four points $A(x,-1,-1)$, $B(4,5,1)$, $C(3,9,4)$ and $D(-4,4,4)$ are coplanar. [4 marks]

Solution:

Concept:

- Four points are coplanar if the scalar triple product of three vectors drawn from one of them is zero. This product measures the signed volume of their parallelepiped.
- If B, C, D are non-collinear, then $\overrightarrow{BC} \times \overrightarrow{BD}$ is a nonzero normal to their plane. A point A lies on that plane exactly when $\overrightarrow{BA} \cdot (\overrightarrow{BC} \times \overrightarrow{BD}) = 0$.

Step 1: Form three vectors with the common starting point B The given points are $A(x,-1,-1)$, $B(4,5,1)$, $C(3,9,4)$ and $D(-4,4,4)$. Subtracting the coordinates of B gives

$$\begin{aligned} \overrightarrow{BA} &= (x-4,-6,-2), \\ \overrightarrow{BC} &= (-1,4,3), \\ \overrightarrow{BD} &= (-8,-1,3). \end{aligned}$$

Use vectors with the same initial point in the scalar triple product.

Quick Tip to Solve: Choose the fixed point B as the common starting point, so the cross product contains no unknown; the final dot product is then a linear equation in x.

Step 2: Calculate a normal to the plane through B, C and D Calculate the cross product:

$$\begin{aligned}\overrightarrow{BC} \times \overrightarrow{BD} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} \\ &= \hat{i}(12+3) - \hat{j}(-3+24) + \hat{k}(1+32) \\ &= 15\hat{i} - 21\hat{j} + 33\hat{k}.\end{aligned}$$

The nonzero cross product shows that B, C, D are non-collinear and determine a unique plane.

Step 3: Impose the coplanarity condition The scalar triple product must vanish:

$$\begin{aligned}\overrightarrow{BA} \cdot (\overrightarrow{BC} \times \overrightarrow{BD}) &= 0, \\ 15(x-4) + (-6)(-21) + (-2)(33) &= 0, \\ 15x - 60 + 126 - 66 &= 0, \\ 15x &= 0.\end{aligned}$$

To check the coordinate arithmetic independently, divide the normal vector by 3. The plane through B, with variable coordinates (X, Y, Z), is

$$5(X-4) - 7(Y-5) + 11(Z-1) = 0,$$

or $5X - 7Y + 11Z + 4 = 0$. Substituting $A(x, -1, -1)$ reduces its left side to $5x + 7 - 11 + 4 = 5x$, agreeing with the scalar triple product condition.

Step 4: Conclusion Hence the required value is $\boxed{x = 0}$. With this value, A lies in the plane determined by B, C, D, so all four points are coplanar.

Quick Tip to Solve: Choose the fixed point B as the common starting point, so the cross product contains no unknown; the final dot product is then a linear equation in x.

16. A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall? [4 marks]

Solution:

Concept:

- Related rates express how two changing quantities are linked. If $x(t)$ is the distance of the foot of the ladder from the wall and $y(t)$ is its height on the wall, then dx/dt and dy/dt are their rates of change with respect to time t.
- The wall and ground are perpendicular. The ladder is the hypotenuse of a right triangle, so Pythagoras' theorem gives $x^2 + y^2 = 13^2$, with x and y measured in metres.
- The ladder has fixed length. Differentiate the relation with respect to time using the chain rule: $2x\,dx/dt + 2y\,dy/dt = 0$. A negative value of dy/dt indicates that the height decreases.

Quick Tip to Solve: Use the 5-12-13 right triangle to get the height immediately. Then $dy/dt = -(x/y)\,dx/dt$; the dimensionless ratio lets you retain the given speed in centimetres per second.

Step 1: Express the given speed in metres per second The bottom moves away from the wall, so x increases and its rate is positive. Since the length and the horizontal distance are given in metres, convert the given speed to the same length unit:

$$\frac{dx}{dt} = 2\,\text{cm s}^{-1} = \frac{2}{100}\,\text{m s}^{-1} = \frac{1}{50}\,\text{m s}^{-1}.$$

Let t be measured in seconds. The unknown vertical rate will then also be in metres per second.

Step 2: Find the height at the specified instant The relation between the distances holds throughout the motion:

$$x^2 + y^2 = 169.$$

At the instant when the foot is $5\,\text{m}$ from the wall,

$$y^2 = 169 - 25 = 144, \qquad y = 12\,\text{m}.$$

We take the positive root because y is a height above the ground.

Step 3: Differentiate and substitute the instantaneous values First differentiate the relation between the variable distances; only then substitute $x = 5$ and $y = 12$. Treating these instantaneous values as constants before differentiation would lose the rates we need.

$$\begin{aligned} 2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0, \\ \frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt}, \\ \left.\frac{dy}{dt}\right|_{x=5} &= -\frac{5}{12}\cdot\frac{1}{50} = -\frac{1}{120}\,\text{m s}^{-1}. \end{aligned}$$

Converting back to centimetres per second multiplies the numerical rate by 100:

$$\frac{dy}{dt} = -\frac{100}{120}\,\text{cm s}^{-1} = -\frac{5}{6}\,\text{cm s}^{-1}.$$

The negative sign agrees with the physical motion: as the foot moves away from the wall, the top moves downward.

Step 4: Conclusion The height on the wall is decreasing at $\boxed{\frac{5}{6}\,\text{cm s}^{-1}}$ when the foot is $5\,\text{m}$ from the wall. The signed vertical rate is $dy/dt = -5/6\,\text{cm s}^{-1}$.

Quick Tip to Solve: Use the 5-12-13 right triangle to get the height immediately. Then $dy/dt = -(x/y)\,dx/dt$; the dimensionless ratio lets you retain the given speed in centimetres per second.

17. Find the vector equation of the plane determined by the points $A(3,-1,2)$, $B(5,2,4)$ and $C(-1,-1,6)$. Hence, find the distance of the plane, thus obtained, from the origin. [4 marks]

Solution:

Concept:

- A normal vector to a plane is perpendicular to two non-parallel vectors lying in it. For the plane through A, B and C, a normal is $\overrightarrow{AB} \times \overrightarrow{AC}$.
- If $\vec{a}$ is the position vector of a known point in a plane and $\vec{n}$ is a nonzero normal, its vector equation is $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$, where $\vec{r}$ is the position vector of an arbitrary point of the plane.
- The perpendicular distance from the origin to $ax + by + cz = d$ is $D = |d|/\sqrt{a^2 + b^2 + c^2}$. The denominator is the magnitude of the normal vector.

Step 1: Form two vectors in the plane Subtract the coordinates of $A(3,-1,2)$ from those of $B(5,2,4)$ and $C(-1,-1,6)$:

$$\overrightarrow{AB} = (5-3)\hat{i} + (2+1)\hat{j} + (4-2)\hat{k} = 2\hat{i} + 3\hat{j} + 2\hat{k},$$
$$\overrightarrow{AC} = (-1-3)\hat{i} + (-1+1)\hat{j} + (6-2)\hat{k} = -4\hat{i} + 4\hat{k}.$$

These displacement vectors lie in the required plane because their endpoints are the given points.

Step 2: Calculate a normal vector Take their cross product to obtain a vector perpendicular to both:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix}$$
$$= \hat{i}(12-0) - \hat{j}(8+8) + \hat{k}(0+12)$$
$$= 12\hat{i} - 16\hat{j} + 12\hat{k}.$$

This vector is nonzero, so the points are non-collinear and determine a unique plane. Dividing by 4, use the simpler normal

$$\vec{n} = 3\hat{i} - 4\hat{j} + 3\hat{k}.$$

Multiplying a normal by a nonzero scalar does not change the plane's direction.

Step 3: Write the vector equation through the given point The position vector of A is $\vec{a} = 3\hat{i} - \hat{j} + 2\hat{k}$. Therefore,

$$(\vec{r} - (3\hat{i} - \hat{j} + 2\hat{k})) \cdot (3\hat{i} - 4\hat{j} + 3\hat{k}) = 0.$$

Since $\vec{a} \cdot \vec{n} = 9 + 4 + 6 = 19$, this is equivalently

$$\vec{r} \cdot (3\hat{i} - 4\hat{j} + 3\hat{k}) = 19.$$

With $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, its Cartesian form is $3x - 4y + 3z = 19$. As a check, substituting B gives $15 - 8 + 12 = 19$, and substituting C gives $-3 + 4 + 18 = 19$; both lie on the plane.

