CBSE Class 12 Mathematics Question Paper 2019 with Solution PDF-Set 65-5-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 100**

General Instructions

(i) All questions are compulsory.

(ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. Find the order of the differential equation of the family of circles of radius 3 units. [1 mark]

Solution:

Concept:

- A family with two independent arbitrary constants gives a differential equation of order two after their elimination.
- The order is the order of the highest derivative present, not its exponent.

Step 1: Write the family Let the centre be (a, b), where both coordinates can vary independently. Since the radius is fixed at 3, the family is

$$(x-a)^2 + (y-b)^2 = 9.$$

There is no arbitrary radius parameter.

Step 2: Eliminate the centre coordinates Differentiating gives $(x-a) + (y-b)y\prime = 0$. Differentiating again gives $1 + (y\prime)^2 + (y-b)y\prime\prime = 0$. Eliminating a, b yields

$$9(y\prime\prime)^2 = \left(1 + (y\prime)^2\right)^3.$$

The highest derivative is the second derivative, even though it is squared.

Step 3: Conclusion The order of the differential equation is $\boxed{2}$.

Quick Tip to Solve: Count the two variable coordinates of the centre; a fixed radius adds no arbitrary constant.

2(a). Find the angle between the line $\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} - \hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 3$. [1 mark]

Solution:

Concept:

- For angle θ between a line of direction $\vec{d}$ and a plane of normal $\vec{n}$, $\sin\theta = |\vec{d} \cdot \vec{n}|/(|\vec{d}||\vec{n}|)$.
- The line-plane angle is the complement of the angle with the normal.

Step 1: Identify the vectors The coefficient of λ gives $\vec{d} = 3\hat{i} - \hat{j} + 2\hat{k}$. The plane has normal $\vec{n} = \hat{i} + \hat{j} + \hat{k}$. Thus $|\vec{d}| = \sqrt{14}$ and $|\vec{n}| = \sqrt{3}$. The position vector on the line does not affect its direction.

Step 2: Apply the angle formula The dot product is $\vec{d} \cdot \vec{n} = 3 - 1 + 2 = 4$. Therefore

$$\sin\theta = \frac{|4|}{\sqrt{14}\sqrt{3}} = \frac{4}{\sqrt{42}}.$$

The acute angle with the plane is obtained by taking the inverse sine.

Quick Tip to Solve: Use sine for the line-plane angle; cosine gives the angle with the plane normal.

Step 3: Conclusion The required angle is $\boxed{\sin^{-1}(4/\sqrt{42})}$.

Quick Tip to Solve: Use sine for the line-plane angle; cosine gives the angle with the plane normal.

OR

2(b). Find the co-ordinates of the point, where the line $\frac{x+2}{1} = \frac{y-5}{3} = \frac{z+1}{5}$ cuts the yz-plane. [1 mark]

Solution:

Concept:

- Every point on the yz-plane has $x = 0$.
- Equating the symmetric ratios to a parameter gives the coordinates of a point on the line.

Step 1: Write the line parametrically Let $(x+2)/1 = (y-5)/3 = (z+1)/5 = t$. Then

$$x = t - 2, \qquad y = 5 + 3t, \qquad z = 5t - 1.$$

Each coordinate uses the same parameter, so the resulting point necessarily lies on the given line.

Step 2: Impose the coordinate-plane condition At the yz-plane, $x = 0$. Hence $t - 2 = 0$, giving $t = 2$. Substituting gives $y = 5 + 6 = 11$ and $z = 10 - 1 = 9$. As a check, all three ratios at $(0, 11, 9)$ equal 2.

Step 3: Conclusion The line cuts the yz-plane at $\boxed{(0, 11, 9)}$.

Quick Tip to Solve: Set the coordinate absent from the plane name to zero before finding the common ratio.

3. If $A = \begin{bmatrix} 8 & 2 \\ 3 & 2 \end{bmatrix}$, then find $|\operatorname{adj} A|$. [1 mark]

Solution:

Concept:

- For an $n \times n$ matrix, $|\operatorname{adj} A| = |A|^{n-1}$.
- For a 2×2 matrix this reduces to $|\operatorname{adj} A| = |A|$.

Step 1: Compute the determinant The matrix has order two. Its determinant is

$$|A| = 8 \cdot 2 - 2 \cdot 3 = 16 - 6 = 10.$$

The off-diagonal product must be subtracted from the main-diagonal product.

Step 2: Find the adjoint determinant Using the identity gives $|\operatorname{adj} A| = 10^{2-1} = 10$. Directly, $\operatorname{adj} A = \begin{bmatrix} 2 & -2 \\ -3 & 8 \end{bmatrix}$, whose determinant is $2 \cdot 8 - (-2)(-3) = 16 - 6 = 10$, confirming the result.

Step 3: Conclusion $\boxed{|\operatorname{adj} A| = 10}$.

Quick Tip to Solve: For a two by two matrix, the adjoint has the same determinant as the original matrix.

4. If $y = 2\sqrt{\sec(e^{2x})}$, then find $\frac{dy}{dx}$. [1 mark]

Solution:

Concept:

- The chain rule differentiates each nested function and multiplies their derivatives.
- $\frac{d}{du}\sec u = \sec u \tan u$ and $\frac{d}{dx}e^{2x} = 2e^{2x}$.

Step 1: Differentiate the outer square root Write $y = 2[\sec(e^{2x})]^{1/2}$. Differentiating the square-root layer gives

$$\frac{dy}{dx} = 2 \cdot \frac{1}{2}[\sec(e^{2x})]^{-1/2}\frac{d}{dx}\sec(e^{2x}).$$

The prefactor 2 cancels the factor one-half.

Step 2: Differentiate both inner layers By the chain rule, $\frac{d}{dx}\sec(e^{2x}) = \sec(e^{2x})\tan(e^{2x})\,2e^{2x}$. Multiplying and simplifying $\sec(e^{2x})/[\sec(e^{2x})]^{1/2}$ leaves its square root, at points where the given real-valued function is defined.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = 2e^{2x}\sqrt{\sec(e^{2x})}\tan(e^{2x})}.$$

Quick Tip to Solve: Follow the three layers in order: square root, secant, then the exponential.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5(a). Four cards are drawn one by one with replacement from a well-shuffled deck of playing cards. Find the probability that at least three cards are of diamonds. [2 marks]

Solution:

Concept:

- Replacement makes the four card draws independent, with constant diamond probability $p = 13/52 = 1/4$.
- For $X \sim \text{Bin}(n, p)$, $P(X = r) = \binom{n}{r}p^r(1-p)^{n-r}$.

Step 1: Identify the distribution and event Let X denote the number of diamonds in four draws. A standard deck has 13 diamonds and 39 non-diamonds. Since the card is replaced after each draw, $X \sim \text{Bin}(4, 1/4)$. At least three diamonds means exactly three or exactly four; these two events are disjoint.

Step 2: Evaluate the two probabilities For exactly three diamonds, choose the three diamond positions

Quick Tip to Solve: For at least three successes in four trials, add the exactly-three and exactly-four terms.

in $\binom{4}{3} = 4$ ways:

$$P(X = 3) = \binom{4}{3}\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right) = \frac{12}{256}.$$

For four diamonds there is only one pattern:

$$P(X = 4) = \left(\frac{1}{4}\right)^4 = \frac{1}{256}.$$

Independence permits multiplication within each pattern, and disjointness permits addition between the two counts.

Step 3: Conclusion The required probability is

$$\boxed{P(X \geq 3) = \frac{12+1}{256} = \frac{13}{256}}.$$

Quick Tip to Solve: For at least three successes in four trials, add the exactly-three and exactly-four terms.

OR

5(b). The probability of two students A and B coming to school on time are $\frac{2}{7}$ and $\frac{4}{7}$, respectively. Assuming that the events 'A coming on time' and 'B coming on time' are independent, find the probability of only one of them coming to school on time. [2 marks]

Solution:

Concept:

- Independent events satisfy $P(A \cap B) = P(A)P(B)$, and their complements are independent as well.
- Exactly one event occurs through the disjoint cases $A \cap B^c$ and $A^c \cap B$.

Step 1: Form the two exclusive cases Let A and B denote the respective students coming on time. Then $P(A) = 2/7$, $P(B) = 4/7$, so $P(A^c) = 5/7$ and $P(B^c) = 3/7$. Only one is on time when A is on time and B is late, or when A is late and B is on time. Neither both-on-time nor both-late belongs to the event.

Step 2: Use independence and add By independence,

$$P(A \cap B^c) = \frac{2}{7} \cdot \frac{3}{7} = \frac{6}{49}, \qquad P(A^c \cap B) = \frac{5}{7} \cdot \frac{4}{7} = \frac{20}{49}.$$

These cases cannot occur together, so their probabilities add. Equivalently, subtracting twice the both-on-time probability from $P(A) + P(B)$ gives the same result: $6/7 - 2(8/49) = 26/49$.

Step 3: Conclusion The probability that only one student comes on time is $\boxed{26/49}$.

Quick Tip to Solve: Exactly one requires a success paired with a failure; use both complementary cases.

6(a). Using vectors, prove that the points $(2, -1, 3)$, $(3, -5, 1)$ and $(-1, 11, 9)$ are collinear. [2 marks]

Solution:

Concept:

- Three points are collinear if two displacement vectors from the same point are scalar multiples.
- A negative scalar means that the two displacement vectors point in opposite directions on the same line.

Step 1: Calculate vectors from a common point Name the points $A = (2, -1, 3)$, $B = (3, -5, 1)$, and $C = (-1, 11, 9)$. Subtract the coordinates of A from those of B and C:

$$\overrightarrow{AB} = (3 - 2, -5 + 1, 1 - 3) = (1, -4, -2),$$
$$\overrightarrow{AC} = (-1 - 2, 11 + 1, 9 - 3) = (-3, 12, 6).$$

Using the same starting point ensures these are the vectors needed to test the three points.

Step 2: Compare the vectors Every coordinate of $\overrightarrow{AC}$ is minus three times the corresponding coordinate of $\overrightarrow{AB}$:

$$\overrightarrow{AC} = -3\overrightarrow{AB}.$$

Both vectors are nonzero, and scalar multiples are parallel. They therefore lie along one line through A. The negative factor places C on the side of A opposite to B; it does not prevent collinearity.

Step 3: Conclusion Hence the three given points $\boxed{A, B, C \text{ are collinear}}$.

Quick Tip to Solve: Subtract one chosen point from both others, then check whether every coordinate has the same ratio.

OR

6(b). For any two vectors $\vec{a}$ and $\vec{b}$, prove that $(\vec{a} \times \vec{b})^2 = \vec{a}^2\,\vec{b}^2 - (\vec{a} \cdot \vec{b})^2$. [2 marks]

Solution:

Concept:

- For vectors, the notation $\vec{a}^{\,2}$ means $\vec{a} \cdot \vec{a} = |\vec{a}|^2$, the square of the magnitude.
- If θ is the angle between nonzero vectors, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$ and $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$.

Step 1: Express both sides using the angle Let $a = |\vec{a}|$, $b = |\vec{b}|$, and let θ be the angle between the vectors. The square on a vector product denotes its dot product with itself. Thus

$$(\vec{a} \times \vec{b})^2 = |\vec{a} \times \vec{b}|^2 = a^2 b^2 \sin^2\theta.$$

Also, $\vec{a}^{\,2}\vec{b}^{\,2} = a^2 b^2$ and $(\vec{a} \cdot \vec{b})^2 = a^2 b^2 \cos^2\theta$.

