

CBSE Class 12 Mathematics Question Paper 2019 with Solution PDF-Set 65-5-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of 4 questions of **one mark** each, Section B comprises of 8 questions of **two marks** each, Section C comprises of 11 questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. If A is a square matrix of order 2 and $|A| = 4$, then find the value of $|2 \cdot AA'|$, where A' is the transpose of matrix A . [1 mark]

Solution:

Concept:

- For an order- n matrix M , $|kM| = k^n|M|$; the scalar multiplies every row.
- Determinants satisfy $|PQ| = |P||Q|$ and $|A'| = |A|$.

Step 1: Apply the scalar factor The matrix AA' has order 2, just as A does. Therefore multiplying it by 2 multiplies both rows by 2, giving

$$|2AA'| = 2^2|AA'|.$$

Step 2: Use the product and transpose properties Transposing does not change the determinant, so $|A'| = 4$. Hence

$$|AA'| = |A||A'| = 4 \times 4 = 16.$$

Substituting this into the scalar-multiple formula gives $2^2 \times 16 = 64$.

Step 3: Conclusion The required determinant is $\boxed{64}$.

Quick Tip to Solve: Count the order before taking a scalar outside a determinant: here the factor is 2^2 .

2(a). Find the angle between the line $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} - \hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 3$. [1 mark]

Solution:

Concept:

- The direction vector of a line is its parameter coefficient, and the normal vector of a plane is its vector-equation coefficient.
- For the acute angle θ between a line and a plane, $\sin \theta = |\vec{d} \cdot \vec{n}| / (|\vec{d}||\vec{n}|)$.

Step 1: Identify the two vectors From the given equations, $\vec{d} = 3\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{n} = \hat{i} + \hat{j} + \hat{k}$. The fixed point on the line and the constant 3 in the plane do not affect their angle.

Step 2: Compute the angle with the plane We have $\vec{d} \cdot \vec{n} = 3 - 1 + 2 = 4$, $|\vec{d}| = \sqrt{14}$, and $|\vec{n}| = \sqrt{3}$. Thus

$$\sin \theta = \frac{4}{\sqrt{14}\sqrt{3}} = \frac{4}{\sqrt{42}}.$$

The angle with a plane is complementary to the angle with its normal, which is why sine is used here.

Step 3: Conclusion The angle is $\boxed{\theta = \sin^{-1}\left(4/\sqrt{42}\right)}$.

Quick Tip to Solve: For a line-plane angle use sine; cosine gives the angle with the plane normal.

2(b). Find the co-ordinates of the point, where the line $\frac{x+2}{1} = \frac{y-5}{3} = \frac{z+1}{5}$ cuts the yz-plane. [1 mark]

Solution:

Concept:

- Every point on the yz-plane has $x = 0$.
- Equating the symmetric ratios to a parameter t gives the coordinates of points on the line.

Step 1: Write the parametric coordinates Put

$$\frac{x+2}{1} = \frac{y-5}{3} = \frac{z+1}{5} = t.$$

Then $x = t - 2$, $y = 3t + 5$, and $z = 5t - 1$. These expressions preserve each numerator sign in the printed line.

Step 2: Impose the coordinate-plane condition At the intersection with the yz-plane, $x = 0$. Thus $t - 2 = 0$, giving $t = 2$. Substitution yields $y = 3(2) + 5 = 11$ and $z = 5(2) - 1 = 9$. All three original ratios then equal 2, verifying that the point lies on the line.

Step 3: Conclusion The required point is $(0, 11, 9)$.

Quick Tip to Solve: On the yz-plane set $x = 0$ first, then use the common ratio to find the other coordinates.

3. Find the differential equation representing the family of curves $y = -A \cos 3x + B \sin 3x$. [1 mark]

Solution:

Concept:

- Two independent arbitrary constants A, B are eliminated by differentiating twice.
- The second derivative of either $\sin 3x$ or $\cos 3x$ is -9 times that function.

Step 1: Differentiate the family For $y = -A \cos 3x + B \sin 3x$, treat A and B as constants. The chain rule gives

$$y' = 3A \sin 3x + 3B \cos 3x.$$

The factor 3 arises from differentiating the argument $3x$.

Step 2: Eliminate both constants Differentiating again gives

$$y'' = 9A \cos 3x - 9B \sin 3x = -9(-A \cos 3x + B \sin 3x) = -9y.$$

This relation contains neither arbitrary constant and is second order, as expected for this two-constant family.

Step 3: Conclusion The required differential equation is $\frac{d^2y}{dx^2} + 9y = 0$.

Quick Tip to Solve: For a sine-cosine family of frequency 3, two derivatives multiply the entire function by -9 .

4. Find the differential of the function $\cos^{-1}(\sin 2x)$ w.r.t. x .

[1 mark]

Solution:

Concept:

- For $-1 < u < 1$, $\frac{d}{dx} \cos^{-1} u = -u' / \sqrt{1 - u^2}$.
- The nonnegative square root satisfies $\sqrt{\cos^2 2x} = |\cos 2x|$, so the sign depends on the interval.

Step 1: Apply the chain rule Let $y = \cos^{-1}(\sin 2x)$. Taking $u = \sin 2x$ gives $u' = 2 \cos 2x$. Hence, wherever $\cos 2x \neq 0$,

$$\frac{dy}{dx} = -\frac{2 \cos 2x}{\sqrt{1 - \sin^2 2x}}.$$

Step 2: Simplify with the principal square root Using $1 - \sin^2 2x = \cos^2 2x$, the derivative is $-2 \cos 2x / |\cos 2x|$. It is -2 when $\cos 2x > 0$ and 2 when $\cos 2x < 0$. At $x = \pi/4 + k\pi/2$, $k \in \mathbb{Z}$, the function has a corner and its derivative does not exist.

Step 3: Conclusion $y' = -2 \cos 2x / |\cos 2x|$ for $\cos 2x \neq 0$; the differential is $dy = y' dx$. In particular, $y' = -2$ on $-\pi/4 < x < \pi/4$.

Quick Tip to Solve: Keep the absolute value in $\sqrt{\cos^2 2x}$; simplifying it to $\cos 2x$ requires a positive-sign interval.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5. If an operation $*$ on the set of integers $\mathbb{Z}$ is defined by $a * b = 2a^2 + b$, then find (i) whether it is a binary or not, and (ii) if a binary, then is it commutative or not.

[2 marks]

Solution:

Concept:

- A binary operation on $\mathbb{Z}$ is a well-defined mapping $\mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$; its result must be an integer for every pair.
- Commutativity requires $a * b = b * a$ for every $a, b \in \mathbb{Z}$. A single counterexample disproves it.

Step 1: Check closure on the integers Let a, b be any integers. Then a^2 , $2a^2$, and $2a^2 + b$ are all integers, because integers are closed under multiplication and addition. The rule also produces one uniquely determined result for each ordered pair. Therefore $a * b = 2a^2 + b$ defines a binary operation on $\mathbb{Z}$.

Step 2: Test the order of the operands Reversing the inputs gives $b * a = 2b^2 + a$, which need not equal $2a^2 + b$. For instance, choose $a = 1$, $b = 2$. Then

$$1 * 2 = 2(1)^2 + 2 = 4, \quad 2 * 1 = 2(2)^2 + 1 = 9.$$

Since both inputs lie in $\mathbb{Z}$ and the results differ, the universal commutativity condition fails. Equality for some other pair would not repair this counterexample.

Quick Tip to Solve: Check closure using arbitrary integers, then use a small unequal pair to disprove commutativity.

Step 3: Conclusion (i) $*$ is a binary operation on $\mathbb{Z}$. (ii) It is not commutative.

Quick Tip to Solve: Check closure using arbitrary integers, then use a small unequal pair to disprove commutativity.

6(a). Four cards are drawn one by one with replacement from a well-shuffled deck of playing cards. Find the probability that at least three cards are of diamonds. [2 marks]

Solution:

Concept:

- With replacement, each draw is independent and has diamond probability $p = 13/52 = 1/4$, with non-diamond probability $q = 3/4$.
- For four independent draws, $P(X = r) = \binom{4}{r} p^r q^{4-r}$, where X counts diamonds.

Step 1: Identify the favourable cases At least three diamonds means exactly three or exactly four diamonds. These events are disjoint. Replacement restores the full deck before each draw, so the probabilities stay fixed and the binomial model applies. For exactly three diamonds, the one non-diamond can occupy any of four positions.

Step 2: Evaluate and add the probabilities Thus

$$P(X = 3) = \binom{4}{3} \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right) = \frac{12}{256},$$
$$P(X = 4) = \binom{4}{4} \left(\frac{1}{4}\right)^4 = \frac{1}{256}.$$

Adding these disjoint possibilities gives $P(X \geq 3) = 12/256 + 1/256 = 13/256$. No case with two diamonds belongs to the requested event.

Step 3: Conclusion The required probability is $\frac{13}{256}$.

Quick Tip to Solve: For at least three successes in four trials, add only the binomial terms for three and four.

OR

6(b). The probability of two students A and B coming to school on time are $\frac{2}{7}$ and $\frac{4}{7}$, respectively. Assuming that the events 'A coming on time' and 'B coming on time' are independent, find the probability of only one of them coming to school on time. [2 marks]

Solution:

Concept:

- If A and B are independent, their complementary combinations are independent too.
- Exactly one occurs means the disjoint union $(A \cap B') \cup (A' \cap B)$, excluding the case where both

Quick Tip to Solve: For exactly one of two independent events, use $p(1 - q) + (1 - p)q$, keeping both exclusive cases.

occur.

Step 1: Form the two exclusive cases Here A denotes the first student arriving on time and B denotes the second. Given $P(A) = 2/7$ and $P(B) = 4/7$, their complementary probabilities are $P(A') = 5/7$ and $P(B') = 3/7$. Exactly one student is on time when either A occurs and B does not, or the reverse happens.

Step 2: Use independence and add Independence gives

$$P(A \cap B') = \frac{2}{7} \cdot \frac{3}{7} = \frac{6}{49},$$
$$P(A' \cap B) = \frac{5}{7} \cdot \frac{4}{7} = \frac{20}{49}.$$

These two events cannot occur together, so their probabilities add to $26/49$. Simply adding $P(A)$ and $P(B)$ would count the both-on-time case twice; that case must be excluded completely.

Step 3: Conclusion The probability of only one arriving on time is $\boxed{\frac{26}{49}}$.

Quick Tip to Solve: For exactly one of two independent events, use $p(1 - q) + (1 - p)q$, keeping both exclusive cases.

7. Find the value of $(x - y)$ from the matrix equation

$$2 \begin{bmatrix} x & 5 \\ 7 & y - 3 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}.$$

[2 marks]

Solution:

Concept:

- Scalar multiplication multiplies every entry of a matrix; matrices of the same order are added entry by entry.
- Two matrices are equal exactly when their corresponding entries are equal. Thus the diagonal entries determine the unknowns x and y .

Step 1: Expand the matrix on the left First multiply the first matrix by 2, including the whole expression $y - 3$. Adding the second matrix gives

$$2 \begin{bmatrix} x & 5 \\ 7 & y - 3 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2x - 3 & 10 - 4 \\ 14 + 1 & 2y - 6 + 2 \end{bmatrix} = \begin{bmatrix} 2x - 3 & 6 \\ 15 & 2y - 4 \end{bmatrix}.$$

The off-diagonal entries already agree with those on the right.