Step 4: Find the perpendicular distance from the origin The Cartesian equation is $3x-4y+3z-19 = 0$.

Quick Tip to Solve: Divide the cross-product normal by its common factor before writing the plane. For distance, use the constant and the normal from the same equation, so any scaling cancels correctly.

Apply the point-to-plane distance formula at $(0,0,0)$:

$$D = \frac{|3(0) - 4(0) + 3(0) - 19|}{\sqrt{3^2 + (-4)^2 + 3^2}}$$
$$= \frac{19}{\sqrt{9+16+9}} = \frac{19}{\sqrt{34}}.$$

The absolute value ensures a positive distance, independent of which orientation is chosen for the normal.

Step 5: Conclusion The required vector equation is $\boxed{\vec{r} \cdot (3\hat{i} - 4\hat{j} + 3\hat{k}) = 19}$. Its perpendicular distance from the origin is $\boxed{\frac{19}{\sqrt{34}} \text{ units}}$.

Quick Tip to Solve: Divide the cross-product normal by its common factor before writing the plane. For distance, use the constant and the normal from the same equation, so any scaling cancels correctly.

18(a). Solve the differential equation $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$, given that $y = 1$ when $x = 0$. [4 marks]

Solution:

Concept:

- For a separable equation, collect the y terms with dy and the x terms with dx, then integrate.
- The identity $1+x^2+y^2+x^2y^2 = (1+x^2)(1+y^2)$ separates the variables. Also, $\int \frac{dy}{1+y^2} = \tan^{-1} y$.
- The initial condition fixes the integration constant. A solution must remain finite on the interval containing the initial point.

Step 1: Factor and separate the variables The given equation is

$$\frac{dy}{dx} = (1 + x^2)(1 + y^2).$$

Since $1 + y^2 > 0$ for every real y, division by this factor loses no real solution. Therefore,

$$\frac{dy}{1 + y^2} = (1 + x^2)\, dx.$$

The equation is now ready for integration.

Step 2: Integrate and use the initial value Integrating both sides gives

$$\tan^{-1} y = x + \frac{x^3}{3} + C.$$

Substituting $x = 0$ and $y = 1$,

$$\tan^{-1} 1 = C = \frac{\pi}{4}.$$

Quick Tip to Solve: Factor the four terms into $(1 + x^2)(1 + y^2)$ before integrating; then apply the initial value to the inverse-tangent relation.

Hence the relation satisfied by the required solution is

$$\tan^{-1} y = x + \frac{x^3}{3} + \frac{\pi}{4}.$$

Taking the tangent gives an explicit expression for y near $x = 0$.

Step 3: Check the differential equation and initial condition Put $\theta = x + x^3/3 + \pi/4$. Differentiating $y = \tan\theta$ yields

$$\frac{dy}{dx} = \sec^2\theta\,(1 + x^2) = (1 + y^2)(1 + x^2),$$

which equals the printed right-hand side. At $x = 0$, the expression gives $y = \tan(\pi/4) = 1$. The initial branch satisfies $-\pi/2 < \theta < \pi/2$.

Step 4: Conclusion The required particular solution is

$$\boxed{y = \tan\left(x + \frac{x^3}{3} + \frac{\pi}{4}\right)},$$

on the interval containing 0 where $-\pi/2 < x + x^3/3 + \pi/4 < \pi/2$.

Quick Tip to Solve: Factor the four terms into $(1 + x^2)(1 + y^2)$ before integrating; then apply the initial value to the inverse-tangent relation.

OR

18(b). Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$, given that $y = 1$ when $x = 0$. [4 marks]

Solution:

Concept:

- The ratio $xy/(x^2 + y^2)$ is homogeneous of degree zero.
- Use $x = vy$, where $v = x/y$ is defined at the initial point $(0, 1)$.
- Use $dx = v\,dy + y\,dv$ and $\int dy/y = \ln y$ on the branch $y > 0$.

Step 1: Rewrite the equation using a homogeneous substitution Write the given equation in differential form:

$$(x^2 + y^2)\,dy = xy\,dx.$$

Set $x = vy$. Differentiating this identity gives $dx = v\,dy + y\,dv$. Substitution yields

$$y^2(v^2 + 1)\,dy = vy^2(v\,dy + y\,dv).$$

Expanding the right-hand side and cancelling the common $v^2y^2\,dy$ terms gives

$$y^2\,dy = vy^3\,dv, \qquad \frac{dy}{y} = v\,dv.$$

Quick Tip to Solve: With an initial point on $x = 0$, use $x = vy$ rather than $y = vx$; the ratio x/y is defined at the given point.

Division by y is valid near $y = 1$; no division by x is needed.

Step 2: Integrate and determine the constant Integrating the separated relation on the positive branch gives

$$\ln y = \frac{v^2}{2} + C = \frac{x^2}{2y^2} + C.$$

Apply $x = 0$, $y = 1$: $0 = 0 + C$, so $C = 0$. Thus $\ln y = x^2/(2y^2)$, which also implies $y \geq 1$. The positive branch is selected by the given initial value; the logarithm is therefore well defined.

Step 3: Verify the implicit solution Differentiate $x^2 = 2y^2 \ln y$ with respect to x:

$$2x = (4y \ln y + 2y)\frac{dy}{dx}.$$

Consequently,

$$\frac{dy}{dx} = \frac{x}{y(2\ln y + 1)} = \frac{xy}{x^2 + y^2},$$

using $2\ln y = x^2/y^2$. At $x = 0$, it gives $\ln y = 0$, hence $y = 1$. The derivative denominator is positive for $y \geq 1$.

Step 4: Conclusion The required particular solution is

$$\boxed{x^2 = 2y^2 \ln y, \qquad y > 0},$$

Quick Tip to Solve: With an initial point on $x = 0$, use $x = vy$ rather than $y = vx$; the ratio x/y is defined at the given point.

19(a). Let $A = \mathbb{R} - \{2\}$ and $B = \mathbb{R} - \{1\}$. If $f : A \to B$ is a function defined by $f(x) = \dfrac{x-1}{x-2}$, show that f is one-one and onto. Hence, find f^{-1}. [4 marks]

Solution:

Concept:

- A function is one-one if equal images force equal preimages: $f(u) = f(v) \Rightarrow u = v$.
- A function $f : A \to B$ is onto if every $y \in B$ has a preimage $x \in A$. A one-one and onto function has an inverse $f^{-1} : B \to A$.
- In a rational function, denominators must remain nonzero. Here $x \neq 2$ in A, and $y \neq 1$ in B.

Step 1: Check that the function takes values in its stated codomain For every $x \in A$, the denominator $x - 2$ is nonzero, so the value of $f(x)$ is defined. Moreover,

$$f(x) = \frac{x-1}{x-2} = 1 + \frac{1}{x-2}.$$

Since $1/(x-2) \neq 0$, we have $f(x) \neq 1$. Thus $f(x) \in B$, as required by the definition of $f : A \to B$.

Quick Tip to Solve: Write the original function as $1 + 1/(x-2)$ to see immediately why its image excludes 1; when finding the inverse, check that the computed preimage excludes 2.

Step 2: Prove that the function is one-one Let $u, v \in A$ and suppose $f(u) = f(v)$. Then

$$\begin{aligned} \frac{u-1}{u-2} &= \frac{v-1}{v-2}, \\ (u-1)(v-2) &= (v-1)(u-2), \\ uv - 2u - v + 2 &= uv - 2v - u + 2, \\ -u + v &= 0. \end{aligned}$$

Cross multiplication is valid because both denominators are nonzero. Consequently $u = v$, so f is one-one.

