CBSE Class 12 Mathematics Question Paper 2019 with Solution PDF-Set 65-4-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 100**

General Instructions

(i) All questions are compulsory.

(ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1(a). Find the acute angle between the planes $\vec{r}\cdot(\hat{i}-2\hat{j}-2\hat{k})=1$ and $\vec{r}\cdot(3\hat{i}-6\hat{j}+2\hat{k})=0$. [1 mark]

Solution:

Concept:

- The acute angle between two planes is the acute angle between their normal vectors.
- For normals $\vec{n}_1, \vec{n}_2$, $\cos\theta = |\vec{n}_1 \cdot \vec{n}_2|/(|\vec{n}_1||\vec{n}_2|)$.

Step 1: Identify the normals The coefficients of $\vec{r}$ give $\vec{n}_1 = (1,-2,-2)$ and $\vec{n}_2 = (3,-6,2)$. The constants on the right affect the positions of the planes, but not the angle between them.

Step 2: Evaluate the cosine

$$\vec{n}_1 \cdot \vec{n}_2 = 3 + 12 - 4 = 11, \qquad |\vec{n}_1| = 3, \qquad |\vec{n}_2| = 7.$$

Thus $\cos\theta = 11/21$; the absolute value selects the acute angle.

Step 3: Conclusion The required acute angle is $\boxed{\theta = \cos^{-1}(11/21)}$.

Quick Tip to Solve: Use the plane coefficients as normals; ignore the right-hand constants when calculating the angle.

OR

1(b). Find the length of the intercept, cut off by the plane $2x+y-z=5$ on the x-axis. [1 mark]

Solution:

Concept:

- Every point on the x-axis has coordinates $(x,0,0)$.
- The length of an intercept on this axis is the distance of the intersection point from the origin, namely $|x|$.

Step 1: Find the point on the axis Substitute $y=0$ and $z=0$ into the plane equation $2x+y-z=5$. This gives $2x=5$, so the plane meets the x-axis at $P=(5/2,0,0)$.

Step 2: Calculate the intercept length The intercept is the segment from $O=(0,0,0)$ to P. Its length is $OP=\sqrt{(5/2)^2}=|5/2|$. Since the coordinate here is positive, the distance equals the coordinate itself.

Step 3: Conclusion The length of the x-axis intercept is $\boxed{5/2 \text{ units}}$.

Quick Tip to Solve: For an x-axis intercept, first set both y and z to zero, then take the absolute value of x.

2. If $y = \log(\cos e^x)$, then find $\frac{dy}{dx}$. [1 mark]

Solution:

Concept:

- The chain rule differentiates nested functions from the outside inward.
- Use $d(\log u)/dx = u'/u$, $d(\cos v)/dx = -\sin v\, v'$, and $d(e^x)/dx = e^x$.

Step 1: Differentiate the logarithm Here the logarithm is the natural logarithm used in calculus. With $u = \cos(e^x)$, the derivative is

$$\frac{dy}{dx} = \frac{1}{\cos(e^x)} \frac{d}{dx} \cos(e^x).$$

The expression is considered where $\cos(e^x) > 0$, its real logarithmic domain.

Step 2: Apply the inner chain rule

$$\frac{d}{dx} \cos(e^x) = -\sin(e^x)e^x.$$

Hence the ratio of sine to cosine becomes $\tan(e^x)$, with the negative sign retained.

Step 3: Conclusion $\boxed{\frac{dy}{dx} = -e^x \tan(e^x)}$.

Quick Tip to Solve: Count all three layers: logarithm, cosine, and exponential; retain the factor e to the power x.

3. A is a square matrix with $|A| = 4$. Then find the value of $|A \cdot (\text{adj } A)|$. [1 mark]

Solution:

Concept:

- For a square matrix A of order n, the adjoint identity is $A \,\text{adj}\, A = |A| I_n$.
- The determinant of cI_n is c^n, because all n diagonal entries equal c.

Step 1: Use the adjoint identity The printed question gives $|A| = 4$ but does not specify the order of A. Denote its order by n. Then $A \,\text{adj}\, A = 4I_n$; since the determinant is nonzero, the adjoint identity applies without any singularity concern.

Step 2: Take the determinant

$$|A \,\text{adj}\, A| = |4I_n| = 4^n.$$

Each of the n diagonal factors contributes a factor of four. Therefore a particular numerical value requires the order of the matrix.

Step 3: Conclusion $\boxed{|A \,\text{adj}\, A| = 4^n}$, where n is the order of A. The order is not specified in the question.

Quick Tip to Solve: Keep the matrix order explicit: multiplying an identity matrix by 4 makes its determinant 4 to that order.

4. Form the differential equation representing the family of curves $y = A\sin x$, by eliminating the arbitrary constant A. [1 mark]

Solution:

Concept:

- A family with one arbitrary constant normally yields a first-order differential equation.
- Differentiate once and eliminate the constant between the original equation and its derivative.

Step 1: Differentiate the given family From $y = A\sin x$, where A is constant with respect to x, differentiation gives $dy/dx = A\cos x$. Both equations involve exactly the same arbitrary constant.

Step 2: Eliminate the constant without division Multiply the derivative equation by $\sin x$ and the original equation by $\cos x$. Both right sides become $A\sin x\cos x$, giving

$$\sin x\frac{dy}{dx} = y\cos x.$$

This form avoids division by $\sin x$, which could vanish.

Step 3: Conclusion The required differential equation is $\boxed{\sin x\frac{dy}{dx} - y\cos x = 0}$.

Quick Tip to Solve: Cross-multiply the original and differentiated equations to eliminate A without dividing by a trigonometric factor.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5(a). Find: $\int x\cdot\tan^{-1}x\,dx$. [2 marks]

Solution:

Concept:

- Integration by parts states $\int u\,dv = uv - \int v\,du$.
- Choose the inverse trigonometric factor as u, since $d(\tan^{-1}x)/dx = 1/(1+x^2)$.

Step 1: Apply integration by parts Let $u = \tan^{-1}x$ and $dv = x\,dx$. Then $du = dx/(1+x^2)$ and $v = x^2/2$. Therefore

$$I = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2}{1+x^2}\,dx.$$

The remaining integral is rational and can be simplified directly.

Step 2: Reduce and integrate the rational expression Use $x^2/(1+x^2) = 1 - 1/(1+x^2)$. Thus

$$\begin{aligned} I &= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x) + C \\ &= \frac{x^2+1}{2}\tan^{-1}x - \frac{x}{2} + C. \end{aligned}$$

Quick Tip to Solve: After integration by parts, rewrite x squared over one plus x squared as one minus its reciprocal.

As a check, differentiating this expression gives $x\tan^{-1}x + 1/2 - 1/2$, exactly the original integrand; C is an arbitrary real constant.

Step 3: Conclusion $\boxed{\int x\tan^{-1}x\,dx = \frac{x^2+1}{2}\tan^{-1}x - \frac{x}{2} + C}$.

Quick Tip to Solve: After integration by parts, rewrite x squared over one plus x squared as one minus its reciprocal.

OR

5(b). Find: $\int \frac{dx}{\sqrt{5-4x-2x^2}}$. [2 marks]

Solution:

Concept:

- Complete the square to convert the quadratic radicand into $a^2 - u^2$.
- The standard formula is $\int du/\sqrt{a^2-u^2} = \sin^{-1}(u/a) + C$, with $a > 0$ and $|u| < a$.

Step 1: Complete the square

$$5 - 4x - 2x^2 = 7 - 2(x+1)^2.$$

Put $u = \sqrt{2}(x+1)$, so $du = \sqrt{2}\,dx$. The translation centres the quadratic at $x = -1$, and the scaling makes the coefficient of u^2 equal to one.

Step 2: Use the inverse-sine integral

$$\begin{aligned} I &= \frac{1}{\sqrt{2}}\int \frac{du}{\sqrt{7-u^2}} \\ &= \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{u}{\sqrt{7}}\right) + C. \end{aligned}$$

The real integrand requires $7 - 2(x+1)^2 > 0$, or $|x+1| < \sqrt{7/2}$. Differentiation on this interval cancels the factor $\sqrt{2}$ from the substitution and recovers $1/\sqrt{7-2(x+1)^2}$.

Step 3: Conclusion $\boxed{I = \frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{\frac{2}{7}}(x+1)\right) + C}$.

Quick Tip to Solve: Complete the square before choosing a substitution; keep the one over square root two factor from dx.

6. Solve the following differential equation: $\dfrac{dy}{dx} + y = \cos x - \sin x$. [2 marks]

Solution:

Concept:

- A linear equation $y' + P(x)y = Q(x)$ has integrating factor $e^{\int P(x)dx}$.
- Multiplying by it converts the left side into the derivative of the product of the integrating factor and y.

Step 1: Calculate and apply the integrating factor Here $P(x) = 1$ and $Q(x) = \cos x - \sin x$. Hence the integrating factor is e^x. Multiplying throughout gives

$$e^x \frac{dy}{dx} + e^x y = e^x(\cos x - \sin x),$$

so the left side is $d(e^x y)/dx$ by the product rule.

Step 2: Integrate and isolate the dependent variable Since $d(e^x \cos x)/dx = e^x(\cos x - \sin x)$, integration gives

$$e^x y = e^x \cos x + C.$$

Divide by e^x, which is never zero for real x, to obtain the general solution. To verify it, $y' = -\sin x - Ce^{-x}$; adding $y = \cos x + Ce^{-x}$ cancels the terms containing C and yields the prescribed right side.

Step 3: Conclusion The general solution is $\boxed{y = \cos x + Ce^{-x}}$, where C is an arbitrary constant.

Quick Tip to Solve: Recognise e to the power x times (cos x minus sin x) as the derivative of e to the power x times cos x.

7. Find: $\displaystyle\int_{-\pi/4}^{0} \frac{1 + \tan x}{1 - \tan x}\, dx$. [2 marks]

Solution:

Concept:

- The tangent addition formula gives $\tan(x + \pi/4) = (1 + \tan x)/(1 - \tan x)$.
- For a definite integral, a substitution changes both the integrand and the limits. Also, $\int \tan u\, du = -\log|\cos u| + C$.

Step 1: Simplify and change the limits Write the given integrand as $\tan(x + \pi/4)$. Put $u = x + \pi/4$, so $du = dx$. At $x = -\pi/4$, $u = 0$; at $x = 0$, $u = \pi/4$. Thus the required integral is

$$I = \int_0^{\pi/4} \tan u\, du.$$

The transformed interval lies where $\cos u > 0$, so the logarithm below is well defined.

Quick Tip to Solve: Recognise the tangent addition formula before attempting a rational substitution in $\tan x$.

Step 2: Evaluate the definite integral Using the antiderivative of tangent,

$$\begin{aligned} I &= [-\log(\cos u)]_0^{\pi/4} \\ &= -\log(1/\sqrt{2}) + \log 1 \\ &= \log\sqrt{2} = \tfrac{1}{2}\log 2. \end{aligned}$$

The value is positive, consistent with the nonnegative integrand on this interval.