Step 2: Compare the diagonal entries Equating the first diagonal entry gives $2x - 3 = 7$, so $2x = 10$ and $x = 5$. Equating the second diagonal entry gives $2y - 4 = 14$, so $2y = 18$ and $y = 9$. These values satisfy all four entries of the matrix equation, so the required difference is obtained by subtracting 9 from 5.

Step 3: Conclusion Hence $\boxed{x - y = 5 - 9 = -4}$.

Quick Tip to Solve: Distribute the scalar over the entire entry $y - 3$ before comparing the diagonal entries.

8. Solve the following differential equation :

$$(y + 3x^2) \frac{dx}{dy} = x.$$

[2 marks]

Solution:

Concept:

- A first-order linear equation $dy/dx + P(x)y = Q(x)$ has integrating factor $IF = e^{\int P(x) dx}$.
- After multiplication by the integrating factor, the left side is the derivative of $y IF$. Here C denotes an arbitrary real constant.

Step 1: Express the equation with y as dependent variable For a nonzero branch of x , rewrite the given equation $(y + 3x^2) dx/dy = x$ as

$$\frac{dy}{dx} = \frac{y + 3x^2}{x}, \quad \frac{dy}{dx} - \frac{y}{x} = 3x.$$

This is linear in y , with $P(x) = -1/x$ and $Q(x) = 3x$. On an interval not crossing zero we may choose the integrating factor $1/x$, since a nonzero constant multiple of an integrating factor gives the same method.

Step 2: Multiply by the integrating factor and integrate Multiplying by $1/x$ gives

$$\frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = 3, \quad \frac{d}{dx} \left(\frac{y}{x} \right) = 3.$$

Integration therefore yields $y/x = 3x + C$. To check, differentiate $y = 3x^2 + Cx$: then $dy/dx = 6x + C$, while $(y + 3x^2)/x = 6x + C$, confirming the equation wherever the inverse derivative exists.

Step 3: Check the branch excluded by division The original equation is written for x as a function of y . The constant function $x(y) = 0$ has $dx/dy = 0$, so both sides of the original equation vanish. It is an additional solution that must be retained because division by x excluded it.

Step 4: Conclusion The nonzero solution branches satisfy $y = 3x^2 + Cx$; the original equation also admits $x = 0$.

Quick Tip to Solve: Treat y as the dependent variable to obtain a linear equation, and check $x=0$ before dividing by x .

9(a). Using vectors, prove that the points $(2, -1, 3)$, $(3, -5, 1)$ and $(-1, 11, 9)$ are collinear.

[2 marks]

Solution:

Concept:

- For points with position vectors $\vec{a}$ and $\vec{b}$, the displacement vector is $\overrightarrow{AB} = \vec{b} - \vec{a}$.
- Three distinct points are collinear if two displacement vectors from the same point are scalar multiples of each other. A negative scalar means opposite directions.

Step 1: Form two displacement vectors Let $A = (2, -1, 3)$, $B = (3, -5, 1)$ and $C = (-1, 11, 9)$.

Quick Tip to Solve: Subtract one common point from both others, then check the same multiplier in all three components.

Subtract the coordinates of A from those of each of the other two points:

$$\begin{aligned}\overrightarrow{AB} &= (3-2)\hat{i} + (-5+1)\hat{j} + (1-3)\hat{k} \\ &= \hat{i} - 4\hat{j} - 2\hat{k}, \\ \overrightarrow{AC} &= (-1-2)\hat{i} + (11+1)\hat{j} + (9-3)\hat{k} \\ &= -3\hat{i} + 12\hat{j} + 6\hat{k}.\end{aligned}$$

All signs follow from subtracting the same starting point.

Step 2: Compare the displacement vectors Each component of $\overrightarrow{AC}$ is -3 times the corresponding component of $\overrightarrow{AB}$. Thus $\overrightarrow{AC} = -3\overrightarrow{AB}$. Since $\overrightarrow{AB} \neq \vec{0}$, the two vectors have the same supporting line. The negative multiplier puts C on the opposite side of A from B , but does not affect collinearity.

Step 3: Conclusion Hence the points $(2, -1, 3)$, $(3, -5, 1)$, $(-1, 11, 9)$ are collinear.

Quick Tip to Solve: Subtract one common point from both others, then check the same multiplier in all three components.

OR

9(b). For any two vectors $\vec{a}$ and $\vec{b}$, prove that $(\vec{a} \times \vec{b})^2 = \vec{a}^2 \vec{b}^2 - (\vec{a} \cdot \vec{b})^2$.

[2 marks]

Solution:

Concept:

- For nonzero vectors making angle θ , $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$ and $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$.
- The square of a vector means its scalar square: $\vec{v}^2 = \vec{v} \cdot \vec{v} = |\vec{v}|^2$. Use $\sin^2\theta + \cos^2\theta = 1$.

Step 1: Evaluate the square of the cross product First suppose both vectors are nonzero and let $\theta \in [0, \pi]$ be the angle between them. By the magnitude formula for the cross product,

$$(\vec{a} \times \vec{b})^2 = |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2 \theta.$$

This square is a scalar, not another cross product. The formula includes parallel and antiparallel vectors because their sine factor is zero.

Step 2: Replace the sine square using the dot product Using the trigonometric identity and then the dot-product formula,

$$\begin{aligned}|\vec{a}|^2 |\vec{b}|^2 \sin^2 \theta &= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2 \theta) \\ &= |\vec{a}|^2 |\vec{b}|^2 - (|\vec{a}||\vec{b}|\cos\theta)^2 \\ &= \vec{a}^2 \vec{b}^2 - (\vec{a} \cdot \vec{b})^2.\end{aligned}$$

If either vector is zero, the cross product and dot product are zero and both sides vanish. Hence the statement also holds in that case without requiring an angle.

Step 3: Conclusion Therefore, for any two vectors, $(\vec{a} \times \vec{b})^2 = \vec{a}^2 \vec{b}^2 - (\vec{a} \cdot \vec{b})^2$.

Quick Tip to Solve: Write dot and cross products using the same angle, then use sin squared plus cos squared equals one.

10(a). Find :

$$\int \frac{x-1}{(x-2)(x-3)} dx.$$

[2 marks]

Solution:

Concept:

- A proper rational function with distinct linear factors can be separated into partial fractions.
- The logarithmic integral is $\int dx/(x-a) = \ln|x-a| + C$, valid on intervals avoiding $x=a$.

Step 1: Determine the partial fractions For $x \neq 2, 3$, write

$$\frac{x-1}{(x-2)(x-3)} = \frac{A}{x-2} + \frac{B}{x-3}.$$

Multiplying by the denominator gives the identity $x-1 = A(x-3) + B(x-2)$. Setting $x=2$ in this polynomial identity gives $1 = -A$, so $A = -1$. Setting $x=3$ gives $2 = B$, so $B = 2$. As a check, the resulting numerator is $-(x-3) + 2(x-2) = x-1$.

Step 2: Integrate the two simple fractions By linearity of integration,

$$\begin{aligned} \int \frac{x-1}{(x-2)(x-3)} dx &= - \int \frac{dx}{x-2} + 2 \int \frac{dx}{x-3} \\ &= -\ln|x-2| + 2\ln|x-3| + C. \end{aligned}$$

The absolute values make the logarithms meaningful on every interval where the original integrand is defined. Differentiating these logarithmic terms recovers $-1/(x-2) + 2/(x-3)$, which is the original fraction.

Step 3: Conclusion $\int \frac{x-1}{(x-2)(x-3)} dx = 2\ln|x-3| - \ln|x-2| + C$, for $x \neq 2, 3$.

Quick Tip to Solve: After clearing denominators, substitute the two denominator roots to find A and B without simultaneous equations.

OR

10(b). Integrate $\frac{e^x}{\sqrt{5-4e^x-e^{2x}}}$ with respect to x .

[2 marks]

Solution:

Concept:

- Use substitution when the numerator is the derivative of an expression in the denominator:
 $d(e^x) = e^x dx$.
- The standard integral $\int du/\sqrt{a^2-u^2} = \sin^{-1}(u/a) + C$ holds for $|u| < a$, with $a > 0$.

Step 1: Complete the square and substitute The denominator can be rewritten by completing the square in e^x :

$$5 - 4e^x - e^{2x} = 9 - (e^x + 2)^2.$$

Quick Tip to Solve: Complete the square in e to the x first; adding 2 makes its differential exactly the numerator.

Put $u = e^x + 2$; then $du = e^x dx$. Thus the entire numerator is absorbed into the differential, and the integral becomes

$$\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}} dx = \int \frac{du}{\sqrt{9 - u^2}}.$$

The radius parameter in the standard formula is 3, since $9 = 3^2$.

Step 2: Apply the inverse-sine integral and check the domain Using the standard formula gives $\sin^{-1}(u/3) + C$. Substituting back yields $\sin^{-1}((e^x + 2)/3) + C$. For a real integrand the radicand must be positive. With $t = e^x > 0$, it is $(1 - t)(t + 5) > 0$, equivalent to $0 < t < 1$, or $x < 0$. On this interval, differentiation gives

$$\frac{d}{dx} \sin^{-1} \left(\frac{e^x + 2}{3} \right) = \frac{e^x}{\sqrt{9 - (e^x + 2)^2}},$$

which verifies the result exactly.

Step 3: Conclusion $\int \frac{e^x}{\sqrt{5 - 4e^x - e^{2x}}} dx = \sin^{-1} \left(\frac{e^x + 2}{3} \right) + C$, on $x < 0$.

Quick Tip to Solve: Complete the square in e to the x first; adding 2 makes its differential exactly the numerator.

11. Find :

$$\int e^x \left(\frac{2 + \sin 2x}{2 \cos^2 x} \right) dx.$$

[2 marks]

Solution:

Concept:

- The product rule gives $\frac{d}{dx}[e^x f(x)] = e^x[f(x) + f'(x)]$. Thus the integral of this expression is $e^x f(x) + C$.
- Use $\sin 2x = 2 \sin x \cos x$, $\tan x = \sin x / \cos x$, and $\frac{d}{dx} \tan x = \sec^2 x$ on intervals where $\cos x \neq 0$.

Step 1: Simplify the trigonometric factor Splitting the numerator and using the double-angle identity gives

$$\frac{2 + \sin 2x}{2 \cos^2 x} = \frac{1}{\cos^2 x} + \frac{2 \sin x \cos x}{2 \cos^2 x} = \sec^2 x + \tan x.$$

Hence the required integral becomes $\int e^x (\tan x + \sec^2 x) dx$. This simplification holds wherever the original integrand is defined.

Step 2: Recognise the derivative of a product Take $f(x) = \tan x$, so that $f'(x) = \sec^2 x$. By the product rule,

$$\frac{d}{dx}(e^x \tan x) = e^x \tan x + e^x \sec^2 x.$$

The derivative agrees with the simplified integrand term by term. Therefore the product-rule integration identity applies directly, with an arbitrary constant on each interval of definition.

Quick Tip to Solve: Split the numerator first: identify $\tan x$ and its derivative before applying the product rule in reverse.