Step 3: Prove that every element of the codomain has a preimage Take an arbitrary $y \in B$, so $y \neq 1$. Solving $y = f(x)$ gives

$$\begin{aligned} y(x-2) &= x - 1, \\ x(y-1) &= 2y - 1, \\ x &= \frac{2y-1}{y-1}. \end{aligned}$$

This is a real number because $y - 1 \neq 0$, and

$$x - 2 = \frac{2y - 1 - 2(y-1)}{y-1} = \frac{1}{y-1} \neq 0.$$

Hence this preimage belongs to A. Substitution confirms that

$$f(x) = \frac{x-1}{x-2} = \frac{y/(y-1)}{1/(y-1)} = y.$$

Thus every $y \in B$ is attained, and f is onto.

Step 4: Determine the inverse and its domain The one-one property makes the preimage found above unique. Therefore the same expression gives the inverse, with input in B and output in A:

$$f^{-1} : B \to A, \qquad f^{-1}(y) = \frac{2y-1}{y-1}.$$

The restriction $y \neq 1$ is essential: it is precisely the value excluded from the original codomain. Our calculation of $x - 2$ also verifies that the inverse never outputs the excluded value 2.

Step 5: Conclusion Hence f is one-one and onto, and its inverse is

$$\boxed{f^{-1}(x) = \frac{2x-1}{x-1}, \quad x \in \mathbb{R} \setminus \{1\}.}$$

Quick Tip to Solve: Write the original function as $1 + 1/(x-2)$ to see immediately why its image excludes 1; when finding the inverse, check that the computed preimage excludes 2.

OR

19(b). Show that the relation S in the set $A = \{x \in \mathbb{Z} : 0 \leq x \leq 12\}$ given by $S = \{(a, b) : a, b \in \mathbb{Z}, |a - b| \text{ is divisible by } 3\}$ is an equivalence relation. [4 marks]

Solution:

Concept:

- A relation on A is an equivalence relation when it is reflexive, symmetric and transitive.
- An integer n is divisible by 3 if $n = 3k$ for some $k \in \mathbb{Z}$. Thus $3 \mid |a - b|$ is equivalent to $3 \mid (a - b)$, because changing the sign preserves divisibility.
- Reflexivity requires aSa for every $a \in A$; symmetry requires $aSb \Rightarrow bSa$; transitivity requires aSb and $bSc \Rightarrow aSc$.

Step 1: Verify reflexivity The underlying set is $A = \{0, 1, 2, \ldots, 12\}$, and the relation is considered between elements of this set. Let a be any element of A. Then

$$|a - a| = 0 = 3 \cdot 0.$$

Since 0 is an integer multiple of 3, it is divisible by 3. Therefore $(a, a) \in S$ for every $a \in A$. This proves that S is reflexive, including at the endpoints 0 and 12.

Step 2: Verify symmetry Let $a, b \in A$ and assume $(a, b) \in S$. By the defining condition, $|a - b|$ is divisible by 3. Reversing the order does not change an absolute difference:

$$|b - a| = |-(a - b)| = |a - b|.$$

Hence $|b - a|$ is also divisible by 3, which implies $(b, a) \in S$. The argument applies to any related pair in A, so S is symmetric.

Step 3: Verify transitivity using signed differences Let $a, b, c \in A$ with $(a, b) \in S$ and $(b, c) \in S$. The absolute differences are divisible by 3; consequently their signed versions are integer multiples of 3. Thus there exist integers m, n such that

$$a - b = 3m, \qquad b - c = 3n.$$

Adding these equations eliminates the intermediate element b:

$$a - c = (a - b) + (b - c) = 3(m + n).$$

Because $m + n \in \mathbb{Z}$, we obtain

$$|a - c| = 3|m + n|,$$

which is divisible by 3. Therefore $(a, c) \in S$, establishing transitivity. Using signed differences here avoids incorrectly assuming that absolute differences always add.

Step 4: Identify the groups of related elements The defining condition says that related integers have

Quick Tip to Solve: For transitivity, replace divisibility of absolute differences by signed equations $a - b = 3m$ and $b - c = 3n$, then add them; do not add absolute values directly.

the same remainder on division by 3. In this set the corresponding equivalence classes are

$$[0] = \{0, 3, 6, 9, 12\},$$
$$[1] = \{1, 4, 7, 10\},$$
$$[2] = \{2, 5, 8, 11\}.$$

They are disjoint and together contain every element of A. These classes give an additional concrete description of the relation proved above: two elements are related exactly when they lie in the same displayed class.

Step 5: Conclusion Since S is reflexive, symmetric and transitive on A, it is an equivalence relation on A.

Quick Tip to Solve: For transitivity, replace divisibility of absolute differences by signed equations $a - b = 3m$ and $b - c = 3n$, then add them; do not add absolute values directly.

20(a). **If $x = \sin t,\ y = \sin pt$, prove that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2y = 0$.** [4 marks]

Solution:

Concept:

- For parametric equations $x = x(t)$ and $y = y(t)$, the chain rule gives $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, provided $dx/dt \neq 0$.
- For the second derivative, differentiate dy/dx with respect to t, then divide by dx/dt once more.
- Since $x = \sin t$, the identity $1 - \sin^2 t = \cos^2 t$ converts the coefficient $1 - x^2$ into a parameter expression. The quantity p is a constant.

Step 1: Find the first derivative Differentiate the given equations with respect to t. Working locally where $\cos t \neq 0$, x is a valid independent variable and we obtain

$$\frac{dx}{dt} = \cos t, \qquad \frac{dy}{dt} = p\cos(pt).$$

Hence the chain rule gives

$$\frac{dy}{dx} = \frac{p\cos(pt)}{\cos t}.$$

Step 2: Calculate the second derivative Apply the quotient rule to the first derivative, then divide by $dx/dt = \cos t$. The numerator derivative is $-p^2\sin(pt)$, and the denominator derivative is $-\sin t$. Therefore,

$$\frac{d^2y}{dx^2} = \frac{1}{\cos t}\frac{d}{dt}\left(\frac{p\cos(pt)}{\cos t}\right)$$
$$= \frac{-p^2\sin(pt)\cos t + p\cos(pt)\sin t}{\cos^3 t}.$$

Step 3: Substitute into the required expression Use $1 - x^2 = \cos^2 t$, $x = \sin t$, and $y = \sin(pt)$. This

Quick Tip to Solve: Keep both derivatives in terms of t and substitute $1 - x^2 = \cos^2 t$ before simplifying; the two terms containing $p\sin t\cos(pt)/\cos t$ cancel immediately.

gives

$$(1-x^2)\frac{d^2y}{dx^2} = -p^2\sin(pt) + \frac{p\cos(pt)\sin t}{\cos t},$$
$$x\frac{dy}{dx} = \frac{p\sin t\cos(pt)}{\cos t}.$$

The fractional terms are identical and cancel when the second expression is subtracted from the first. Thus,

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = -p^2\sin(pt) + p^2\sin(pt)$$
$$= 0.$$

Step 4: Conclusion Hence, wherever the parametric curve defines the indicated derivatives,

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0}.$$

Quick Tip to Solve: Keep both derivatives in terms of t and substitute $1-x^2 = \cos^2 t$ before simplifying; the two terms containing $p\sin t\cos(pt)/\cos t$ cancel immediately.

OR

20(b). Differentiate $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ **with respect to** $\cos^{-1}x^2$**.** [4 marks]

Solution:

Concept:

- For differentiation with respect to $u = \cos^{-1}(x^2)$, express the given function directly in terms of u.
- The half-angle identities are $1+\cos\theta = 2\cos^2(\theta/2)$ and $1-\cos\theta = 2\sin^2(\theta/2)$. The tangent subtraction identity gives $\tan(\pi/4 - v) = \dfrac{1-\tan v}{1+\tan v}$.
- On its principal range $(-\pi/2, \pi/2)$, $\tan^{-1}(\tan v) = v$.

Step 1: Introduce the variable of differentiation Denote the given inverse tangent by y, and put

$$\theta = \cos^{-1}(x^2), \qquad x^2 = \cos\theta.$$

The radicals require $-1 \le x \le 1$. Consequently, $0 \le x^2 \le 1$, so the principal inverse cosine gives $0 \le \theta \le \pi/2$. In this range, $\sin(\theta/2)$ and $\cos(\theta/2)$ are nonnegative, which fixes the signs when taking square roots.

