

CBSE Class 12 Mathematics Question Paper 2019 with Solution PDF-Set 65-2-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 100

General Instructions

- (i) All questions are compulsory.
- (ii) The question paper consists of **29** questions divided into four sections A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.
- (iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.
- (iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only one of the alternatives in all such questions.
- (v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Question numbers 1 to 4 carry 1 mark each.

1. Find the differential equation representing the family of curves $y = ae^{2x} + 5$, where a is an arbitrary constant. [1 mark]

Solution:

Concept:

- To form a differential equation for a family with one arbitrary constant, differentiate once and eliminate that constant.
- The exponential rule gives $\frac{d}{dx}e^{2x} = 2e^{2x}$, while the derivative of the constant 5 is zero.

Step 1: Differentiate the family Treat a as constant and differentiate both sides of the given equation with respect to x :

$$\frac{dy}{dx} = 2ae^{2x}.$$

Step 2: Eliminate the arbitrary constant The original equation gives $ae^{2x} = y - 5$. Substituting this into the differentiated equation yields

$$\frac{dy}{dx} = 2(y - 5).$$

This relation involves only y and its derivative; the arbitrary constant no longer appears.

Step 3: Conclusion The required differential equation is $\boxed{\frac{dy}{dx} - 2y + 10 = 0}$.

Quick Tip to Solve: Replace the whole factor ae^{2x} by $y - 5$ after differentiating; there is no need to solve separately for a .

2. If $y = \cos(\sqrt{3x})$, then find $\frac{dy}{dx}$. [1 mark]

Solution:

Concept:

- For a composite function $y = \cos u$, the chain rule gives $\frac{dy}{dx} = -\sin u \frac{du}{dx}$.
- The inner function is $u = \sqrt{3x} = (3x)^{1/2}$; its derivative includes the derivative 3 of $3x$.

Step 1: Differentiate the inner function For $x > 0$, applying the power rule followed by the chain rule gives

$$\frac{du}{dx} = \frac{1}{2}(3x)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{3x}}.$$

Step 2: Apply the outer derivative Multiply the derivative of the cosine by the inner derivative:

$$\frac{dy}{dx} = -\sin(\sqrt{3x}) \frac{3}{2\sqrt{3x}}.$$

Quick Tip to Solve: Work from the outer cosine through the square root to $3x$, multiplying all three derivatives so that the factor 3 is retained.

The minus sign comes from differentiating cosine, and the factor 3 comes from differentiating the expression inside the square root.

Step 3: Conclusion $\frac{dy}{dx} = -\frac{3 \sin(\sqrt{3x})}{2\sqrt{3x}}$, for $x > 0$.

Quick Tip to Solve: Work from the outer cosine through the square root to $3x$, multiplying all three derivatives so that the factor 3 is retained.

3. If A is a square matrix satisfying $A'A = I$, write the value of $|A|$.

[1 mark]

Solution:

Concept:

- For square matrices of the same order, the determinant of a product satisfies $|BC| = |B||C|$.
- The transpose A' has the same determinant as A , and the identity matrix has determinant $|I| = 1$.

Step 1: Take determinants on both sides Starting with the given relation $A'A = I$, use the product rule:

$$|A'A| = |I| \implies |A'||A| = 1.$$

Step 2: Use invariance under transposition Since $|A'| = |A|$, the determinant must satisfy

$$|A|^2 = 1.$$

Both signs are possible: the identity matrix has determinant 1, whereas changing one of its diagonal entries to -1 gives a matrix satisfying the same condition with determinant -1 .

Step 3: Conclusion The possible values are $|A| = \pm 1$.

Quick Tip to Solve: Take determinants immediately in $A'A = I$, and retain both square roots of 1; orthogonality does not force a positive determinant.

4(a). Find the direction cosines of a line which makes equal angles with the coordinate axes.

[1 mark]

Solution:

Concept:

- If an oriented line makes angles α, β, γ with the positive coordinate axes, its direction cosines are $l = \cos \alpha, m = \cos \beta, n = \cos \gamma$.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$. Equal angles imply equal direction cosines.