Step 3: Conclusion The required integral is $\boxed{\frac{1}{2}\log 2}$.

Quick Tip to Solve: Recognise the tangent addition formula before attempting a rational substitution in $\tan x$.

8. Let $*$ be an operation defined as $* : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that $a * b = 2a + b,\ a, b \in \mathbb{R}$. Check if $*$ is a binary operation. If yes, find if it is associative too. [2 marks]

Solution:

Concept:

- A binary operation on $\mathbb{R}$ assigns a unique real output to every ordered pair of real numbers.
- Associativity requires $(a * b) * c = a * (b * c)$ for every $a, b, c \in \mathbb{R}$.

Step 1: Check closure and uniqueness For any real numbers a, b, the expression $2a + b$ is a real number and has exactly one value. Hence the stated rule defines a function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. This verifies that $*$ is a binary operation on $\mathbb{R}$.

Step 2: Compare the two bracketings Apply the same rule to the outer operation in each expression:

$$\begin{aligned} (a * b) * c &= (2a + b) * c = 2(2a + b) + c = 4a + 2b + c, \\ a * (b * c) &= a * (2b + c) = 2a + 2b + c. \end{aligned}$$

Their difference is $2a$, which does not vanish for every real a. For example, with $a = 1, b = 0, c = 0$, the two expressions are 4 and 2, respectively. A single such counterexample disproves associativity.

Step 3: Conclusion The rule defines a binary operation on $\mathbb{R}$, but it is $\boxed{\text{not associative}}$.

Quick Tip to Solve: When nesting this operation, double the entire first input, including its brackets.

9(a). X and Y are two points with position vectors $3\vec{a} + \vec{b}$ and $\vec{a} - 3\vec{b}$ respectively. Write the position vector of a point Z which divides the line segment XY in the ratio $2 : 1$ externally. [2 marks]

Solution:

Concept:

- If points have position vectors $\vec{x}, \vec{y}$, their external division in the ratio $m : n$ gives $\vec{z} = (m\vec{y} - n\vec{x})/(m - n)$, where $m \neq n$.
- The ratio $XZ : ZY = 2 : 1$ means that Z lies beyond Y on the line through X, Y.

Quick Tip to Solve: For external division use subtraction in both numerator and denominator; the weight of each endpoint comes from the opposite part of the ratio.

Step 1: Apply the external section formula Here $\vec{x} = 3\vec{a} + \vec{b}$, $\vec{y} = \vec{a} - 3\vec{b}$, $m = 2$ and $n = 1$. External division uses a difference in the denominator, so

$$\vec{z} = \frac{2(\vec{a} - 3\vec{b}) - (3\vec{a} + \vec{b})}{2 - 1}.$$

The weight 2 multiplies the position vector of Y, opposite to the first endpoint named in the ratio.

Step 2: Simplify and verify the ratio Collecting the vector coefficients gives

$$\vec{z} = 2\vec{a} - 6\vec{b} - 3\vec{a} - \vec{b} = -\vec{a} - 7\vec{b}.$$

As a check, $\vec{z} - \vec{x} = -4\vec{a} - 8\vec{b}$ and $\vec{z} - \vec{y} = -2\vec{a} - 4\vec{b}$. Thus $\vec{z} - \vec{x} = 2(\vec{z} - \vec{y})$, which verifies the external ratio $XZ : ZY = 2 : 1$.

Step 3: Conclusion The position vector of Z is $\boxed{-\vec{a} - 7\vec{b}}$.

Quick Tip to Solve: For external division use subtraction in both numerator and denominator; the weight of each endpoint comes from the opposite part of the ratio.

OR

9(b). Let $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$ be two vectors. Show that the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular to each other. [2 marks]

Solution:

Concept:

- Two nonzero vectors are perpendicular if and only if their scalar product is zero.
- In Cartesian components, $(x_1, y_1, z_1) \cdot (x_2, y_2, z_2) = x_1x_2 + y_1y_2 + z_1z_2$.

Step 1: Compute the sum and difference Add and subtract corresponding components of the given vectors:

$$\vec{a} + \vec{b} = 4\hat{i} + \hat{j} - \hat{k},$$
$$\vec{a} - \vec{b} = -2\hat{i} + 3\hat{j} - 5\hat{k}.$$

Both vectors are nonzero, so the zero scalar-product criterion can be used to establish an angle of $90°$.

Step 2: Evaluate their scalar product The unit vectors along distinct coordinate axes have zero scalar product, while each unit vector has scalar product one with itself. Therefore

$$(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 4(-2) + (1)(3) + (-1)(-5)$$
$$= -8 + 3 + 5 = 0.$$

Equivalently, $|\vec{a}|^2 = 1 + 4 + 9 = 14$ and $|\vec{b}|^2 = 9 + 1 + 4 = 14$, so the identity $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$ gives the same result.

Step 3: Conclusion Hence $\boxed{(\vec{a} + \vec{b}) \perp (\vec{a} - \vec{b})}$, as required.

Quick Tip to Solve: Equal magnitudes of the original vectors immediately make their sum and difference perpendicular.

10(a). Out of 8 outstanding students of a school, in which there are 3 boys and 5 girls, a team of 4 students is to be selected for a quiz competition. Find the probability that 2 boys and 2 girls are selected. [2 marks]

Solution:

Concept:

- For equally likely outcomes, $P(E) = n(E)/n(S)$, where E is the desired event and S the sample space.
- The number of unordered selections of r objects from n distinct objects is $\binom{n}{r} = n!/[r!(n-r)!]$.

Step 1: Count all possible teams A team is an unordered selection of four different students from the eight available students. Each such team is equally likely under random selection. Hence

$$n(S) = \binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70.$$

No ordering factor is required, since changing the order in which team members are listed does not produce a new team.

Step 2: Count the teams with the required composition There are $\binom{3}{2} = 3$ choices of two boys and $\binom{5}{2} = 10$ choices of two girls. Every boy-pair can be combined with every girl-pair, giving

$$n(E) = \binom{3}{2}\binom{5}{2} = 3 \cdot 10 = 30.$$

Therefore

$$P(E) = \frac{30}{70} = \frac{3}{7}.$$

Each favourable team is counted exactly once because its boy-pair and girl-pair are uniquely determined.

Step 3: Conclusion The probability of selecting two boys and two girls is $\boxed{\frac{3}{7}}$.

Quick Tip to Solve: Use combinations separately for boys and girls, then divide their product by the total number of four-student teams.

OR

10(b). In a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing? [2 marks]

Solution:

Concept:

- Each independent random guess succeeds with probability $p = 1/3$ and fails with probability $q = 2/3$.

Quick Tip to Solve: For four or more successes out of five trials, add the exactly-four and exactly-five binomial terms.

- The binomial formula for five trials is $P(X = r) = \binom{5}{r} p^r q^{5-r}$.

Step 1: Identify the required binomial events Let X denote the number of correct answers. Assuming one correct option among the three possible answers for each question, independent guessing gives $X \sim B(5, 1/3)$. Four or more correct answers means exactly four correct or exactly five correct. These events are mutually exclusive, so

$$P(X \geq 4) = P(X = 4) + P(X = 5).$$

Step 2: Calculate and add the two probabilities For exactly four correct answers, choose the one incorrectly answered question in $\binom{5}{4} = 5$ ways. For all five correct, there is only one pattern. Thus

$$\begin{aligned} P(X \geq 4) &= \binom{5}{4} \left(\frac{1}{3}\right)^4 \frac{2}{3} + \binom{5}{5} \left(\frac{1}{3}\right)^5 \\ &= \frac{10}{243} + \frac{1}{243} = \frac{11}{243}. \end{aligned}$$

The factor $2/3$ belongs only to the exactly-four case, because that case contains one incorrect guess.

Step 3: Conclusion The required probability is $\boxed{\frac{11}{243}}$.

Quick Tip to Solve: For four or more successes out of five trials, add the exactly-four and exactly-five binomial terms.

11. The probabilities of solving a specific problem independently by A and B are $\frac{1}{3}$ and $\frac{1}{5}$ respectively. If both try to solve the problem independently, find the probability that the problem is solved.

[2 marks]

Solution:

Concept:

- For independent events E and F, $P(E \cap F) = P(E)P(F)$. Here E means that A solves the problem and F means that B solves it.
- The problem is solved when at least one person succeeds. Its complement is that both fail, so $P(E \cup F) = 1 - P(E^c \cap F^c)$.

Step 1: Find each probability of failure The given probabilities of success are $P(E) = \frac{1}{3}$ and $P(F) = \frac{1}{5}$. Therefore,

$$P(E^c) = 1 - \frac{1}{3} = \frac{2}{3}, \qquad P(F^c) = 1 - \frac{1}{5} = \frac{4}{5}.$$

Since the attempts are independent, the failure events are also independent. Hence the probability that neither person solves the problem is

$$P(E^c \cap F^c) = \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}.$$

Quick Tip to Solve: For independent attempts, calculate the chance that everyone fails and subtract it from 1; this automatically includes joint success.

Step 2: Use the complement of total failure The event that the problem is solved includes A alone solving it, B alone solving it, and both solving it. Thus it is the complement of neither solving it:

$$P(E \cup F) = 1 - \frac{8}{15} = \frac{7}{15}.$$

As a check, the addition rule gives $\frac{1}{3} + \frac{1}{5} - \frac{1}{3} \cdot \frac{1}{5} = \frac{7}{15}$; subtracting the intersection avoids counting joint success twice.

Step 3: Conclusion The probability that the problem is solved is $\boxed{\frac{7}{15}}$.

Quick Tip to Solve: For independent attempts, calculate the chance that everyone fails and subtract it from 1; this automatically includes joint success.

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12. For the matrix $A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$, find $(A + A')$ and verify that it is a symmetric matrix. [2 marks]

Solution:

Concept:

- The transpose A' is obtained by interchanging the rows and columns of A. Matrix addition adds entries in the same position.
- A square matrix S is symmetric if $S' = S$, equivalently if $s_{ij} = s_{ji}$ for every pair of indices.

Step 1: Form the transpose and add For the given matrix,

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}, \qquad A' = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}.$$

The diagonal entries remain fixed under transposition, while the entries 3 and 5 exchange positions. Adding corresponding entries gives

$$S = A + A' = \begin{bmatrix} 2+2 & 3+5 \\ 5+3 & 7+7 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 14 \end{bmatrix}.$$

Step 2: Verify symmetry by transposing the sum Taking the transpose of the matrix just obtained,

$$S' = \begin{bmatrix} 4 & 8 \\ 8 & 14 \end{bmatrix} = S.$$

In particular, the two off-diagonal entries are equal, both being 8, and the diagonal entries are unchanged. This proves the required symmetry directly. It also agrees with the identity $(A + A')' = A' + A$, since transposing twice recovers the original matrix.

Step 3: Conclusion $\boxed{A + A' = \begin{bmatrix} 4 & 8 \\ 8 & 14 \end{bmatrix}}$, and this matrix is symmetric because its transpose equals

Quick Tip to Solve: When forming a transpose, swap only the off-diagonal entries here; their equal sums immediately verify symmetry.

itself.