Step 2: Simplify the radicals with half-angle identities Substitute $x^2 = \cos\theta$ into the expression inside the inverse tangent. The two radicals become

$$\sqrt{1+x^2} = \sqrt{2}\cos\frac{\theta}{2}, \qquad \sqrt{1-x^2} = \sqrt{2}\sin\frac{\theta}{2}.$$

Quick Tip to Solve: Set the variable of differentiation $\cos^{-1}(x^2)$ equal to θ; the two square roots then become half-angle sine and cosine, reducing the entire inverse tangent to a linear function of θ.

Cancel $\sqrt{2}$ and divide both parts by the positive $\cos(\theta/2)$:

$$\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \frac{\cos(\theta/2)-\sin(\theta/2)}{\cos(\theta/2)+\sin(\theta/2)}$$
$$= \tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right).$$

Step 3: Resolve the inverse tangent and differentiate Since $0 \le \pi/4 - \theta/2 \le \pi/4$, the angle is within the principal range of the inverse tangent. Hence there is no additional multiple of π, and

$$y = \tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right] = \frac{\pi}{4}-\frac{\theta}{2}.$$

Differentiate this linear relation directly with respect to θ:

$$\frac{dy}{d\theta} = 0 - \frac{1}{2} = -\frac{1}{2}.$$

At endpoints, the same relation gives the one-sided derivative.

Step 4: Conclusion Hence, the derivative with respect to $\cos^{-1}(x^2)$ is $\boxed{-\frac{1}{2}}$.

Quick Tip to Solve: Set the variable of differentiation $\cos^{-1}(x^2)$ equal to θ; the two square roots then become half-angle sine and cosine, reducing the entire inverse tangent to a linear function of θ.

21. Using properties of determinants, find the value of k if

$$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix} = k(x^3+y^3).$$ [4 marks]

Solution:

Concept:

- Replacing one column by that column plus other columns leaves a determinant unchanged. A common factor in a column can be taken outside the determinant.
- Replacing a row by its difference with another row leaves the determinant unchanged. Expansion along a column containing two zeros reduces a third-order determinant to a second-order determinant.
- The sum-of-cubes identity is $x^3+y^3 = (x+y)(x^2-xy+y^2)$; it lets us compare the resulting polynomial with the given expression.

Step 1: Combine the columns and extract their common factor Denote the given determinant by D. Apply $C_3 \to C_3 + C_1 + C_2$. Each entry of the new third column is $2(x+y)$, so

$$D = \begin{vmatrix} x & y & 2(x+y) \\ y & x+y & 2(x+y) \\ x+y & x & 2(x+y) \end{vmatrix} = 2(x+y)\begin{vmatrix} x & y & 1 \\ y & x+y & 1 \\ x+y & x & 1 \end{vmatrix}.$$

Quick Tip to Solve: Notice that all row sums equal $2(x+y)$: add the columns first, then subtract the first row from the other two.

The first equality uses column addition, while the second uses linearity in the third column. No division by $x+y$ is needed, so the calculation also holds when $x+y=0$.

Step 2: Create zeros and evaluate the remaining minor In the determinant on the right, use $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$. Hence

$$D = 2(x+y)\begin{vmatrix} x & y & 1 \\ y-x & x & 0 \\ y & x-y & 0 \end{vmatrix}.$$

Expand along the third column. The only nonzero entry is in position $(1,3)$, whose cofactor sign is $(-1)^{1+3} = +1$. Therefore

$$\begin{aligned} D &= 2(x+y)\big[(y-x)(x-y) - xy\big] \\ &= 2(x+y)\big[-(x-y)^2 - xy\big] \\ &= -2(x+y)(x^2 - xy + y^2) \\ &= -2(x^3+y^3). \end{aligned}$$

This is the required form of the determinant as a polynomial in x and y.

Step 3: Conclusion Comparing $D = -2(x^3+y^3)$ with the given identity $D = k(x^3+y^3)$, the constant is $\boxed{k=-2}$.

Quick Tip to Solve: Notice that all row sums equal $2(x+y)$: add the columns first, then subtract the first row from the other two.

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22. Find: $\displaystyle\int \frac{1}{\sin(x-a)\cos(x-b)}\,dx.$ [4 marks]

Solution:

Concept:

- Use $\cos(u-v) = \cos u\cos v + \sin u \sin v$ to rewrite the reciprocal product.
- The logarithmic antiderivatives are $\int \cot u\,du = \ln|\sin u| + C$ and $\int \tan u\,du = -\ln|\cos u| + C$.
- The constants are a, b. Work on an interval where $\sin(x-a)\cos(x-b) \neq 0$.

Step 1: Rewrite the integrand using the difference of the angles Let $I = \displaystyle\int \frac{dx}{\sin(x-a)\cos(x-b)}$. By the cosine subtraction formula,

$$\begin{aligned} \cot(x-a) + \tan(x-b) &= \frac{\cos(x-a)\cos(x-b) + \sin(x-a)\sin(x-b)}{\sin(x-a)\cos(x-b)} \\ &= \frac{\cos(b-a)}{\sin(x-a)\cos(x-b)}. \end{aligned}$$

Since $\cos(b-a) = \cos(a-b)$, for $\cos(a-b) \neq 0$ this gives

$$I = \sec(a-b)\int \big[\cot(x-a) + \tan(x-b)\big], dx.$$

Quick Tip to Solve: Add $\cot(x-a)$ and $\tan(x-b)$ over a common denominator; their numerator is the constant $\cos(a-b)$.

Step 2: Integrate and check the logarithmic expression Integrating the two terms separately yields

$$I = \sec(a-b)\big[\ln|\sin(x-a)| - \ln|\cos(x-b)|\big] + C.$$

Combine the logarithms. To verify, differentiate:

$$\sec(a-b)\big[\cot(x-a) + \tan(x-b)\big] = \frac{1}{\sin(x-a)\cos(x-b)},$$

Step 3: Handle the exceptional constant difference If $\cos(a-b) = 0$, the preceding division is unavailable. Put $u = x - a$ and $\delta = a - b$. Then $\sin\delta = \pm 1$ and

$$\cos(x-b) = \cos(u+\delta) = -\sin\delta\sin u.$$

Consequently,

$$I = -\frac{1}{\sin\delta}\int \csc^2 u\, du = \sin\delta\cot u + C.$$

Here $1/\sin\delta = \sin\delta$ and $d(\cot u)/du = -\csc^2 u$.

Step 4: Conclusion For $\cos(a-b) \neq 0$,

$$\boxed{I = \sec(a-b)\ln\left|\frac{\sin(x-a)}{\cos(x-b)}\right| + C}.$$

If $\cos(a-b) = 0$, the corresponding antiderivative is $\boxed{I = \sin(a-b)\cot(x-a) + C}$.

Quick Tip to Solve: Add $\cot(x-a)$ and $\tan(x-b)$ over a common denominator; their numerator is the constant $\cos(a-b)$.

23. Prove that: $\sin^{-1}\left(\frac{8}{17}\right) + \cos^{-1}\left(\frac{4}{5}\right) = \cot^{-1}\left(\frac{36}{77}\right)$. [4 marks]

Solution:

Concept:

- The principal values of $\sin^{-1}x$ and $\cos^{-1}x$ lie in $[-\pi/2, \pi/2]$ and $[0, \pi]$, respectively. For the positive arguments here, both angles are acute.
- Use $\sin^2\theta + \cos^2\theta = 1$ and the addition formulae for sine and cosine. The principal value of $\cot^{-1}x$ lies in $(0, \pi)$; a positive cotangent corresponds to an acute angle.

Step 1: Introduce the two principal angles Put $\alpha = \sin^{-1}(8/17)$ and $\beta = \cos^{-1}(4/5)$. Thus $0 < \alpha < \pi/2$ and $0 < \beta < \pi/2$. Since the complementary trigonometric ratios are positive in the first quadrant,

$$\cos\alpha = \sqrt{1 - \frac{64}{289}} = \frac{15}{17}, \qquad \sin\beta = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}.$$

Choosing the positive square roots is justified by the principal ranges, rather than by the squared

Quick Tip to Solve: Recognise the 8-15-17 and 3-4-5 triangles, then compute the cosine of the sum to establish the correct inverse-trigonometric branch.

identities alone.