Step 1: Express the equal direction cosines Since all three angles are equal, write $l = m = n = t$. Substitute into the identity for direction cosines:

$$t^2 + t^2 + t^2 = 1 \implies 3t^2 = 1.$$

Quick Tip to Solve: Put $l = m = n$ before using $l^2 + m^2 + n^2 = 1$; the three signs cannot be chosen independently when the angles are equal.

Step 2: Determine the common value Hence $t^2 = \frac{1}{3}$, so the common value is $t = \pm \frac{1}{\sqrt{3}}$. All three signs must agree because $l = m = n$. Reversing the orientation of the same geometric line changes all three signs together.

Step 3: Conclusion The direction cosines are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, or all three negatives for the opposite orientation.

Quick Tip to Solve: Put $l = m = n$ before using $l^2 + m^2 + n^2 = 1$; the three signs cannot be chosen independently when the angles are equal.

OR

4(b). A line passes through the point with position vector $2\hat{i} - \hat{j} + 4\hat{k}$ and is in the direction of the vector $\hat{i} + \hat{j} - 2\hat{k}$. Find the equation of the line in cartesian form. [1 mark]

Solution:

Concept:

- A line through (x_0, y_0, z_0) with direction ratios (a, b, c) has parametric equations $x = x_0 + \lambda a$, $y = y_0 + \lambda b$, $z = z_0 + \lambda c$.
- Eliminating λ gives the cartesian form $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$ when the direction ratios are nonzero.

Step 1: Identify the point and direction The given position vector specifies the point $(2, -1, 4)$. The direction vector $\hat{i} + \hat{j} - 2\hat{k}$ supplies direction ratios $(1, 1, -2)$; normalising them is unnecessary.

Step 2: Eliminate the line parameter A general point of the line therefore satisfies

$$x = 2 + \lambda, \quad y = -1 + \lambda, \quad z = 4 - 2\lambda.$$

Solving each equation for the same parameter gives $\lambda = x - 2 = y + 1 = (z - 4)/(-2)$.

Step 3: Conclusion The cartesian equation is $\boxed{\frac{x - 2}{1} = \frac{y + 1}{1} = \frac{z - 4}{-2}}$.

Quick Tip to Solve: Use the direction-vector components as denominators and subtract the point coordinates in the numerators; subtracting -1 produces $y + 1$.

Section B (16 marks)

Question numbers 5 to 12 carry 2 marks each.

5(a). Show that the points $A(-2\hat{i} + 3\hat{j} + 5\hat{k})$, $B(\hat{i} + 2\hat{j} + 3\hat{k})$ and $C(7\hat{i} - \hat{k})$ are collinear. [2 marks]

Solution:

Concept:

- For points with position vectors $\vec{a}$ and $\vec{b}$, the displacement from the first to the second is $\vec{AB} = \vec{b} - \vec{a}$.
- Three distinct points are collinear if two displacement vectors along consecutive point pairs are scalar multiples. The common point then ensures that the parallel directions belong to the same line.

Step 1: Compute the displacement from A to B Subtract the corresponding components of the position vectors:

$$\begin{aligned}\vec{AB} &= (1 - (-2))\hat{i} + (2 - 3)\hat{j} + (3 - 5)\hat{k} \\ &= 3\hat{i} - \hat{j} - 2\hat{k}.\end{aligned}$$

Step 2: Compare the displacement from B to C The position vector of C is $7\hat{i} + 0\hat{j} - \hat{k}$, so its missing $\hat{j}$ coefficient is zero. Hence

$$\begin{aligned}\vec{BC} &= (7 - 1)\hat{i} + (0 - 2)\hat{j} + (-1 - 3)\hat{k} \\ &= 6\hat{i} - 2\hat{j} - 4\hat{k} \\ &= 2\vec{AB}.\end{aligned}$$

Thus the two nonzero displacements have the same direction and meet at B .

Step 3: Conclusion Since $\vec{BC} = 2\vec{AB}$, the points A, B, C are collinear. In fact, B lies between A and C , with $AB : BC = 1 : 2$.

Quick Tip to Solve: Insert the zero $\hat{j}$ component of C , then check whether every component of $\vec{BC}$ is twice the matching component of $\vec{AB}$.