Step 2: Apply the trigonometric identity Using $1 - \cos^2\theta = \sin^2\theta$, the right-hand side becomes

$$\vec{a}^{\,2}\vec{b}^{\,2} - (\vec{a} \cdot \vec{b})^2 = a^2 b^2 (1 - \cos^2\theta) = a^2 b^2 \sin^2\theta.$$

Quick Tip to Solve: Replace the dot product by a cosine and the cross-product magnitude by a sine; then use $\sin^2\theta + \cos^2\theta = 1$.

This equals the expression obtained for the left-hand side. If either vector is zero, both sides are zero directly, so this argument covers any two vectors.

Step 3: Conclusion $\boxed{(\vec{a}\times\vec{b})^2=\vec{a}^2\vec{b}^2-(\vec{a}\cdot\vec{b})^2}$. Hence proved.

Quick Tip to Solve: Replace the dot product by a cosine and the cross-product magnitude by a sine; then use $\sin^2\theta+\cos^2\theta=1$.

7(a). Find :

$$\int \frac{x-1}{(x-2)(x-3)}\,dx.$$

[2 marks]

Solution:

Concept:

- For distinct linear factors, decompose a proper rational function into partial fractions before integrating.
- Use $\int (x-a)^{-1}\,dx=\ln|x-a|+C$ on intervals that exclude $x=a$.

Step 1: Find the partial fractions Write

$$\frac{x-1}{(x-2)(x-3)}=\frac{A}{x-2}+\frac{B}{x-3}.$$

Multiplying by $(x-2)(x-3)$ gives $x-1=A(x-3)+B(x-2)$. Putting $x=2$ gives $1=-A$, so $A=-1$. Putting $x=3$ gives $2=B$. Thus the integrand equals $-1/(x-2)+2/(x-3)$.

Step 2: Integrate and check Integrating the two terms separately,

$$I=-\ln|x-2|+2\ln|x-3|+C.$$

The absolute values make the logarithms valid on each interval of the original domain $x\neq 2,3$. To check, differentiate the expression: $-1/(x-2)+2/(x-3)$ has numerator $-(x-3)+2(x-2)=x-1$ over the original denominator. Hence it reproduces the given integrand.

Step 3: Conclusion $\boxed{\int \frac{x-1}{(x-2)(x-3)}\,dx=2\ln|x-3|-\ln|x-2|+C}$.

Quick Tip to Solve: For the coefficients, substitute the roots 2 and 3 after clearing denominators; no simultaneous equations are needed.

OR

7(b). Integrate $\dfrac{e^x}{\sqrt{5-4e^x-e^{2x}}}$ with respect to x. [2 marks]

Solution:

Concept:

- The substitution $t = e^x$ gives $dt = e^x\,dx$, matching the numerator exactly.
- Complete the square and use $\int du/\sqrt{a^2-u^2} = \sin^{-1}(u/a) + C$.

Step 1: Substitute and complete the square Let $t = e^x$, so $dt = e^x\,dx$. The integral becomes

$$I = \int \frac{dt}{\sqrt{5-4t-t^2}}.$$

Since $5 - 4t - t^2 = 9 - (t+2)^2$, set $u = t + 2$, giving $du = dt$. Therefore

$$I = \int \frac{du}{\sqrt{9-u^2}}.$$

This is the standard inverse-sine integral with $a = 3$.

Step 2: Use the standard integral and verify The antiderivative in the substituted variable is $\sin^{-1}(u/3) + C$. Substitute $u = e^x + 2$ back. For a real integrand, $5 - 4e^x - e^{2x} = (1-e^x)(e^x+5) > 0$, so $x < 0$. On this domain, differentiating $\sin^{-1}((e^x+2)/3)$ gives

$$\frac{e^x/3}{\sqrt{1-(e^x+2)^2/9}} = \frac{e^x}{\sqrt{5-4e^x-e^{2x}}},$$

verifying the result.

Step 3: Conclusion $\boxed{I = \sin^{-1}\left(\dfrac{e^x+2}{3}\right) + C}$, on the real domain $x < 0$.

Quick Tip to Solve: First absorb $e^x\,dx$ with $t = e^x$; the radicand then becomes $9 - (t+2)^2$.

8. If $P(A) = 0.6$, $P(B) = 0.5$ and $P(B \mid A) = 0.4$, find $P(A \cup B)$ and $P(A \mid B)$. [2 marks]

Solution:

Concept:

- Conditional probability satisfies $P(B \mid A) = P(A \cap B)/P(A)$ when $P(A) > 0$.
- The addition theorem is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; similarly $P(A \mid B) = P(A \cap B)/P(B)$.

Step 1: Find the common probability The given conditional probability concerns event B after restricting to event A. Thus its denominator is $P(A)$, not $P(B)$. Using the definition,

$$P(A \cap B) = P(A)P(B \mid A) = 0.6 \times 0.4 = 0.24.$$

Quick Tip to Solve: Find $P(A \cap B)$ first; the union and either conditional probability then use the same overlap.

This intersection probability will be used in both required calculations; it is less than each individual event probability, as it should be.

Step 2: Compute the union and the reverse conditional probability The union counts outcomes belonging to either event. Adding $P(A)$ and $P(B)$ counts their overlap twice, so subtract it once:

$$P(A \cup B) = 0.6 + 0.5 - 0.24 = 0.86.$$

To reverse the conditioning, divide the same intersection probability by $P(B)$:

$$P(A \mid B) = \frac{0.24}{0.5} = 0.48.$$

Both values lie between zero and one, consistent with their being probabilities.

Step 3: Conclusion $\boxed{P(A \cup B) = 0.86, \qquad P(A \mid B) = 0.48}$.

Quick Tip to Solve: Find $P(A \cap B)$ first; the union and either conditional probability then use the same overlap.

9. Find the value of $(x - y)$ from the matrix equation

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 \\ 15 & 14 \end{bmatrix}.$$

[2 marks]

Solution:

Concept:

- Scalar multiplication multiplies every matrix entry; matrix addition adds corresponding entries.
- Equal matrices have equal corresponding entries, so each unknown is determined by its own position.

Step 1: Simplify the left-hand matrix Multiply the first matrix by 2 and add the second entry by entry:

$$2\begin{bmatrix} x & 5 \\ 7 & y-3 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 2x-3 & 10-4 \\ 14+1 & 2y-6+2 \end{bmatrix}$$
$$= \begin{bmatrix} 2x-3 & 6 \\ 15 & 2y-4 \end{bmatrix}.$$

The off-diagonal entries already agree with the right-hand matrix, namely 6 and 15.

Step 2: Compare the diagonal entries Equality of the top-left entries gives $2x - 3 = 7$, hence $2x = 10$ and $x = 5$. Equality of the bottom-right entries gives $2y - 4 = 14$, hence $2y = 18$ and $y = 9$. Notice that the factor 2 acts on all of $y - 3$, so its contribution is $2y - 6$. Substitution gives the diagonal entries $2(5) - 3 = 7$ and $2(9) - 4 = 14$, confirming both unknowns.

Step 3: Conclusion $\boxed{x - y = 5 - 9 = -4}$.

Quick Tip to Solve: Use only the two diagonal entries to find the unknowns, but remember to distribute 2 over $y - 3$.

10. Find :

$\int e^x(\cos x - \sin x)\ \text{cosec}^2\, x\, dx.$ [2 marks]

Solution:

Concept:

- The product rule gives $\frac{d}{dx}[e^x f(x)] = e^x[f(x) + f\prime(x)]$.
- Use $\text{cosec}\, x = 1/\sin x$ and $\frac{d}{dx}(\text{cosec}\, x) = -\,\text{cosec}\, x \cot x$.

Step 1: Rewrite the trigonometric factor Using $\text{cosec}^2 x = 1/\sin^2 x$,

$$(\cos x - \sin x)\,\text{cosec}^2 x = \frac{\cos x}{\sin^2 x} - \frac{1}{\sin x}$$
$$= \text{cosec}\, x \cot x - \text{cosec}\, x.$$

Therefore the integrand is $e^x(\text{cosec}\, x \cot x - \text{cosec}\, x)$. The original expression is defined only when $\sin x \neq 0$.

Step 2: Recognise the product derivative Take $f(x) = -\,\text{cosec}\, x$. Its derivative is $f\prime(x) = \text{cosec}\, x \cot x$, so the rewritten integrand is exactly $e^x[f(x) + f\prime(x)]$. Explicitly,

$$\frac{d}{dx}[-e^x\,\text{cosec}\, x] = -e^x\,\text{cosec}\, x + e^x\,\text{cosec}\, x \cot x.$$

This matches the preceding expression, including its sign. Integrating a derivative restores the function plus an arbitrary constant on each interval where $\sin x \neq 0$.

Step 3: Conclusion $\boxed{\int e^x(\cos x - \sin x)\,\text{cosec}^2 x\, dx = -e^x\,\text{cosec}\, x + C}$.

Quick Tip to Solve: Divide the bracket by $\sin^2 x$, then recognise the derivative of $-e^x\,\text{cosec}\, x$.

11. Find the differential equation of the family of curves represented by $y^2 = a(b^2 - x^2)$. [2 marks]

Solution:

Concept:

- Eliminate the arbitrary constants by differentiating the family with respect to x. Since a and b are two constants, two differentiations are required.
- With $y' = dy/dx$ and $y'' = d^2y/dx^2$, the chain and product rules give $d(y^2)/dx = 2yy'$ and $d(yy')/dx = (y')^2 + yy''$.

Step 1: Differentiate the family once Starting with $y^2 = a(b^2 - x^2)$, differentiate while treating a, b as constants:

$$2yy' = -2ax, \qquad yy' = -ax.$$

The constant b has disappeared, but a remains, so this relation still depends on an arbitrary constant.

Quick Tip to Solve: Multiply the second derivative equation by x and substitute yy'=-ax to eliminate a without dividing by x.

Step 2: Differentiate again and eliminate a Differentiating the first derivative gives

$$(y')^2 + yy'' = -a.$$

Multiply by x and use $-ax = yy'$ from the preceding equation:

$$x\big((y')^2 + yy''\big) = yy'.$$

This removes the remaining constant without dividing by x. Thus no unnecessary restriction at x=0 has been introduced, and the resulting equation contains neither a nor b.

Step 3: Conclusion The required differential equation is

$$\boxed{xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0}.$$

Quick Tip to Solve: Multiply the second derivative equation by x and substitute yy'=-ax to eliminate a without dividing by x.

12. Let an operation $*$ on the set of natural numbers $\mathbb{N}$ be defined by $a * b = a^b$. Find (i) whether $*$ is a binary or not, and (ii) if it is a binary, then is it commutative or not. [2 marks]

Solution:

Concept:

- A binary operation on a set S assigns a unique element of S to every ordered pair in $S \times S$. Thus its defining rule must be well-defined and closed on S.
- A binary operation is commutative if $a*b = b*a$ for every pair of elements. A single counterexample suffices to disprove commutativity.

Step 1: Check whether the operation is binary For any $a, b \in \mathbb{N} = \{1, 2, 3, \ldots\}$, the value a^b is a uniquely determined positive integer. Indeed, it is the product of b copies of the positive integer a. Hence

$$a * b = a^b \in \mathbb{N}.$$

The rule is defined for every ordered pair of natural numbers and its output belongs to the same set. Therefore it meets both the uniqueness and closure requirements.