Step 2: Evaluate the sine and cosine of the sum Applying the angle-addition formulae gives

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\\ &= \frac{8}{17}\cdot\frac{4}{5}+\frac{15}{17}\cdot\frac{3}{5}=\frac{32+45}{85}=\frac{77}{85},\\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta-\sin\alpha\sin\beta\\ &= \frac{15}{17}\cdot\frac{4}{5}-\frac{8}{17}\cdot\frac{3}{5}=\frac{60-24}{85}=\frac{36}{85}.\end{aligned}$$

These values also satisfy $(77/85)^2+(36/85)^2=1$, which checks the arithmetic.

Step 3: Check the branch before taking the inverse The sum of the two acute angles lies in $(0,\pi)$. Its cosine is positive, so it actually lies in $(0,\pi/2)$. Therefore it belongs to the principal range of inverse cotangent. Moreover,

$$\cot(\alpha+\beta)=\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}=\frac{36/85}{77/85}=\frac{36}{77}.$$

Cotangent is one-to-one on $(0,\pi)$, so applying its principal inverse introduces no extra multiple of π.

Step 4: Conclusion Hence $\alpha+\beta=\cot^{-1}(36/77)$, proving

$$\boxed{\sin^{-1}\left(\frac{8}{17}\right)+\cos^{-1}\left(\frac{4}{5}\right)=\cot^{-1}\left(\frac{36}{77}\right)}.$$

Quick Tip to Solve: Recognise the 8-15-17 and 3-4-5 triangles, then compute the cosine of the sum to establish the correct inverse-trigonometric branch.

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24(a). Using elementary row transformations, find the inverse of the matrix $\begin{bmatrix}2&-3&5\\3&2&-4\\1&1&-2\end{bmatrix}$. [6 marks]

Solution:

Concept:

- To find A^{-1} by elementary row transformations, apply each operation to the entire augmented matrix $[A \mid I]$, where I is the identity matrix of the same order.
- If row reduction yields $[I \mid B]$, then $B=A^{-1}$. Row interchanges, nonzero row scaling and addition of a multiple of another row are reversible operations.

Step 1: Form the augmented matrix and choose a unit pivot Let A be the given matrix. Augment it

Quick Tip to Solve: Swap the first and third rows to start with a unit pivot, and apply every row operation to all six columns of the augmented matrix.

with I_3, keeping the right block aligned with the corresponding rows. First interchange R_1 and R_3:

$$\left[\begin{array}{ccc|ccc} 2 & -3 & 5 & 1 & 0 & 0 \\ 3 & 2 & -4 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 0 & 1 \\ 3 & 2 & -4 & 0 & 1 & 0 \\ 2 & -3 & 5 & 1 & 0 & 0 \end{array}\right].$$

This gives the first pivot as 1 and avoids fractions during the elimination.

Step 2: Clear the first column below the pivot Use $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$, using the unchanged first row in both operations:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & -3 \\ 0 & -5 & 9 & 1 & 0 & -2 \end{array}\right].$$

Now apply $R_2 \to -R_2$, obtaining

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & 3 \\ 0 & -5 & 9 & 1 & 0 & -2 \end{array}\right].$$

The second row now has a unit pivot in the second column.

Step 3: Obtain the third pivot Add five times the new second row to the third row:

$$R_3 \to R_3 + 5R_2 : \quad \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & -1 & 1 & -5 & 13 \end{array}\right].$$

Then $R_3 \to -R_3$ gives

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & -1 & 5 & -13 \end{array}\right].$$

All three pivots are nonzero, so reduction to the identity is possible and the inverse exists.

Step 4: Reduce the left block to the identity Clear the third column using $R_1 \to R_1 + 2R_3$ and $R_2 \to R_2 + 2R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 10 & -25 \\ 0 & 1 & 0 & -2 & 9 & -23 \\ 0 & 0 & 1 & -1 & 5 & -13 \end{array}\right].$$

Finally, $R_1 \to R_1 - R_2$ yields

$$[I_3 \mid A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & -2 & 9 & -23 \\ 0 & 0 & 1 & -1 & 5 & -13 \end{array}\right].$$

Each operation has been performed on both blocks; the final right block is therefore the required inverse.

Quick Tip to Solve: Swap the first and third rows to start with a unit pivot, and apply every row operation to all six columns of the augmented matrix.

Step 5: Verify the inverse by multiplication As a check on the signs and entries, multiply the original matrix by the right block:

$$\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} = \begin{bmatrix} 6-5 & 2-27+25 & -4+69-65 \\ -4+4 & 3+18-20 & -6-46+52 \\ -2+2 & 1+9-10 & -2-23+26 \end{bmatrix} = I_3.$$

The product is the identity, confirming that the row transformations and the final inverse are consistent.

Step 6: Conclusion The inverse found using elementary row transformations is

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}}.$$

Quick Tip to Solve: Swap the first and third rows to start with a unit pivot, and apply every row operation to all six columns of the augmented matrix.

OR

24(b). Using matrices, solve the following system of linear equations:

$x + 2y - 3z = -4$

$2x + 3y + 2z = 2$

$3x - 3y - 4z = 11$ [6 marks]

Solution:

Concept:

- A linear system can be written as $AX = B$, where A is the coefficient matrix, X is the column of unknowns and B is the column of constants. Preserve the order x, y, z in every row.
- If $|A| \neq 0$, the system has a unique solution $X = A^{-1}B$. The inverse is $A^{-1} = \dfrac{\text{adj } A}{|A|}$, where the adjugate is the transpose of the cofactor matrix.

Step 1: Express the equations as a matrix equation Collecting the coefficients and constants in their printed order gives

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad B = \begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix}.$$

The three equations are therefore exactly $AX = B$.

Step 2: Calculate the determinant and establish invertibility Expanding along the first row, with

Quick Tip to Solve: After forming the cofactors, transpose them to get the adjugate; then check the resulting values in all three printed equations.

alternating cofactor signs, yields

$$\begin{aligned}|A| &= 1\{3(-4) - 2(-3)\} - 2\{2(-4) - 2(3)\} \\ &\quad + (-3)\{2(-3) - 3(3)\} \\ &= -6 + 28 + 45 = 67.\end{aligned}$$

Since this is nonzero, multiplication by A^{-1} is valid and the solution is unique.

Step 3: Form the adjugate and inverse For each entry use $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant remaining after deleting row i and column j. For instance, $C_{12} = -\{2(-4) - 2(3)\} = 14$ and $C_{21} = -\{2(-4) - (-3)(-3)\} = 17$. Computing all nine cofactors gives

$$C = \begin{pmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{pmatrix}.$$

Transpose this matrix before dividing by the determinant:

$$\operatorname{adj} A = C^T = \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}, \qquad A^{-1} = \frac{1}{67} \operatorname{adj} A.$$

Step 4: Multiply the inverse by the constants and verify Multiplying $AX = B$ on the left by A^{-1}, we obtain

$$\begin{aligned}X &= \frac{1}{67} \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix} \\ &= \frac{1}{67} \begin{pmatrix} 24 + 34 + 143 \\ -56 + 10 - 88 \\ 60 + 18 - 11 \end{pmatrix} = \frac{1}{67} \begin{pmatrix} 201 \\ -134 \\ 67 \end{pmatrix}.\end{aligned}$$

Check the resulting values in each original equation:

$$\begin{aligned}3 + 2(-2) - 3(1) &= -4, \\ 2(3) + 3(-2) + 2(1) &= 2, \\ 3(3) - 3(-2) - 4(1) &= 11.\end{aligned}$$

All three constants are reproduced, confirming the matrix calculation.

Step 5: Conclusion The unique solution is $\boxed{x = 3,\ y = -2,\ z = 1}$.

Quick Tip to Solve: After forming the cofactors, transpose them to get the adjugate; then check the resulting values in all three printed equations.

25(a). Using integration, find the area of the region bounded by the parabola $y^2 = 4x$ and the circle $4x^2 + 4y^2 = 9$. [6 marks]

Solution:

Concept:

- With horizontal strips, the area between two curves is $\int (x_{\text{right}} - x_{\text{left}})\, dy$. Here the parabola is $x = y^2/4$ and the circle has radius $3/2$.
- Both curves are symmetric about the x-axis, so the required enclosed area is twice its portion above that axis. For $a > 0$, $\int \sqrt{a^2 - y^2}\, dy = \frac{1}{2}\{y\sqrt{a^2 - y^2} + a^2 \sin^{-1}(y/a)\} + C$.