OR

5(b). Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} + 5\hat{j} - 2\hat{k}$. [2 marks]

Solution:

Concept:

- The vector product is computed by expanding $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$, with cofactor signs $+, -, +$.
- For a vector $p\hat{i} + q\hat{j} + r\hat{k}$, its magnitude is $\sqrt{p^2 + q^2 + r^2}$. A magnitude is always nonnegative.

Step 1: Expand the vector product Substitute the components of the two given vectors, including the

Quick Tip to Solve: Check the middle cofactor sign before squaring. As a check here, $|\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = 14 \cdot 38 - 5^2 = 507$.

positive $\hat{i}$ component 3 of $\vec{b}$:

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 3 \\ 3 & 5 & -2 \end{vmatrix} \\ &= \hat{i}(-2 - 15) - \hat{j}(-4 - 9) + \hat{k}(10 - 3) \\ &= -17\hat{i} + 13\hat{j} + 7\hat{k}.\end{aligned}$$

The middle coefficient is positive because the negative minor is multiplied by the negative cofactor sign.

Step 2: Take the magnitude Square all three components and add:

$$|\vec{a} \times \vec{b}| = \sqrt{(-17)^2 + 13^2 + 7^2} = \sqrt{289 + 169 + 49} = \sqrt{507}.$$

To simplify the root, use $507 = 169 \cdot 3 = 13^2 \cdot 3$.

Step 3: Conclusion The required magnitude is $|\vec{a} \times \vec{b}| = 13\sqrt{3}$.

Quick Tip to Solve: Check the middle cofactor sign before squaring. As a check here, $|\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2 = 14 \cdot 38 - 5^2 = 507$.

6. Find: $\int \frac{x-5}{(x-3)^3} e^x dx$.

[2 marks]

Solution:

Concept:

- The product rule gives $\frac{d}{dx}[e^x f(x)] = e^x[f(x) + f'(x)]$. Therefore $\int e^x[f(x) + f'(x)] dx = e^x f(x) + C$.
- Rewrite the numerator relative to the denominator's factor: $x - 5 = (x - 3) - 2$. The derivative of $(x - 3)^{-2}$ is $-2(x - 3)^{-3}$.

Step 1: Split the rational factor For $x \neq 3$, separate the integrand into the two terms suggested by the numerator:

$$\begin{aligned}\frac{x-5}{(x-3)^3} &= \frac{(x-3)-2}{(x-3)^3} \\ &= \frac{1}{(x-3)^2} - \frac{2}{(x-3)^3}.\end{aligned}$$

These terms have the form $f + f'$ with $f(x) = (x - 3)^{-2}$.

Step 2: Recognise the product derivative Applying the product rule confirms the required antiderivative exactly:

$$\begin{aligned}\frac{d}{dx} \left[\frac{e^x}{(x-3)^2} \right] &= e^x \left[\frac{1}{(x-3)^2} - \frac{2}{(x-3)^3} \right] \\ &= e^x \frac{x-5}{(x-3)^3}.\end{aligned}$$

Quick Tip to Solve: Split $x - 5$ as $(x - 3) - 2$; this exposes $f + f'$ and avoids a lengthy integration-by-parts calculation.

Thus the integral follows directly, with an arbitrary additive constant on any interval not containing 3.

Step 3: Conclusion $\int \frac{x-5}{(x-3)^3} e^x dx = \frac{e^x}{(x-3)^2} + C$, where $x \neq 3$.

Quick Tip to Solve: Split $x-5$ as $(x-3)-2$; this exposes $f + f'$ and avoids a lengthy integration-by-parts calculation.

7. Examine whether the operation $*$ defined on $\mathbb{R}$, the set of all real numbers, by $a * b = \sqrt{a^2 + b^2}$ is a binary operation or not, and if it is, find whether it is associative or not. [2 marks]

Solution:

Concept:

- A binary operation on a set assigns exactly one element of that set to every ordered pair of elements of the set. Thus closure and a well-defined output are required.
- An operation is associative if $(a * b) * c = a * (b * c)$ for every a, b, c in its domain. The square-root symbol denotes the unique non-negative square root.

Step 1: Check closure on the real numbers For arbitrary $a, b \in \mathbb{R}$, both squares are non-negative, so $a^2 + b^2 \geq 0$. Therefore $\sqrt{a^2 + b^2}$ exists and is a unique real number. The rule is defined for every ordered pair $(a, b) \in \mathbb{R} \times \mathbb{R}$ and always gives an element of $\mathbb{R}$. Its outputs need not include all real numbers for it to be a binary operation.