Quick Tip to Solve: When forming a transpose, swap only the off-diagonal entries here; their equal sums immediately verify symmetry.

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13. A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall? [4 marks]

Solution:

Concept:

- Let x be the horizontal distance of the foot from the wall and y the vertical height of the top above the ground, both in metres. Both depend on time t, measured in seconds.
- The wall and ground are perpendicular. The fixed ladder length gives the Pythagorean constraint $x^2 + y^2 = 13^2$. Differentiating this relation connects the instantaneous rates.

Step 1: Express the given motion in consistent units The foot moves away from the wall, so x increases. Converting its positive rate to metres per second gives

$$\frac{dx}{dt} = 2\,\text{cm/s} = \frac{1}{50}\,\text{m/s}.$$

Use metres for both distances, since the rigid ladder has length 13 m.

Step 2: Find the height at the specified instant The right triangle satisfies $x^2 + y^2 = 169$. At $x = 5\,\text{m}$,

$$y = \sqrt{169 - 25} = 12\,\text{m}.$$

The positive root is required because y is a physical height.

Step 3: Differentiate the geometric relation Differentiate with respect to time, treating both distances as functions of t. The fixed ladder length has zero derivative:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0, \qquad \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

The negative sign shows that the top moves down when the foot moves away: the vertical side must shorten as the horizontal side grows.

Step 4: Substitute the instantaneous distances and rate Using $x = 5$, $y = 12$, and $dx/dt = 1/50$ in metres per second,

$$\frac{dy}{dt} = -\frac{5}{12} \times \frac{1}{50} = -\frac{1}{120}\,\text{m/s} = -\frac{5}{6}\,\text{cm/s}.$$

This is the signed rate of height change. The requested rate of decrease is its positive magnitude, since the word decreasing already specifies the direction.

Quick Tip to Solve: Use the 5-12-13 right triangle at the specified instant, then apply $dy/dt = -(x/y)dx/dt$ with consistent distance units.

Step 5: Conclusion The height on the wall is decreasing at $\boxed{\frac{5}{6}\text{ cm/s}}$ (equivalently, $\frac{1}{120}$ m/s).

Quick Tip to Solve: Use the 5-12-13 right triangle at the specified instant, then apply $dy/dt = -(x/y)dx/dt$ with consistent distance units.

14. Prove that: $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$. [4 marks]

Solution:

Concept:

- Principal values of $\cos^{-1} t$ belong to $[0, \pi]$; principal values of $\sin^{-1} t$ belong to $[-\pi/2, \pi/2]$.
- Use $\sin^2\theta + \cos^2\theta = 1$ and the sine and cosine addition formulae. The angle's quadrant determines the square-root sign.
- Equality of sines implies equality of angles only after the relevant principal range has been checked.

Step 1: Find the remaining trigonometric ratios Let $\alpha = \cos^{-1}(12/13)$ and $\beta = \sin^{-1}(3/5)$. Since both arguments lie strictly between zero and one, $0 < \alpha < \pi/2$ and $0 < \beta < \pi/2$. Thus both angles are acute, giving

$$\sin\alpha = \sqrt{1 - \frac{144}{169}} = \frac{5}{13}, \qquad \cos\beta = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}.$$

The positive square roots are essential because the principal angles are in the first quadrant.

Step 2: Evaluate the sine of the sum and establish its quadrant Using the addition formula gives

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \frac{5}{13}\frac{4}{5} + \frac{12}{13}\frac{3}{5} = \frac{20+36}{65} = \frac{56}{65}.\end{aligned}$$

To apply sine inverse, we must exclude a supplementary-angle possibility. The acute-angle ranges show that $0 < \alpha + \beta < \pi$. Also,

$$\cos(\alpha+\beta) = \frac{12}{13}\frac{4}{5} - \frac{5}{13}\frac{3}{5} = \frac{48-15}{65} = \frac{33}{65} > 0.$$

In $(0, \pi)$, positive cosine places the angle in $(0, \pi/2)$. Hence the sum belongs to the principal range of sine inverse, and $\sin^{-1}(\sin(\alpha+\beta)) = \alpha + \beta$.

Step 3: Conclusion Taking the principal sine inverse therefore proves

$$\boxed{\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)}.$$

Quick Tip to Solve: Compute the cosine of the angle sum as well as its sine to exclude the supplementary angle before taking sine inverse.

15. Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, and hence evaluate $\int_0^1 x^2(1-x)^n\,dx$. [4 marks]

Solution:

Concept:

- In a definite integral, substitute the variable, differential and limits together. Reversing the order of limits reverses the sign.
- The reflection substitution $t = a - x$ proves $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, whenever the integral exists.
- For $r > -1$, $\int_0^1 t^r\,dt = 1/(r+1)$. Here the integral converges for real $n > -1$, including the usual nonnegative integral values of n.

Step 1: Prove the reflection property In $I = \int_0^a f(x)\,dx$, put $t = a - x$. Then $dx = -dt$. The old lower limit $x = 0$ becomes $t = a$, while the old upper limit $x = a$ becomes $t = 0$. Therefore

$$I = -\int_a^0 f(a-t)\,dt = \int_0^a f(a-t)\,dt.$$

The integration variable is a dummy variable, so it may be renamed x. This proves $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.

Step 2: Apply the property and expand the polynomial factor Let $J = \int_0^1 x^2(1-x)^n\,dx$. Apply the proved property with $a = 1$ and $f(x) = x^2(1-x)^n$. Substituting $1 - x$ in place of x gives

$$\begin{aligned} J &= \int_0^1 (1-x)^2 x^n\,dx \\ &= \int_0^1 \left(x^n - 2x^{n+1} + x^{n+2}\right)\,dx. \end{aligned}$$

For $n > -1$, the lower-end contributions are zero.

Step 3: Integrate and simplify to a single fraction Termwise integration gives

$$\begin{aligned} J &= \left[\frac{x^{n+1}}{n+1} - \frac{2x^{n+2}}{n+2} + \frac{x^{n+3}}{n+3}\right]_0^1 \\ &= \frac{1}{n+1} - \frac{2}{n+2} + \frac{1}{n+3} \\ &= \frac{(n+2)(n+3) - 2(n+1)(n+3) + (n+1)(n+2)}{(n+1)(n+2)(n+3)}. \end{aligned}$$

The numerator reduces to $(n^2+5n+6) - (2n^2+8n+6) + (n^2+3n+2) = 2$.

Step 4: Conclusion Thus the reflection property is proved, and

$$\boxed{\int_0^1 x^2(1-x)^n\,dx = \frac{2}{(n+1)(n+2)(n+3)}}$$

for $n > -1$, in particular for every nonnegative integer n.

Quick Tip to Solve: Reflect the integral first so that the fixed quadratic factor becomes $(1-x)^2$; expanding three terms avoids repeated integration by parts.

16(a). **If** $x = \sin t,\ y = \sin pt$**, prove that** $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + p^2y = 0$**.** [4 marks]

Solution:

Concept:

- For parametric functions $x = x(t)$, $y = y(t)$, the chain rule gives $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt}$, wherever $dx/dt \neq 0$.
- The second derivative is $\dfrac{d^2y}{dx^2} = \dfrac{1}{dx/dt}\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$. Here p is a constant and $1 - \sin^2 t = \cos^2 t$.

Step 1: Calculate the first derivative Given $x = \sin t$ and $y = \sin(pt)$, differentiating both with respect to the parameter t gives

$$\frac{dx}{dt} = \cos t, \qquad \frac{dy}{dt} = p\cos(pt).$$

Thus, on any interval where $\cos t \neq 0$, the parametric derivative is

$$\frac{dy}{dx} = \frac{p\cos(pt)}{\cos t}.$$

The factor p appears from differentiating the argument pt; it must be retained in the next differentiation as well.

Step 2: Calculate the second derivative using the parameter Use the quotient rule with respect to t, and then divide by $dx/dt = \cos t$:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{1}{\cos t}\frac{d}{dt}\left(\frac{p\cos(pt)}{\cos t}\right)\\ &= \frac{1}{\cos t}\frac{-p^2\sin(pt)\cos t + p\cos(pt)\sin t}{\cos^2 t}\\ &= \frac{-p^2\sin(pt)\cos t + p\cos(pt)\sin t}{\cos^3 t}.\end{aligned}$$

The positive sign of the second numerator term follows because the derivative of the denominator $\cos t$ is $-\sin t$. Dividing once more by $\cos t$ is essential: differentiating dy/dx with respect to t alone does not give d^2y/dx^2.

Step 3: Substitute into the proposed differential expression Since $1 - x^2 = \cos^2 t$, substitute the two derivatives into the left-hand side:

$$\begin{aligned}&(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y\\ &= \frac{-p^2\sin(pt)\cos t + p\cos(pt)\sin t}{\cos t} - \frac{p\sin t\cos(pt)}{\cos t} + p^2\sin(pt)\\ &= -p^2\sin(pt) + \frac{p\cos(pt)\sin t}{\cos t} - \frac{p\cos(pt)\sin t}{\cos t} + p^2\sin(pt)\\ &= 0.\end{aligned}$$

Both pairs of terms cancel exactly. The proof applies wherever the parametrisation defines the derivatives with respect to x.

Quick Tip to Solve: For the second parametric derivative, differentiate dy/dx with respect to t, then divide by dx/dt again; use $1 - x^2 = \cos^2 t$ before simplifying.

Step 4: Conclusion Hence $\boxed{(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0}$, as required.

Quick Tip to Solve: For the second parametric derivative, differentiate dy/dx with respect to t, then divide by dx/dt again; use $1 - x^2 = \cos^2 t$ before simplifying.

OR

16(b). Differentiate $\tan^{-1}\left[\dfrac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right]$ **with respect to** $\cos^{-1} x^2$**.** [4 marks]

Solution:

Concept:

- To differentiate a function $Y(x)$ with respect to $U(x)$, use $\dfrac{dY}{dU} = \dfrac{dY/dx}{dU/dx}$ wherever $dU/dx \neq 0$. An exact relation between Y and U is even simpler.
- For $\theta \in [0, \pi/2]$, the positive square roots satisfy $\sqrt{1+\cos\theta} = \sqrt{2}\cos(\theta/2)$ and $\sqrt{1-\cos\theta} = \sqrt{2}\sin(\theta/2)$.
- The tangent subtraction formula is $\tan(\pi/4 - v) = \dfrac{1-\tan v}{1+\tan v}$. The principal value of $\tan^{-1}(\tan z)$ equals z for $-\pi/2 < z < \pi/2$.

Step 1: Introduce the variable with respect to which differentiation is required Put

$$U = \cos^{-1}(x^2), \qquad Y = \tan^{-1}\left(\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right).$$

For the real expression, $-1 \le x \le 1$, so $0 \le x^2 \le 1$ and $0 \le U \le \pi/2$. Therefore $x^2 = \cos U$, and the half-angle formulae have nonnegative sine and cosine. In particular, the denominator of the given fraction is positive throughout the real domain.