Step 3: Conclusion

$$\int e^x \left(\frac{2 + \sin 2x}{2 \cos^2 x} \right) dx = e^x \tan x + C.$$

Quick Tip to Solve: Split the numerator first: identify $\tan x$ and its derivative before applying the product rule in reverse.

12. If A and B are independent events with $P(A) = \frac{3}{7}$ and $P(B) = \frac{2}{5}$, then find $P(A' \cap B')$. [2 marks]

Solution:**Concept:**

- For an event E , its complement has probability $P(E') = 1 - P(E)$.
- Independence means $P(A \cap B) = P(A)P(B)$. The complements of two independent events are also independent.

Step 1: Find the complement probabilities The complement A' means that A does not occur, and similarly for B' . From the given probabilities,

$$P(A') = 1 - \frac{3}{7} = \frac{4}{7}, \quad P(B') = 1 - \frac{2}{5} = \frac{3}{5}.$$

Both complement probabilities lie between zero and one, as required.

Step 2: Use independence for the intersection To justify using the product for the complements, apply inclusion-exclusion and the independence of A and B :

$$\begin{aligned} P(A' \cap B') &= 1 - P(A \cup B) \\ &= 1 - P(A) - P(B) + P(A \cap B) \\ &= 1 - P(A) - P(B) + P(A)P(B) \\ &= (1 - P(A))(1 - P(B)). \end{aligned}$$

Substituting the complement probabilities gives $P(A' \cap B') = \frac{4}{7} \times \frac{3}{5} = \frac{12}{35}$. The intersection requires both events to be absent, so addition of the complement probabilities would be inappropriate.

Step 3: Conclusion

$$P(A' \cap B') = \frac{12}{35}.$$

Quick Tip to Solve: For independent events, complement each probability first and then multiply to find the probability that neither event occurs.

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13. Find the equation of the plane passing through the point $(-1, 3, 2)$ and perpendicular to the planes $x + 2y + 3z = 5$ and $3x + 3y + z = 0$. [4 marks]

Solution:

Concept:

- A plane $ax + by + cz = d$ has a normal vector (a, b, c) . Two planes are perpendicular when their normal vectors have zero scalar product.
- A plane through (x_0, y_0, z_0) with normal $(a, b, c) \neq (0, 0, 0)$ has equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.
- The cross product of two nonparallel normal vectors gives a vector perpendicular to both, which may be used as the normal of the required plane.

Step 1: Identify the required normal direction The given planes have normal vectors $\mathbf{n}_1 = (1, 2, 3)$ and $\mathbf{n}_2 = (3, 3, 1)$. If $\mathbf{n} = (a, b, c)$ is the normal of the required plane, its perpendicularity to each given plane requires

$$a + 2b + 3c = 0, \quad 3a + 3b + c = 0.$$

Thus $\mathbf{n}$ must be perpendicular to both given normals. Since those normals are not parallel, their cross product is nonzero and provides the required direction.

Step 2: Calculate a normal vector Using the cross product,

$$\begin{aligned}\mathbf{n}_1 \times \mathbf{n}_2 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 3 & 1 \end{vmatrix} \\ &= (2 - 9)\mathbf{i} - (1 - 9)\mathbf{j} + (3 - 6)\mathbf{k} \\ &= -7\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}.\end{aligned}$$

We may therefore choose $\mathbf{n} = (-7, 8, -3)$. Any nonzero scalar multiple yields the same plane; changing all three signs does not change its geometry.

Step 3: Use the given point and check perpendicularity The required plane passes through $(-1, 3, 2)$. Its point-normal form is

$$-7(x + 1) + 8(y - 3) - 3(z - 2) = 0.$$

Expanding gives $-7x + 8y - 3z - 25 = 0$, or equivalently $7x - 8y + 3z + 25 = 0$. Substitution of the point gives $-7 - 24 + 6 + 25 = 0$, confirming incidence. The normal $(7, -8, 3)$ also satisfies

$$(7, -8, 3) \cdot (1, 2, 3) = 7 - 16 + 9 = 0,$$

$$(7, -8, 3) \cdot (3, 3, 1) = 21 - 24 + 3 = 0.$$

Hence the constructed plane has both required perpendicularities.

Quick Tip to Solve: Cross the normals of the two given planes, then insert the given point using point-normal form and verify both dot products.

Step 4: Conclusion

$$7x - 8y + 3z + 25 = 0.$$

Quick Tip to Solve: Cross the normals of the two given planes, then insert the given point using point-normal form and verify both dot products.

14. Evaluate :

$$\int_1^5 (|x-1| + |x-2| + |x-4|) dx.$$

[4 marks]

Solution:**Concept:**

- The absolute value satisfies $|t| = t$ for $t \geq 0$, and $|t| = -t$ for $t < 0$. Split the interval at every interior point where an argument changes sign.
- Definite integrals are additive across adjacent intervals: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$. Linear functions are integrated using $\int x dx = x^2/2$.

Step 1: Remove the absolute values on each interval Let $I = \int_1^5 (|x-1| + |x-2| + |x-4|) dx$. Throughout this interval $x-1 \geq 0$, while the other arguments change sign at 2 and 4. Split the interval into $[1, 2]$, $[2, 4]$ and $[4, 5]$. Where an argument is zero, the adjacent formulae agree.

$$|x-1| + |x-2| + |x-4| = \begin{cases} (x-1) + (2-x) + (4-x) = 5-x, & 1 \leq x \leq 2, \\ (x-1) + (x-2) + (4-x) = x+1, & 2 \leq x \leq 4, \\ (x-1) + (x-2) + (x-4) = 3x-7, & 4 \leq x \leq 5. \end{cases}$$

Step 2: Integrate the three pieces By interval additivity,

$$\begin{aligned} I &= \int_1^2 (5-x) dx + \int_2^4 (x+1) dx + \int_4^5 (3x-7) dx \\ &= \left[5x - \frac{x^2}{2} \right]_1^2 + \left[\frac{x^2}{2} + x \right]_2^4 + \left[\frac{3x^2}{2} - 7x \right]_4^5 \\ &= (8 - \frac{9}{2}) + (12 - 4) + (\frac{5}{2} - (-4)) \\ &= \frac{7}{2} + 8 + \frac{13}{2} = 18. \end{aligned}$$

As a separate check, the integrals of $|x-1|$, $|x-2|$ and $|x-4|$ are respectively 8, $\frac{1}{2} + \frac{9}{2} = 5$, and $\frac{9}{2} + \frac{1}{2} = 5$. Their sum agrees with the piecewise calculation. All contributions are non-negative, as expected for absolute values.

Step 3: Conclusion The required integral is $\boxed{18}$.

Quick Tip to Solve: Split at 2 and 4 only: the zero of $x-1$ is already the lower endpoint, so it creates no extra interval.

15. If x, y, z are different and $\Delta = \begin{vmatrix} x & x^2 & x^3 - 1 \\ y & y^2 & y^3 - 1 \\ z & z^2 & z^3 - 1 \end{vmatrix} = 0$, then using properties of determinants, show that $xyz = 1$. [4 marks]

Solution:

Concept:

- A determinant is linear in each column. Therefore the third column $(x^3 - 1, y^3 - 1, z^3 - 1)^T$ can be split into its cubic column minus its constant column.
- Factoring a scalar from a row factors it from the determinant. Subtracting a multiple of one row from another leaves the determinant unchanged; two column swaps also preserve its sign.

Step 1: Split the third column Linearity in the third column gives

$$\Delta = \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} - \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}.$$

Write $V = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$. Factor x, y, z from the respective rows of the first determinant to obtain $xyzV$. This is a factorisation, so no assumption that these numbers are nonzero is required. Move the constant column of the second determinant to first place using two column swaps; its sign is unchanged, giving V . Therefore $\Delta = (xyz - 1)V$.

Step 2: Factor the remaining determinant In V , apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$. Since $y^2 - x^2 = (y - x)(y + x)$ and $z^2 - x^2 = (z - x)(z + x)$, factor $y - x$ and $z - x$ from the last two rows:

$$V = (y - x)(z - x) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & y + x \\ 0 & 1 & z + x \end{vmatrix}.$$

Expanding along the first column gives

$$V = (y - x)(z - x)((z + x) - (y + x)) = (y - x)(z - x)(z - y).$$

Substitute into the given equation $\Delta = 0$:

$$0 = (xyz - 1)(y - x)(z - x)(z - y).$$

The numbers x, y, z are pairwise different, so all three difference factors are nonzero. Their product is therefore nonzero, and we can cancel it to obtain $xyz - 1 = 0$.

Step 3: Conclusion Thus, using determinant properties and the distinctness of x, y, z , we obtain $xyz = 1$.

Quick Tip to Solve: Split the cubic-minus-one column first; the two determinants then share the same Vandermonde factor.

16. Prove that :

$$\sin^{-1} \frac{4}{5} + \tan^{-1} \frac{5}{12} + \cos^{-1} \frac{63}{65} = \frac{\pi}{2}.$$

[4 marks]

Solution:

Concept:

- Principal values of $\sin^{-1} t$, $\tan^{-1} t$ and $\cos^{-1} t$ lie in $[-\pi/2, \pi/2]$, $(-\pi/2, \pi/2)$ and $[0, \pi]$, respectively. These ranges determine the signs and remove ambiguity in inverse trigonometric identities.
- Use $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ and $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$, followed by the complementary-angle identity.

Step 1: Find the trigonometric ratios of the principal angles Put $\alpha = \sin^{-1}(4/5)$ and $\beta = \tan^{-1}(5/12)$. Both arguments are positive, so $0 < \alpha, \beta < \pi/2$. In particular, all their sine and cosine values are positive. Since $\sin \alpha = 4/5$,

$$\cos \alpha = \sqrt{1 - \sin^2 \alpha} = \sqrt{1 - 16/25} = 3/5.$$

For β , the positive hypotenuse corresponding to opposite and adjacent sides 5 and 12 is $\sqrt{5^2 + 12^2} = 13$. Hence $\sin \beta = 5/13$ and $\cos \beta = 12/13$. The positive signs are justified by the principal ranges, rather than by the squared identity alone.

Step 2: Compute the sum and establish its range The addition formulae give

$$\begin{aligned}\sin(\alpha + \beta) &= \frac{4}{5} \frac{12}{13} + \frac{3}{5} \frac{5}{13} = \frac{48 + 15}{65} = \frac{63}{65}, \\ \cos(\alpha + \beta) &= \frac{3}{5} \frac{12}{13} - \frac{4}{5} \frac{5}{13} = \frac{36 - 20}{65} = \frac{16}{65}.\end{aligned}$$

As both individual angles are acute, their sum is between 0 and π . Its cosine is positive, so the sum must actually lie in $(0, \pi/2)$. This is essential: knowledge of its sine alone would also permit a supplementary angle, which would not give the required principal-value equality.

Step 3: Identify the third angle as the complement Let $\theta = \pi/2 - (\alpha + \beta)$. The established range gives $0 < \theta < \pi/2$, which is contained in the principal range $[0, \pi]$ of inverse cosine. Therefore

$$\cos \theta = \cos(\pi/2 - (\alpha + \beta)) = \sin(\alpha + \beta) = \frac{63}{65}.$$

Since cosine is one-to-one on $[0, \pi]$, this implies

$$\cos^{-1} \frac{63}{65} = \theta = \frac{\pi}{2} - \alpha - \beta.$$

Adding $\alpha + \beta$ to both sides now proves the identity for exactly the principal values appearing in the question.