Step 2: Compare the two groupings Take any $a, b, c \in \mathbb{R}$. Applying the same definition twice gives

$$\begin{aligned}(a * b) * c &= \sqrt{\left(\sqrt{a^2 + b^2}\right)^2 + c^2} = \sqrt{a^2 + b^2 + c^2}, \\ a * (b * c) &= \sqrt{a^2 + \left(\sqrt{b^2 + c^2}\right)^2} = \sqrt{a^2 + b^2 + c^2}.\end{aligned}$$

These expressions agree for all real triples, without any sign restriction on the inputs.

Step 3: Conclusion The given rule defines a binary operation on $\mathbb{R}$, and this operation is associative.

Quick Tip to Solve: To test associativity here, write both nested radicals: squaring each inner square root leaves the same sum $a^2 + b^2 + c^2$.

8. If $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$, show that $(A - 2I)(A - 3I) = 0$.

[2 marks]

Solution:

Concept:

- For a matrix of order two, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Subtracting kI subtracts k from each diagonal entry only.
- In a product of two matrices, entry (i, j) is the dot product of row i of the first matrix with column

Quick Tip to Solve: After subtracting the diagonal entries, each row of the first factor has equal entries and each column of the second has opposite entries, so every dot product vanishes.

j of the second. The zero matrix has every entry equal to zero.

Step 1: Form the two matrix factors Using $A = \begin{pmatrix} 4 & 2 \\ -1 & 1 \end{pmatrix}$, subtract the required multiples of the identity:

$$A - 2I = \begin{pmatrix} 4-2 & 2 \\ -1 & 1-2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix},$$

$$A - 3I = \begin{pmatrix} 4-3 & 2 \\ -1 & 1-3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}.$$

Notice that the off-diagonal entries are unaffected by the subtraction.

Step 2: Multiply in the stated order Compute all four entries by row-by-column multiplication:

$$\begin{aligned} (A - 2I)(A - 3I) &= \begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 2(1) + 2(-1) & 2(2) + 2(-2) \\ (-1)(1) + (-1)(-1) & (-1)(2) + (-1)(-2) \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Step 3: Conclusion Hence $(A - 2I)(A - 3I) = 0_{2 \times 2}$, as required.

Quick Tip to Solve: After subtracting the diagonal entries, each row of the first factor has equal entries and each column of the second has opposite entries, so every dot product vanishes.

9(a). Find: $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx.$

[2 marks]

Solution:

Concept:

- A fraction with a sum in the numerator can be split into separate fractions, after which common factors can be cancelled wherever the original denominator is nonzero.
- The derivatives $\frac{d}{dx}(\sec x) = \sec x \tan x$ and $\frac{d}{dx}(\csc x) = -\csc x \cot x$ identify the two antiderivatives needed here.

Step 1: Separate and simplify the integrand On any interval where $\sin x \cos x \neq 0$, divide each term in the numerator by the common denominator:

$$\begin{aligned} \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} &= \frac{\sin x}{\cos^2 x} + \frac{\cos x}{\sin^2 x} \\ &= \sec x \tan x + \csc x \cot x. \end{aligned}$$

This form uses standard derivatives directly; expanding $\sin^3 x + \cos^3 x$ as a sum of cubes would not

Quick Tip to Solve: Split this numerator before integrating, and remember that the positive term $\csc x \cot x$ integrates to $-\csc x$.

simplify the integration as efficiently.

Step 2: Integrate the standard derivative forms By linearity of integration,

$$\begin{aligned}\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx &= \int \sec x \tan x dx + \int \csc x \cot x dx \\ &= \sec x - \csc x + C.\end{aligned}$$

The second term has a minus sign because differentiating $\csc x$ gives $-\csc x \cot x$. Differentiating the expression obtained therefore recovers the simplified integrand exactly.

Step 3: Conclusion The required integral is $\boxed{\sec x - \csc x + C}$, valid on intervals where $\sin x \cos x \neq 0$.