Step 1: Find the points where the curves meet Substitute $y^2 = 4x$ in the circle equation:

$$4x^2 + 16x = 9, \qquad 4x^2 + 16x - 9 = 0.$$

The roots are

$$x = \frac{-16 \pm \sqrt{256 + 144}}{8} = \frac{-16 \pm 20}{8} = \frac{1}{2},\ -\frac{9}{2}.$$

A real point on the parabola has $x = y^2/4 \geq 0$, so the negative root is inadmissible. The intersections are consequently $(1/2, \sqrt{2})$ and $(1/2, -\sqrt{2})$, since $y^2 = 2$.

Step 2: Set up horizontal strips across the enclosed region The circle can be written as $x^2 + y^2 = 9/4$. Its right semicircle is $x = \sqrt{9/4 - y^2}$. Between the two intersections, the parabola forms the left boundary of the region enclosed with this circle arc, and the right semicircle forms the right boundary. At $y = 0$, these boundaries have $x = 0$ and $x = 3/2$, respectively. Thus each strip has width

$$\sqrt{\frac{9}{4} - y^2} - \frac{y^2}{4}.$$

Using symmetry about the x-axis, the area is

$$A = 2\int_0^{\sqrt{2}} \left(\sqrt{\frac{9}{4} - y^2} - \frac{y^2}{4}\right) dy.$$

Horizontal strips avoid splitting the region into multiple integrals with different upper boundaries.

Step 3: Evaluate the circle and parabola contributions Applying the stated square-root integral formula with $a = 3/2$,

$$A = 2\left[\frac{y}{2}\sqrt{\frac{9}{4} - y^2} + \frac{9}{8}\sin^{-1}\left(\frac{2y}{3}\right) - \frac{y^3}{12}\right]_0^{\sqrt{2}}.$$

At the upper limit, $\sqrt{9/4 - 2} = 1/2$ and $(\sqrt{2})^3 = 2\sqrt{2}$. At the lower limit all three terms vanish. Therefore

$$A = 2\left\{\frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{\sqrt{2}}{6}\right\}$$
$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right).$$

Quick Tip to Solve: Write both boundaries as functions of y and use symmetry: horizontal strips give one integral from 0 to the intersection height sqrt(2).

The inverse sine is its principal value in radians; its argument lies between zero and one. The positive area is consistent with the nonnegative strip widths throughout the interval.

Step 4: Conclusion The required area is $\boxed{\frac{\sqrt{2}}{6}+\frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)}$ square units, with the angle measured in radians.

Quick Tip to Solve: Write both boundaries as functions of y and use symmetry: horizontal strips give one integral from 0 to the intersection height sqrt(2).

OR

25(b). Using the method of integration, find the area of the region bounded by the lines $3x-2y+1=0$, $2x+3y-21=0$ and $x-5y+9=0$. [6 marks]

Solution:

Concept:

- The pairwise intersections of three non-parallel lines identify the vertices of their bounded triangular region.
- For vertical strips between $y=u(x)$ and $y=l(x)$, the area is $\int_a^b[u(x)-l(x)]\,dx$, where $u(x)$ is the upper boundary and $l(x)$ the lower boundary.
- Split an area integral wherever the upper or lower boundary changes. The integrand must be non-negative on each interval.

Step 1: Express the three lines as functions of x Denote the given lines by L_1, L_2, L_3, respectively. Solving each equation for y gives

$$L_1:\ y=\frac{3x+1}{2}, \qquad L_2:\ y=\frac{21-2x}{3}, \qquad L_3:\ y=\frac{x+9}{5}.$$

Their distinct slopes ensure that each pair has a unique intersection. These intersections determine the limits for integrating the bounded region.

Step 2: Find the three vertices For $L_1\cap L_3$, equate the two expressions for y:

$$\frac{3x+1}{2}=\frac{x+9}{5} \implies 15x+5=2x+18 \implies x=1, \quad y=2.$$

Thus one vertex is $A(1,2)$. For $L_1\cap L_2$,

$$3(3x+1)=2(21-2x) \implies 13x=39 \implies x=3, \quad y=5,$$

giving $B(3,5)$. Finally, for $L_2\cap L_3$,

$$5(21-2x)=3(x+9) \implies 13x=78 \implies x=6, \quad y=3,$$

giving $C(6,3)$.

Quick Tip to Solve: Order the vertices by their x-coordinates first; the middle x-coordinate is where the upper boundary changes and the area integral must be split.

Step 3: Identify the boundaries of each vertical strip The segment AC, lying on L_3, is the lower boundary throughout $1 \le x \le 6$. From $x = 1$ to $x = 3$, the upper boundary is AB, on L_1. From $x = 3$ to $x = 6$, it is BC, on L_2. Hence the required area S is

$$S = \int_1^3 \left(\frac{3x+1}{2} - \frac{x+9}{5} \right) dx + \int_3^6 \left(\frac{21-2x}{3} - \frac{x+9}{5} \right) dx.$$

The respective strip heights simplify to $13(x-1)/10$ and $13(6-x)/15$, both non-negative on their stated intervals.

Step 4: Evaluate and add the two integrals The area between $x = 1$ and $x = 3$ is

$$S_1 = \frac{13}{10} \int_1^3 (x-1)\, dx = \frac{13}{10} \left[\frac{(x-1)^2}{2} \right]_1^3 = \frac{13}{5}.$$

The area between $x = 3$ and $x = 6$ is

$$S_2 = \frac{13}{15} \int_3^6 (6-x)\, dx = \frac{13}{15} \left[6x - \frac{x^2}{2} \right]_3^6 = \frac{13}{15} \left(18 - \frac{27}{2} \right) = \frac{39}{10}.$$

Adding the two contributions gives $S_1 + S_2 = 26/10 + 39/10 = 65/10$. The limits include exactly the three bounding segments and no exterior region.

Step 5: Conclusion The area of the region bounded by the three given lines is $\boxed{\frac{13}{2} \text{ square units}}$.

Quick Tip to Solve: Order the vertices by their x-coordinates first; the middle x-coordinate is where the upper boundary changes and the area integral must be split.

26. A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contains at least 8 units of vitamin A and 10 units of vitamin C. Food I contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C. It costs Rs. 50 per kg to produce food I. Food II contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C and it costs Rs. 70 per kg to produce food II. Formulate this problem as a LPP to minimise the cost of a mixture that will produce the required diet. Also find the minimum cost. [6 marks]

Solution:

Concept:

- A linear programming problem expresses a linear objective function and linear constraints in non-negative decision variables.
- An "at least" requirement gives a greater-than-or-equal-to inequality. The nutrients from both foods add to give the total nutrient content.
- A finite optimum on a polygonal feasible region is attained at a corner point. For an unbounded feasible region, existence of the proposed minimum must also be checked.

Step 1: Define the amounts of the two foods and the cost Let x kg of Food I and y kg of Food II be used in the mixture. Since these represent quantities of food, $x \geq 0$ and $y \geq 0$. Food I costs Rs. 50 per kg and Food II costs Rs. 70 per kg. Therefore the total production cost, measured in rupees, is

$$Z = 50x + 70y.$$

The aim is to minimise this cost while meeting both vitamin requirements. No fixed total mass of the mixture is specified, so there is no additional mass equality constraint.

Step 2: Formulate the nutrient constraints and the LPP Each kilogram of Food I supplies 2 units of vitamin A and each kilogram of Food II supplies 1 unit. Thus the mixture supplies $2x + y$ units of vitamin A, which must be at least 8. Similarly, its vitamin C content is $x + 2y$ units and must be at least 10. The complete LPP is therefore

$$\begin{aligned} &\text{Minimise } Z = 50x + 70y \\ &\text{subject to } 2x + y \geq 8, \\ &\qquad\qquad x + 2y \geq 10, \\ &\qquad\qquad x \geq 0, \quad y \geq 0. \end{aligned}$$

Both inequalities point towards larger nutrient totals because extra vitamin content is allowed.