Step 4: Conclusion Hence $\sin^{-1} \frac{4}{5} + \tan^{-1} \frac{5}{12} + \cos^{-1} \frac{63}{65} = \frac{\pi}{2}$, as required.

Quick Tip to Solve: Use the 3-4-5 and 5-12-13 ratios, then check the sum's positive cosine before identifying the complementary inverse cosine.

17. Using vectors, find the value of x such that the four points $A(x, 5, -1)$, $B(3, 2, 1)$, $C(4, 5, 5)$ and $D(4, 2, -2)$ are coplanar. [4 marks]

Solution:

Concept:

- Four points A, B, C, D are coplanar if three vectors drawn from the same point have zero scalar triple product: $\vec{BA} \cdot (\vec{BC} \times \vec{BD}) = 0$.
- The vector $\vec{BC} \times \vec{BD}$ is perpendicular to both vectors. When nonzero, it is a normal to the plane through B, C, D ; a point lies in that plane precisely when its displacement from B is perpendicular to this normal.

Step 1: Form three vectors from the common point B Taking $B(3, 2, 1)$ as the common initial point avoids introducing x into two of the vectors. Subtracting coordinates component by component,

$$\begin{aligned}\vec{BA} &= (x - 3)\hat{i} + 3\hat{j} - 2\hat{k}, \\ \vec{BC} &= \hat{i} + 3\hat{j} + 4\hat{k}, \\ \vec{BD} &= \hat{i} + 0\hat{j} - 3\hat{k}.\end{aligned}$$

For example, the last component of $\vec{BD}$ is $-2 - 1 = -3$, whereas that of $\vec{BA}$ is $-1 - 1 = -2$. Keeping all vectors based at the same point makes the scalar triple-product condition applicable directly.

Step 2: Compute a normal vector to the plane through B, C and D Evaluate the cross product, remembering the negative sign of the middle cofactor:

$$\begin{aligned}\vec{BC} \times \vec{BD} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 4 \\ 1 & 0 & -3 \end{vmatrix} \\ &= \hat{i}\{3(-3) - 4(0)\} - \hat{j}\{1(-3) - 4(1)\} + \hat{k}\{1(0) - 3(1)\} \\ &= -9\hat{i} + 7\hat{j} - 3\hat{k}.\end{aligned}$$

This vector is nonzero, so B, C, D are noncollinear and define a unique plane. Thus setting the scalar triple product to zero is both necessary and sufficient for A to lie in that plane.

Step 3: Apply coplanarity and verify the candidate The required condition becomes

$$\begin{aligned}\vec{BA} \cdot (\vec{BC} \times \vec{BD}) &= 0, \\ -9(x - 3) + 7(3) + (-3)(-2) &= 0, \\ -9x + 27 + 21 + 6 &= 0, \\ -9x + 54 &= 0.\end{aligned}$$

Solving this linear equation gives the candidate $x = 6$. To check it directly, $\vec{BA}$ then has components $(3, 3, -2)$, and its dot product with the normal is $-27 + 21 + 6 = 0$. Hence the point lies in the plane through the other three points. The nonzero coefficient of x also shows that no second value satisfies the condition.

Step 4: Conclusion The four points are coplanar precisely when $x = 6$.

Quick Tip to Solve: Choose B as the common initial point: only BA contains x , while the cross product of BC and BD supplies a fixed normal.

18(a). Differentiate $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, $|x| < \frac{1}{\sqrt{3}}$, **w.r.t.** $\tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right)$.

[4 marks]

Solution:

Concept:

- For functions $u(x)$ and $v(x)$, the chain rule gives $du/dv = (du/dx)/(dv/dx)$ when $dv/dx \neq 0$.
- The triple-angle identity is $\tan 3\theta = (3 \tan \theta - \tan^3 \theta)/(1 - 3 \tan^2 \theta)$. The principal range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$.
- The derivatives of $\tan^{-1} x$ and $\sin^{-1} x$ are $1/(1 + x^2)$ and $1/\sqrt{1 - x^2}$, respectively.

Step 1: Identify the first function on its principal branch Write $u = \tan^{-1}((3x - x^3)/(1 - 3x^2))$ and set $\theta = \tan^{-1} x$. Since $|x| < 1/\sqrt{3}$, we have $-\pi/6 < \theta < \pi/6$. Thus 3θ lies in the principal range of the inverse tangent. Using the triple-angle formula gives

$$u = \tan^{-1}(\tan 3\theta) = 3\theta = 3 \tan^{-1} x, \quad \frac{du}{dx} = \frac{3}{1 + x^2}.$$

Checking this range prevents an incorrect inversion of the tangent function, whose unrestricted values repeat after adding multiples of π .

Step 2: Identify and differentiate the second function Write $v = \tan^{-1}(x/\sqrt{1 - x^2})$. Let $\phi = \sin^{-1} x$, so that $-\pi/2 < \phi < \pi/2$. In this interval cosine is positive, and therefore $\cos \phi = \sqrt{1 - x^2}$. Consequently

$$\tan \phi = \frac{x}{\sqrt{1 - x^2}}, \quad v = \tan^{-1}(\tan \phi) = \phi = \sin^{-1} x, \quad \frac{dv}{dx} = \frac{1}{\sqrt{1 - x^2}}.$$

The given interval lies inside $(-1, 1)$, so this derivative exists and never vanishes.

Step 3: Differentiate the first expression with respect to the second Both quantities depend on x . Using the chain rule and the nonzero derivative of v ,

$$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{3}{1 + x^2} \div \frac{1}{\sqrt{1 - x^2}} = \frac{3\sqrt{1 - x^2}}{1 + x^2}.$$

The factor $\sqrt{1 - x^2}$ enters because we divide by the derivative of the second expression; differentiating only the first expression would not answer the question.

Step 4: Conclusion The required derivative is $\boxed{\frac{3\sqrt{1 - x^2}}{1 + x^2}}$, for $|x| < 1/\sqrt{3}$.

Quick Tip to Solve: Reduce the expressions to $3 \tan^{-1} x$ and $\sin^{-1} x$, verify their principal ranges, and then divide their derivatives.

18(b). If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, $|x| < 1$, $|y| < 1$, show that $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$.

[4 marks]

Solution:**Concept:**

- Implicit differentiation treats y as a function of x . Thus $d\sqrt{1-y^2}/dx = -y(dy/dx)/\sqrt{1-y^2}$.
- The difference of squares identity simplifies products of conjugate radicals: $(A+B)(A-B) = A^2 - B^2$.

Step 1: Set up positive radicals and simplify the given relation Put $A = \sqrt{1-x^2}$ and $B = \sqrt{1-y^2}$. The restrictions $|x| < 1$ and $|y| < 1$ ensure $A > 0$ and $B > 0$. The equation becomes $A + B = a(x-y)$. Since its left side is positive, $a \neq 0$ and $x - y \neq 0$. Multiplying by $A - B$ gives

$$a(x-y)(A-B) = A^2 - B^2 = y^2 - x^2 = -(x-y)(x+y).$$

Cancelling the nonzero factor $x - y$ yields $a(A - B) = -(x + y)$, or

$$aA + x = aB - y.$$

This relation will simplify the derivative without requiring us to solve explicitly for y .

Step 2: Differentiate implicitly and collect derivative terms Differentiate $A + B = a(x - y)$, regarding a as constant and writing $y' = dy/dx$:

$$-\frac{x}{A} - \frac{y}{B}y' = a(1 - y').$$

On collecting all terms containing y' , we obtain

$$\left(a - \frac{y}{B}\right)y' = a + \frac{x}{A}, \quad \frac{aB - y}{B}y' = \frac{aA + x}{A}.$$

Both sides now contain the equal quantities found from the original equation.

Step 3: Cancel the common nonzero factor Let $K = aA + x = aB - y$. This factor cannot vanish: if $K = 0$, then $a = -x/A = y/B$, giving $xB = -yA$. Squaring this equality yields $x^2 = y^2$; since $x \neq y$, we must have $y = -x$ and hence $A = B$. The original relation would then give $2A = 2ax = -2x^2/A$, which is impossible because its left side is positive and its right side is nonpositive. We may therefore cancel K in

$$\frac{K}{B}y' = \frac{K}{A}, \quad y' = \frac{B}{A}.$$

Step 4: Conclusion Hence $\frac{dy}{dx} = \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = \sqrt{\frac{1-y^2}{1-x^2}}$, as required.

Quick Tip to Solve: Multiply the original radical equation by its conjugate first to obtain $aA + x = aB - y$; this makes the implicit derivative simplify immediately.

19(a). Prove that the relation R in the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ given by $R = \{(a, b) : |a - b| \text{ is even}\}$ is an equivalence relation. [4 marks]

Solution:

Concept:

- An equivalence relation on a set must be reflexive, symmetric and transitive. Each property must hold for every relevant element of the set.
- For integers, $|a - b|$ is even exactly when $a - b$ is even, because reversing the sign preserves divisibility by 2.
- An even difference means that the two integers have the same parity: both are odd or both are even.

Step 1: Verify reflexivity Take any $a \in A = \{1, 2, 3, 4, 5, 6, 7\}$. Then

$$|a - a| = 0 = 2 \cdot 0.$$

Zero is an even integer. Hence $(a, a) \in R$ for every $a \in A$, proving that R is reflexive on A .

Step 2: Verify symmetry Let $a, b \in A$ and suppose $(a, b) \in R$. Thus $|a - b|$ is even. Since

$$|b - a| = |-(a - b)| = |a - b|,$$

$|b - a|$ is also even. Therefore $(b, a) \in R$. This implication applies to every pair in R , so the relation is symmetric.

Step 3: Verify transitivity Let $a, b, c \in A$, with $(a, b) \in R$ and $(b, c) \in R$. Thus $|a - b|$ and $|b - c|$ are even. Their corresponding signed differences are even as well, so there are integers m, n such that

$$a - b = 2m, \quad b - c = 2n.$$

Adding the signed differences yields

$$a - c = (a - b) + (b - c) = 2(m + n).$$

Since $m + n$ is an integer, $a - c$ is even. Taking its absolute value preserves evenness, so $|a - c|$ is even and $(a, c) \in R$. Therefore R is transitive. This uses signed differences; the sum of two absolute differences need not equal the absolute value of the total difference.

Step 4: Conclusion The relation R is reflexive, symmetric and transitive, and hence is an equivalence relation on A . Its equivalence classes are $\{1, 3, 5, 7\}$ and $\{2, 4, 6\}$, consisting of odd and even elements respectively.

Quick Tip to Solve: Use even signed differences for transitivity and add them through the common element; do not add the absolute values directly.

19(b). Show that the function f in $A = \mathbb{R} - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is one-one and onto. Hence, find f^{-1} . [4 marks]

Solution:

Concept:

- A function $f : A \rightarrow A$ is one-one if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, and onto if each $y \in A$ has a preimage in A .
- To obtain the inverse, solve $y = f(x)$ for x . Check that all denominators are nonzero and that the resulting preimage belongs to the specified domain.