Quick Tip to Solve: Split this numerator before integrating, and remember that the positive term $\csc x \cot x$ integrates to $-\csc x$.

OR

9(b). Find: $\int \frac{x-3}{(x-1)^3} e^x dx$.

[2 marks]

Solution:

Concept:

- The product rule gives $\frac{d}{dx}\{e^x f(x)\} = e^x\{f(x) + f'(x)\}$. Thus an integrand of this form has antiderivative $e^x f(x)$.
- For $x \neq 1$, the power rule gives $\frac{d}{dx}(x-1)^{-2} = -2(x-1)^{-3}$. Splitting the numerator reveals this derivative pair.

Step 1: Rewrite the rational factor Since $x-3 = (x-1) - 2$, the integrand can be written as

$$\begin{aligned}e^x \frac{x-3}{(x-1)^3} &= e^x \left(\frac{x-1}{(x-1)^3} - \frac{2}{(x-1)^3} \right) \\ &= e^x \left(\frac{1}{(x-1)^2} - \frac{2}{(x-1)^3} \right).\end{aligned}$$

Choose $f(x) = (x-1)^{-2}$. The expression in parentheses is precisely $f(x) + f'(x)$.

Step 2: Apply and verify the product rule Differentiate the candidate product to check both terms and their signs:

$$\begin{aligned}\frac{d}{dx} \left(\frac{e^x}{(x-1)^2} \right) &= \frac{e^x}{(x-1)^2} - \frac{2e^x}{(x-1)^3} \\ &= \frac{e^x\{(x-1) - 2\}}{(x-1)^3} \\ &= \frac{(x-3)e^x}{(x-1)^3}.\end{aligned}$$

This is exactly the printed integrand, so integration reverses this differentiation. A constant of integration is allowed on each interval avoiding $x = 1$.

Quick Tip to Solve: Write $x-3 = (x-1) - 2$; the resulting two rational terms are f and f' for $f = (x-1)^{-2}$.

Step 3: Conclusion $\int \frac{x-3}{(x-1)^3} e^x dx = \boxed{\frac{e^x}{(x-1)^2} + C}$, for $x \neq 1$.

Quick Tip to Solve: Write $x-3 = (x-1) - 2$; the resulting two rational terms are f and f' for $f = (x-1)^{-2}$.

10(a). A coin is tossed 5 times. What is the probability of getting (i) 3 heads, (ii) at most 3 heads?

[2 marks]

Solution:

Concept:

- For five independent tosses of a fair coin, the number of heads X has a binomial distribution with $n = 5$ and $p = \frac{1}{2}$.
- The binomial formula is $P(X = r) = \binom{5}{r} \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{5-r} = \binom{5}{r}/32$. The phrase “at most 3” includes 0, 1, 2 and 3 heads.

Step 1: Calculate the probability of exactly three heads There are $2^5 = 32$ equally likely sequences of five tosses. Choosing which three positions contain heads gives $\binom{5}{3} = 10$ sequences. Thus, for part (i),

$$P(X = 3) = \binom{5}{3} \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 = \frac{10}{32} = \frac{5}{16}.$$

The other two positions must contain tails, so no additional choice is required.

Step 2: Include every outcome with at most three heads For part (ii), the events $X = 0, 1, 2, 3$ are mutually exclusive. Add their probabilities:

$$\begin{aligned} P(X \leq 3) &= \frac{\binom{5}{0} + \binom{5}{1} + \binom{5}{2} + \binom{5}{3}}{32} \\ &= \frac{1 + 5 + 10 + 10}{32} = \frac{26}{32} = \frac{13}{16}. \end{aligned}$$

As a check, the excluded possibilities are four and five heads, with combined probability $(5 + 1)/32$. Subtracting this from 1 gives the same result.

Step 3: Conclusion The required probabilities are (i) $\boxed{\frac{5}{16}}$ for exactly 3 heads, and (ii) $\boxed{\frac{13}{16}}$ for at most 3 heads.

Quick Tip to Solve: For at most three heads in five tosses, use $1 - P(X = 4) - P(X = 5)$ to reduce the number of binomial terms.