Step 2: Simplify the inverse tangent with half-angle identities Substituting $x^2 = \cos U$ gives

$$\begin{aligned}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} &= \frac{\cos(U/2)-\sin(U/2)}{\cos(U/2)+\sin(U/2)} \\ &= \frac{1-\tan(U/2)}{1+\tan(U/2)} \\ &= \tan\left(\frac{\pi}{4}-\frac{U}{2}\right).\end{aligned}$$

Since $0 \le U \le \pi/2$, the angle $\pi/4 - U/2$ lies in $[0, \pi/4]$. It is inside the principal range of inverse tangent, so no additional multiple of π is needed. Consequently,

$$Y = \frac{\pi}{4} - \frac{U}{2}.$$

Checking the angle range is necessary before cancelling tangent with inverse tangent.

Quick Tip to Solve: Set the given inverse cosine equal to U; the two square roots become half-angle sine and cosine, revealing a linear relation before differentiation.

Step 3: Differentiate the resulting linear relation Now regard Y directly as a function of U. The term $\pi/4$ is constant and the coefficient of U is $-1/2$, so

$$\frac{dY}{dU} = \frac{d}{dU}\left(\frac{\pi}{4} - \frac{U}{2}\right) = -\frac{1}{2}.$$

For $0 < |x| < 1$, this also agrees with the ordinary derivative quotient: differentiating the identity gives $dY/dx = -(1/2)dU/dx$. At boundary values of U, the same linear relation gives the corresponding one-sided slope. Using the identity avoids an unnecessary quotient of zero derivatives at $x = 0$.

Step 4: Conclusion The required derivative with respect to $\cos^{-1}(x^2)$ is $\boxed{-\frac{1}{2}}$.

Quick Tip to Solve: Set the given inverse cosine equal to U; the two square roots become half-angle sine and cosine, revealing a linear relation before differentiation.

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17. Integrate the function $\dfrac{\cos(x+a)}{\sin(x+b)}$ with respect to x. [4 marks]

Solution:

Concept:

- The identity $\cos(u+v) = \cos u \cos v - \sin u \sin v$ expresses the numerator in terms of the denominator's angle.
- Use $\int \cot u \, du = \log|\sin u| + C$. Here a, b are constants and C is an arbitrary integration constant.

Step 1: Rewrite the numerator Since $x + a = (x + b) + (a - b)$, the compound-angle identity gives

$$\cos(x+a) = \cos(x+b)\cos(a-b) - \sin(x+b)\sin(a-b).$$

On any interval where the printed integrand is defined, $\sin(x+b) \neq 0$. Dividing the identity by this sine gives

$$\frac{\cos(x+a)}{\sin(x+b)} = \cos(a-b)\cot(x+b) - \sin(a-b).$$

Both coefficients are constants with respect to x, so the integral now separates into two standard integrals.

Step 2: Apply substitution and integrate By linearity, the required integral is

$$I = \cos(a-b)\int \frac{\cos(x+b)}{\sin(x+b)}\,dx - \sin(a-b)\int dx.$$

Put $t = \sin(x+b)$, so $dt = \cos(x+b)\,dx$. The first integral is therefore $\int dt/t = \log|t|$. Substituting back yields

$$I = \cos(a-b)\log|\sin(x+b)| - x\sin(a-b) + C.$$

The absolute value is necessary because the sine can be negative as well as positive. The formula is valid on each interval that contains no zero of the denominator.

Quick Tip to Solve: Expand the numerator about $x + b$ to turn the quotient into a cotangent term plus a constant before integrating.

Step 3: Verify the antiderivative Differentiating the expression obtained yields

$$\frac{dI}{dx} = \cos(a-b)\frac{\cos(x+b)}{\sin(x+b)} - \sin(a-b).$$

Putting the terms over their common denominator, the numerator becomes $\cos(a-b)\cos(x+b) - \sin(a-b)\sin(x+b) = \cos(x+a)$. Thus differentiation recovers exactly the printed integrand, including its sign. The arbitrary constant may differ between the disconnected intervals of the domain.

Step 4: Conclusion

$$\boxed{\int \frac{\cos(x+a)}{\sin(x+b)}\,dx = \cos(a-b)\log|\sin(x+b)| - x\sin(a-b) + C.}$$

Quick Tip to Solve: Expand the numerator about $x+b$ to turn the quotient into a cotangent term plus a constant before integrating.

18(a). Let $A = \mathbb{R} - \{2\}$ and $B = \mathbb{R} - \{1\}$. If $f : A \to B$ is a function defined by $f(x) = \dfrac{x-1}{x-2}$, show that f is one-one and onto. Hence, find f^{-1}. [4 marks]

Solution:

Concept:

- A function is one-one if equal images imply equal inputs, and onto if every element of its codomain has a preimage in its domain.
- A bijection has an inverse function. To find it, solve $y = f(x)$ for x, taking care of the excluded values in both sets.

Step 1: Check the specified domain and codomain For $x \in A = \mathbb{R} \setminus \{2\}$, the denominator $x-2$ is nonzero, so the formula defines a real value. Further,

$$f(x) - 1 = \frac{x-1}{x-2} - 1 = \frac{1}{x-2} \neq 0.$$

Hence $f(x) \neq 1$, so every image lies in $B = \mathbb{R} \setminus \{1\}$. This verifies that the given formula does define a function from A to B.

Step 2: Prove that the function is one-one Let $x_1, x_2 \in A$ and suppose $f(x_1) = f(x_2)$. Since both denominators are nonzero, cross multiplication is valid:

$$\frac{x_1-1}{x_1-2} = \frac{x_2-1}{x_2-2},$$
$$(x_1-1)(x_2-2) = (x_2-1)(x_1-2),$$
$$x_1x_2 - 2x_1 - x_2 + 2 = x_1x_2 - 2x_2 - x_1 + 2.$$

Cancelling common terms gives $-x_1 + x_2 = 0$, hence $x_1 = x_2$. Therefore no two distinct inputs have the same image, and f is one-one.

Quick Tip to Solve: After solving for the preimage, subtract the excluded domain value 2; a nonzero result proves that the candidate is admissible.

Step 3: Prove that the function is onto Take an arbitrary $y \in B$, so $y \neq 1$. Solving $y = (x-1)/(x-2)$ gives

$$yx - 2y = x - 1, \qquad x(y-1) = 2y - 1, \qquad x = \frac{2y-1}{y-1}.$$

This candidate is a real number and belongs to A, since

$$x - 2 = \frac{2y - 1 - 2(y-1)}{y-1} = \frac{1}{y-1} \neq 0.$$

Also $x - 1 = y/(y-1)$, so direct substitution yields

$$f(x) = \frac{y/(y-1)}{1/(y-1)} = y.$$

Thus every chosen element of B has a preimage in A; therefore f is onto.

Step 4: Obtain the inverse with its domain Since f is both one-one and onto, it is bijective. The preceding calculation expresses its unique preimage as $x = (2y-1)/(y-1)$. Replacing the inverse's input symbol y by x, we obtain $f^{-1}(x) = (2x-1)/(x-1)$. Its domain is B, where $x \neq 1$, and its values lie in A, as the preimage calculation established.

Step 5: Conclusion The function f is one-one and onto, and

$$\boxed{f^{-1} : B \to A, \qquad f^{-1}(x) = \frac{2x-1}{x-1}, \quad x \neq 1.}$$

Quick Tip to Solve: After solving for the preimage, subtract the excluded domain value 2; a nonzero result proves that the candidate is admissible.

OR

18(b). Show that the relation S in the set $A = \{x \in \mathbb{Z} : 0 \leq x \leq 12\}$ given by $S = \{(a,b) : a, b \in \mathbb{Z}, |a-b|$ is divisible by $3\}$ is an equivalence relation. [4 marks]

Solution:

Concept:

- An equivalence relation on A must be reflexive, symmetric and transitive. These properties must hold for arbitrary elements of the set.
- For integers a, b, the condition $3 \mid |a-b|$ is equivalent to $3 \mid (a-b)$, because changing the sign does not change divisibility.

Step 1: Prove reflexivity The underlying set is $A = \{0, 1, 2, \ldots, 12\}$, and both entries of a related pair belong to A. For an arbitrary $a \in A$,

$$|a - a| = 0 = 3 \cdot 0.$$

Zero is divisible by 3, so $(a, a) \in S$. This holds for every member of A; hence S is reflexive.

Quick Tip to Solve: Use signed differences for transitivity: the middle element cancels when $a-b$ and $b-c$ are added.

Step 2: Prove symmetry Take arbitrary $a, b \in A$ and suppose $(a, b) \in S$. Then 3 divides $|a - b|$. Since

$$|b - a| = |-(a - b)| = |a - b|,$$

3 also divides $|b - a|$. Thus $(b, a) \in S$ whenever $(a, b) \in S$, establishing symmetry.

Step 3: Prove transitivity Let $a, b, c \in A$, with $(a, b) \in S$ and $(b, c) \in S$. Divisibility of the absolute differences implies divisibility of the signed differences. Therefore, for some integers m, n,

$$a - b = 3m, \qquad b - c = 3n.$$

Adding gives

$$a - c = (a - b) + (b - c) = 3(m + n).$$

Since $m + n \in \mathbb{Z}$, it follows that $|a - c| = 3|m + n|$ is divisible by 3. Hence $(a, c) \in S$, proving transitivity. We used signed differences here: an equality such as $|a - c| = |a - b| + |b - c|$ would not hold for arbitrary elements.

Step 4: Conclusion All three defining properties hold. Therefore, S is an equivalence relation on A.

Quick Tip to Solve: Use signed differences for transitivity: the middle element cancels when $a - b$ and $b - c$ are added.

19(a). Solve the differential equation $\frac{dy}{dx} = 1 + x^2 + y^2 + x^2y^2$, given that $y = 1$ when $x = 0$. [4 marks]

Solution:

Concept:

- A differential equation with separated variables can be integrated by putting all functions of y with dy and all functions of x with dx.
- The standard integral is $\int dy/(1 + y^2) = \tan^{-1} y + C$. The initial value fixes the constant of integration and selects the solution branch through the given point.

Step 1: Factor and separate the variables The right-hand side factors as

$$1 + x^2 + y^2 + x^2y^2 = (1 + x^2)(1 + y^2).$$

Hence the given differential equation becomes

$$\frac{dy}{dx} = (1 + x^2)(1 + y^2), \qquad \frac{dy}{1 + y^2} = (1 + x^2)\,dx.$$

Division by $1 + y^2$ is valid for every real y, since it is strictly positive. Thus no real solution is lost by this separation.

Quick Tip to Solve: Group the four terms into $(1 + x^2)(1 + y^2)$ before integrating; apply the initial condition to the inverse-tangent equation to fix the constant immediately.

Step 2: Integrate both sides Integrating the separated equation gives

$$\int \frac{dy}{1+y^2} = \int (1+x^2)\,dx,$$
$$\tan^{-1} y = x + \frac{x^3}{3} + C.$$

Here C is an arbitrary real constant. The inverse tangent is taken on its principal range $(-\pi/2, \pi/2)$; a solution through the specified initial point remains on the corresponding continuous branch.