Step 1: Check that the function maps into the stated set Here $A = \mathbb{R} \setminus \{2/3\}$. For $x \in A$, $6x - 4 \neq 0$, so $f(x)$ is defined. Also $f(x)$ cannot equal $2/3$: that equality would give $3(4x + 3) = 2(6x - 4)$, or $12x + 9 = 12x - 8$, which is impossible. Therefore $f(x) \in A$, and $f : A \rightarrow A$ is well defined.

Step 2: Prove that the function is one-one Let $x_1, x_2 \in A$ and suppose $f(x_1) = f(x_2)$. Their denominators are nonzero, so cross multiplication gives

$$\begin{aligned}(4x_1 + 3)(6x_2 - 4) &= (4x_2 + 3)(6x_1 - 4), \\ 24x_1x_2 - 16x_1 + 18x_2 - 12 &= 24x_1x_2 - 16x_2 + 18x_1 - 12, \\ 34x_2 &= 34x_1.\end{aligned}$$

Hence $x_1 = x_2$. Thus f is one-one.

Step 3: Prove that the function is onto Choose any $y \in A$. Solving $y = (4x + 3)/(6x - 4)$ gives

$$\begin{aligned}6xy - 4y &= 4x + 3, \\ (6y - 4)x &= 4y + 3, \\ x &= \frac{4y + 3}{6y - 4}.\end{aligned}$$

Since $y \neq 2/3$, this value is defined. It also lies in A , since assuming it equals $2/3$ again gives $12y + 9 = 12y - 8$, a contradiction. To verify the preimage explicitly, substitution yields $f(x) = \frac{(34y)/(6y-4)}{34/(6y-4)} = y$. Thus every element of the codomain has a preimage in the domain, proving that f is onto.

Step 4: Conclusion The function $f : A \rightarrow A$ is one-one and onto, so its inverse exists. The unique preimage above, written with input symbol x , gives

$$\boxed{f^{-1}(x) = \frac{4x + 3}{6x - 4}}, \quad x \in \mathbb{R} \setminus \{2/3\}.$$

Quick Tip to Solve: When solving for the preimage, check both its denominator and the excluded value; here the inverse repeats the original formula.

20(a). Find the particular solution of the differential equation :

$$(1 + e^{2x}) dy + (1 + y^2) e^x dx = 0, \text{ given that } y(0) = 1.$$

[4 marks]

Solution:

Concept:

- Separate variables by putting the terms involving y with dy and those involving x with dx .
- Use $\int dt/(1+t^2) = \tan^{-1} t + C$. In the exponential integral the substitution $t = e^x$ gives $dt = e^x dx$.
- The initial condition determines the arbitrary constant and selects the particular solution.

Step 1: Separate the variables The given equation is $(1 + e^{2x}) dy + (1 + y^2) e^x dx = 0$. Divide by $(1 + y^2)(1 + e^{2x})$, whose factors are strictly positive for real x, y . Thus no real solution is lost and

$$\frac{dy}{1 + y^2} = -\frac{e^x dx}{1 + e^{2x}}.$$

Step 2: Integrate both sides Put $t = e^x$. Then $dt = e^x dx$, and the right-hand denominator becomes $1 + t^2$. The separated equation yields

$$\int \frac{dy}{1 + y^2} = -\int \frac{dt}{1 + t^2},$$
$$\tan^{-1} y = -\tan^{-1}(e^x) + C.$$

Equivalently, $\tan^{-1} y + \tan^{-1}(e^x) = C$, where the constants from the two integrations have been combined into one constant.

Step 3: Apply the given initial value At $x = 0$, we have $y = 1$ and $e^x = 1$. Consequently,

$$C = \tan^{-1} 1 + \tan^{-1} 1 = \frac{\pi}{2}.$$

Hence $\tan^{-1} y = \pi/2 - \tan^{-1}(e^x)$. Since $e^x > 0$, the right-hand angle lies between zero and $\pi/2$. Taking tangents gives $y = 1/e^x$. To verify, its derivative is $-e^{-x}$, so substitution gives

$$-(1 + e^{2x})e^{-x} + (1 + e^{-2x})e^x = 0.$$

The value at $x = 0$ is also exactly 1, as required.

Step 4: Conclusion The particular solution is $y = e^{-x}$, valid for every real x .

Quick Tip to Solve: Recognise the two inverse-tangent integrals and use $t = e^x$ to handle the exponential fraction in one substitution.

20(b). Find the particular solution of the differential equation :

$$x \frac{dy}{dx} \sin\left(\frac{y}{x}\right) + x - y \sin\left(\frac{y}{x}\right) = 0, \text{ given that } y(1) = \frac{\pi}{2}.$$

[4 marks]

Solution:

Concept:

- A differential equation involving y/x is reduced by the homogeneous substitution $y = vx$, where $v = y/x$.
- Differentiation gives $dy/dx = v + x dv/dx$. The resulting equation can be separated and integrated using $\int \sin v dv = -\cos v$.
- The initial value fixes the integration constant. The original equation requires $x \neq 0$ because it contains y/x .

Step 1: Make the homogeneous substitution The printed equation is

$$x \frac{dy}{dx} \sin\left(\frac{y}{x}\right) + x - y \sin\left(\frac{y}{x}\right) = 0.$$

Put $y = vx$, so that $dy/dx = v + x dv/dx$. Substituting and cancelling the opposite terms $xv \sin v$ gives

$$\begin{aligned} x(v + x dv/dx) \sin v + x - vx \sin v &= 0, \\ x^2 \sin v \frac{dv}{dx} + x &= 0. \end{aligned}$$

Step 2: Separate and integrate Since the original equation has $x \neq 0$, divide by x^2 and rearrange:

$$\sin v dv = -\frac{dx}{x}.$$

Integrating yields $-\cos v = -\log|x| + C_1$. Rename the arbitrary constant after multiplying by -1 . The general implicit relation is therefore

$$\cos\left(\frac{y}{x}\right) = \log|x| + C.$$

Step 3: Determine the constant and check the relation The given condition $y(1) = \pi/2$ gives $v = \pi/2$ at $x = 1$. Thus $0 = 0 + C$, so $C = 0$. On the interval containing $x = 1$, we have $x > 0$ and can write $\log|x| = \log x$. Differentiating the resulting relation gives

$$-\sin\left(\frac{y}{x}\right) \frac{x dy/dx - y}{x^2} = \frac{1}{x}.$$

Multiplication by x^2 and rearrangement recover precisely the printed equation. The initial condition is satisfied because $\cos(\pi/2) = \log 1 = 0$. The local branch is $y = x \cos^{-1}(\log x)$, where $-1 < \log x < 1$; this branch contains the initial point.

Step 4: Conclusion The particular solution is $\boxed{\cos(y/x) = \log x}$. Equivalently, the branch through the given initial point is $y = x \cos^{-1}(\log x)$, for $e^{-1} < x < e$.

Quick Tip to Solve: Group the sine terms as $(x dy/dx - y) \sin(y/x)$; this reveals the derivative of y/x immediately.

21. If $y = (\sin x)^x + \sin^{-1}(\sqrt{1-x^2})$, then find $\frac{dy}{dx}$.

[4 marks]

Solution:

Concept:

- Logarithmic differentiation is useful when both the base and exponent depend on x . If $u = (\sin x)^x$ with $\sin x > 0$, then $\log u = x \log(\sin x)$.
- The inverse-sine chain rule is $\frac{d}{dx} \sin^{-1} v = \frac{v'}{\sqrt{1-v^2}}$, where the square root is non-negative. In particular, $\sqrt{x^2} = |x|$.
- For the usual differentiable real-valued interpretation of the printed variable power, combine $\sin x > 0$ with $|x| \leq 1$. The interior domain is therefore $0 < x < 1$, on which $|x| = x$.

Step 1: Differentiate the variable power Put $u = (\sin x)^x$. Taking natural logarithms on the interior domain gives

$$\log u = x \log(\sin x).$$

Apply the product rule to the right side and the chain rule to the left side:

$$\frac{u'}{u} = \log(\sin x) + x \frac{\cos x}{\sin x} = \log(\sin x) + x \cot x.$$

Thus the derivative of the first term is

$$u' = (\sin x)^x [\log(\sin x) + x \cot x].$$

Both contributions are needed: differentiating just the base would miss the derivative of the exponent.

Step 2: Differentiate the inverse-sine term Let $v = \sqrt{1-x^2}$. The square-root chain rule gives

$$v' = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}.$$

Consequently,

$$\begin{aligned} \frac{d}{dx} \sin^{-1}(\sqrt{1-x^2}) &= \frac{-x}{\sqrt{1-x^2} \sqrt{1-(1-x^2)}} \\ &= -\frac{x}{|x| \sqrt{1-x^2}}. \end{aligned}$$

Since $0 < x < 1$, this simplifies to $-1/\sqrt{1-x^2}$. Equivalently, on this domain the inverse-sine term equals $\cos^{-1} x$, which confirms the negative sign.

Step 3: Apply the sum rule The given function is the sum of u and the inverse-sine expression, so its derivative is the sum of the two derivatives obtained above. The logarithmic term belongs only to the first summand, while the inverse-trigonometric contribution retains its negative sign.

Step 4: Conclusion

$$\boxed{\frac{dy}{dx} = (\sin x)^x [\log(\sin x) + x \cot x] - \frac{1}{\sqrt{1-x^2}}}, \quad 0 < x < 1.$$

Quick Tip to Solve: Log-differentiate the first summand separately; for the second, use $\sin^{-1} \sqrt{1-x^2} = \cos^{-1} x$ on $0 < x < 1$.

22. Find :

$$\int \cos 2x \cdot \cos 4x \cdot \cos 6x \, dx.$$

[4 marks]

Solution:**Concept:**

- The product-to-sum identity $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$ converts products of cosines into terms that can be integrated individually.
- The double-angle identity gives $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$. Also, $\int \cos(kx) \, dx = \frac{\sin(kx)}{k} + C$ for nonzero constant k .

Step 1: Combine the first two cosine factors Write the required integral as I . Choose $A = 2x$ and $B = 4x$ in the product-to-sum identity. Since cosine is even, $\cos(-2x) = \cos 2x$, and hence

$$\cos 2x \cos 4x = \frac{1}{2}(\cos 6x + \cos 2x).$$

Multiplying this identity by the remaining factor $\cos 6x$ yields

$$I = \frac{1}{2} \int \cos^2 6x \, dx + \frac{1}{2} \int \cos 2x \cos 6x \, dx.$$

Step 2: Reduce the remaining products For the squared factor, use $\cos^2 6x = (1 + \cos 12x)/2$. For the other product, use the same product-to-sum identity with $A = 2x$ and $B = 6x$:

$$\cos 2x \cos 6x = \frac{1}{2}(\cos 8x + \cos 4x).$$

Therefore the original integrand becomes

$$\cos 2x \cos 4x \cos 6x = \frac{1}{4}(1 + \cos 12x + \cos 8x + \cos 4x).$$

Step 3: Integrate each term and check the coefficients By linearity of integration,

$$\begin{aligned} I &= \frac{1}{4} \left(x + \frac{\sin 12x}{12} + \frac{\sin 8x}{8} + \frac{\sin 4x}{4} \right) + C \\ &= \frac{x}{4} + \frac{\sin 12x}{48} + \frac{\sin 8x}{32} + \frac{\sin 4x}{16} + C. \end{aligned}$$

To verify, differentiate this expression. Its derivative is $\frac{1}{4} + \frac{1}{4} \cos 12x + \frac{1}{4} \cos 8x + \frac{1}{4} \cos 4x$, which is exactly the reduced integrand from the preceding step.