10(b). Find the probability distribution of X , the number of heads in a simultaneous toss of two coins. [2 marks]

Solution:

Concept:

- For two fair coins tossed independently, the four ordered outcomes HH, HT, TH, TT are equally likely, each having probability $\frac{1}{4}$.
- A probability distribution lists every possible value x of a random variable X together with $P(X = x)$. These probabilities are non-negative and add to 1.

Step 1: List the outcomes and count heads Distinguish the two coins as the first and second coin, even though they are tossed simultaneously. The sample space is

$$S = \{HH, HT, TH, TT\}.$$

The variable X , the number of heads, takes value 0 for TT , value 1 for either HT or TH , and value 2 for HH . Thus its possible values are 0, 1, 2; the two mixed outcomes are separate equally likely outcomes.

Step 2: Assign probabilities to all possible values Count the sample points belonging to each value and divide by the total number, 4:

$$P(X = 0) = \frac{1}{4}, \quad P(X = 1) = \frac{2}{4} = \frac{1}{2}, \quad P(X = 2) = \frac{1}{4}.$$

The complete distribution can therefore be displayed as

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

All entries are non-negative, and $\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$, which verifies that no possible outcome was omitted.

Step 3: Conclusion The probability distribution is $P(X = 0) = \frac{1}{4}, P(X = 1) = \frac{1}{2}, P(X = 2) = \frac{1}{4}$; all other values have probability zero.

Quick Tip to Solve: Keep HT and TH separate when listing two-coin outcomes; one head is twice as likely as either zero heads or two heads.

11. If $P(\text{not } A) = 0.7$, $P(B) = 0.7$ and $P(B/A) = 0.5$, then find $P(A/B)$. [2 marks]

Solution:

Concept:

- Complement rule: $P(A) = 1 - P(\text{not } A)$, since an event and its complement exhaust the sample space.
- Conditional probability: $P(B/A) = P(A \cap B)/P(A)$ and $P(A/B) = P(A \cap B)/P(B)$, provided the

Quick Tip to Solve: First convert the complement to $P(A)$, then use the same intersection to reverse the conditional probability.

respective denominators are positive.

Step 1: Find the probability of event A The given probability 0.7 refers to the complement of A , not to A itself. Therefore,

$$P(A) = 1 - 0.7 = 0.3.$$

Both $P(A) = 0.3$ and the given $P(B) = 0.7$ are nonzero, so both conditional probabilities are defined.

Step 2: Determine the intersection and reverse the conditioning Use the given $P(B/A) = 0.5$ to obtain the common intersection:

$$P(A \cap B) = P(B/A)P(A) = 0.5 \times 0.3 = 0.15.$$

For the required probability, the conditioning event is B ; hence its probability belongs in the denominator:

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.7} = \frac{15}{70} = \frac{3}{14}.$$

Step 3: Conclusion The required conditional probability is $P(A/B) = \frac{3}{14}$.

Quick Tip to Solve: First convert the complement to $P(A)$, then use the same intersection to reverse the conditional probability.

12. Find the differential equation of the family of curves $y = Ae^{2x} + Be^{-2x}$, where A and B are arbitrary constants. [2 marks]

Solution:

Concept:

- A differential equation for a family of curves is formed by differentiating and eliminating the arbitrary constants.
- The chain rule gives $\frac{d}{dx}e^{kx} = ke^{kx}$. With two independent arbitrary constants, differentiate twice to obtain a second-order relation.

Step 1: Differentiate the family once The family is

$$y = Ae^{2x} + Be^{-2x},$$

where A and B are constants with respect to x . Differentiating each term and retaining the negative factor from the second exponential gives

$$\frac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}.$$

Quick Tip to Solve: For exponentials with exponents $2x$ and $-2x$, two derivatives multiply both terms by 4, so compare directly with y .

Step 2: Differentiate again and eliminate both constants Differentiate the first derivative once more:

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x} = 4(Ae^{2x} + Be^{-2x}).$$

The second term is positive because differentiating $-2Be^{-2x}$ supplies another factor -2 . The expression in parentheses is exactly the original y . Therefore,

$$\frac{d^2y}{dx^2} = 4y.$$

Both arbitrary constants have disappeared, as required; the relation holds for every curve in the given family.

Step 3: Conclusion The required differential equation is $\boxed{\frac{d^2y}{dx^2} - 4y = 0}$, of order 2 and degree 1.