Step 3: Find the corner points of the feasible region The feasible region lies in the first quadrant and on the side of both nutrient boundary lines away from the origin. It extends indefinitely towards larger values of x and y. The intersection of the nutrient boundaries satisfies

$$2x + y = 8,$$
$$x + 2y = 10.$$

Substituting $y = 8 - 2x$ in the second equation gives $x + 16 - 4x = 10$, so $x = 2$ and $y = 4$. On the y-axis, the constraints require $y \geq 8$ and $y \geq 5$, giving the boundary corner $(0, 8)$. On the x-axis, they

Quick Tip to Solve: For this mixture, write the cost as 10 times the vitamin A total plus 30 times the vitamin C total; the required nutrient amounts then give the minimum cost directly.

require $x \geq 4$ and $x \geq 10$, giving the boundary corner $(10, 0)$. Hence the three feasible corner points are $(0, 8)$, $(2, 4)$ and $(10, 0)$.

Step 4: Compare the costs and verify a global minimum Evaluating the objective at the feasible vertices gives

(x, y)	$Z = 50x + 70y$ (rupees)
$(0, 8)$	560
$(2, 4)$	380
$(10, 0)$	500

To verify the minimum even though the feasible region is unbounded, rewrite the cost using the two nutrient totals:

$$Z = 10(2x + y) + 30(x + 2y)$$
$$\geq 10(8) + 30(10) = 380.$$

Thus every feasible mixture costs at least 380 rupees. The point $(2, 4)$ attains equality because it gives vitamin A content $2(2) + 4 = 8$ units and vitamin C content $2 + 2(4) = 10$ units. Both nutrient constraints are satisfied exactly, so the cost bound is attained by an admissible mixture.

Step 5: Conclusion The minimum cost is $\boxed{\text{Rs. } 380}$. Use 2 kg of Food I and 4 kg of Food II. The LPP is to minimise $50x + 70y$ subject to $2x + y \geq 8$, $x + 2y \geq 10$ and $x, y \geq 0$.

Quick Tip to Solve: For this mixture, write the cost as 10 times the vitamin A total plus 30 times the vitamin C total; the required nutrient amounts then give the minimum cost directly.

27(a). Find the vector equation of a line passing through the point $(2, 3, 2)$ and parallel to the line $\vec{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$. Also, find the distance between these two lines. [6 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$, parallel to a nonzero vector $\vec{b}$, has equation $\vec{r} = \vec{a} + t\vec{b}$, where $t \in \mathbb{R}$.
- For two distinct parallel lines with common direction vector $\vec{b}$, their distance is $d = \dfrac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$, where $\vec{a}_1$ and $\vec{a}_2$ locate any one point on each line.
- The cross product selects the component of the joining vector perpendicular to the lines. Thus the shortest distance is independent of which points on the parallel lines are selected.

Step 1: Identify the direction and write the required line The given line passes through $A(-2, 3, 0)$ and has direction vector

$$\vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}.$$

A parallel line has the same direction vector (or any nonzero scalar multiple of it). Since the required line passes through $B(2, 3, 2)$, its equation is

$$\vec{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + t(2\hat{i} - 3\hat{j} + 6\hat{k}), \quad t \in \mathbb{R}.$$

Quick Tip to Solve: For parallel lines, first copy their common direction vector; then use the cross-product distance formula, not the distance between the two listed points.

Setting $t = 0$ gives the prescribed point, and the coefficients of t confirm that its direction is parallel to the given line. The parameter t is independent of the parameter on the original line.

Step 2: Calculate the perpendicular component of the joining vector Using the chosen points A and B, we obtain

$$\overrightarrow{AB} = 4\hat{i} + 2\hat{k}.$$

Its length is not directly the distance between the lines, since it is not perpendicular to their direction. Indeed, $\overrightarrow{AB} \cdot \vec{b} = 20 \neq 0$. Therefore use the cross product:

$$\overrightarrow{AB} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 0 & 2 \\ 2 & -3 & 6 \end{vmatrix}$$
$$= 6\hat{i} - 20\hat{j} - 12\hat{k}.$$

Consequently,

$$|\overrightarrow{AB} \times \vec{b}| = \sqrt{36 + 400 + 144} = \sqrt{580} = 2\sqrt{145},$$

while $|\vec{b}| = \sqrt{4 + 9 + 36} = 7$. Dividing the cross-product magnitude by the direction-vector magnitude removes the parallel component of the separation.

Step 3: Find and verify the distance The shortest distance between the two lines is therefore

$$d = \frac{2\sqrt{145}}{7}.$$

As a check, subtract the projection of $\overrightarrow{AB}$ along $\vec{b}$:

$$\vec{v} = \overrightarrow{AB} - \frac{20}{49}\vec{b}.$$

This vector satisfies $\vec{v} \cdot \vec{b} = 20 - \frac{20}{49}(49) = 0$, so it represents a perpendicular joining displacement. Its squared magnitude is

$$|\vec{v}|^2 = |\overrightarrow{AB}|^2 - \frac{(\overrightarrow{AB} \cdot \vec{b})^2}{|\vec{b}|^2} = 20 - \frac{400}{49} = \frac{580}{49},$$

which agrees with the square of the distance calculated above.

Step 4: Conclusion The required line is $\boxed{\vec{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + t(2\hat{i} - 3\hat{j} + 6\hat{k})}$, $t \in \mathbb{R}$. The distance between the lines is $\boxed{\frac{2\sqrt{145}}{7} \text{ units}}$.

Quick Tip to Solve: For parallel lines, first copy their common direction vector; then use the cross-product distance formula, not the distance between the two listed points.

OR

27(b). Find the coordinates of the foot of the perpendicular Q drawn from $P(3,2,1)$ to the plane $2x-y+z+1=0$. Also, find the distance PQ and the image of the point P treating this plane as a mirror. [6 marks]

Solution:

Concept:

- The plane $ax+by+cz+d=0$ has normal vector $\vec{n}=a\hat{i}+b\hat{j}+c\hat{k}$. A perpendicular line through a point P is parallel to this normal.
- The foot of the perpendicular is the intersection of that normal line with the plane. The distance from $P(x_0,y_0,z_0)$ to the plane is $\dfrac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}$.
- Reflection in a plane reverses the normal displacement while keeping its component within the plane unchanged. If P' is the image and Q is the perpendicular foot, Q is the midpoint of PP', so $\vec{OP'}=2\vec{OQ}-\vec{OP}$.

Step 1: Write the line through P perpendicular to the plane The given plane is $2x-y+z+1=0$. Its normal vector is

$$\vec{n}=2\hat{i}-\hat{j}+\hat{k}.$$

Since PQ is perpendicular to the plane, it must be parallel to $\vec{n}$. The normal line through $P(3,2,1)$ is

$$\vec{r}=(3\hat{i}+2\hat{j}+\hat{k})+t(2\hat{i}-\hat{j}+\hat{k}).$$

Thus a general point on this line has coordinates $(3+2t,2-t,1+t)$. The perpendicular foot is obtained by choosing the value of t for which these coordinates satisfy the plane equation.

Step 2: Locate the perpendicular foot Q Substitute the coordinates of the normal line into the equation of the plane:

$$\begin{aligned}2(3+2t)-(2-t)+(1+t)+1&=0,\\ 6+6t&=0,\\ t&=-1.\end{aligned}$$

Hence the foot has coordinates

$$Q=(3-2,2+1,1-1)=(1,3,0).$$

To verify the intersection, substitute Q into the plane: $2(1)-3+0+1=0$. Also, $\overrightarrow{PQ}=(-2,1,-1)=-\vec{n}$, which confirms that the joining segment is normal to the plane. These two checks establish that Q is the required perpendicular foot.

Step 3: Calculate the perpendicular distance PQ Using the coordinates of P and Q,

$$PQ=\sqrt{(1-3)^2+(3-2)^2+(0-1)^2}=\sqrt{4+1+1}=\sqrt{6}.$$

Quick Tip to Solve: Find the foot using a line parallel to the plane's normal, then obtain the reflected point immediately from the midpoint relation P' = 2Q - P.