Step 2: Test commutativity using a counterexample Interchanging the two arguments changes a^b to b^a, and these need not agree. Choose $a = 2$ and $b = 3$:

$$2 * 3 = 2^3 = 8, \qquad 3 * 2 = 3^2 = 9.$$

Since $8 \neq 9$, the equality required for commutativity fails for this pair. Equality for special pairs such as a=b does not establish equality for every pair.

Step 3: Conclusion (i) The operation $*$ is a binary operation on $\mathbb{N}$. (ii) It is not commutative, since

Quick Tip to Solve: First check that powers of positive integers remain natural; then use 2 and 3 to disprove commutativity immediately.

$2 * 3 \neq 3 * 2$.

Quick Tip to Solve: First check that powers of positive integers remain natural; then use 2 and 3 to disprove commutativity immediately.

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13(a). Find the particular solution of the differential equation :

$(1 + e^{2x})\, dy + (1 + y^2)\, e^x\, dx = 0$**, given that** $y(0) = 1$**.** [4 marks]

Solution:

Concept:

- A separable differential equation can be rearranged so that terms involving y multiply dy and terms involving x multiply dx. Integrate both sides and then use the initial value to determine the constant.
- The standard integral $\int du/(1 + u^2) = \tan^{-1} u + C$ applies to both sides here after substituting $u = e^x$ on the x side.

Step 1: Separate the variables The printed differential equation is

$$(1 + e^{2x})\, dy + (1 + y^2)e^x\, dx = 0.$$

Both $1 + y^2$ and $1 + e^{2x}$ are positive for real x,y. Divide by their product and move the x term to the right:

$$\frac{dy}{1 + y^2} = -\frac{e^x}{1 + e^{2x}}\, dx.$$

Thus this separation does not discard any real solution by division by a zero factor.

Step 2: Integrate the separated equation Integrating the left side gives $\tan^{-1} y$. On the right put $u = e^x$, so $du = e^x\, dx$ and $e^{2x} = u^2$. Hence

$$\begin{aligned} \tan^{-1} y &= -\int \frac{du}{1 + u^2} + C \\ &= -\tan^{-1}(e^x) + C. \end{aligned}$$

Equivalently, $\tan^{-1} y + \tan^{-1}(e^x) = C$, with one arbitrary integration constant.

Step 3: Apply the prescribed initial condition Substitute $x = 0$ and $y = 1$, using $e^0 = 1$ and $\tan^{-1} 1 = \pi/4$:

$$C = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

Therefore $\tan^{-1} y = \pi/2 - \tan^{-1}(e^x)$. Since $e^x > 0$, its inverse tangent lies between 0 and $\pi/2$. Taking tangent consequently yields

$$y = \cot\big(\tan^{-1}(e^x)\big) = \frac{1}{e^x}.$$

Quick Tip to Solve: Spot $e^x\, dx$ as the differential of e^x; substituting $u = e^x$ converts the right-hand integral into the same inverse-tangent form as the left.

Step 4: Verify the particular solution For $y = e^{-x}$, $dy/dx = -e^{-x}$. Substitution into the original equation in derivative form gives

$$(1 + e^{2x})(-e^{-x}) + (1 + e^{-2x})e^{x} = -e^{-x} - e^{x} + e^{x} + e^{-x} = 0.$$

Also $y(0) = e^{0} = 1$. Thus both the differential equation and its initial condition hold.

Step 5: Conclusion The required particular solution is $\boxed{y = e^{-x}}$, for all real x. Equivalently, $\tan^{-1} y + \tan^{-1}(e^{x}) = \pi/2$.

Quick Tip to Solve: Spot $e^{x}\,dx$ as the differential of e^{x}; substituting $u = e^{x}$ converts the right-hand integral into the same inverse-tangent form as the left.

OR

13(b). Find the particular solution of the differential equation :

$x\dfrac{dy}{dx}\sin\left(\dfrac{y}{x}\right) + x - y\sin\left(\dfrac{y}{x}\right) = 0$**, given that** $y(1) = \dfrac{\pi}{2}$**.** [4 marks]

Solution:

Concept:

- For a homogeneous differential equation involving y/x, put $v = y/x$, so that $y = vx$ and $dy/dx = v + x\,dv/dx$.
- Separate the variables after substitution. Here $\int \sin v\,dv = -\cos v$ and $\int dx/x = \ln|x|$.
- A particular solution is obtained by using the prescribed initial value to determine the integration constant; the solution interval must contain the initial point.

Step 1: Substitute the ratio variable The differential equation contains the same ratio y/x in both trigonometric terms. Since this ratio requires $x \neq 0$, set $v = y/x$. Differentiating $y = vx$ using the product rule gives

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substitute both expressions into the given equation:

$$x\left(v + x\frac{dv}{dx}\right)\sin v + x - vx\sin v = 0.$$

The two terms involving $xv\sin v$ cancel. Hence

$$x^{2}\sin v\frac{dv}{dx} + x = 0.$$

Step 2: Separate and integrate Dividing by x^{2}, which is nonzero on the solution interval, and separating gives

$$\sin v\,dv = -\frac{dx}{x}.$$

Quick Tip to Solve: Put $y = vx$ before separating: the two $xv\sin v$ terms cancel immediately, leaving a direct sine integral.

Integration therefore yields

$$-\cos v = -\ln|x| + C_1, \qquad \cos v = \ln|x| + C.$$

Returning to the original variables, the implicit family is

$$\cos\left(\frac{y}{x}\right) = \ln|x| + C.$$

Keeping the cosine on the left avoids introducing unnecessary inverse-trigonometric branches before applying the initial condition.

Step 3: Determine the constant and check the initial branch At $x = 1$, the given value $y = \pi/2$ gives $v = \pi/2$. Thus

$$\cos\frac{\pi}{2} = \ln 1 + C, \qquad 0 = 0 + C, \qquad C = 0.$$

The interval containing $x = 1$ has $x > 0$, so $\ln|x| = \ln x$. Differentiating the resulting implicit equation confirms

$$-\sin\left(\frac{y}{x}\right)\frac{x\,dy/dx - y}{x^2} = \frac{1}{x},$$

which rearranges to the printed differential equation. The branch through the initial point can also be written $y = x\cos^{-1}(\ln x)$ for $e^{-1} < x < e$; here the inverse cosine takes values in $(0, \pi)$, and the endpoints are excluded because the differential equation would require $x = 0$ when the sine vanishes.

Step 4: Conclusion The required particular solution is $\boxed{\cos(y/x) = \ln x}$, on the branch through $(1, \pi/2)$. Equivalently, $y = x\cos^{-1}(\ln x)$ for $e^{-1} < x < e$.

Quick Tip to Solve: Put $y = vx$ before separating: the two $xv\sin v$ terms cancel immediately, leaving a direct sine integral.

14(a). Prove that the relation R in the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ given by $R = \{(a, b) : |a - b|$ is even$\}$ is an equivalence relation. [4 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation if it is reflexive, symmetric and transitive.
- An integer and its negative have the same parity. Thus $|a - b|$ is even precisely when $a - b$ is even, or equivalently when a and b have the same parity.

Step 1: Prove reflexivity Let a be any element of $A = \{1, 2, 3, 4, 5, 6, 7\}$. Then

$$|a - a| = 0 = 2 \cdot 0.$$

Zero is an even integer, so the defining condition of R holds for (a, a). Therefore $(a, a) \in R$ for every $a \in A$. This proves that R is reflexive on all of A, rather than only on some of its elements.

Step 2: Prove symmetry Let $a, b \in A$ and suppose $(a, b) \in R$. By definition, $|a - b|$ is even. Since

Quick Tip to Solve: Replace "absolute difference is even" by "same parity" to see the relation quickly, then prove all three properties with arbitrary elements.

reversing a difference changes only its sign,

$$|b-a| = |-(a-b)| = |a-b|.$$

Consequently $|b-a|$ is also even, which gives $(b,a) \in R$. Thus every related ordered pair has its reversed pair in the relation, proving symmetry.

Step 3: Prove transitivity Let $a, b, c \in A$, with $(a,b) \in R$ and $(b,c) \in R$. Then $|a-b|$ and $|b-c|$ are even. Removing the absolute value can only change a sign, so there exist integers m, n such that

$$a-b = 2m, \qquad b-c = 2n.$$

Adding these equalities eliminates the intermediate element b:

$$a-c = (a-b)+(b-c) = 2(m+n).$$

Hence $|a-c| = 2|m+n|$ is even, and therefore $(a,c) \in R$. This proves transitivity for arbitrary elements satisfying the two hypotheses. Notice that the signed differences are added; one need not assume that absolute differences themselves add.

Step 4: Identify the parity classes The defining condition relates exactly the elements of the same parity. For example, every odd element differs from 1 by an even integer, and every even element differs from 2 by an even integer. The resulting classes are

$$[1] = \{1,3,5,7\}, \qquad [2] = \{2,4,6\}.$$

These classes are disjoint and together contain every element of A, illustrating the partition induced by the relation.

Step 5: Conclusion Since R is reflexive, symmetric and transitive, $\boxed{R \text{ is an equivalence relation on } A}$.

Quick Tip to Solve: Replace "absolute difference is even" by "same parity" to see the relation quickly, then prove all three properties with arbitrary elements.

OR

14(b). Show that the function f in $A = \mathbb{R} - \left\{\frac{2}{3}\right\}$ defined as $f(x) = \frac{4x+3}{6x-4}$ is one-one and onto. Hence, find f^{-1}. [4 marks]

Solution:

Concept:

- A function $f : A \to A$ is one-one if $f(u) = f(v)$ implies $u = v$ for all $u, v \in A$.
- It is onto if every $y \in A$ equals $f(x)$ for some $x \in A$. A bijection has an inverse obtained by solving $y = f(x)$ for x.

Step 1: Check that the function maps A into A Here $A = \mathbb{R} \setminus \{2/3\}$. For $x \in A$, the denominator

Quick Tip to Solve: When solving for the preimage, check both that its denominator is nonzero and that it avoids the excluded domain value.

$6x - 4$ is nonzero, so $f(x)$ is defined. Also, $f(x)$ cannot equal the excluded value: if

$$\frac{4x+3}{6x-4} = \frac{2}{3},$$

then $12x + 9 = 12x - 8$, which would give $9 = -8$, a contradiction. Thus $f(x) \in A$, and the stated function is well defined from A to itself.

Step 2: Prove that f is one-one Let $u, v \in A$ and suppose $f(u) = f(v)$. Both denominators are nonzero, so cross multiplication gives

$$\begin{aligned}(4u+3)(6v-4) &= (4v+3)(6u-4),\\ 24uv - 16u + 18v - 12 &= 24uv - 16v + 18u - 12,\\ 34(v-u) &= 0.\end{aligned}$$

Hence $u = v$. This proves injectivity on the whole specified domain, without making any assumption about which side of $2/3$ the inputs lie on.

Step 3: Prove that f is onto Choose any $y \in A$. To find its preimage, solve

$$y = \frac{4x+3}{6x-4} \quad \Longrightarrow \quad (6y-4)x = 4y+3 \quad \Longrightarrow \quad x = \frac{4y+3}{6y-4}.$$

Since $y \neq 2/3$, this expression is defined. Moreover, it cannot equal $2/3$, by the same contradiction $12y + 9 = 12y - 8$. Consequently $x \in A$. Direct substitution confirms the preimage:

$$f(x) = \frac{(34y)/(6y-4)}{34/(6y-4)} = y.$$

Every member of the codomain therefore has a preimage in the domain, establishing surjectivity.