Quick Tip to Solve: For exponentials with exponents $2x$ and $-2x$, two derivatives multiply both terms by 4, so compare directly with y .

Section C (44 marks)

Question numbers 13 to 23 carry 4 marks each.

13. Solve for x : $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$.

[4 marks]

Solution:

Concept:

- Principal values of $\tan^{-1} t$ lie in $(-\pi/2, \pi/2)$. Equal tangents do not alone imply equal angles, so candidates must satisfy the original principal-value equation.
- The tangent addition identity is $\tan(\alpha + \beta) = (\tan \alpha + \tan \beta) / (1 - \tan \alpha \tan \beta)$, when its denominator is nonzero.
- The inverse tangent is an odd, strictly increasing function: $\tan^{-1}(-t) = -\tan^{-1} t$.

Step 1: Establish the necessary sign of x Put $\alpha = \tan^{-1}(x+1)$ and $\beta = \tan^{-1}(x-1)$. Their required sum is $\tan^{-1}(8/31)$, which is positive and less than $\pi/2$. At $x = 0$, the left side is $\tan^{-1} 1 + \tan^{-1}(-1) = 0$. Since both inverse tangent terms increase strictly with x , their sum is positive only when $x > 0$. Thus any solution must be positive.

Step 2: Take tangents and obtain algebraic candidates A solution has a finite tangent for $\alpha + \beta$; hence $1 - (x+1)(x-1) = 2 - x^2$ cannot vanish. Taking tangents on both sides gives

$$\frac{(x+1) + (x-1)}{1 - (x+1)(x-1)} = \frac{8}{31}, \quad \frac{2x}{2 - x^2} = \frac{8}{31}.$$

Quick Tip to Solve: Check the sign of the inverse tangent sum before taking tangents; then verify the surviving root within the principal interval.

Cross-multiplying and simplifying,

$$\begin{aligned}62x &= 16 - 8x^2, \\4x^2 + 31x - 8 &= 0, \\(4x - 1)(x + 8) &= 0.\end{aligned}$$

The algebraic candidates are $x = 1/4$ and $x = -8$. The negative candidate violates the necessary condition $x > 0$ and is rejected.

Step 3: Verify the remaining candidate on the principal branch At $x = 1/4$, the left side becomes

$$S = \tan^{-1} \frac{5}{4} - \tan^{-1} \frac{3}{4}.$$

Both angles are in $(0, \pi/2)$, and the first exceeds the second, so $0 < S < \pi/2$. Moreover,

$$\tan S = \frac{5/4 - 3/4}{1 + (5/4)(3/4)} = \frac{1/2}{31/16} = \frac{8}{31}.$$

There is exactly one angle in $(0, \pi/2)$ with this tangent. Consequently $S = \tan^{-1}(8/31)$, which verifies the original equation and removes any ambiguity introduced by taking tangents.

Step 4: Conclusion The unique real solution is $x = \frac{1}{4}$.

Quick Tip to Solve: Check the sign of the inverse tangent sum before taking tangents; then verify the surviving root within the principal interval.

14(a). If $x = ae^t(\sin t + \cos t)$ and $y = ae^t(\sin t - \cos t)$, then prove that $\frac{dy}{dx} = \frac{x+y}{x-y}$. [4 marks]

Solution:

Concept:

- For a parametric curve $x = x(t)$, $y = y(t)$, the chain rule gives $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, provided $dx/dt \neq 0$.
- The product rule is $\frac{d}{dt}(uv) = u'v + uv'$. Here a is constant, $(e^t)' = e^t$, $(\sin t)' = \cos t$ and $(\cos t)' = -\sin t$.

Step 1: Differentiate the expression for x Apply the product rule to $x = ae^t(\sin t + \cos t)$, differentiating both the exponential and the trigonometric factor:

$$\begin{aligned}\frac{dx}{dt} &= ae^t(\sin t + \cos t) + ae^t(\cos t - \sin t) \\&= ae^t(2 \cos t) = 2ae^t \cos t.\end{aligned}$$

The sine terms cancel, while the two cosine terms add. This derivative will be the denominator in the parametric differentiation formula.