Step 4: Conclusion

$$\int \cos 2x \cos 4x \cos 6x \, dx = \frac{x}{4} + \frac{\sin 4x}{16} + \frac{\sin 8x}{32} + \frac{\sin 12x}{48} + C.$$

Quick Tip to Solve: Pair $\cos 2x$ with $\cos 4x$ first: their sum frequency is $6x$, so multiplying by the third factor immediately creates $\cos^2 6x$.

23. Find the interval in which the function f given by $f(x) = \sin 2x + \cos 2x$, $0 \leq x \leq \pi$, is strictly decreasing. [4 marks]

Solution:

Concept:

- For a differentiable function, $f'(x) < 0$ on an interval implies that the function is strictly decreasing there: as x increases, $f(x)$ decreases.
- The identity $\cos t - \sin t = \sqrt{2} \cos(t + \pi/4)$ converts the sign test into a single trigonometric inequality. The final interval must lie inside the given domain $[0, \pi]$.

Step 1: Differentiate the function Using the chain rule on both terms gives

$$\begin{aligned} f'(x) &= 2 \cos 2x - 2 \sin 2x \\ &= 2\sqrt{2} \cos\left(2x + \frac{\pi}{4}\right). \end{aligned}$$

Thus the factor $2\sqrt{2}$ is positive, so only the cosine factor controls the sign of the derivative. The stationary points satisfy $\cos(2x + \pi/4) = 0$.

Step 2: Solve the derivative sign inequality in the given domain Since $0 \leq x \leq \pi$, the angle $2x + \pi/4$ lies between $\pi/4$ and $9\pi/4$. In this angle range, cosine is negative precisely when

$$\frac{\pi}{2} < 2x + \frac{\pi}{4} < \frac{3\pi}{2}.$$

Subtracting $\pi/4$ and dividing throughout by 2, we obtain

$$\frac{\pi}{8} < x < \frac{5\pi}{8}.$$

Before $\pi/8$ and after $5\pi/8$, within the prescribed domain, the derivative is positive. Hence neither adjacent interval can be added to the decreasing interval.

Step 3: Interpret the stationary endpoints At $x = \pi/8$ and $x = 5\pi/8$, the derivative is zero. It is strictly negative at every point between them. The mean value theorem shows that for any $a < b$ in $[\pi/8, 5\pi/8]$, $f(b) - f(a) = f'(c)(b - a) < 0$ for some $c \in (a, b)$. Thus the endpoint values can also be included when describing the maximal interval of strict decrease.

Step 4: Conclusion The derivative is negative on $(\pi/8, 5\pi/8)$. Consequently, f is strictly decreasing throughout this interval; its maximal decreasing interval including stationary endpoints is $[\pi/8, 5\pi/8]$.

Quick Tip to Solve: Rewrite $\cos 2x - \sin 2x$ as a single cosine before solving its sign inequality; then intersect with $0 \leq x \leq \pi$.

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24(a). Using elementary row transformations, find the inverse of the matrix $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$. [6 marks]

Solution:

Concept:

- To find A^{-1} by elementary row transformations, augment A with the identity matrix I_3 . Apply every row operation to both blocks until $[A \mid I_3]$ becomes $[I_3 \mid A^{-1}]$.
- Adding a multiple of one row to another is reversible. Thus reducing the left block to the identity establishes that A is invertible and supplies its inverse in the right block.

Step 1: Form the augmented matrix For the given matrix, start with

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array} \right].$$

We choose integer row operations to avoid fractions. The entries in the last column of the left block first suggest adding the third row to the first row.

Step 2: Create a unit pivot in the first column First use $R_1 \leftarrow R_1 + R_3$, giving

$$\left[\begin{array}{ccc|ccc} 3 & 4 & 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array} \right].$$

Next use $R_1 \leftarrow R_1 - R_2$, with the current rows:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 1 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array} \right].$$

The first entry is now 1, and the first row has no third-column entry.

Step 3: Eliminate below and above the pivots Apply $R_2 \leftarrow R_2 - 2R_1$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array} \right].$$

Now apply $R_1 \leftarrow R_1 - R_2$ and $R_3 \leftarrow R_3 - 4R_2$. Since R_2 is unchanged by these two operations, both

Quick Tip to Solve: For this matrix, begin with $R_1 \leftarrow R_1 + R_3$ followed by $R_1 \leftarrow R_1 - R_2$; this creates a unit pivot and avoids fractions throughout.

use the same second row. This yields

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & 1 & 8 & -12 & 9 \end{array} \right].$$

The left block is exactly I_3 , so the right block is the inverse of the original matrix.

Step 4: Verify the resulting matrix Let B denote the right block. Multiplying by the original matrix checks every row and column:

$$AB = \begin{bmatrix} 9-8 & -12+12 & 9-9 \\ 6-6 & -8+9 & 6-6 \\ -8+8 & 12-12 & -8+9 \end{bmatrix} = I_3.$$

This independent check confirms the signs and entries obtained by the elementary row transformations.

Step 5: Conclusion Therefore, using elementary row transformations,

$$A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}.$$

Quick Tip to Solve: For this matrix, begin with $R_1 \leftarrow R_1 + R_3$ followed by $R_1 \leftarrow R_1 - R_2$; this creates a unit pivot and avoids fractions throughout.

OR

24(b). Using matrices, solve the following system of linear equations :

$$2x + 3y + 10z = 4$$

$$4x - 6y + 5z = 1$$

$$6x + 9y - 20z = 2.$$

[6 marks]

Solution:

Concept:

- A system of linear equations can be written as $AX = B$, where A is the coefficient matrix, X is the column of unknowns and B is the column of constants.
- Elementary row operations on the augmented matrix $[A \mid B]$ preserve the solution set. Reducing its coefficient block to the identity gives $[I \mid X]$.

Step 1: Write the system in matrix form Keeping the unknowns in the order x, y, z , the given equations

Quick Tip to Solve: Use the first row to eliminate x : the third row immediately becomes $-50z = -10$, making the matrix reduction short.

become

$$\underbrace{\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_X = \underbrace{\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}}_B.$$

We apply every row operation to the constant column as well as to the coefficient block. This is essential because it keeps the transformed equations equivalent to the original system.

Step 2: Eliminate the first-column entries below the pivot Form the augmented matrix and use $R_2 \rightarrow R_2 - 2R_1$, followed by $R_3 \rightarrow R_3 - 3R_1$:

$$\left[\begin{array}{ccc|c} 2 & 3 & 10 & 4 \\ 4 & -6 & 5 & 1 \\ 6 & 9 & -20 & 2 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} 2 & 3 & 10 & 4 \\ 0 & -12 & -15 & -7 \\ 0 & 0 & -50 & -10 \end{array} \right].$$

The three diagonal entries are nonzero, so there is a pivot for each unknown. In particular, the coefficient matrix is invertible and this system has a unique solution.

Step 3: Reduce the last column and the second pivot First use $R_3 \rightarrow -\frac{1}{50}R_3$. Next eliminate the third-column entries by $R_2 \rightarrow R_2 + 15R_3$ and $R_1 \rightarrow R_1 - 10R_3$:

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 2 \\ 0 & -12 & 0 & -4 \\ 0 & 0 & 1 & \frac{1}{5} \end{array} \right].$$

Now $R_2 \rightarrow -\frac{1}{12}R_2$ gives

$$\left[\begin{array}{ccc|c} 2 & 3 & 0 & 2 \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{5} \end{array} \right].$$

Step 4: Complete the identity matrix Apply $R_1 \rightarrow R_1 - 3R_2$ and then $R_1 \rightarrow \frac{1}{2}R_1$. The augmented matrix reduces to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{5} \end{array} \right].$$

Thus the transformed matrix equation is $IX = \left[\frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{5} \right]^T$, where T denotes transpose.

Step 5: Verify in all three original equations Substitution into the left sides gives

$$\begin{aligned} 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{3}\right) + 10\left(\frac{1}{5}\right) &= 1 + 1 + 2 = 4, \\ 4\left(\frac{1}{2}\right) - 6\left(\frac{1}{3}\right) + 5\left(\frac{1}{5}\right) &= 2 - 2 + 1 = 1, \\ 6\left(\frac{1}{2}\right) + 9\left(\frac{1}{3}\right) - 20\left(\frac{1}{5}\right) &= 3 + 3 - 4 = 2. \end{aligned}$$

Quick Tip to Solve: Use the first row to eliminate x: the third row immediately becomes $-50z = -10$, making the matrix reduction short.

Each right-hand side is reproduced, confirming both the row reduction and the unique solution.

Step 6: Conclusion The solution obtained using matrices is $x = \frac{1}{2}, \quad y = \frac{1}{3}, \quad z = \frac{1}{5}$.

Quick Tip to Solve: Use the first row to eliminate x: the third row immediately becomes $-50z = -10$, making the matrix reduction short.

25(a). Using integration, find the area of the following region :

$$\{(x, y) : x^2 + y^2 \leq 16a^2 \text{ and } y^2 \leq 6ax\}.$$

[6 marks]

Solution:

Concept:

- For horizontal strips, the area between curves is $\int (x_{\text{right}} - x_{\text{left}}) dy$. The inequalities determine which portions of the curves bound the region.
- Both inequalities involve y^2 , so the region is symmetric about the x-axis. Its area is twice the area above that axis.
- For $R > 0$, $\int \sqrt{R^2 - y^2} dy = \frac{y}{2} \sqrt{R^2 - y^2} + \frac{R^2}{2} \sin^{-1}\left(\frac{y}{R}\right) + C$, with $|y| \leq R$.

Step 1: Identify the boundaries of the common region First take $a > 0$, the usual positive length parameter. The inequality $x^2 + y^2 \leq 16a^2$ describes the interior of the circle with centre at the origin and radius $4a$. The other inequality, $y^2 \leq 6ax$, requires

$$x \geq \frac{y^2}{6a} \geq 0.$$

Thus the common region lies on the right of the parabola and inside the right half of the circle. For a fixed y , its left boundary is $x = y^2/(6a)$, and its right boundary is $x = \sqrt{16a^2 - y^2}$. A horizontal strip therefore avoids splitting the integral into separate x-intervals.

Step 2: Find the points of intersection At an intersection, substitute $y^2 = 6ax$ into $x^2 + y^2 = 16a^2$:

$$\begin{aligned} x^2 + 6ax - 16a^2 &= 0, \\ (x - 2a)(x + 8a) &= 0. \end{aligned}$$

Since $x \geq 0$, the root $-8a$ is inadmissible. The intersections have $x = 2a$ and $y^2 = 12a^2$, giving $y = \pm 2\sqrt{3}a$. The horizontal strip has nonnegative width exactly between these two y-values.

Step 3: Set up the area integral Using symmetry about the x-axis, the desired area S is

$$S = 2 \int_0^{2\sqrt{3}a} \left(\sqrt{16a^2 - y^2} - \frac{y^2}{6a} \right) dy.$$

The first term integrates the distance from the y-axis to the circular boundary. Subtracting the second term removes the distance from the y-axis to the parabolic boundary, leaving precisely the strip in the common region.