Step 4: Obtain the inverse function The preceding equation expresses the unique input in terms of the output. Interchanging the names of the variables gives the inverse rule $f^{-1}(x) = (4x+3)/(6x-4)$, with domain and codomain both A. The substitution above also shows that applying the same rule twice returns the original element, so the inverse is indeed the original function.

Step 5: Conclusion The function is one-one and onto on A, and $\boxed{f^{-1}(x) = \dfrac{4x+3}{6x-4},\ x \in \mathbb{R} \setminus \{2/3\}}$.

Quick Tip to Solve: When solving for the preimage, check both that its denominator is nonzero and that it avoids the excluded domain value.

15. Find whether the function $f(x) = \cos\left(2x + \frac{\pi}{4}\right)$ is increasing or decreasing in the interval $\frac{3\pi}{8} < x < \frac{5\pi}{8}$. [4 marks]

Solution:

Concept:

- For a differentiable function on an interval, a positive derivative throughout the interval implies that the function is strictly increasing there; a negative derivative implies that it is strictly decreasing.
- The chain rule gives $\frac{d}{dx}\cos u = -\sin u \frac{du}{dx}$. The sign of the sine depends on the quadrant of its argument.

Step 1: Differentiate the given function Write the inner angle as $u = 2x + \pi/4$. Then $du/dx = 2$, and hence

$$f'(x) = \frac{d}{dx}\cos\left(2x + \frac{\pi}{4}\right) = -2\sin\left(2x + \frac{\pi}{4}\right).$$

The factor 2 comes from differentiating the inner angle. To determine the sign of this derivative, we must examine the angle $2x + \pi/4$, rather than the angle x itself. The cosine and sine functions are differentiable for all real arguments, so the derivative test applies on the entire interval given in the question.

Step 2: Determine the quadrant of the inner angle Starting with the specified open interval, multiply each part by the positive number 2, and then add $\pi/4$:

$$\frac{3\pi}{8} < x < \frac{5\pi}{8} \implies \frac{3\pi}{4} < 2x < \frac{5\pi}{4}$$
$$\implies \pi < 2x + \frac{\pi}{4} < \frac{3\pi}{2}.$$

Thus the inner angle lies strictly in the third quadrant. Sine is negative throughout that quadrant, so

$$\sin\left(2x + \frac{\pi}{4}\right) < 0, \qquad -2\sin\left(2x + \frac{\pi}{4}\right) > 0.$$

Neither boundary angle belongs to the given interval; in particular, the zero value of sine at π does not occur at an interior point.

Step 3: Apply the derivative criterion We have established $f'(x) > 0$ at every point of the specified interval. To make the monotonicity implication explicit, take any $a < b$ within it. The mean value theorem supplies a point $c \in (a, b)$ such that

$$f(b) - f(a) = f'(c)(b - a) > 0.$$

Both factors on the right are positive. Hence the function takes a larger value at every larger input in this interval. As a consistency check, the inner angle advances from π towards $3\pi/2$, where cosine rises from -1 towards 0; the endpoints themselves are excluded.

Step 4: Conclusion Since $f'(x) > 0$ throughout $\frac{3\pi}{8} < x < \frac{5\pi}{8}$, the given function is $\boxed{\text{strictly increasing}}$ on this interval.

Quick Tip to Solve: Transform the entire interval into an interval for the inner angle before using a quadrant sign to decide monotonicity.

16. Using vectors, find the value of x such that the four points $A(x, 5, -1)$, $B(3, 2, 1)$, $C(4, 5, 5)$ and $D(4, 2, -2)$ are coplanar. [4 marks]

Solution:

Concept:

- Four points are coplanar precisely when three displacement vectors drawn from the same point have zero scalar triple product. This includes the condition that the fourth point lies in the plane determined by the other three points.
- For vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$, the scalar triple product is $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. The cross product is perpendicular to the plane spanned by $\mathbf{v}$ and $\mathbf{w}$.

Step 1: Form three vectors with a common initial point Take $B(3, 2, 1)$ as the common initial point. Subtract its coordinates from the coordinates of each of the other points:

$$\begin{aligned}\overrightarrow{BA} &= (x-3)\hat{i} + 3\hat{j} - 2\hat{k},\\ \overrightarrow{BC} &= \hat{i} + 3\hat{j} + 4\hat{k},\\ \overrightarrow{BD} &= \hat{i} - 3\hat{k}.\end{aligned}$$

These differences preserve the signs of the given coordinates, particularly the negative third coordinates of A and D.

Step 2: Obtain a normal to the plane through B, C and D Compute the cross product using the standard determinant expansion:

$$\begin{aligned}\overrightarrow{BC} \times \overrightarrow{BD} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 4 \\ 1 & 0 & -3 \end{vmatrix}\\ &= -9\hat{i} + 7\hat{j} - 3\hat{k}.\end{aligned}$$

This vector is nonzero, so B, C, D are not collinear and determine a unique plane. A vector from B to another point lies in this plane when its dot product with this normal is zero.

Step 3: Apply and check the coplanarity condition Therefore the required scalar triple product must vanish:

$$\begin{aligned}\overrightarrow{BA} \cdot (\overrightarrow{BC} \times \overrightarrow{BD}) &= 0,\\ -9(x-3) + 3(7) + (-2)(-3) &= 0,\\ -9x + 54 &= 0.\end{aligned}$$

Thus division by 9 gives the required value. As a check, substituting that value makes $\overrightarrow{BA} = (3, 3, -2)$, and its dot product with $(-9, 7, -3)$ is $-27 + 21 + 6 = 0$, as required.

Step 4: Conclusion The four given points are coplanar if and only if $\boxed{x = 6}$.

Quick Tip to Solve: Choose B as the common initial point: the resulting vector BD has a zero middle component, making the cross product especially simple.

17. Prove that :

$\sin^{-1}\frac{4}{5}+\tan^{-1}\frac{5}{12}+\cos^{-1}\frac{63}{65}=\frac{\pi}{2}.$ [4 marks]

Solution:

Concept:

- The principal values of $\sin^{-1}t$, $\tan^{-1}t$ and $\cos^{-1}t$ lie in $[-\pi/2,\pi/2]$, $(-\pi/2,\pi/2)$ and $[0,\pi]$, respectively. Positive arguments strictly between zero and one give acute angles for inverse sine and inverse cosine.
- For angles α,β, use $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$ and $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$. Check the quadrant before taking an inverse function.

Step 1: Define the two principal angles and find their ratios Put $\alpha=\sin^{-1}(4/5)$ and $\beta=\tan^{-1}(5/12)$. Both are in $(0,\pi/2)$. Hence the positive square roots must be used:

$$\sin\alpha=\frac{4}{5},\qquad \cos\alpha=\sqrt{1-\frac{16}{25}}=\frac{3}{5}.$$

Since $\tan\beta=5/12$, the corresponding hypotenuse is $\sqrt{5^2+12^2}=13$, giving

$$\sin\beta=\frac{5}{13},\qquad \cos\beta=\frac{12}{13}.$$

Step 2: Evaluate the sine and cosine of their sum The angle-addition formula gives

$$\begin{aligned}\sin(\alpha+\beta)&=\frac{4}{5}\frac{12}{13}+\frac{3}{5}\frac{5}{13}\\&=\frac{48+15}{65}=\frac{63}{65}.\end{aligned}$$

To establish the correct branch, also calculate

$$\cos(\alpha+\beta)=\frac{3}{5}\frac{12}{13}-\frac{4}{5}\frac{5}{13}=\frac{36-20}{65}=\frac{16}{65}>0.$$

As each angle is acute, $0<\alpha+\beta<\pi$. The positive cosine therefore puts their sum in $(0,\pi/2)$; the supplementary possibility is excluded.

Step 3: Identify the complementary principal angle Set $\delta=\pi/2-(\alpha+\beta)$. The branch check above shows that δ is acute, so in particular it belongs to the principal range of inverse cosine. Using the complementary-angle identity,

$$\cos\delta=\sin(\alpha+\beta)=\frac{63}{65}.$$

Consequently $\cos^{-1}(63/65)=\delta=\pi/2-(\alpha+\beta)$. Adding $\alpha+\beta$ to both sides proves the stated identity without any ambiguity about inverse-trigonometric branches.

Step 4: Conclusion Therefore

$$\boxed{\sin^{-1}\frac{4}{5}+\tan^{-1}\frac{5}{12}+\cos^{-1}\frac{63}{65}=\frac{\pi}{2}}.$$

Quick Tip to Solve: Recognize the 3-4-5 and 5-12-13 ratios, then compute both sine and cosine of the sum to fix its principal-value branch.

18. If $x^p y^q = (x+y)^{p+q}$, prove that $\frac{dy}{dx} = \frac{y}{x}$ and $\frac{d^2y}{dx^2} = 0$. [4 marks]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x; thus $\frac{d}{dx}\log y = \frac{1}{y}\frac{dy}{dx}$.
- Logarithmic differentiation changes powers and products into multiples and sums. Work on an interval where the given powers and logarithmic derivatives are defined and $x \neq 0$.
- The quotient rule gives $\frac{d}{dx}(y/x) = \frac{x\,dy/dx - y}{x^2}$. Therefore, establishing $x\,dy/dx = y$ also establishes the second derivative.

Step 1: Take logarithms and differentiate implicitly Taking logarithms of the given relation on its usual positive domain, we obtain

$$p\log x + q\log y = (p+q)\log(x+y).$$

Here p and q are constants. Differentiating with respect to x, and writing $y' = dy/dx$, gives

$$\frac{p}{x} + \frac{q}{y}y' = \frac{p+q}{x+y}(1+y').$$

The factor $1 + y'$ is essential because both terms in $x + y$ depend on x.

Step 2: Collect the terms containing the first derivative Multiply the differentiated equation by $xy(x+y)$ and rearrange:

$$\begin{aligned} py(x+y) + qx(x+y)y' &= (p+q)xy(1+y'), \\ \{qx(x+y) - (p+q)xy\}y' &= (p+q)xy - py(x+y), \\ x(qx-py)y' &= y(qx-py). \end{aligned}$$

At a regular point, cancel the common nonzero factor $qx - py$. Thus the required relation for the first derivative is $xy' = y$, or $y' = y/x$. This cancellation is the standard implicit differentiation step. Equivalently, dividing the original equation by x^{p+q} shows that $t = y/x$ satisfies $t^q = (1+t)^{p+q}$, an equation independent of x. On a differentiable branch its isolated admissible root is constant, so $y = tx$. This also covers a branch point at which the cancelled factor vanishes. As usual, the statement concerns a defined, nondegenerate branch of the given relation.

Step 3: Differentiate once more Now apply the quotient rule to $y' = y/x$:

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{x\,dy/dx - y}{x^2} \\ &= \frac{x(y/x) - y}{x^2} \\ &= \frac{y-y}{x^2} = 0. \end{aligned}$$

The same result follows from the straight-line branch $y = tx$, since its slope t is constant.

Quick Tip to Solve: After logarithmic differentiation, clear all denominators together; both sides factor by the same expression before using the quotient rule.

Step 4: Conclusion Hence, $\boxed{\frac{dy}{dx} = \frac{y}{x}}$ and $\boxed{\frac{d^2y}{dx^2} = 0}$, as required.

Quick Tip to Solve: After logarithmic differentiation, clear all denominators together; both sides factor by the same expression before using the quotient rule.

19. Find :

$\int (\sin x \cdot \sin 2x \cdot \sin 3x)\, dx.$ [4 marks]

Solution:

Concept:

- The product-to-sum identity $2\sin A \sin B = \cos(A - B) - \cos(A + B)$ replaces a product of two sines by a difference of cosines.
- The identity $2\sin A \cos B = \sin(A + B) + \sin(A - B)$ then reduces the remaining products to single sine functions.
- For a nonzero constant k, $\int \sin(kx)\, dx = -\frac{\cos(kx)}{k}$. The arbitrary constant of integration is denoted by C.