Step 3: Use the initial condition and check the result Substitute $x = 0$ and $y = 1$:

$$\tan^{-1} 1 = 0 + C \quad \Longrightarrow \quad C = \frac{\pi}{4}.$$

Taking tangent then gives $y = \tan(x+x^3/3+\pi/4)$. To verify it, write $u = x+x^3/3+\pi/4$. Differentiation yields

$$\frac{dy}{dx} = \sec^2 u\,(1+x^2) = (1+y^2)(1+x^2),$$

exactly the given right-hand side, and $y(0) = \tan(\pi/4) = 1$. The solution is defined on the interval containing zero on which $-\pi/2 < u < \pi/2$, so it does not cross a pole of tangent.

Step 4: Conclusion The required particular solution is

$$\boxed{y = \tan\left(x + \frac{x^3}{3} + \frac{\pi}{4}\right)}.$$

Quick Tip to Solve: Group the four terms into $(1+x^2)(1+y^2)$ before integrating; apply the initial condition to the inverse-tangent equation to fix the constant immediately.

OR

19(b). Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{xy}{x^2+y^2}$, given that $y = 1$ when $x = 0$. [4 marks]

Solution:

Concept:

- A differential equation can be integrated after multiplying its differential form by a suitable nonzero integrating factor. Here the initial value $y = 1$ allows division by powers of y on a neighbourhood of the initial point.
- The product rule gives $d(x^2/y^2) = 2x\,dx/y^2 - 2x^2\,dy/y^3$, while $d(\ln|y|) = dy/y$.

Step 1: Rearrange the differential equation The printed equation and initial condition are

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2}, \qquad y(0) = 1.$$

Quick Tip to Solve: For this initial condition, avoid dividing by x; multiplying the differential form by y^{-3} exposes the derivative of x^2/y^2 immediately.

Since y is nonzero near the initial point, multiply by $x^2 + y^2$ and divide the differential form by y^3:

$$\frac{x}{y^2}\,dx - \left(\frac{x^2}{y^3} + \frac{1}{y}\right) dy = 0.$$

This rearrangement avoids dividing by x, which would be inappropriate at the given initial point $x = 0$.

Step 2: Recognise and integrate an exact differential Using the product rule, the left side is

$$\frac{1}{2}\, d\left(\frac{x^2}{y^2}\right) - d(\ln |y|).$$

Hence integration gives

$$\frac{x^2}{2y^2} - \ln |y| = C.$$

Both terms have been integrated together as an exact differential; no assumption that x and y are independent along the solution is needed.

Step 3: Apply the initial condition and check the relation Substituting $x = 0$, $y = 1$ gives $C = 0 - \ln 1 = 0$. The branch through the initial point has $y > 0$, so the relation becomes

$$\frac{x^2}{2y^2} = \ln y.$$

For verification, differentiate its equivalent form $x^2 = 2y^2 \ln y$:

$$2x = (4y \ln y + 2y)\frac{dy}{dx}.$$

As $2y^2 \ln y = x^2$, this reduces to

$$\frac{dy}{dx} = \frac{x}{y(2 \ln y + 1)} = \frac{xy}{x^2 + y^2}.$$

The relation also contains $(0, 1)$, as required.

Step 4: Conclusion The particular solution through $(0, 1)$ is $\boxed{x^2 = 2y^2 \ln y, \quad y > 0}$, equivalently $\boxed{x^2/y^2 = \ln(y^2)}$ on the positive branch.

Quick Tip to Solve: For this initial condition, avoid dividing by x; multiplying the differential form by y^{-3} exposes the derivative of x^2/y^2 immediately.

20. Using properties of determinants, find the value of x for which

$$\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0.$$ [4 marks]

Solution:

Concept:

- Adding other columns to one column leaves a determinant unchanged. A common factor in a column can then be taken outside the determinant.
- Adding a multiple of one row to another row also preserves the determinant. An upper triangular determinant equals the product of its diagonal entries.

Step 1: Add the columns to simplify the first column Let the given determinant be D. Apply $C_1 \leftarrow C_1 + C_2 + C_3$. Each entry in its first column becomes $(4-x)+(4+x)+(4+x) = 12+x$. Thus

$$D = \begin{vmatrix} 12+x & 4+x & 4+x \\ 12+x & 4-x & 4+x \\ 12+x & 4+x & 4-x \end{vmatrix} = (12+x)\begin{vmatrix} 1 & 4+x & 4+x \\ 1 & 4-x & 4+x \\ 1 & 4+x & 4-x \end{vmatrix}.$$

Factoring is an identity valid even when $12+x=0$; we have not divided by this expression, so its possible zero is retained.

Step 2: Subtract the first row from the other rows In the remaining determinant, apply $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$. The first-column entries below the top entry become zero. Also, $(4-x)-(4+x) = -2x$, whereas equal off-diagonal entries cancel. Therefore

$$D = (12+x)\begin{vmatrix} 1 & 4+x & 4+x \\ 0 & -2x & 0 \\ 0 & 0 & -2x \end{vmatrix} = (12+x)(1)(-2x)(-2x) = 4x^2(12+x).$$

This evaluation uses determinant properties throughout and avoids a lengthy expansion of the original three-by-three determinant.

Step 3: Solve the equation and verify the candidate values The given condition now becomes

$$4x^2(12+x) = 0.$$

By the zero-product property, either $x^2 = 0$ or $12+x=0$. These give $x=0$ and $x=-12$, respectively. For $x=0$, every entry of the original determinant equals 4; its rows are identical and its determinant is zero. For $x=-12$, the diagonal entries are 16 and the off-diagonal entries are -8. Each row then sums to zero, so the sum of all three columns is the zero column and the determinant is zero. The factored expression shows that there are no further real values.

Step 4: Conclusion The required values are $\boxed{x=0 \text{ or } x=-12}$.

Quick Tip to Solve: When all diagonal entries agree and all off-diagonal entries agree, add the columns first: their common row sum reveals one factor immediately.

21. Find the vector equation of the plane which contains the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4=0$, $\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5=0$ and which is perpendicular to the plane $\vec{r}\cdot(5\hat{i}+3\hat{j}-6\hat{k})+8=0$.

[4 marks]

Solution:

Concept:

- A plane has vector equation $\vec{r}\cdot\vec{n}+d=0$, where $\vec{r}$ is the position vector of an arbitrary point on the plane and the nonzero vector $\vec{n}$ is normal to it.
- Every plane through the intersection of two distinct intersecting planes $P_1=0$ and $P_2=0$ belongs to the family $\alpha P_1+\beta P_2=0$, with α,β not both zero.
- Two planes are perpendicular precisely when their normal vectors have zero dot product. The constant term changes the position of a plane, but does not affect its orientation.

Step 1: Form the family containing the intersection line Denote the first two left-hand sides by P_1 and P_2. Their normals are $\vec{n}_1=\hat{i}+2\hat{j}+3\hat{k}$ and $\vec{n}_2=2\hat{i}+\hat{j}-\hat{k}$. These vectors are not proportional, so the planes intersect in a line. A general plane through that line is

$$\alpha\{\vec{r}\cdot(\hat{i}+2\hat{j}+3\hat{k})-4\}+\beta\{\vec{r}\cdot(2\hat{i}+\hat{j}-\hat{k})+5\}=0.$$

Any point on the common line makes both braces zero and therefore satisfies this equation. Its normal is $\alpha\vec{n}_1+\beta\vec{n}_2$.

Step 2: Impose perpendicularity to the third plane The third plane has normal $\vec{m}=5\hat{i}+3\hat{j}-6\hat{k}$. Thus the required condition is

$$\begin{aligned}(\alpha\vec{n}_1+\beta\vec{n}_2)\cdot\vec{m}&=0,\\ \alpha(5+6-18)+\beta(10+3+6)&=0,\\ -7\alpha+19\beta&=0.\end{aligned}$$

Hence $\alpha:\beta=19:7$. A common nonzero multiplier does not change a plane, so choose $\alpha=19$ and $\beta=7$.

Step 3: Simplify and check the resulting plane Substitution into the family gives

$$\vec{r}\cdot\{19(\hat{i}+2\hat{j}+3\hat{k})+7(2\hat{i}+\hat{j}-\hat{k})\}-76+35=0.$$

The normal simplifies to $33\hat{i}+45\hat{j}+50\hat{k}$, and the constant is -41. To verify the orientation, its dot product with the third plane's normal is $33(5)+45(3)-50(6)=165+135-300=0$. Containment of the common line follows directly from the combination $19P_1+7P_2$.

Step 4: Conclusion The required vector equation is $\boxed{\vec{r}\cdot(33\hat{i}+45\hat{j}+50\hat{k})-41=0}$.

Quick Tip to Solve: Take the dot products of each given normal with the third normal first; the two numbers immediately determine the coefficients of the plane family.

22. Find the value of x such that the four points with position vectors, $A(3\hat{i}+2\hat{j}+\hat{k})$, $B(4\hat{i}+x\hat{j}+5\hat{k})$, $C(4\hat{i}+2\hat{j}-2\hat{k})$ and $D(6\hat{i}+5\hat{j}-\hat{k})$ are coplanar. [4 marks]

Solution:

Concept:

- Four points A, B, C, D are coplanar if their three displacement vectors from a common point have zero scalar triple product: $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD}) = 0$.
- The scalar triple product is the signed volume of the parallelepiped formed by the three vectors. Zero volume means that they lie in a plane.
- If $\overrightarrow{AC} \times \overrightarrow{AD} \neq \vec{0}$, the points A, C, D determine a unique plane. Their cross product is a normal to that plane, so a fourth point lies on it precisely when its displacement from A is perpendicular to this normal.

Step 1: Find the displacement vectors from a common point The given position vectors give the coordinates $A = (3, 2, 1)$, $B = (4, x, 5)$, $C = (4, 2, -2)$, and $D = (6, 5, -1)$. Subtracting the coordinates of A from each of the other points, including its third coordinate with the correct sign, gives

$$\begin{aligned}\overrightarrow{AB} &= \hat{i} + (x-2)\hat{j} + 4\hat{k},\\ \overrightarrow{AC} &= \hat{i} - 3\hat{k},\\ \overrightarrow{AD} &= 3\hat{i} + 3\hat{j} - 2\hat{k}.\end{aligned}$$

Step 2: Obtain a normal to the plane through A, C and D Evaluate the cross product by expanding the determinant, remembering the negative sign of its middle cofactor:

$$\begin{aligned}\overrightarrow{AC} \times \overrightarrow{AD} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix}\\ &= \{0(-2) - (-3)(3)\}\hat{i} - \{1(-2) - (-3)(3)\}\hat{j} + \{1(3) - 0(3)\}\hat{k}\\ &= 9\hat{i} - 7\hat{j} + 3\hat{k}.\end{aligned}$$

This vector is nonzero; hence A, C, D are noncollinear and determine a plane.