Quick Tip to Solve: Integrate horizontally: the parabola is the left boundary and the circle the right boundary, so symmetry gives a single integral.

Step 4: Evaluate the circular and parabolic contributions With $R = 4a$, the circular contribution on the upper half is

$$\begin{aligned} I_1 &= \left[\frac{y}{2} \sqrt{16a^2 - y^2} + 8a^2 \sin^{-1} \left(\frac{y}{4a} \right) \right]_0^{2\sqrt{3}a} \\ &= 2\sqrt{3}a^2 + 8a^2 \left(\frac{\pi}{3} \right). \end{aligned}$$

Here the square root at the upper limit is $2a$, and $\sin^{-1}(\sqrt{3}/2) = \pi/3$. The parabolic contribution is

$$I_2 = \left[\frac{y^3}{18a} \right]_0^{2\sqrt{3}a} = \frac{24\sqrt{3}a^3}{18a} = \frac{4\sqrt{3}}{3}a^2.$$

Step 5: Combine the contributions and account for the parameter Therefore,

$$\begin{aligned} S &= 2(I_1 - I_2) \\ &= 2 \left(2\sqrt{3}a^2 + \frac{8\pi}{3}a^2 - \frac{4\sqrt{3}}{3}a^2 \right) \\ &= \frac{4a^2}{3}(4\pi + \sqrt{3}). \end{aligned}$$

This also gives the area for $a < 0$: replacing a by $|a|$ and reflecting the region in the y-axis preserves area. If $a = 0$, the circle reduces to the origin and the area is zero, again consistent with the expression.

Step 6: Conclusion The area of the given region is $\boxed{\frac{4a^2}{3}(4\pi + \sqrt{3})}$ square units.

Quick Tip to Solve: Integrate horizontally: the parabola is the left boundary and the circle the right boundary, so symmetry gives a single integral.

OR

25(b). Using integration, find the area of triangle ABC bounded by the lines $4x - y + 5 = 0$, $x + y - 5 = 0$ and $x - 4y + 5 = 0$. [6 marks]

Solution:

Concept:

- The intersection of two boundary lines gives a vertex of the triangle. A vertical strip has area $(y_{\text{upper}} - y_{\text{lower}}) dx$.
- The required area is the integral of the strip heights. Split the integral at a vertex where the upper boundary changes, so that every strip height remains non-negative.

Step 1: Express the boundaries and find the vertices Write the three lines as $y = 4x + 5$, $y = 5 - x$ and $y = (x + 5)/4$. The intersection of the first two satisfies $4x + 5 = 5 - x$, giving $x = 0$ and $y = 5$; call this vertex $B(0, 5)$. For the first and third lines,

$$4x + 5 = \frac{x + 5}{4} \implies 16x + 20 = x + 5 \implies x = -1, \quad y = 1.$$

Quick Tip to Solve: Find all three intersection points first; the middle x-coordinate, here zero, tells you exactly where to split the vertical-strip integral.

Thus their intersection is $A(-1, 1)$. For the second and third lines,

$$5 - x = \frac{x + 5}{4} \implies 20 - 4x = x + 5 \implies x = 3, \quad y = 2.$$

The remaining vertex is $C(3, 2)$. Hence the full horizontal extent of the triangle is from $x = -1$ to $x = 3$.

Step 2: Choose the correct vertical strip heights The side AC , given by $y = (x + 5)/4$, is the lower boundary throughout the triangle. For $-1 \leq x \leq 0$, the upper boundary is the side AB , namely $y = 4x + 5$. For $0 \leq x \leq 3$, it is the side BC , namely $y = 5 - x$. Therefore the area S is

$$S = \int_{-1}^0 \left(4x + 5 - \frac{x + 5}{4} \right) dx + \int_0^3 \left(5 - x - \frac{x + 5}{4} \right) dx.$$

The first integrand is $15(x + 1)/4$, and the second is $5(3 - x)/4$; both are non-negative on their respective intervals.

Step 3: Evaluate the two definite integrals Integrating the simplified expressions gives

$$\begin{aligned} S &= \frac{15}{4} \left[\frac{x^2}{2} + x \right]_{-1}^0 + \frac{5}{4} \left[3x - \frac{x^2}{2} \right]_0^3 \\ &= \frac{15}{4} \left(0 - \left(\frac{1}{2} - 1 \right) \right) + \frac{5}{4} \left(9 - \frac{9}{2} \right) \\ &= \frac{15}{8} + \frac{45}{8}. \end{aligned}$$

As an independent check, the triangle determinant gives $\frac{1}{2}|(-1)(5 - 2) + 0(2 - 1) + 3(1 - 5)| = 15/2$, consistent with the integral calculation.

Step 4: Conclusion The area of triangle ABC is $\boxed{\frac{15}{2} \text{ square units}}$.

Quick Tip to Solve: Find all three intersection points first; the middle x-coordinate, here zero, tells you exactly where to split the vertical-strip integral.

26. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit for type A souvenirs is ₹ 100 each and for type B souvenirs, profit is ₹ 120 each. How many souvenirs of each type should the company manufacture in order to maximise the profit? Formulate the problem as a LPP and then solve it graphically. [6 marks]

Solution:

Concept:

- A linear programming problem expresses the available resources as linear inequalities and the quantity to maximise as a linear objective function.

Quick Tip to Solve: Convert both time limits to minutes before forming inequalities, then test the common-region vertices rather than all four axis intercepts.

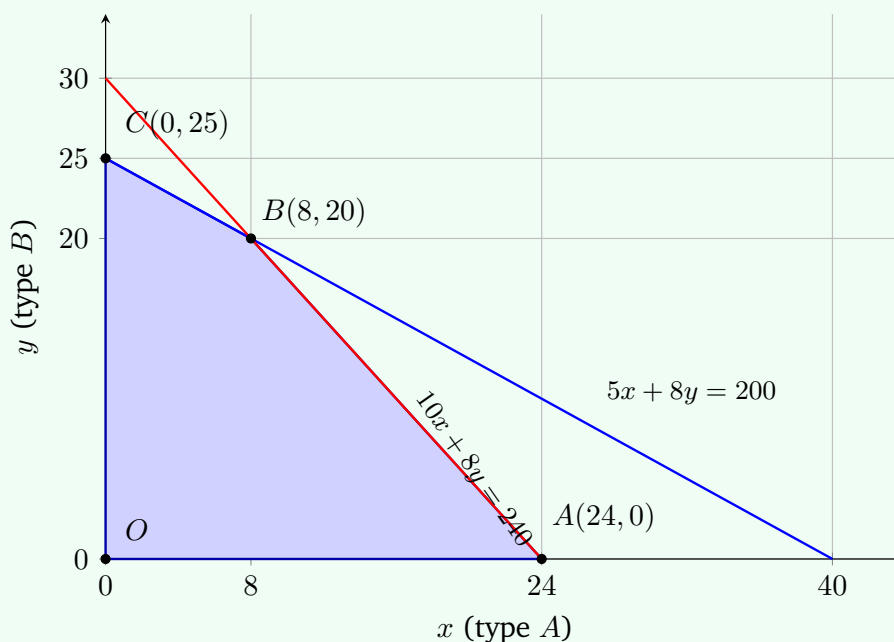
- For a bounded feasible region, the maximum of a linear objective function occurs at a corner point. Non-negative variables represent the numbers of souvenirs produced.

Step 1: Define the variables and formulate the constraints Let x and y be the numbers of souvenirs of types A and B , respectively. The cutting time available is $3 \times 60 + 20 = 200$ minutes, and the assembling time available is $4 \times 60 = 240$ minutes. Each type A uses 5 minutes for cutting and 10 minutes for assembling; each type B uses 8 minutes for each operation. Thus

$$5x + 8y \leq 200, \quad 10x + 8y \leq 240, \quad x \geq 0, \quad y \geq 0.$$

The profit, in rupees, is $Z = 100x + 120y$. The LPP is to maximise Z subject to these four inequalities.

Step 2: Draw the boundary lines and identify the feasible region The cutting boundary $5x + 8y = 200$ passes through $(40, 0)$ and $(0, 25)$. The assembling boundary $10x + 8y = 240$ passes through $(24, 0)$ and $(0, 30)$. Both resource inequalities include the origin, so shade their common region in the first quadrant, below both lines. The graph shows the resulting bounded quadrilateral $OABC$. On the x -axis the more restrictive intercept is 24; on the y -axis it is 25. The intersection of the resource boundaries provides the fourth vertex.



The shaded quadrilateral is the feasible production region.

Step 3: Calculate the intersection and the corner points At the intersection, solve

$$\begin{aligned} 5x + 8y &= 200, \\ 10x + 8y &= 240. \end{aligned}$$

Subtraction gives $5x = 40$, so $x = 8$. Substituting in the first equation gives $40 + 8y = 200$, hence $y = 20$. The vertices of the feasible region are therefore $O(0, 0)$, $A(24, 0)$, $B(8, 20)$, and $C(0, 25)$.

Quick Tip to Solve: Convert both time limits to minutes before forming inequalities, then test the common-region vertices rather than all four axis intercepts.

Step 4: Evaluate the profit at every corner By the corner-point method,

Corner point (x, y)	$Z = 100x + 120y$ (rupees)
$(0, 0)$	0
$(24, 0)$	2400
$(8, 20)$	$800 + 2400 = 3200$
$(0, 25)$	3000

The largest value occurs at $(8, 20)$. Both coordinates are whole numbers, so the graphical optimum is a valid production plan. It uses $5(8) + 8(20) = 200$ minutes for cutting and $10(8) + 8(20) = 240$ minutes for assembling, exactly matching both available resources.

Step 5: Conclusion The company should manufacture $\boxed{8}$ souvenirs of type A and $\boxed{20}$ souvenirs of type B . The maximum profit is $\boxed{\text{₹ } 3200}$.

Quick Tip to Solve: Convert both time limits to minutes before forming inequalities, then test the common-region vertices rather than all four axis intercepts.

27(a). Find the vector equation of the line passing through $(2, 1, -1)$ and parallel to the line $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$. Also, find the distance between these two lines. [6 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$, parallel to a nonzero vector $\vec{b}$, has vector equation $\vec{r} = \vec{a} + t\vec{b}$, where $t \in \mathbb{R}$.
- For parallel lines with common direction vector $\vec{b}$, their distance is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$. This is the perpendicular component of the vector joining one point on each line.

Step 1: Identify the point and the common direction The given line passes through $A(1, 1, 0)$, whose position vector is $\vec{a}_1 = \hat{i} + \hat{j}$, and has direction vector

$$\vec{b} = 2\hat{i} - \hat{j} + \hat{k}.$$

The required line must pass through $B(2, 1, -1)$, whose position vector is $\vec{a}_2 = 2\hat{i} + \hat{j} - \hat{k}$. Since the two lines are parallel, we use precisely the same direction vector; the line parameter is independent of the parameter on the given line.

Step 2: Form the required vector equation Every point on the required line is obtained by starting at B and adding a real multiple of $\vec{b}$. Thus its equation is

$$\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + t(2\hat{i} - \hat{j} + \hat{k}), \quad t \in \mathbb{R}.$$

For $t = 0$ the point is $(2, 1, -1)$, as required. Its component equations are $x = 2 + 2t$, $y = 1 - t$, $z = -1 + t$, confirming the common direction ratios $2, -1, 1$.