Step 1: Pair the first and third sine factors Let

$$I = \int \sin x \sin 2x \sin 3x\, dx.$$

Apply the first product-to-sum identity with $A = x$ and $B = 3x$:

$$\sin x \sin 3x = \frac{1}{2}\{\cos(-2x) - \cos 4x\} = \frac{1}{2}(\cos 2x - \cos 4x).$$

Here we use the fact that cosine is even. Consequently,

$$I = \frac{1}{2}\int \{\sin 2x \cos 2x - \sin 2x \cos 4x\}\, dx.$$

Pairing the factors this way produces frequencies that are easy to integrate after one further identity.

Step 2: Express the integrand as a sum of sines For the first product, the double-angle identity gives

$$\sin 2x \cos 2x = \frac{1}{2}\sin 4x.$$

For the second product, use the sine-cosine identity:

$$\begin{aligned}\sin 2x \cos 4x &= \frac{1}{2}\{\sin 6x + \sin(-2x)\} \\ &= \frac{1}{2}(\sin 6x - \sin 2x).\end{aligned}$$

Since sine is odd, the term involving the difference of the angles has a negative sign. Substituting

Quick Tip to Solve: Pair sin x with sin 3x first; remember that sin(-2x) is negative when expanding sin 2x cos 4x.

both expressions yields

$$I = \frac{1}{4}\int (\sin 4x - \sin 6x + \sin 2x)\, dx.$$

Thus there is no substitution left to perform: integrate each sine separately.

Step 3: Integrate term by term and check Keeping the frequency in each denominator, we get

$$I = \frac{1}{4}\left(-\frac{\cos 4x}{4} + \frac{\cos 6x}{6} - \frac{\cos 2x}{2}\right) + C$$
$$= -\frac{\cos 4x}{16} + \frac{\cos 6x}{24} - \frac{\cos 2x}{8} + C.$$

To verify the signs and coefficients, differentiate this expression:

$$I' = \frac{1}{4}\sin 4x - \frac{1}{4}\sin 6x + \frac{1}{4}\sin 2x.$$

This equals the reduced integrand obtained above, and reversing the product-to-sum steps gives exactly $\sin x \sin 2x \sin 3x$.

Step 4: Conclusion Therefore, $\boxed{\int \sin x \sin 2x \sin 3x\, dx = \frac{\cos 6x}{24} - \frac{\cos 4x}{16} - \frac{\cos 2x}{8} + C}$.

Quick Tip to Solve: Pair sin x with sin 3x first; remember that sin(-2x) is negative when expanding sin 2x cos 4x.

20(a). **Differentiate** $\tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$, $|x| < \frac{1}{\sqrt{3}}$, **w.r.t.** $\tan^{-1}\left(\frac{x}{\sqrt{1 - x^2}}\right)$. [4 marks]

Solution:

Concept:

- For differentiable functions $u(x)$ and $v(x)$, differentiation of u with respect to v gives $\frac{du}{dv} = \frac{du/dx}{dv/dx}$, provided $dv/dx \neq 0$.
- The triple-angle identity is $\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$. The principal value of $\tan^{-1}$ lies in $(-\pi/2, \pi/2)$.
- On $(-1, 1)$, $\tan^{-1}\left(x/\sqrt{1 - x^2}\right) = \sin^{-1} x$, whose derivative is $1/\sqrt{1 - x^2}$.

Step 1: Choose a substitution and check the principal branch Put $\theta = \tan^{-1} x$, so that $x = \tan\theta$. The stated restriction $|x| < 1/\sqrt{3}$, together with the monotonicity of the inverse tangent, gives $-\pi/6 < \theta < \pi/6$. Hence $-\pi/2 < 3\theta < \pi/2$. This check is needed before cancelling an inverse tangent with a tangent. Let $u = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right)$. The denominator is positive throughout the given interval, and the triple-angle identity now gives

$$u = \tan^{-1}(\tan 3\theta) = 3\theta = 3\tan^{-1} x.$$

Quick Tip to Solve: First use the bound on x to justify $\tan^{-1}(\tan 3\theta) = 3\theta$, then divide the two derivatives.

Consequently,

$$\frac{du}{dx} = \frac{3}{1+x^2}.$$

Step 2: Differentiate the independent inverse-trigonometric expression Let $v = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$. Set $\phi = \sin^{-1} x$. Since $|x| < 1/\sqrt{3} < 1$, we have $-\pi/2 < \phi < \pi/2$, $\cos\phi = \sqrt{1-x^2} > 0$, and $\tan\phi = x/\sqrt{1-x^2}$. Therefore $v = \phi = \sin^{-1} x$, and

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}}.$$

This derivative is finite and nonzero at every point of the prescribed open interval.

Step 3: Apply the derivative ratio The question asks for differentiation with respect to v, rather than merely with respect to x. Thus use the chain rule in ratio form:

$$\frac{du}{dv} = \frac{du/dx}{dv/dx} = \frac{3}{1+x^2}\sqrt{1-x^2}.$$

Both derivatives used above apply on the full given interval, so the ratio needs no additional branch restriction.

Step 4: Conclusion The required derivative is $\boxed{\frac{3\sqrt{1-x^2}}{1+x^2}}$, for $|x| < 1/\sqrt{3}$.

Quick Tip to Solve: First use the bound on x to justify $\tan^{-1}(\tan 3\theta) = 3\theta$, then divide the two derivatives.

OR

20(b). **If** $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, $|x| < 1$, $|y| < 1$, **show that** $\frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$. [4 marks]

Solution:

Concept:

- Implicit differentiation treats y as a function of x. In particular, $\frac{d}{dx}\sqrt{1-y^2} = -\frac{y}{\sqrt{1-y^2}}\frac{dy}{dx}$.
- The restrictions $|x| < 1$, $|y| < 1$ make both square roots strictly positive. The parameter a is constant.
- The difference of squares identity connects $\sqrt{1-x^2} - \sqrt{1-y^2}$ to $x+y$ using the given relation.

Step 1: Introduce positive square roots and differentiate Write $P = \sqrt{1-x^2} > 0$, $Q = \sqrt{1-y^2} > 0$, and $y' = dy/dx$. The given equation becomes $P + Q = a(x-y)$. Differentiating with respect to x, with a constant, gives

$$-\frac{x}{P} - \frac{y}{Q}y' = a(1-y').$$

Quick Tip to Solve: Multiply the sum of the radicals by their difference to obtain $a(P-Q) = -(x+y)$; this cancels the implicit derivative factors.

Collecting the terms containing y' and multiplying by PQ, we obtain

$$P(aQ - y)y' = Q(aP + x).$$

Step 2: Use the original equation to simplify the factors Since $P+Q > 0$, the equation $P+Q = a(x-y)$ implies $x - y \neq 0$ and $a \neq 0$. Now

$$\begin{aligned}(P - Q)(P + Q) &= P^2 - Q^2 \\ &= (1 - x^2) - (1 - y^2) \\ &= -(x - y)(x + y).\end{aligned}$$

Substituting $P + Q = a(x - y)$ and cancelling the nonzero factor $x - y$ yields

$$a(P - Q) = -(x + y).$$

Rearranging this identity gives $aP + x = aQ - y$.

Step 3: Justify cancellation and obtain the derivative The common factor cannot vanish. Indeed, if $aP + x = aQ - y = 0$, then $x = -aP$ and $y = aQ$. This would give $x - y = -a(P + Q)$, and substitution into the original relation would imply $(1 + a^2)(P + Q) = 0$, impossible because $P + Q > 0$ for real a. Therefore we may cancel $aQ - y = aP + x$ in

$$P(aQ - y)y' = Q(aQ - y),$$

obtaining $y' = Q/P$.

Step 4: Conclusion Hence $\boxed{\frac{dy}{dx} = \frac{\sqrt{1 - y^2}}{\sqrt{1 - x^2}} = \sqrt{\frac{1 - y^2}{1 - x^2}}}$, as required.

Quick Tip to Solve: Multiply the sum of the radicals by their difference to obtain $a(P - Q) = -(x + y)$; this cancels the implicit derivative factors.

21. Using properties of determinants, prove the following :

$$\begin{vmatrix} a^2 & bc & ac + c^2 \\ a^2 + ab & b^2 & ac \\ ab & b^2 + bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$$ [4 marks]

Solution:

Concept:

- A common factor from a column can be taken outside the determinant. Taking factors from three columns multiplies the remaining determinant by their product.
- Adding a multiple of one column to another, or of one row to another, leaves the determinant unchanged. Expansion along a column containing zeros then simplifies the calculation.

Step 1: Take the common factors from the columns Denote the given determinant by D. The

Quick Tip to Solve: After extracting column factors, use $C_3 - C_1 - C_2$; its two equal entries disappear under $R_3 - R_2$.

first, second and third columns have common factors a, b and c, respectively. Hence, using the column-factor property,

$$D = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}.$$

This is a factorisation of the column entries, so it does not require division by any of the variables. It remains valid even when a variable is zero.

Step 2: Create a zero in the third column Apply $C_3 \longrightarrow C_3 - C_1 - C_2$, keeping the first two columns unchanged. Subtracting these columns leaves the determinant unchanged. The entries in the third column become 0, $-2b$ and $-2b$. Thus,

$$D = abc\begin{vmatrix} a & c & 0 \\ a+b & b & -2b \\ b & b+c & -2b \end{vmatrix}.$$

Step 3: Create a second zero and expand Now apply $R_3 \longrightarrow R_3 - R_2$. This gives

$$D = abc\begin{vmatrix} a & c & 0 \\ a+b & b & -2b \\ -a & c & 0 \end{vmatrix}.$$

Expand along the third column, whose only nonzero entry is in position $(2,3)$. Its cofactor sign is $(-1)^{2+3} = -1$, so

$$\begin{aligned} D &= abc(-2b)(-1)\begin{vmatrix} a & c \\ -a & c \end{vmatrix} \\ &= 2ab^2c\big(ac - c(-a)\big) \\ &= 2ab^2c(2ac). \end{aligned}$$

The negative entry and the negative cofactor sign cancel, which is essential for obtaining the positive value.

Step 4: Conclusion Therefore, the given determinant equals $\boxed{4a^2b^2c^2}$, as required.

Quick Tip to Solve: After extracting column factors, use $C_3 - C_1 - C_2$; its two equal entries disappear under $R_3 - R_2$.

22. Evaluate :

$\int_{-1}^{2} |\, x^3 - x \,|\, dx.$ [4 marks]

Solution:

Concept:

- For a real-valued expression $f(x)$, the modulus is $|f(x)| = f(x)$ where $f(x) \geq 0$, and $|f(x)| = -f(x)$ where $f(x) < 0$.

Quick Tip to Solve: Factor the expression first and split at 0 and 1; negate only the integral on $(0,1)$.

- A definite integral can be split at the zeros where the sign changes. For the polynomial here, use the antiderivative $F(x) = \dfrac{x^4}{4} - \dfrac{x^2}{2}$, for which $F'(x) = x^3 - x$.

Step 1: Find the zeros and determine the signs Factor the expression inside the modulus:

$$x^3 - x = x(x-1)(x+1).$$

Its zeros in the integration interval are -1, 0 and 1. On $(-1, 0)$, the factors have signs negative, negative and positive, respectively, so their product is positive. On $(0, 1)$, only $x - 1$ is negative, so the product is negative. On $(1, 2)$, all three factors are positive. At the zeros the modulus vanishes, so the endpoint values do not affect the sign-based splitting.