Step 3: Apply the coplanarity condition and verify the value For B to lie in the same plane, impose

$$\begin{aligned}\overrightarrow{AB} \cdot (9\hat{i} - 7\hat{j} + 3\hat{k}) &= 0,\\ 9 - 7(x-2) + 3(4) &= 0,\\ 35 - 7x &= 0.\end{aligned}$$

Thus $x = 5$. Checking directly, $\overrightarrow{AB} = (1, 3, 4)$ then has dot product $9 - 21 + 12 = 0$ with the normal, confirming that B lies in the plane through A, C, D.

Step 4: Conclusion The four given points are coplanar when $\boxed{x = 5}$.

Quick Tip to Solve: Use A as the common starting point and cross AC with AD: neither vector contains x, so only the final dot product requires solving an equation.

23. If $y = (\log x)^x + x^{\log x}$, find $\dfrac{dy}{dx}$. [4 marks]

Solution:

Concept:

- Logarithmic differentiation is useful when both the base and the exponent depend on x. In this calculus question, $\log x$ denotes the natural logarithm.
- For a positive differentiable function u, $\dfrac{d}{dx}\log u = \dfrac{u'}{u}$. Also, $\dfrac{d}{dx}\log x = \dfrac{1}{x}$; apply the product rule to the logarithm of a variable power.
- The usual real-valued logarithmic differentiation of the first term is taken on $x > 1$, where $\log x > 0$. This also makes every logarithm used below well defined.

Step 1: Differentiate the first variable power Let $u = (\log x)^x$. Taking logarithms of the positive quantity u, we obtain

$$\log u = x \log(\log x).$$

The right-hand side is a product, so its derivative has two terms. The derivative of the inner logarithm is $\dfrac{1}{x \log x}$. Therefore,

$$\begin{aligned} \frac{u'}{u} &= \log(\log x) + x\frac{1}{x\log x} \\ &= \log(\log x) + \frac{1}{\log x}, \\ u' &= (\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right]. \end{aligned}$$

Multiplying by u is essential: differentiating $\log u$ gives u'/u, rather than u'.

Step 2: Differentiate the second variable power Let $v = x^{\log x}$. Here both occurrences of $\log x$ contribute on differentiation, since

$$\log v = (\log x)(\log x) = (\log x)^2.$$

Hence, by the chain rule,

$$\frac{v'}{v} = 2\log x \frac{1}{x}, \qquad v' = \frac{2\log x}{x}\, x^{\log x}.$$

Since the original function is $y = u + v$, its derivative is the sum $u' + v'$. Neither variable power can be differentiated by the constant-exponent power rule directly.

Step 3: Conclusion Thus, for $x > 1$,

$$\boxed{\frac{dy}{dx} = (\log x)^x \left[\log(\log x) + \frac{1}{\log x}\right] + \frac{2\log x}{x}\, x^{\log x}.}$$

Quick Tip to Solve: Differentiate the two powers separately after taking their logarithms; the second becomes the simple square $(\log x)^2$.

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24(a). Find the vector equation of a line passing through the point $(2, 3, 2)$ and parallel to the line $\vec{r} = (-2\hat{i} + 3\hat{j}) + \lambda(2\hat{i} - 3\hat{j} + 6\hat{k})$. Also, find the distance between these two lines. [6 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$ and parallel to a nonzero vector $\vec{b}$ has equation $\vec{r} = \vec{a} + t\vec{b}$, where $t \in \mathbb{R}$.
- For two parallel lines with common direction $\vec{b}$, their distance is $d = \dfrac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$. This is the perpendicular component of the vector joining a point on each line.
- Equivalently, the squared perpendicular component is $|\vec{a}_2 - \vec{a}_1|^2 - \dfrac{[(\vec{a}_2 - \vec{a}_1) \cdot \vec{b}]^2}{|\vec{b}|^2}$, which provides an independent check.

Step 1: Identify the common direction and form the required line From the given equation, a point on the original line is $A = (-2, 3, 0)$, and its direction vector is

$$\vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}.$$

The required line passes through $B = (2, 3, 2)$, whose position vector is $\vec{a}_2 = 2\hat{i} + 3\hat{j} + 2\hat{k}$. Since it must be parallel to the given line, use the same direction vector. Its equation is therefore

$$\vec{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + t(2\hat{i} - 3\hat{j} + 6\hat{k}), \quad t \in \mathbb{R}.$$

Substituting $t = 0$ gives precisely the prescribed point. The direction coefficients $2, -3, 6$ agree with those of the original line, verifying the parallelism.

Step 2: Obtain the perpendicular separation using a cross product The vector from A to B is

$$\vec{AB} = \vec{a}_2 - \vec{a}_1 = 4\hat{i} + 2\hat{k}.$$

Its length alone is not the distance between the lines, because it need not be perpendicular to them. Compute its cross product with their common direction:

$$\vec{AB} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 0 & 2 \\ 2 & -3 & 6 \end{vmatrix} = 6\hat{i} - 20\hat{j} - 12\hat{k}.$$

Thus,

$$|\vec{AB} \times \vec{b}| = \sqrt{36 + 400 + 144} = 2\sqrt{145}, \qquad |\vec{b}| = \sqrt{4 + 9 + 36} = 7.$$

Dividing the area of the parallelogram by the length of its direction side yields the perpendicular distance between the parallel lines.

Quick Tip to Solve: For parallel lines, copy the direction vector and change only the point vector; use their cross-product distance formula, not the skew-line scalar-triple-product formula.

Step 3: Check the distance by resolving the joining vector Here $|\vec{AB}|^2 = 20$ and $\vec{AB} \cdot \vec{b} = 4(2)+2(6) = 20$. Removing the squared component along the lines gives

$$d^2 = 20 - \frac{20^2}{49} = \frac{580}{49}.$$

Its positive square root agrees with the cross-product calculation. It is nonzero, so these are distinct parallel lines.

Step 4: Conclusion The required line and the distance between the two lines are

$$\boxed{\vec{r} = (2\hat{i} + 3\hat{j} + 2\hat{k}) + t(2\hat{i} - 3\hat{j} + 6\hat{k})}$$

and

$$\boxed{d = \frac{2\sqrt{145}}{7} \text{ units.}}$$

Quick Tip to Solve: For parallel lines, copy the direction vector and change only the point vector; use their cross-product distance formula, not the skew-line scalar-triple-product formula.

OR

24(b). Find the coordinates of the foot of the perpendicular Q drawn from $P(3, 2, 1)$ to the plane $2x - y + z + 1 = 0$. Also, find the distance PQ and the image of the point P treating this plane as a mirror. [6 marks]

Solution:

Concept:

- For a plane $ax + by + cz + d = 0$, the vector (a, b, c) is normal to the plane. A perpendicular from a point therefore has this direction.
- If Q is the perpendicular foot of P on a mirror plane and P' is its image, then P, Q, P' are collinear and Q is the midpoint of PP'. Thus $P' = 2Q - P$.

Step 1: Parametrise the perpendicular through P The coefficients of x, y, z in the plane give its normal vector $\mathbf{n} = (2, -1, 1)$, whose length is $\sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$. Since the required perpendicular passes through $P(3, 2, 1)$, a general point on it is

$$(x, y, z) = (3, 2, 1) + t(2, -1, 1).$$

Hence its coordinates are $x = 3 + 2t$, $y = 2 - t$ and $z = 1 + t$. This uses the plane's normal direction, so the line is perpendicular to the plane, not merely a line meeting it.

Step 2: Locate the foot on the plane The foot Q must satisfy the plane equation as well as lie on the

Quick Tip to Solve: First move along the plane's normal to find the foot; then double the foot's coordinates and subtract the original point to obtain its image.

perpendicular. Substituting the parametric coordinates gives

$$\begin{aligned} 2(3+2t)-(2-t)+(1+t)+1 &= 0, \\ 6+6t &= 0, \\ t &= -1. \end{aligned}$$

Therefore the foot has coordinates $(3-2, 2+1, 1-1) = (1, 3, 0)$. As a direct check, $2(1)-3+0+1 = 0$, so this point is on the specified plane. Also $\overrightarrow{PQ} = (-2, 1, -1) = -\mathbf{n}$, confirming perpendicularity.

Step 3: Calculate the perpendicular distance Apply the distance formula to the two points:

$$PQ = \sqrt{(1-3)^2 + (3-2)^2 + (0-1)^2} = \sqrt{4+1+1} = \sqrt{6}.$$

Independently, the point-to-plane formula gives

$$\frac{|2(3) - 2 + 1 + 1|}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{6}{\sqrt{6}} = \sqrt{6},$$

which agrees with the coordinate calculation.

Step 4: Reflect P across the mirror plane Let the image be $P'(x', y', z')$. Since the mirror plane bisects the perpendicular segment joining a point and its image, the midpoint condition gives

$$\begin{aligned} x' &= 2(1) - 3 = -1, \\ y' &= 2(3) - 2 = 4, \\ z' &= 2(0) - 1 = -1. \end{aligned}$$

The midpoint of $(3, 2, 1)$ and $(-1, 4, -1)$ is indeed $(1, 3, 0)$. The plane expression has values 6 at P and -6 at P', so the points lie on opposite sides at equal perpendicular distances.

Step 5: Conclusion The perpendicular foot is $\boxed{Q = (1, 3, 0)}$, the distance is $\boxed{PQ = \sqrt{6} \text{ units}}$, and the mirror image of P is $\boxed{P' = (-1, 4, -1)}$.

Quick Tip to Solve: First move along the plane's normal to find the foot; then double the foot's coordinates and subtract the original point to obtain its image.

25(a). Using elementary row transformations, find the inverse of the matrix $\begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$. [6 marks]

Solution:

Concept:

- To find the inverse by elementary row transformations, augment the square matrix A with the identity matrix I, forming $[A \mid I]$.
- Apply each row operation to the whole augmented row. If the left block becomes I, the right block

Quick Tip to Solve: Swap the row beginning with 1 into the first position; this keeps every row operation integral and makes the inverse easier to check.

becomes A^{-1}, since the same transformations send A to I.

Step 1: Form the augmented matrix and choose a convenient pivot For the given matrix, start with

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 2 & -3 & 5 & 1 & 0 & 0 \\ 3 & 2 & -4 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right].$$

Interchange R_1 and R_3 to obtain a unit first pivot without introducing fractions:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 0 & 1 \\ 3 & 2 & -4 & 0 & 1 & 0 \\ 2 & -3 & 5 & 1 & 0 & 0 \end{array}\right].$$

The same interchange is performed on the identity block; omitting this would produce an incorrect inverse.

Step 2: Eliminate the first column below the pivot Apply $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 2R_1$. The resulting augmented matrix is

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & -3 \\ 0 & -5 & 9 & 1 & 0 & -2 \end{array}\right].$$

Next apply $R_2 \to -R_2$, giving

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & 3 \\ 0 & -5 & 9 & 1 & 0 & -2 \end{array}\right].$$

The second pivot is now one, making the remaining elimination straightforward.