This can also be checked directly using the point-to-plane formula:

$$PQ = \frac{|2(3) - 2 + 1 + 1|}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{6}{\sqrt{6}} = \sqrt{6}.$$

The absolute value ensures a nonnegative distance regardless of which side of the plane contains the point.

Step 4: Find and check the mirror image of P Let the image be $P'(x', y', z')$. Reflection makes Q the midpoint of PP', giving

$$x' = 2(1) - 3 = -1,$$
$$y' = 2(3) - 2 = 4,$$
$$z' = 2(0) - 1 = -1.$$

Therefore $P' = (-1, 4, -1)$. The displacement $\overrightarrow{QP'} = (-2, 1, -1)$ equals $\overrightarrow{PQ}$, so the image is the same distance beyond the foot on the opposite side. Moreover, the plane expression equals 6 at P and -6 at P'. Thus the two points are at equal perpendicular distances on opposite sides, as required for reflection.

Step 5: Conclusion The foot of the perpendicular is $\boxed{Q(1, 3, 0)}$, its length is $\boxed{PQ = \sqrt{6} \text{ units}}$, and the mirror image of P is $\boxed{P'(-1, 4, -1)}$.

Quick Tip to Solve: Find the foot using a line parallel to the plane's normal, then obtain the reflected point immediately from the midpoint relation P' = 2Q - P.

28. Using integration, find the area of the region bounded by the line $y = 3x + 2$, the x-axis and the ordinates $x = -2$ and $x = 1$. [6 marks]

Solution:

Concept:

- The geometric area between a curve $y = f(x)$ and the x-axis over $[a, b]$ is $A = \int_a^b |f(x)|\, dx$. It is always non-negative.
- When a curve crosses the x-axis, split the interval at every zero. Integrate $-f(x)$ where the curve lies below the axis, and $f(x)$ where it lies above the axis.
- An antiderivative of the line $y = 3x + 2$ is $F(x) = \frac{3}{2}x^2 + 2x$; each definite integral is evaluated by subtracting its endpoint values.

Step 1: Locate where the line crosses the axis The region extends from the ordinate $x = -2$ to the ordinate $x = 1$. Set $y = 0$ to find the point at which its upper and lower boundaries exchange roles:

$$3x + 2 = 0 \quad \Longrightarrow \quad x = -\frac{2}{3}.$$

This value lies inside the prescribed interval. Since $3(-2) + 2 = -4$, the line lies below the x-axis on $[-2, -2/3]$. Since $3(1) + 2 = 5$, it lies above the axis on $[-2/3, 1]$. Thus there are two positive contributions to the total area. A single signed integral over the whole interval would subtract one

Quick Tip to Solve: Before integrating a line against the axis, solve for its zero: here splitting at $x = -2/3$ prevents cancellation of the two areas.

region from the other, so it would not give the required geometric area.

Step 2: Integrate the part below the axis On the left interval, the upper boundary is $y = 0$ and the lower boundary is $y = 3x + 2$. The height of a vertical strip is therefore $0 - (3x + 2)$. Hence

$$\begin{aligned} A_1 &= \int_{-2}^{-2/3} [-(3x+2)]\,dx \\ &= \left[-\frac{3}{2}x^2 - 2x\right]_{-2}^{-2/3} \\ &= \left(-\frac{2}{3} + \frac{4}{3}\right) - (-6+4) \\ &= \frac{2}{3} + 2 = \frac{8}{3}. \end{aligned}$$

The positive value agrees with the fact that this integral measures the strip height, rather than the negative ordinate of the line.

Step 3: Integrate the part above the axis and add On the right interval, the line is the upper boundary and the axis is the lower boundary. Thus

$$\begin{aligned} A_2 &= \int_{-2/3}^{1} (3x+2)\,dx \\ &= \left[\frac{3}{2}x^2 + 2x\right]_{-2/3}^{1} \\ &= \left(\frac{3}{2} + 2\right) - \left(\frac{2}{3} - \frac{4}{3}\right) \\ &= \frac{7}{2} + \frac{2}{3} = \frac{25}{6}. \end{aligned}$$

Adding the two disjoint regions gives

$$A = A_1 + A_2 = \frac{8}{3} + \frac{25}{6} = \frac{16+25}{6} = \frac{41}{6}.$$

Both contributions are positive, and their sum is greater than either contribution alone, as required for the entire bounded region.

Step 4: Conclusion The required area is $\boxed{\frac{41}{6} \text{ square units}}$.

Quick Tip to Solve: Before integrating a line against the axis, solve for its zero: here splitting at $x = -2/3$ prevents cancellation of the two areas.

29. There are three coins. One is a coin having tails on both faces, another is a biased coin that comes up tails 70% of the time and the third is an unbiased coin. One of the coins is chosen at random and tossed, it shows tail. Find the probability that it was a coin with tail on both the faces. [6 marks]

Solution:

Concept:

- Bayes' theorem updates the probability of an initial choice after an outcome is observed. For mutually exclusive and exhaustive events C_1, C_2, C_3, $P(C_1 \mid T) = \dfrac{P(C_1)P(T \mid C_1)}{\sum_{i=1}^{3} P(C_i)P(T \mid C_i)}$.
- The law of total probability gives $P(T) = \sum_{i=1}^{3} P(C_i)P(T \mid C_i)$, where T denotes the observed tail. It accounts for all the possible coins that could have produced that tail.
- Choosing one of three coins at random gives equal prior probabilities of $1/3$. These priors need not remain equal after a tail is seen, because the coins have different probabilities of producing tails.

Step 1: Define the events and their probabilities Let C_1 denote choosing the coin with tails on both faces, C_2 choosing the biased coin, and C_3 choosing the unbiased coin. Let T denote the event that the subsequent toss shows a tail. Exactly one coin is chosen, so C_1, C_2, C_3 are mutually exclusive and exhaustive. As the choice is random,

$$P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}.$$

The double-tailed coin always gives a tail, the biased coin gives a tail with probability $70/100$, and the fair coin gives a tail with probability $1/2$. Therefore,

$$P(T \mid C_1) = 1, \qquad P(T \mid C_2) = \frac{7}{10}, \qquad P(T \mid C_3) = \frac{1}{2}.$$

Step 2: Find the total probability of seeing a tail A tail can arise from any of the three choices. Weight each coin's tail probability by the probability of selecting that coin:

$$\begin{aligned} P(T) &= P(C_1)P(T \mid C_1) + P(C_2)P(T \mid C_2) + P(C_3)P(T \mid C_3) \\ &= \frac{1}{3} \cdot 1 + \frac{1}{3} \cdot \frac{7}{10} + \frac{1}{3} \cdot \frac{1}{2} \\ &= \frac{10}{30} + \frac{7}{30} + \frac{5}{30} = \frac{22}{30} = \frac{11}{15}. \end{aligned}$$

In particular, the denominator of the desired conditional probability must include tails from both the biased coin and the unbiased coin as well as those from the double-tailed coin.

Step 3: Apply Bayes' theorem to the observed tail We need the probability of C_1 given T, which reverses the order of conditioning from the known probability $P(T \mid C_1)$. The joint probability of selecting the double-tailed coin and observing a tail is

$$P(C_1 \cap T) = P(C_1)P(T \mid C_1) = \frac{1}{3}.$$

Quick Tip to Solve: With equal selection probabilities, normalize the three tail likelihoods directly: the desired probability is $1/(1 + 0.7 + 0.5) = 5/11$.

Therefore, Bayes' theorem gives

$$P(C_1 \mid T) = \frac{P(C_1)P(T \mid C_1)}{P(T)} = \frac{1/3}{11/15} = \frac{1}{3} \cdot \frac{15}{11} = \frac{5}{11}.$$

This value exceeds the original probability $1/3$ because the double-tailed coin is more likely to produce the observed outcome than either of the other two coins. It remains below 1, since the other coins can also produce tails.

Step 4: Conclusion Given that a tail was observed, the probability that the selected coin has tails on both faces is $\boxed{\frac{5}{11}}$.

Quick Tip to Solve: With equal selection probabilities, normalize the three tail likelihoods directly: the desired probability is $1/(1 + 0.7 + 0.5) = 5/11$.