Step 2: Remove the modulus on each subinterval Denote the required integral by I. By additivity of definite integrals and the signs just found,

$$I = \int_{-1}^{0} (x^3 - x)\,dx - \int_{0}^{1} (x^3 - x)\,dx + \int_{1}^{2} (x^3 - x)\,dx.$$

The minus sign in front of the middle integral makes its contribution positive; it accounts for the portion where the original polynomial lies below zero. Using one polynomial integral over the entire interval would incorrectly cancel positive and negative contributions.

Step 3: Evaluate the three contributions With $F(x) = x^4/4 - x^2/2$, the endpoint values are

$$F(-1) = -\frac{1}{4}, \quad F(0) = 0, \quad F(1) = -\frac{1}{4}, \quad F(2) = 2.$$

Therefore,

$$\begin{aligned} I &= [F(x)]_{-1}^{0} - [F(x)]_{0}^{1} + [F(x)]_{1}^{2} \\ &= \left(0 + \frac{1}{4}\right) - \left(-\frac{1}{4} - 0\right) + \left(2 + \frac{1}{4}\right) \\ &= \frac{1}{4} + \frac{1}{4} + \frac{9}{4}. \end{aligned}$$

All three contributions are nonnegative, as expected for an integral of an absolute value.

Step 4: Conclusion Hence, $\displaystyle\int_{-1}^{2} |x^3 - x|\,dx = \boxed{\frac{11}{4}}$.

Quick Tip to Solve: Factor the expression first and split at 0 and 1; negate only the integral on $(0, 1)$.

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23. Find the equation of planes passing through the intersection of the planes $\vec{r} \cdot (2\hat{i} + 6\hat{j}) + 12 = 0$ and $\vec{r} \cdot (3\hat{i} - \hat{j} + 4\hat{k}) = 0$ and are at a unit distance from origin. [4 marks]

Solution:

Concept:

- If two planes have equations $P_1 = 0$ and $P_2 = 0$, every plane $P_1 + \lambda P_2 = 0$ contains their line of intersection. The separately omitted member is $P_2 = 0$.

Quick Tip to Solve: Keep the constant term 12 unchanged in the plane family; the mixed terms in the squared normal length cancel, making the distance equation immediate.

- The perpendicular distance of the origin from $ax+by+cz+d=0$ is $D=|d|/\sqrt{a^2+b^2+c^2}$, where (a,b,c) is the normal vector.

Step 1: Form the family of planes Writing $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, the given planes become

$$P_1=2x+6y+12=0, \qquad P_2=3x-y+4z=0.$$

A plane containing their common line is therefore

$$(2x+6y+12)+\lambda(3x-y+4z)=0,$$

or

$$(2+3\lambda)x+(6-\lambda)y+4\lambda z+12=0.$$

At every point on the intersection, both parent expressions vanish, so this combination also vanishes. The member $P_2=0$, which cannot be written with the coefficient of P_1 equal to one, passes through the origin and has distance zero; it cannot be a required plane.

Step 2: Apply the unit-distance condition The normal has components $2+3\lambda$, $6-\lambda$, and 4λ. Since the required distance is one,

$$\frac{12}{\sqrt{(2+3\lambda)^2+(6-\lambda)^2+16\lambda^2}}=1.$$

Both sides are positive, so squaring introduces no sign ambiguity. Expanding gives

$$\begin{aligned}(4+12\lambda+9\lambda^2)+(36-12\lambda+\lambda^2)+16\lambda^2&=144,\\ 40+26\lambda^2&=144,\\ \lambda^2&=4.\end{aligned}$$

Thus $\lambda=2$ or $\lambda=-2$. The linear terms cancel because the normals of the two given planes have zero scalar product.

Step 3: Substitute and verify both members For $\lambda=2$, the family gives $8x+4y+8z+12=0$, which simplifies to $2x+y+2z+3=0$. Its distance from the origin is $3/\sqrt{4+1+4}=1$. For $\lambda=-2$, the family gives $-4x+8y-8z+12=0$, which simplifies to $x-2y+2z-3=0$. Its distance is also $3/\sqrt{1+4+4}=1$. Each is a linear combination of the given equations and hence contains their intersection.

Step 4: Conclusion The required planes are $\boxed{2x+y+2z+3=0}$ and $\boxed{x-2y+2z-3=0}$, equivalently $\vec{r}\cdot(2\hat{i}+\hat{j}+2\hat{k})+3=0$ and $\vec{r}\cdot(\hat{i}-2\hat{j}+2\hat{k})-3=0$.

Quick Tip to Solve: Keep the constant term 12 unchanged in the plane family; the mixed terms in the squared normal length cancel, making the distance equation immediate.

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit for type A souvenirs is ₹100 each and for type B souvenirs, profit is ₹120 each. How many souvenirs of each type should the company manufacture in order to maximise the profit? Formulate the problem as a LPP and then solve it graphically. [6 marks]

Solution:

Concept:

- A linear programming problem maximises a linear objective subject to linear resource inequalities and non-negative decision variables.
- For a bounded feasible polygon, a linear objective attains its maximum at a vertex. Each vertex must satisfy every constraint, including non-negativity.

Step 1: Define the variables and formulate the LPP Let x and y denote the numbers of souvenirs of types A and B, respectively. The cutting time available is $3 \times 60 + 20 = 200$ minutes, and the assembling time is $4 \times 60 = 240$ minutes. As each type A souvenir uses 5 minutes for cutting and each type B uses 8 minutes, the cutting constraint is $5x + 8y \leq 200$. The assembling constraint is $10x + 8y \leq 240$, since the respective requirements are 10 and 8 minutes. The profit is $Z = 100x + 120y$, measured in rupees. Thus the LPP is

$$\text{Maximise } Z = 100x + 120y,$$
$$5x + 8y \leq 200, \qquad 10x + 8y \leq 240,$$
$$x \geq 0, \qquad y \geq 0.$$

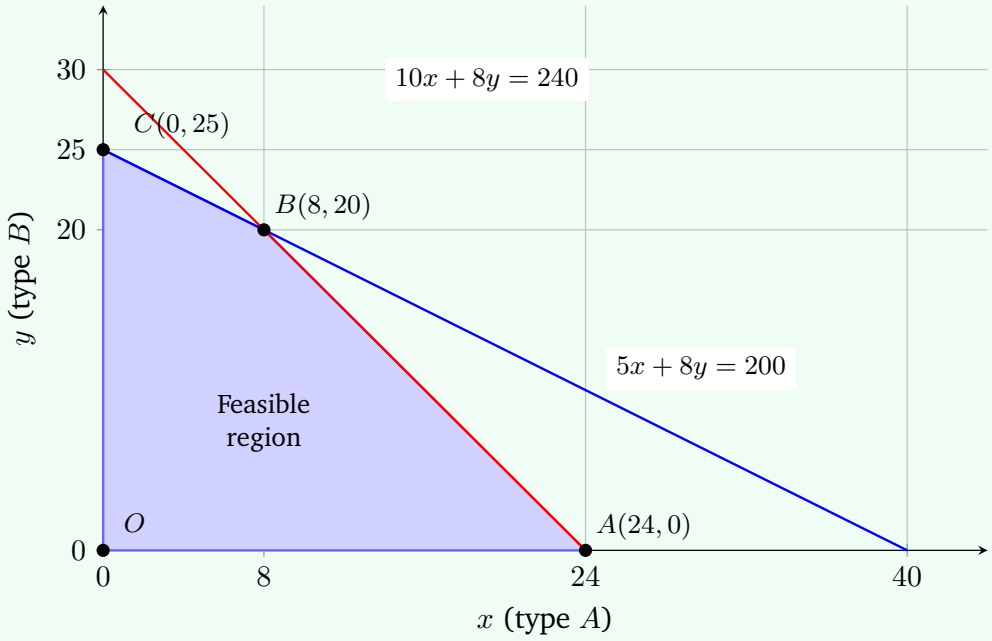
The inequalities allow unused time but do not allow either resource limit to be exceeded.

Step 2: Plot the boundary lines and identify the feasible region The cutting boundary $5x + 8y = 200$ passes through $(40, 0)$ and $(0, 25)$. The assembling boundary $10x + 8y = 240$ passes through $(24, 0)$ and $(0, 30)$. Since the origin satisfies both time inequalities, retain the half-plane containing the origin for each line and restrict attention to the first quadrant. To locate their intersection, subtract the cutting equation from the assembling equation:

$$5x = 40 \quad \Rightarrow \quad x = 8.$$

Substitution gives $40 + 8y = 200$, so $y = 20$. The shaded common region is the bounded quadrilateral with vertices $O(0, 0)$, $A(24, 0)$, $B(8, 20)$, and $C(0, 25)$. The intercepts $(40, 0)$ and $(0, 30)$ are excluded because each violates the other resource constraint.

Quick Tip to Solve: Convert both resource times to minutes before forming inequalities; subtract the two boundary equations to find the shared vertex quickly.



Feasible region for the cutting and assembling constraints; both shaded boundaries are included.

Step 3: Evaluate profit at every feasible vertex Since the feasible region is nonempty and bounded, compare the objective at its four vertices:

(x, y)	$Z = 100x + 120y$
$(0, 0)$	0
$(24, 0)$	2400
$(8, 20)$	$800 + 2400 = 3200$
$(0, 25)$	3000

The largest entry is 3200 at $(8, 20)$. The adjacent vertices have strictly smaller profits, so the maximum is uniquely attained at this intersection. The optimal quantities are whole numbers and therefore also meet the practical requirement that souvenirs be counted in complete units.

Step 4: Check the resource use at the optimum With 8 souvenirs of type A and 20 of type B, cutting requires $5(8) + 8(20) = 200$ minutes and assembling requires $10(8) + 8(20) = 240$ minutes. Both limits are met exactly. Their total profit is $100(8) + 120(20) = 3200$ rupees, consistent with the vertex comparison.

Step 5: Conclusion The company should manufacture $\boxed{8}$ souvenirs of type A and $\boxed{20}$ souvenirs of type B. The maximum profit is $\boxed{₹\,3200}$.

Quick Tip to Solve: Convert both resource times to minutes before forming inequalities; subtract the two boundary equations to find the shared vertex quickly.

25(a). Using integration, find the area of the following region :

$\{(x, y) : x^2 + y^2 \leq 16a^2$ **and** $y^2 \leq 6ax\}$. [6 marks]

Solution:

Concept:

- The area between a right boundary $x = R(y)$ and a left boundary $x = L(y)$ is $\int [R(y) - L(y)]\, dy$.
- Both inequalities involve y^2, so the region is symmetric about the x-axis. Its total area is twice its area above that axis.
- For $r > 0$, $\int \sqrt{r^2 - y^2}\, dy = \frac{y}{2}\sqrt{r^2 - y^2} + \frac{r^2}{2}\sin^{-1}(y/r) + C$.

Step 1: Determine the boundaries and intersections First take $a > 0$. The circle has centre at the origin and radius $4a$. The parabola is $x = y^2/(6a)$, opening towards positive x. Thus the required region lies inside the circle and to the right of the parabola. At an intersection, substitute $y^2 = 6ax$ into the circle:

$$x^2 + 6ax - 16a^2 = 0, \qquad (x - 2a)(x + 8a) = 0.$$

Since $y^2 = 6ax \geq 0$ forces $x \geq 0$, the root $-8a$ is inadmissible. The intersections therefore have $x = 2a$ and $y = \pm 2\sqrt{3}\, a$. On the upper half of the region, the right boundary is $x = \sqrt{16a^2 - y^2}$ and the left boundary is $x = y^2/(6a)$.