Step 3: Obtain the third pivot Use the current second row in $R_3 \to R_3 + 5R_2$. Then multiply the third row by -1:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & -1 & 1 & -5 & 13 \end{array}\right]$$

$$\xrightarrow{R_3 \to -R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & -1 & 5 & -13 \end{array}\right].$$

Every diagonal entry in the left block is now one. All three pivots are nonzero, so the process can reduce this block to the identity.

Step 4: Clear the entries above the pivots Apply $R_1 \to R_1 + 2R_3$ and $R_2 \to R_2 + 2R_3$, using the current third row:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 10 & -25 \\ 0 & 1 & 0 & -2 & 9 & -23 \\ 0 & 0 & 1 & -1 & 5 & -13 \end{array}\right].$$

Quick Tip to Solve: Swap the row beginning with 1 into the first position; this keeps every row operation integral and makes the inverse easier to check.

Finally, $R_1 \to R_1 - R_2$ gives

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & -2 & 9 & -23 \\ 0 & 0 & 1 & -1 & 5 & -13 \end{array}\right] = [I \mid B].$$

To check the result, multiply the original matrix by the right block:

$$AB = \begin{bmatrix} 6-5 & 2-27+25 & -4+69-65 \\ -4+4 & 3+18-20 & -6-46+52 \\ -2+2 & 1+9-10 & -2-23+26 \end{bmatrix} = I.$$

This verifies all nine entries and confirms that the row operations have produced the inverse.

Step 5: Conclusion Thus, by elementary row transformations,

$$\boxed{A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}}.$$

Quick Tip to Solve: Swap the row beginning with 1 into the first position; this keeps every row operation integral and makes the inverse easier to check.

OR

25(b). Using matrices, solve the following system of linear equations:

$x + 2y - 3z = -4$

$2x + 3y + 2z = 2$

$3x - 3y - 4z = 11$ [6 marks]

Solution:

Concept:

- A system of three linear equations can be written as $AX = B$, where A is the coefficient matrix, X is the column of unknowns and B is the constant column.
- If $|A| \neq 0$, the inverse exists and the unique solution is $X = A^{-1}B$, with $A^{-1} = \dfrac{\text{adj}\, A}{|A|}$. The adjugate is the transpose of the cofactor matrix.

Step 1: Express the equations in matrix form Keeping the coefficients in the order x, y, z, write

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 2 \\ 3 & -3 & -4 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix}.$$

The given system is therefore $AX = B$. The negative signs in the first and third equations are retained

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the constant column; verify the final values in all three original equations.

in the corresponding entries of A and B.

Step 2: Check that the coefficient matrix is invertible Expanding the determinant along the first row gives

$$\begin{aligned} |A| &= 1\{3(-4) - 2(-3)\} - 2\{2(-4) - 2(3)\} \\ &\quad + (-3)\{2(-3) - 3(3)\} \\ &= -6 + 28 + 45 = 67. \end{aligned}$$

Since this determinant is nonzero, multiplying by A^{-1} is valid and will produce a unique solution.

Step 3: Form the adjugate and inverse Computing the signed minors gives the cofactor matrix

$$C = \begin{pmatrix} -6 & 14 & -15 \\ 17 & 5 & 9 \\ 13 & -8 & -1 \end{pmatrix}.$$

For example, $C_{12} = -\{2(-4) - 2(3)\} = 14$, while $C_{23} = -\{1(-3) - 2(3)\} = 9$. Transposing this matrix yields

$$\operatorname{adj} A = \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix}.$$

Hence $A^{-1} = \dfrac{1}{67} \operatorname{adj} A$.

Step 4: Multiply by the constant column and verify Multiplication on the left by the inverse gives

$$\begin{aligned} X &= \frac{1}{67} \begin{pmatrix} -6 & 17 & 13 \\ 14 & 5 & -8 \\ -15 & 9 & -1 \end{pmatrix} \begin{pmatrix} -4 \\ 2 \\ 11 \end{pmatrix} \\ &= \frac{1}{67} \begin{pmatrix} 24 + 34 + 143 \\ -56 + 10 - 88 \\ 60 + 18 - 11 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}. \end{aligned}$$

Substituting these entries into the original left sides gives $3+2(-2)-3(1) = -4$, $2(3)+3(-2)+2(1) = 2$, and $3(3) - 3(-2) - 4(1) = 11$. Thus the computed column satisfies all three equations.

Step 5: Conclusion The unique solution of the given system is $\boxed{x = 3,\ y = -2,\ z = 1}$.

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the constant column; verify the final values in all three original equations.

26(a). Using integration, find the area of the region bounded by the parabola $y^2 = 4x$ and the circle $4x^2 + 4y^2 = 9$. [6 marks]

Solution:

Concept:

- For a horizontal strip, the area between two curves is $\int (x_{\text{right}} - x_{\text{left}})\, dy$. The intersection points determine the limits.
- The parabola $y^2 = 4x$ and the circle $x^2 + y^2 = 9/4$ are symmetric about the x-axis, so the desired area is twice the area above that axis.
- The standard integral is $\int \sqrt{a^2 - y^2}\, dy = \frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}(y/a) + C$, where $a > 0$ and $|y| \le a$.

Step 1: Determine the intersection points Divide the circle equation by 4 to obtain $x^2 + y^2 = 9/4$. Substituting $y^2 = 4x$ gives

$$x^2 + 4x = \frac{9}{4},$$
$$4x^2 + 16x - 9 = 0,$$
$$(2x - 1)(2x + 9) = 0.$$

A real point of the parabola must have $x = y^2/4 \ge 0$, so the negative root $-9/2$ cannot occur. Therefore $x = 1/2$ and $y^2 = 2$. The intersections are $(1/2, \sqrt{2})$ and $(1/2, -\sqrt{2})$.

Step 2: Identify the boundaries of a horizontal strip For $-\sqrt{2} \le y \le \sqrt{2}$, the parabola is the left boundary $x = y^2/4$, and the right semicircle is the right boundary $x = \sqrt{9/4 - y^2}$. The arc on the left of the circle does not bound the enclosed region with the parabola: the parabola lies entirely in $x \ge 0$. Thus a strip of thickness dy has area $\{\sqrt{9/4 - y^2} - y^2/4\}\, dy$. Symmetry now gives

$$A = 2\int_0^{\sqrt{2}} \left(\sqrt{\frac{9}{4} - y^2} - \frac{y^2}{4}\right) dy.$$

The integrand is nonnegative throughout this interval and becomes zero at the upper endpoint, as required for the closing boundary.

Step 3: Integrate the two contributions Use the standard circle integral with $a = 3/2$, and integrate the polynomial term directly:

$$A = 2\left[\frac{y}{2}\sqrt{\frac{9}{4} - y^2} + \frac{9}{8}\sin^{-1}\left(\frac{2y}{3}\right) - \frac{y^3}{12}\right]_0^{\sqrt{2}}$$
$$= \left[y\sqrt{\frac{9}{4} - y^2} + \frac{9}{4}\sin^{-1}\left(\frac{2y}{3}\right) - \frac{y^3}{6}\right]_0^{\sqrt{2}}.$$

The inverse sine is its principal value in radians. Its argument lies between 0 and $2\sqrt{2}/3 < 1$, so the displayed antiderivative is well defined on the entire integration interval.

Step 4: Apply the limits and simplify At $y = \sqrt{2}$, the square root is $\sqrt{9/4 - 2} = 1/2$. At $y = 0$, every

Quick Tip to Solve: Use horizontal strips here: both boundaries give a single expression for x in terms of y, avoiding a split integral at x=1/2.

term in the bracket is zero. Consequently,

$$A = \frac{\sqrt{2}}{2} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) - \frac{2\sqrt{2}}{6}$$
$$= \frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right).$$

This includes both symmetric halves of the region; no additional factor of two is needed.

Step 5: Conclusion The required area is $\boxed{\frac{\sqrt{2}}{6} + \frac{9}{4}\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)}$ square units, with the angle measured in radians.

Quick Tip to Solve: Use horizontal strips here: both boundaries give a single expression for x in terms of y, avoiding a split integral at x=1/2.

OR

26(b). Using the method of integration, find the area of the region bounded by the lines $3x-2y+1=0$, $2x+3y-21=0$ and $x-5y+9=0$. [6 marks]

Solution:

Concept:

- The area between an upper curve $y=f(x)$ and a lower curve $y=g(x)$, from $x=a$ to $x=b$, is $\int_a^b [f(x)-g(x)]\,dx$. The difference must be non-negative over the interval used.
- Three non-parallel lines enclosing a finite region form a triangle. Their pairwise intersections determine its vertices and the limits of integration. Split the integral wherever the upper boundary changes.

Step 1: Express the boundaries in terms of x Write the three given lines as

$$L_1 : y = \frac{3x+1}{2}, \qquad L_2 : y = \frac{21-2x}{3}, \qquad L_3 : y = \frac{x+9}{5}.$$

These expressions let us measure each vertical strip by subtracting its lower ordinate from its upper ordinate.

Step 2: Find the three vertices For L_1 and L_2, substitution gives $2x+3(3x+1)/2=21$, so $13x=39$. Thus their intersection is $A(3,5)$. For L_1 and L_3, equating ordinates gives $5(3x+1)=2(x+9)$, hence $13x=13$. Their intersection is $B(1,2)$. For L_2 and L_3, we obtain $5(21-2x)=3(x+9)$, hence $13x=78$. Their intersection is $C(6,3)$. The enclosed region therefore extends from $x=1$ to $x=6$, with the upper boundary changing at $x=3$.

Step 3: Identify the strip heights and integrate The line BC, namely L_3, is the lower boundary throughout. From $x=1$ to $x=3$, the upper boundary is BA, namely L_1; from $x=3$ to $x=6$, it is

Quick Tip to Solve: Sort the vertices by x-coordinate first; split at the middle vertex because the upper boundary changes there.

AC, namely L_2. Consequently,

$$\begin{aligned}\mathcal{A} &= \int_1^3 \left(\frac{3x+1}{2} - \frac{x+9}{5}\right) dx + \int_3^6 \left(\frac{21-2x}{3} - \frac{x+9}{5}\right) dx \\ &= \frac{13}{10}\int_1^3 (x-1)\,dx + \frac{13}{15}\int_3^6 (6-x)\,dx \\ &= \frac{13}{10}\left[\frac{(x-1)^2}{2}\right]_1^3 + \frac{13}{15}\left[6x - \frac{x^2}{2}\right]_3^6 \\ &= \frac{13}{5} + \frac{39}{10}.\end{aligned}$$

Both strip heights are non-negative on their respective intervals, so these integrals directly give geometrical area without any sign correction.

Step 4: Conclusion The required area of the region bounded by the three lines is $\boxed{\frac{13}{2} \text{ square units}}$.

Quick Tip to Solve: Sort the vertices by x-coordinate first; split at the middle vertex because the upper boundary changes there.