Step 3: Compute the perpendicular separation The joining vector is $\vec{AB} = \vec{a}_2 - \vec{a}_1 = \hat{i} - \hat{k}$. The length

Quick Tip to Solve: For parallel lines, use their common direction vector and take the cross product of the joining vector with it; the joining vector's length alone usually overestimates the distance.

of this joining vector itself is not the distance between the lines, because it need not be perpendicular to their direction. Its cross product with $\vec{b}$ gives

$$\begin{aligned}\vec{AB} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 2 & -1 & 1 \end{vmatrix} \\ &= -\hat{i} - 3\hat{j} - \hat{k}.\end{aligned}$$

Hence $|\vec{AB} \times \vec{b}| = \sqrt{1+9+1} = \sqrt{11}$ and $|\vec{b}| = \sqrt{4+1+1} = \sqrt{6}$. The perpendicular separation is therefore $\sqrt{11}/\sqrt{6}$.

Step 4: Check the distance by projection Independently, $|\vec{AB}|^2 = 2$ and $\vec{AB} \cdot \vec{b} = 1$. Removing the squared component parallel to the lines gives

$$d^2 = |\vec{AB}|^2 - \frac{(\vec{AB} \cdot \vec{b})^2}{|\vec{b}|^2} = 2 - \frac{1}{6} = \frac{11}{6},$$

agreeing with the cross-product calculation.

Step 5: Conclusion The required line is $\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + t(2\hat{i} - \hat{j} + \hat{k}), t \in \mathbb{R}$. The distance between the two parallel lines is $\sqrt{\frac{11}{6}}$ units.

Quick Tip to Solve: For parallel lines, use their common direction vector and take the cross product of the joining vector with it; the joining vector's length alone usually overestimates the distance.

OR

27(b). Find the coordinates of the foot Q of the perpendicular drawn from the point $P(1, 3, 4)$ to the plane $2x - y + z + 3 = 0$. Find the distance PQ and the image of P treating the plane as a mirror. [6 marks]

Solution:

Concept:

- A plane $ax + by + cz + d = 0$ has normal vector (a, b, c) . The perpendicular from a point to the plane lies along this normal.
- The distance from (x_0, y_0, z_0) to the plane is $\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$.
- If Q is the foot of the perpendicular from P , then its mirror image P' satisfies $\vec{OP'} = 2\vec{OQ} - \vec{OP}$, because Q is the midpoint of PP' .

Step 1: Write the perpendicular through the point The plane $2x - y + z + 3 = 0$ has normal vector $\vec{n} = 2\hat{i} - \hat{j} + \hat{k}$. Since PQ is perpendicular to this plane, its direction ratios are 2, -1, 1. The line

Quick Tip to Solve: First find the foot by intersecting the normal through the point with the plane; then reflect using the coordinate rule $P' = 2Q - P$, avoiding a second line-plane calculation.

through $P(1, 3, 4)$ in this direction is

$$(x, y, z) = (1, 3, 4) + t(2, -1, 1).$$

Thus a point on this perpendicular has coordinates $x = 1 + 2t$, $y = 3 - t$, $z = 4 + t$. We find the foot by imposing the additional condition that this point lies in the plane.

Step 2: Locate the foot of the perpendicular Substitute the coordinates of the perpendicular line into the plane equation:

$$\begin{aligned} 2(1 + 2t) - (3 - t) + (4 + t) + 3 &= 0, \\ 6 + 6t &= 0, \\ t &= -1. \end{aligned}$$

Hence the foot has coordinates

$$Q = (1 - 2, 3 + 1, 4 - 1) = (-1, 4, 3).$$

Check that $2(-1) - 4 + 3 + 3 = 0$, so Q lies on the plane. Also $\overrightarrow{PQ} = (-2, 1, -1) = -\vec{n}$, confirming that the joining segment is perpendicular to the plane.

Step 3: Evaluate the distance from the point to the plane Using the coordinates just found, the length of the perpendicular is

$$PQ = \sqrt{(-1 - 1)^2 + (4 - 3)^2 + (3 - 4)^2} = \sqrt{4 + 1 + 1} = \sqrt{6}.$$

The point-to-plane formula provides an independent check:

$$PQ = \frac{|2(1) - 3 + 4 + 3|}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{6}{\sqrt{6}} = \sqrt{6}.$$

The absolute value ensures that the result is a nonnegative length regardless of which side of the plane contains the point.

Step 4: Reflect the point in the plane Let the image be $P'(x', y', z')$. Reflection in the plane keeps the perpendicular foot fixed and places the image at the same distance on the opposite side. Therefore Q is the midpoint of PP' , giving

$$\begin{aligned} x' &= 2(-1) - 1 = -3, \\ y' &= 2(4) - 3 = 5, \\ z' &= 2(3) - 4 = 2. \end{aligned}$$

As a check, $\overrightarrow{QP'} = (-2, 1, -1) = \overrightarrow{PQ}$, while the plane expression at P' is $2(-3) - 5 + 2 + 3 = -6$, the negative of its value 6 at P . Thus the points are equally distant on opposite sides.

Step 5: Conclusion The foot of the perpendicular is $Q(-1, 4, 3)$, the distance is $PQ = \sqrt{6}$ units, and the mirror image of P is $P'(-3, 5, 2)$.

Quick Tip to Solve: First find the foot by intersecting the normal through the point with the plane; then reflect using the coordinate rule $P' = 2Q - P$, avoiding a second line-plane calculation.

28. A card from a pack of 52 playing cards is lost. From the remaining cards of the pack, two cards are drawn at random (without replacement) and both are found to be spades. Find the probability of the lost card being a spade. [6 marks]

Solution:

Concept:

- Bayes' theorem updates the probability of an event using observed evidence. For a partition S, N and evidence E , $P(S | E) = \frac{P(S)P(E | S)}{P(S)P(E | S) + P(N)P(E | N)}$.
- A standard pack contains 13 spades and 39 non-spades. When drawing without replacement, the second draw uses the reduced number of cards and spades.

Step 1: Define the lost-card events and the evidence Let S be the event that the lost card is a spade, N the event that it is a non-spade, and E the event that both subsequently drawn cards are spades. Before observing the draws, each of the 52 cards is equally likely to be the lost card. Thus

$$P(S) = \frac{13}{52} = \frac{1}{4}, \quad P(N) = \frac{39}{52} = \frac{3}{4}.$$

The events S and N are mutually exclusive and exhaustive, so they form the partition required for Bayes' theorem. We seek $P(S | E)$, since the two spades have already been observed.

Step 2: Find the likelihood of the observed draws in each case If the lost card is a spade, the remaining pack has 51 cards, including 12 spades. After the first spade is drawn, 11 spades remain among 50 cards. Hence

$$P(E | S) = \frac{12}{51} \cdot \frac{11}{50} = \frac{132}{2550}.$$

If the lost card is a non-spade, all 13 spades are still present among the 51 remaining cards. After drawing one spade, there are 12 among 50 cards. Therefore

$$P(E | N) = \frac{13}{51} \cdot \frac{12}{50} = \frac{156}{2550}.$$

These probabilities differ because losing a spade reduces the number available for the two draws; the draws are not independent.

Step 3: Apply total probability and Bayes' theorem The total probability of the evidence is

$$P(E) = \frac{1}{4} \frac{132}{2550} + \frac{3}{4} \frac{156}{2550}.$$

Consequently,

$$\begin{aligned} P(S | E) &= \frac{\frac{1}{4} \frac{132}{2550}}{\frac{1}{4} \frac{132}{2550} + \frac{3}{4} \frac{156}{2550}} \\ &= \frac{132}{132 + 3(156)} \\ &= \frac{132}{600} = \frac{11}{50}. \end{aligned}$$

As a check, observing two spades makes a lost spade less likely than its prior probability $1/4$, because the observed pair is more likely when no spade was lost.

Quick Tip to Solve: Form the two lost-card cases first; their remaining spade counts are 12 and 13, while both packs contain 51 cards.

Step 4: Conclusion The probability that the lost card is a spade, given that both drawn cards are spades, is $\frac{11}{50}$.

Quick Tip to Solve: Form the two lost-card cases first; their remaining spade counts are 12 and 13, while both packs contain 51 cards.

29. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. [6 marks]

Solution:

Concept:

- The curved surface area of a right circular cylinder of radius r and height h is $A = 2\pi rh$. Its two circular ends are excluded.
- For a cylinder inscribed coaxially in a cone, similar triangles relate its radius and height to the cone's base radius R and height H .
- A stationary value satisfies $A'(r) = 0$. If $A''(r) < 0$, it is a local maximum; comparison with the endpoints establishes the greatest value on the feasible interval.

Step 1: Relate the cylinder's dimensions to the cone Let the fixed cone have base radius $R > 0$ and height $H > 0$. Let an inscribed right circular cylinder have radius r and height h , with its base in the plane of the cone's base and its axis along the cone's axis. The cylinder's upper circular rim touches the cone. For any fixed radius, taking the full available height gives the greatest curved surface area, since $2\pi rh$ increases with h . In an axial cross-section, the small triangle above the cylinder is similar to the entire triangular cross-section of the cone. The corresponding horizontal distances are r and R , and the corresponding heights are $H - h$ and H . Therefore

$$\frac{r}{R} = \frac{H - h}{H}, \quad h = H \left(1 - \frac{r}{R}\right).$$

A non-degenerate inscribed cylinder satisfies $0 < r < R$ and $0 < h < H$.

Step 2: Express the curved surface area as a function of radius Substituting the height relation into the curved surface area formula gives

$$\begin{aligned} A(r) &= 2\pi rh \\ &= 2\pi Hr \left(1 - \frac{r}{R}\right) \\ &= 2\pi H \left(r - \frac{r^2}{R}\right). \end{aligned}$$

Here R and H are constants because the cone is given. Maximising the curved surface area is thus a one-variable optimisation problem. Including the limiting endpoints $r = 0$ and $r = R$ allows the maximum to be compared with degenerate cylinders.

Quick Tip to Solve: Use similar triangles to eliminate the height, then maximise $2\pi rh$; curved surface area excludes both circular ends.

Step 3: Find and verify the greatest area Differentiation gives

$$A'(r) = 2\pi H \left(1 - \frac{2r}{R}\right).$$

At an interior maximum, $A'(r) = 0$, so

$$1 - \frac{2r}{R} = 0 \implies r = \frac{R}{2}.$$

Moreover,

$$A''(r) = -\frac{4\pi H}{R} < 0.$$

Thus the area function is strictly concave throughout the feasible interval, and its only stationary point is a maximum. At each limiting endpoint the area is zero, whereas at $r = R/2$, the height is $H/2$ and the area is $\pi RH/2 > 0$. Hence this stationary point gives the greatest curved surface area among all such inscribed cylinders.

Step 4: Conclusion The right circular cylinder of greatest curved surface area has radius $r = \frac{R}{2}$, half the base radius of the given cone. Its corresponding height is $H/2$.

Quick Tip to Solve: Use similar triangles to eliminate the height, then maximise $2\pi r h$; curved surface area excludes both circular ends.