Step 2: Express the area as a definite integral A horizontal strip of thickness dy has width equal to the circle's right boundary minus the parabola's boundary. For $0 \leq y \leq 2\sqrt{3}\, a$, this width is nonnegative and becomes zero at the intersection. Consequently,

$$A = 2\int_0^{2\sqrt{3}a} \left(\sqrt{16a^2 - y^2} - \frac{y^2}{6a}\right) dy.$$

This single integral includes the entire upper half; symmetry supplies the lower half without counting any area twice.

Step 3: Evaluate the two terms An antiderivative of the expression in parentheses is

$$\frac{y}{2}\sqrt{16a^2 - y^2} + 8a^2 \sin^{-1}\left(\frac{y}{4a}\right) - \frac{y^3}{18a}.$$

At the upper endpoint, $\sqrt{16a^2 - 12a^2} = 2a$ and $\sin^{-1}(\sqrt{3}/2) = \pi/3$. The three terms are respectively $2\sqrt{3}a^2$, $8\pi a^2/3$, and $4\sqrt{3}a^2/3$. All terms vanish at the lower endpoint. Hence

$$A = 2\left(2\sqrt{3}a^2 + \frac{8\pi a^2}{3} - \frac{4\sqrt{3}a^2}{3}\right).$$

If $a < 0$, reflection in the y-axis gives exactly the same area with $|a|$ in place of a; if $a = 0$, the region is only the origin and has zero area.

Step 4: Conclusion The required area is $\boxed{\frac{4a^2}{3}(4\pi + \sqrt{3})}$ square units.

Quick Tip to Solve: Use horizontal strips: the parabola is immediately a left boundary in terms of y, so one integral and symmetry suffice.

OR

25(b). Using integration, find the area of triangle ABC bounded by the lines $4x-y+5=0$, $x+y-5=0$ and $x-4y+5=0$. [6 marks]

Solution:

Concept:

- For a vertical strip between $y=U(x)$ and $y=L(x)$, its area is $[U(x)-L(x)]\,dx$. Integrate this nonnegative height over the horizontal extent of the region.
- The vertices of a triangle bounded by three lines are their pairwise intersections. Split the integral wherever the upper or lower boundary changes.

Step 1: Find the three vertices Write the lines as $y=4x+5$, $y=5-x$, and $y=(x+5)/4$. For the intersection of the first two,

$$4x+5=5-x \quad \Rightarrow \quad x=0,\ y=5.$$

For the second and third,

$$5-x=\frac{x+5}{4} \quad \Rightarrow \quad 20-4x=x+5 \quad \Rightarrow \quad x=3,\ y=2.$$

For the first and third,

$$4x+5=\frac{x+5}{4} \quad \Rightarrow \quad 16x+20=x+5 \quad \Rightarrow \quad x=-1,\ y=-1.$$

We may label these vertices $A(-1,-1)$, $B(0,5)$, and $C(3,2)$.

Step 2: Identify the strip heights on the two intervals The side AC, namely $y=(x+5)/4$, is the lower boundary throughout the triangle. Between -1 and 0, the upper boundary is the side AB, namely $y=4x+5$. Between 0 and 3, it is the side BC, namely $y=5-x$. Thus integration gives

$$A_1=\int_{-1}^{0}\left(4x+5-\frac{x+5}{4}\right)dx,$$
$$A_2=\int_{0}^{3}\left(5-x-\frac{x+5}{4}\right)dx.$$

The first height is $15(x+1)/4$ and the second is $5(3-x)/4$. Both are nonnegative on their intervals.

Step 3: Evaluate and add the areas Integrate each linear strip height separately:

$$A_1=\frac{15}{4}\left[\frac{x^2}{2}+x\right]_{-1}^{0}=\frac{15}{4}\left(0-\left(\frac{1}{2}-1\right)\right)=\frac{15}{8},$$
$$A_2=\frac{5}{4}\left[3x-\frac{x^2}{2}\right]_{0}^{3}=\frac{5}{4}\left(9-\frac{9}{2}\right)=\frac{45}{8}.$$

The two integrals cover the entire triangle, so its area is their sum.

Step 4: Conclusion The required area is $\boxed{\frac{15}{8}+\frac{45}{8}=\frac{15}{2}}$ square units.

Quick Tip to Solve: Find all pairwise intersections first, then split at the middle vertex's x-coordinate so the upper boundary stays fixed in each integral.

26(a). Find the vector equation of the line passing through $(2, 1, -1)$ and parallel to the line $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$. Also, find the distance between these two lines. [6 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$, parallel to a non-zero vector $\vec{b}$, has equation $\vec{r} = \vec{a} + t\vec{b}$, where $t \in \mathbb{R}$.
- For two parallel lines with common direction $\vec{b}$, their perpendicular separation is $d = \dfrac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$. Here $\vec{a}_1, \vec{a}_2$ are position vectors of points on the respective lines.

Step 1: Identify the point and common direction The given line is $\vec{r} = (\hat{i} + \hat{j}) + \lambda(2\hat{i} - \hat{j} + \hat{k})$. Thus it passes through $A(1, 1, 0)$ and has direction vector $\vec{b} = 2\hat{i} - \hat{j} + \hat{k}$. The required line passes through $B(2, 1, -1)$. Since it is parallel to the given line, we can use exactly the same direction vector. Its vector equation is therefore

$$\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + t(2\hat{i} - \hat{j} + \hat{k}), \qquad t \in \mathbb{R}.$$

The parameter t is independent of the parameter on the given line. At $t = 0$, the equation yields the prescribed point, and its coefficient of t confirms parallelism.

Step 2: Calculate the perpendicular separation A vector joining the selected points on the two lines is

$$\overrightarrow{AB} = \vec{a}_2 - \vec{a}_1 = \hat{i} - \hat{k}.$$

Its full length is not the distance between the lines, since this joining vector need not be perpendicular to their direction. The cross product extracts the component perpendicular to $\vec{b}$, multiplied by $|\vec{b}|$. Compute

$$\begin{aligned}\overrightarrow{AB} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 2 & -1 & 1 \end{vmatrix} \\ &= -\hat{i} - 3\hat{j} - \hat{k}.\end{aligned}$$

Hence $|\overrightarrow{AB} \times \vec{b}| = \sqrt{1 + 9 + 1} = \sqrt{11}$ and $|\vec{b}| = \sqrt{4 + 1 + 1} = \sqrt{6}$. Substitution into the parallel-line distance formula gives

$$d = \frac{\sqrt{11}}{\sqrt{6}} = \sqrt{\frac{11}{6}}.$$

Step 3: Check that the separation is a minimum As an independent check, the squared length of a joining vector after moving along the common direction is

$$\begin{aligned}|\overrightarrow{AB} + s\vec{b}|^2 &= |(1 + 2s, -s, -1 + s)|^2 \\ &= 6s^2 + 2s + 2 \\ &= 6\left(s + \frac{1}{6}\right)^2 + \frac{11}{6}.\end{aligned}$$

Quick Tip to Solve: For parallel lines, retain the given direction vector and divide the magnitude of the joining-vector cross product by its length; do not use the skew-line formula.

Its least value is $11/6$, agreeing with the squared perpendicular distance. The positive result also shows that these parallel lines are distinct.

Step 4: Conclusion The required line is $\boxed{\vec{r} = (2\hat{i} + \hat{j} - \hat{k}) + t(2\hat{i} - \hat{j} + \hat{k}),\ t \in \mathbb{R}}$, and the distance between the two lines is $\boxed{\sqrt{11/6}}$ units.

Quick Tip to Solve: For parallel lines, retain the given direction vector and divide the magnitude of the joining-vector cross product by its length; do not use the skew-line formula.

OR

26(b). Find the coordinates of the foot Q of the perpendicular drawn from the point $P(1, 3, 4)$ to the plane $2x - y + z + 3 = 0$. Find the distance PQ and the image of P treating the plane as a mirror. [6 marks]

Solution:

Concept:

- The vector (a, b, c) is normal to the plane $ax + by + cz + d = 0$. A perpendicular from a point to this plane must be parallel to that normal.
- The distance of (x_1, y_1, z_1) from the plane is $\dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$. Reflection in the plane makes the perpendicular foot the midpoint of the original point and its image.

Step 1: Write the normal line through the point The plane is $2x - y + z + 3 = 0$, so a normal vector is $\vec{n} = 2\hat{i} - \hat{j} + \hat{k}$. The line through $P(1, 3, 4)$ perpendicular to the plane has parametric equations

$$x = 1 + 2t, \qquad y = 3 - t, \qquad z = 4 + t.$$

The foot Q lies both on this line and in the plane. Substituting these coordinates into the plane equation yields

$$\begin{aligned} 2(1 + 2t) - (3 - t) + (4 + t) + 3 &= 0, \\ 6 + 6t &= 0, \\ t &= -1. \end{aligned}$$

Therefore the three coordinates of the foot are

$$Q = (1 - 2,\ 3 + 1,\ 4 - 1) = (-1, 4, 3).$$

Step 2: Find and verify the perpendicular distance The signed plane expression at P is $2(1) - 3 + 4 + 3 = 6$. Thus the perpendicular distance is

$$PQ = \frac{|6|}{\sqrt{2^2 + (-1)^2 + 1^2}} = \frac{6}{\sqrt{6}} = \sqrt{6}.$$

Quick Tip to Solve: First find the foot using the normal direction; then use image equals twice the foot minus the original point, rather than solving a second intersection problem.

This also follows directly from the displacement along the normal:

$$\overrightarrow{PQ} = (-2, 1, -1) = -(2, -1, 1).$$

Its length is $\sqrt{6}$, as expected. Moreover, substitution of Q gives $-2 - 4 + 3 + 3 = 0$, so the calculated foot is indeed in the plane. These two checks establish both conditions for a perpendicular foot.

Step 3: Reflect the point across the plane Let the image of P be P'. Reflection across a plane places P' on the same perpendicular line, on the opposite side, with $QP' = QP$. Hence Q is the midpoint of PP', and coordinatewise $P' = 2Q - P$. We obtain

$$\begin{aligned} x' &= 2(-1) - 1 = -3, \\ y' &= 2(4) - 3 = 5, \\ z' &= 2(3) - 4 = 2. \end{aligned}$$

As a final check, the plane expression at this image is

$$2(-3) - 5 + 2 + 3 = -6.$$

It is the negative of the value at P, confirming that the image lies on the opposite side at the same perpendicular distance.

Step 4: Conclusion The foot is $\boxed{Q(-1, 4, 3)}$, the distance is $\boxed{PQ = \sqrt{6}}$ units, and the image of P is $\boxed{P'(-3, 5, 2)}$.

Quick Tip to Solve: First find the foot using the normal direction; then use image equals twice the foot minus the original point, rather than solving a second intersection problem.

27(a). Using elementary row transformations, find the inverse of the matrix $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$. [6 marks]

Solution:

Concept:

- To find the inverse by elementary row transformations, augment the given matrix A with the identity matrix I_3. Apply every row operation to the entire augmented matrix.
- If the left block becomes I_3, the right block is A^{-1}: the same elementary operations transform $[A \mid I_3]$ into $[I_3 \mid A^{-1}]$.