27. An insurance company insured 3000 cyclists, 6000 scooter drivers and 9000 car drivers. The probability of an accident involving a cyclist, a scooter driver and a car driver are 0.3, 0.05 and 0.02 respectively. One of the insured persons meets with an accident. What is the probability that he is a cyclist? [6 marks]

Solution:

Concept:

- Bayes' theorem reverses a conditional probability. If E_1, E_2, E_3 partition the sample space and A is an observed event, then $P(E_1 \mid A) = P(E_1)P(A \mid E_1)/\sum_{i=1}^{3} P(E_i)P(A \mid E_i)$.
- The prior probability of belonging to a category is its number of insured persons divided by the total number insured. The accident rates are conditional probabilities within the respective categories.

Step 1: Define the events and their prior probabilities Let C, S and D denote the events that an insured person is a cyclist, a scooter driver and a car driver, respectively. Let A denote the event that the person meets with an accident. The total number of insured persons is

$$3000 + 6000 + 9000 = 18000.$$

The three categories exhaust the insured population and are treated as disjoint. Thus

$$P(C) = \frac{3000}{18000} = \frac{1}{6}, \quad P(S) = \frac{6000}{18000} = \frac{1}{3}, \quad P(D) = \frac{9000}{18000} = \frac{1}{2}.$$

These probabilities add to one, as required for a partition.

Quick Tip to Solve: Multiply each category size by its accident rate, then divide the cyclist weight by the sum of all three weights.

Step 2: Calculate the overall accident probability The given accident probabilities mean

$$P(A \mid C) = 0.3, \quad P(A \mid S) = 0.05, \quad P(A \mid D) = 0.02.$$

By the law of total probability,

$$\begin{aligned} P(A) &= P(C)P(A \mid C) + P(S)P(A \mid S) + P(D)P(A \mid D) \\ &= \frac{1}{6} \cdot \frac{3}{10} + \frac{1}{3} \cdot \frac{1}{20} + \frac{1}{2} \cdot \frac{1}{50} \\ &= \frac{1}{20} + \frac{1}{60} + \frac{1}{100} = \frac{15+5+3}{300} = \frac{23}{300}. \end{aligned}$$

Each term weights an accident rate by the proportion of insured persons in that category; adding the rates alone would ignore the unequal category sizes.

Step 3: Apply Bayes' theorem to the observed accident We require the probability of the category given that the accident has already occurred, namely $P(C \mid A)$. The joint probability that the insured person is a cyclist and has an accident is

$$P(C \cap A) = P(C)P(A \mid C) = \frac{1}{6} \cdot \frac{3}{10} = \frac{1}{20}.$$

Therefore Bayes' theorem gives

$$P(C \mid A) = \frac{P(C \cap A)}{P(A)} = \frac{1/20}{23/300} = \frac{15}{23}.$$

As a check, the accident-weighted numbers for the three categories are $3000(0.3) = 900$, $6000(0.05) = 300$ and $9000(0.02) = 180$. The cyclist fraction among these is $900/(900+300+180) = 15/23$, agreeing with the probability calculation.

Step 4: Conclusion Given that the insured person has met with an accident, the probability that he is a cyclist is $\boxed{\frac{15}{23}}$.

Quick Tip to Solve: Multiply each category size by its accident rate, then divide the cyclist weight by the sum of all three weights.

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28. Using integration, find the area of the smaller region bounded by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the line $\frac{x}{3} + \frac{y}{2} = 1$. [6 marks]

Solution:

Concept:

- The area between an upper curve $y = f(x)$ and a lower curve $y = g(x)$, between their intersection abscissae a, b, is $\int_a^b [f(x) - g(x)]\,dx$.
- For this ellipse the upper half is $y = 2\sqrt{1 - x^2/9}$. A chord divides the ellipse into two regions;

Quick Tip to Solve: The chord joins the axis intercepts: integrate the first-quadrant elliptical arc and subtract the right triangle of area $\frac{1}{2}(3)(2) = 3$.

comparing their areas identifies the smaller region.

Step 1: Find the points of intersection From the line, $y = 2 - 2x/3$. Substituting into the ellipse gives

$$\frac{x^2}{9} + \left(1 - \frac{x}{3}\right)^2 = 1 \quad \Longrightarrow \quad \frac{2x^2}{9} - \frac{2x}{3} = 0.$$

Hence $x(x-3) = 0$, so the intersections are $(0, 2)$ and $(3, 0)$. The chord joining these points lies in the first quadrant. Between $x = 0$ and $x = 3$, the upper arc of the ellipse lies above this chord.

Step 2: Set up the area of the first-quadrant segment The region enclosed by the chord and the first-quadrant arc has area

$$A = \int_0^3 \left[2\sqrt{1 - \frac{x^2}{9}} - \left(2 - \frac{2x}{3}\right)\right] dx.$$

At both endpoints the vertical height is zero, as required. The integral subtracts the area under the straight line from the area under the upper elliptical arc. In the interior the height is positive: writing $u = x/3$, $\sqrt{1-u^2} \geq 1 - u$ for $0 \leq u \leq 1$, since squaring reduces this to $2u(1-u) \geq 0$.

Step 3: Evaluate the elliptical-arc integral Put $x = 3\sin\theta$, so $dx = 3\cos\theta\, d\theta$. The limits become $\theta = 0$ and $\theta = \pi/2$; cosine is nonnegative on this interval. Thus

$$\begin{aligned}\int_0^3 2\sqrt{1 - x^2/9}\, dx &= 6\int_0^{\pi/2} \cos^2\theta\, d\theta \\ &= 3\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta \\ &= 3\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{3\pi}{2}.\end{aligned}$$

Step 4: Subtract the chord area and identify the smaller region The area under the chord is

$$\int_0^3 \left(2 - \frac{2x}{3}\right) dx = \left[2x - \frac{x^2}{3}\right]_0^3 = 3.$$

Therefore the first-quadrant segment has area $A = 3\pi/2 - 3$. The ellipse has semiaxes 3 and 2, hence total area 6π. The other region has area $6\pi - A = 9\pi/2 + 3$, which exceeds A. This comparison confirms that the integrated segment is the smaller of the two regions bounded by the line and the ellipse.

Step 5: Conclusion The required smaller area is $\boxed{\frac{3\pi}{2} - 3 = \frac{3}{2}(\pi - 2) \text{ square units}}$.

Quick Tip to Solve: The chord joins the axis intercepts: integrate the first-quadrant elliptical arc and subtract the right triangle of area $\frac{1}{2}(3)(2) = 3$.

29. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs. 35 per package of nuts and Rs. 14 per package of bolts. How many packages of each should be produced each day so as to maximise his profit, if he operates each machine for at most 12 hours a day? Convert it into an LPP and solve graphically. [6 marks]

Solution:

Concept:

- Let x and y be the daily numbers of packages of nuts and bolts, respectively. Machine-hour limits give linear inequalities, while daily profit is the objective function.
- For a linear objective on a bounded polygonal feasible region, a maximum occurs at a corner point. Graph the constraints, retain their common feasible side, and compare the objective at every vertex.

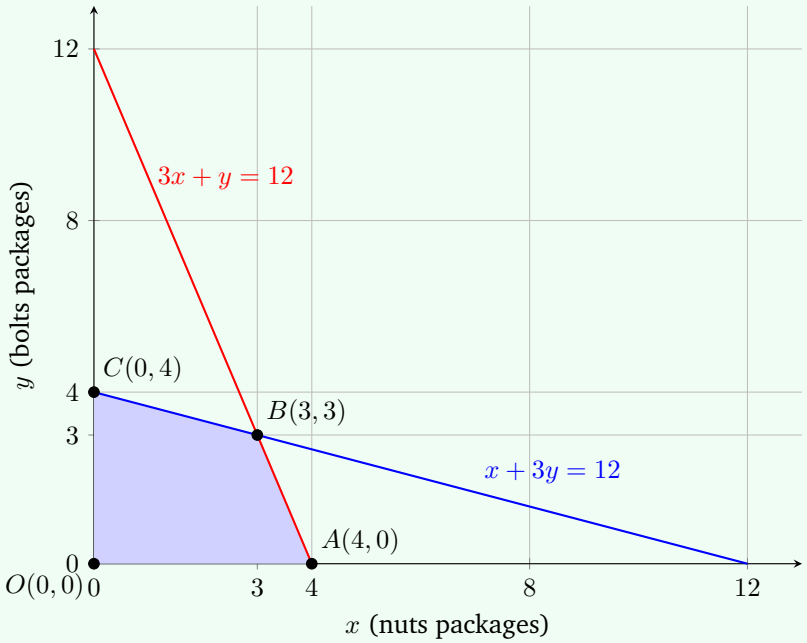
Step 1: Formulate the linear programming problem One package of nuts uses 1 hour on A and 3 hours on B; one package of bolts uses 3 hours on A and 1 hour on B. Since each machine can operate for at most 12 hours per day, the LPP is

$$\begin{aligned} \text{Maximise} \quad & Z = 35x + 14y, \\ \text{subject to} \quad & x + 3y \leq 12, \\ & 3x + y \leq 12, \\ & x \geq 0, \quad y \geq 0. \end{aligned}$$

Here Z is the daily profit in rupees. Nonnegativity is necessary because the package counts cannot be negative.

Step 2: Graph the inequalities and obtain the feasible polygon The boundary $x + 3y = 12$ passes through $(12, 0)$ and $(0, 4)$, while $3x + y = 12$ passes through $(4, 0)$ and $(0, 12)$. For both inequalities the origin satisfies the constraint, so retain the side containing the origin and restrict to the first quadrant. The shaded common region in the graph is the quadrilateral $OABC$. To find the intersection B, subtract the two boundary equations: $2x - 2y = 0$, giving $x = y$. Substitution into $x + 3y = 12$ gives $x = y = 3$, hence $B = (3, 3)$. The other vertices are $O = (0, 0)$, $A = (4, 0)$, and $C = (0, 4)$.

Quick Tip to Solve: Find the intersection of the two machine limits first, but compare its profit with both axis vertices before choosing the production schedule.



Graph of the two machine-hour constraints; the shaded polygon OABC is the feasible region.

Step 3: Compare the daily profit at all corner points The feasible region is closed and bounded, so the corner-point method applies. Evaluate the objective at each vertex:

Corner point (x, y)	$Z = 35x + 14y$ (rupees)
$(0, 0)$	0
$(4, 0)$	140
$(3, 3)$	147
$(0, 4)$	56

The greatest of these values is 147, attained at $(3, 3)$. In particular, producing only nuts gives 140, which is less than the profit at the intersection; the higher profit per nut package alone does not justify ignoring the two separate machine constraints.

Step 4: Check the optimal production against both machine limits At $x = 3$, $y = 3$, machine A is used for $3 + 3(3) = 12$ hours and machine B for $3(3) + 3 = 12$ hours. Both limits are satisfied exactly. The proposed package counts are nonnegative whole numbers, and the daily profit is $35(3) + 14(3) = 105 + 42 = 147$ rupees. Thus the graphical optimum also gives a feasible daily package schedule.

Step 5: Conclusion Produce $\boxed{3 \text{ packages of nuts and } 3 \text{ packages of bolts per day}}$. The maximum daily profit is $\boxed{\text{Rs. } 147}$.

Quick Tip to Solve: Find the intersection of the two machine limits first, but compare its profit with both axis vertices before choosing the production schedule.