Step 1: Set up the augmented matrix For the given matrix, begin with

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right].$$

Quick Tip to Solve: Adding the third row to the first and then subtracting the second creates a unit pivot immediately and avoids fractions throughout this problem.

We seek pivots equal to one and zeros above and below each pivot. Combining the first and third rows removes the negative entry in the third column without introducing fractions.

Step 2: Create the first pivot First apply $R_1 \leftarrow R_1 + R_3$, and then $R_1 \leftarrow R_1 - R_2$. These operations are sequential, so the second operation uses the newly obtained first row:

$$\left[\begin{array}{ccc|ccc} 3 & 4 & 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 1 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right].$$

Step 3: Reduce the first two columns Apply $R_2 \leftarrow R_2 - 2R_1$. Then use the resulting second row in $R_1 \leftarrow R_1 - R_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 4 & 1 & 0 & 0 & 1 \end{array}\right].$$

In particular, the right block must also be changed; reducing only the left block would not determine the inverse.

Step 4: Complete the reduction and verify Finally apply $R_3 \leftarrow R_3 - 4R_2$:

$$[I_3 \mid A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & 1 & 8 & -12 & 9 \end{array}\right].$$

The right block is therefore the required inverse. A direct multiplication checks the result:

$$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 5: Conclusion Hence $A^{-1} = \begin{bmatrix} 3 & -4 & 3 \\ -2 & 3 & -2 \\ 8 & -12 & 9 \end{bmatrix}$.

Quick Tip to Solve: Adding the third row to the first and then subtracting the second creates a unit pivot immediately and avoids fractions throughout this problem.

OR

27(b). Using matrices, solve the following system of linear equations :

$2x + 3y + 10z = 4$

$4x - 6y + 5z = 1$

$6x + 9y - 20z = 2.$ [6 marks]

Solution:

Concept:

- A system of three linear equations can be written as $AX = B$, where A is the coefficient matrix, X is the column of unknowns, and B is the column of constants.
- If $|A| \neq 0$, then A is invertible and the unique solution is $X = A^{-1}B$, with $A^{-1} = \text{adj } A/|A|$.

Step 1: Write the equations in matrix form Keep the unknowns in the same order in all three rows. The given system is

$$\underbrace{\begin{bmatrix} 2 & 3 & 10 \\ 4 & -6 & 5 \\ 6 & 9 & -20 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}}_{B}.$$

Thus we must find the inverse of the coefficient matrix and multiply it by the constant column.

Step 2: Check that the coefficient matrix is invertible Expanding its determinant along the first row gives

$$\begin{aligned} |A| &= 2\{(-6)(-20) - 5(9)\} \\ &\quad - 3\{4(-20) - 5(6)\} + 10\{4(9) - (-6)(6)\} \\ &= 2(75) - 3(-110) + 10(72) \\ &= 150 + 330 + 720 = 1200. \end{aligned}$$

Since this determinant is nonzero, the inverse exists and the system has exactly one solution.

Step 3: Calculate the adjugate and the inverse The cofactors are obtained by deleting the relevant row and column and applying the sign $(-1)^{i+j}$. For example, $C_{12} = -\{4(-20) - 5(6)\} = 110$. The full cofactor matrix is

$$C = \begin{bmatrix} 75 & 110 & 72 \\ 150 & -100 & 0 \\ 75 & 30 & -24 \end{bmatrix}.$$

Transposing this matrix, rather than using the cofactors in their original positions, gives

$$\text{adj } A = C^T = \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}.$$

Therefore

$$A^{-1} = \frac{1}{1200} \begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}.$$

Quick Tip to Solve: Check the inverse calculation by noting that the third equation minus three times the first gives $-50z = -10$, immediately confirming $z = 1/5$.

Step 4: Multiply by the constant column Multiplying $AX = B$ on the left by A^{-1} gives

$$X = \frac{1}{1200}\begin{bmatrix} 75 & 150 & 75 \\ 110 & -100 & 30 \\ 72 & 0 & -24 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$
$$= \frac{1}{1200}\begin{bmatrix} 300+150+150 \\ 440-100+60 \\ 288-48 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/3 \\ 1/5 \end{bmatrix}.$$

As a check, substitution in the three left-hand sides gives, respectively,

$$1+1+2=4, \qquad 2-2+1=1, \qquad 3+3-4=2.$$

All three printed equations are satisfied, confirming the matrix calculation.

Step 5: Conclusion The unique solution is $\boxed{x=\frac{1}{2},\quad y=\frac{1}{3},\quad z=\frac{1}{5}}$.

Quick Tip to Solve: Check the inverse calculation by noting that the third equation minus three times the first gives $-50z = -10$, immediately confirming $z = 1/5$.

28. The sum of the perimeters of a circle and a square is K, where K is some constant. Prove that the sum of their areas is least when the side of the square is twice the radius of the circle. [6 marks]

Solution:

Concept:

- A circle of radius r has perimeter $2\pi r$ and area πr^2; a square of side x has perimeter $4x$ and area x^2.
- Under a fixed perimeter constraint, express the total area as a function of one variable. An interior stationary point satisfies $A'(r) = 0$.
- If $A''(r) > 0$ throughout the admissible interval, the area function is strictly convex and its stationary point is the unique absolute minimum.

Step 1: Use the fixed sum of perimeters Let the radius of the circle be r and the side of the square be x, with $r, x > 0$. Since the total perimeter is the positive constant K,

$$2\pi r + 4x = K, \qquad x = \frac{K - 2\pi r}{4}.$$

Thus $0 < r < K/(2\pi)$. This constraint is essential: changing the circle's radius also changes the side of the square, so their areas cannot be minimised independently.

Step 2: Express and differentiate the total area The sum of the areas, as a function of the radius, is

$$A(r) = \pi r^2 + \left(\frac{K - 2\pi r}{4}\right)^2 = \pi r^2 + \frac{(K - 2\pi r)^2}{16}.$$

Quick Tip to Solve: After using the perimeter constraint, rewrite the derivative as $A'(r) = \pi(2r - x)$; its zero immediately gives the required relation, then verify a minimum.

Differentiating with K held constant gives

$$A'(r) = 2\pi r - \frac{\pi}{4}(K - 2\pi r)$$
$$= \frac{\pi}{2}(\pi + 4)r - \frac{\pi K}{4}.$$

In particular, since $(K - 2\pi r)/4 = x$, the same derivative can be written as $A'(r) = 2\pi r - \pi x$.

Step 3: Locate the stationary dimensions At a stationary point, $A'(r) = 0$, so

$$\frac{\pi}{2}(\pi + 4)r = \frac{\pi K}{4}, \qquad r_0 = \frac{K}{2(\pi + 4)}.$$

Substituting this radius into the perimeter constraint yields

$$x_0 = \frac{K - 2\pi r_0}{4} = \frac{K}{\pi + 4}.$$

Both dimensions are positive, and $r_0 < K/(2\pi)$, so this stationary point is inside the admissible interval.

Step 4: Establish that the stationary point is the least area Differentiating once more,

$$A''(r) = \frac{\pi}{2}(\pi + 4) > 0.$$

This is positive for every admissible radius. Therefore $A'(r)$ increases strictly, is negative before r_0, and is positive after r_0. The total area decreases up to r_0 and increases thereafter, proving an absolute minimum. Equivalently, completing the square verifies the global comparison directly:

$$A(r) = \frac{K^2}{4(\pi + 4)} + \frac{\pi(\pi + 4)}{4}\left(r - \frac{K}{2(\pi + 4)}\right)^2.$$

The second term is nonnegative and vanishes precisely at the stationary radius.

Step 5: Conclusion The least total area occurs at $r = K/[2(\pi+4)]$ and $x = K/(\pi+4)$. Hence $\boxed{x = 2r}$: the side of the square is twice the radius of the circle, as required.

Quick Tip to Solve: After using the perimeter constraint, rewrite the derivative as $A'(r) = \pi(2r - x)$; its zero immediately gives the required relation, then verify a minimum.

29. A bag contains 5 red and 3 black balls and another bag contains 2 red and 6 black balls. Two balls are drawn at random (without replacement) from one of the bags and both are found to be red. Find the probability that balls are drawn from the first bag. [6 marks]

Solution:

Concept:

- Bayes' theorem reverses a conditional probability: for mutually exclusive and exhaustive events B_1, B_2, $P(B_1 \mid E) = P(B_1)P(E \mid B_1)/P(E)$.

Quick Tip to Solve: With equally likely bags, cancel the equal prior factors in Bayes' theorem: the posterior is the first bag's two-red likelihood divided by the sum of both likelihoods.

- The total probability of the observation is $P(E) = P(B_1)P(E \mid B_1) + P(B_2)P(E \mid B_2)$.
- In sampling without replacement, the first draw changes both the number of red balls and the total number available for the second draw.

Step 1: Define the selection and observation events Let B_1 denote selection of the first bag and B_2 selection of the second bag. Let E denote the event that both balls drawn are red. A bag is chosen at random, so the two bags are equally likely to be selected:

$$P(B_1) = P(B_2) = \frac{1}{2}.$$

The events B_1 and B_2 are mutually exclusive and together cover every possible bag selection. We seek $P(B_1 \mid E)$, the probability of the first bag after observing two red balls, rather than the probability of two red balls from that bag.

Step 2: Find the likelihood of two red balls from the first bag The first bag contains 5 red and 3 black balls, giving 8 balls altogether. The first draw is red with probability $5/8$. Since this ball is not replaced, after a red first draw there are 4 red balls among the 7 remaining balls. Therefore

$$P(E \mid B_1) = \frac{5}{8} \times \frac{4}{7} = \frac{20}{56} = \frac{5}{14}.$$

As a check, unordered sampling gives the same value, $\binom{5}{2}/\binom{8}{2} = 10/28 = 5/14$. Using $(5/8)^2$ would incorrectly treat the draws as sampling with replacement.

Step 3: Find the likelihood from the second bag The second bag contains 2 red and 6 black balls, again a total of 8. The chance of the first ball being red is $2/8$, and then only 1 red ball remains among 7 balls. Thus

$$P(E \mid B_2) = \frac{2}{8} \times \frac{1}{7} = \frac{2}{56} = \frac{1}{28}.$$

This also equals $\binom{2}{2}/\binom{8}{2} = 1/28$. The observation is ten times as likely under the first bag as under the second bag, although both bags had equal probabilities before the draws.

Step 4: Apply total probability and Bayes' theorem The probability of observing two red balls, accounting for either selected bag, is

$$P(E) = \frac{1}{2} \times \frac{5}{14} + \frac{1}{2} \times \frac{1}{28} = \frac{5}{28} + \frac{1}{56} = \frac{11}{56}.$$

Since this is positive, Bayes' theorem applies:

$$P(B_1 \mid E) = \frac{P(B_1)P(E \mid B_1)}{P(E)} = \frac{\frac{1}{2} \times \frac{5}{14}}{\frac{11}{56}} = \frac{10/56}{11/56} = \frac{10}{11}.$$

For consistency, the corresponding posterior probability of the second bag is $(1/56)/(11/56) = 1/11$, so the two posterior probabilities sum to 1.

Step 5: Conclusion Given that both balls drawn are red, the probability that they were drawn from

Quick Tip to Solve: With equally likely bags, cancel the equal prior factors in Bayes' theorem: the posterior is the first bag's two-red likelihood divided by the sum of both likelihoods.

the first bag is $\boxed{\frac{10}{11}}$.

Quick Tip to Solve: With equally likely bags, cancel the equal prior factors in Bayes' theorem: the posterior is the first bag's two-red likelihood divided by the sum of both likelihoods.