Quick Tip to Solve: Compute $x + y$ and $x - y$ early: they equal dy/dt and dx/dt , respectively, so their ratio gives the slope immediately.

Step 2: Differentiate the expression for y Next apply the same rule to $y = ae^t(\sin t - \cos t)$. The derivative of $-\cos t$ is $+\sin t$, so

$$\begin{aligned}\frac{dy}{dt} &= ae^t(\sin t - \cos t) + ae^t(\cos t + \sin t) \\ &= 2ae^t \sin t.\end{aligned}$$

Therefore, at a point where $2ae^t \cos t \neq 0$, the slope is

$$\frac{dy}{dx} = \frac{2ae^t \sin t}{2ae^t \cos t} = \tan t.$$

Step 3: Express the proposed ratio in terms of t Add and subtract the original expressions for the coordinates, keeping the entire expression for y inside brackets when subtracting:

$$\begin{aligned}x + y &= ae^t[(\sin t + \cos t) + (\sin t - \cos t)] \\ &= 2ae^t \sin t, \\ x - y &= ae^t[(\sin t + \cos t) - (\sin t - \cos t)] \\ &= 2ae^t \cos t.\end{aligned}$$

Thus $(x + y)/(x - y) = \tan t$ wherever its denominator is nonzero. In fact, the two identities also show directly that $dy/dt = x + y$ and $dx/dt = x - y$.

Step 4: Conclusion Hence, by parametric differentiation,

$$\boxed{\frac{dy}{dx} = \frac{x + y}{x - y}}$$

at every regular point where $x - y \neq 0$, as required.

Quick Tip to Solve: Compute $x + y$ and $x - y$ early: they equal dy/dt and dx/dt , respectively, so their ratio gives the slope immediately.

OR

14(b). Differentiate $x^{\sin x} + (\sin x)^{\cos x}$ with respect to x .

[4 marks]

Solution:

Concept:

- When both the base and exponent vary, logarithmic differentiation uses $\ln(u^v) = v \ln u$ for $u > 0$. Differentiate this product and then multiply by the original function.
- Use the sum rule $(u + v)' = u' + v'$, the product rule, and $\frac{d}{dx} \ln f(x) = \frac{f'(x)}{f(x)}$ for positive $f(x)$.

Step 1: Separate the two variable powers Let

$$y = u + v, \quad u = x^{\sin x}, \quad v = (\sin x)^{\cos x}.$$

Quick Tip to Solve: For each variable power $f(x)^{g(x)}$, multiply the original term by $g' \ln f + gf'/f$; apply this separately to the two terms before adding.

We differentiate on intervals where $x > 0$ and $\sin x > 0$, so both real variable powers and the logarithms used below are defined. Each term has a variable base and a variable exponent, so the ordinary power rule with a constant exponent cannot be applied directly. Take logarithms of the two terms separately; the logarithm of their sum does not split into two logarithms.

Step 2: Differentiate the first term logarithmically Since $u = x^{\sin x} > 0$, taking natural logarithms gives $\ln u = \sin x \ln x$. Differentiate both sides with respect to x , using the chain rule on the left and product rule on the right:

$$\begin{aligned}\frac{u'}{u} &= \cos x \ln x + \frac{\sin x}{x}, \\ u' &= x^{\sin x} \left(\cos x \ln x + \frac{\sin x}{x} \right).\end{aligned}$$

The first contribution comes from the changing exponent and the second from the changing base.

Step 3: Differentiate the second term logarithmically Similarly, $\ln v = \cos x \ln(\sin x)$. Here the derivative of the exponent is $-\sin x$, while the derivative of $\ln(\sin x)$ is $\cos x / \sin x$. Thus

$$\begin{aligned}\frac{v'}{v} &= -\sin x \ln(\sin x) + \cos x \frac{\cos x}{\sin x}, \\ v' &= (\sin x)^{\cos x} \left(\frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x) \right).\end{aligned}$$

We now use $y' = u' + v'$, preserving the negative sign from differentiating $\cos x$.

Step 4: Conclusion The required derivative, for $x > 0$ and $\sin x > 0$, is

$$\begin{aligned}\frac{dy}{dx} &= x^{\sin x} \left(\cos x \ln x + \frac{\sin x}{x} \right) \\ &\quad + (\sin x)^{\cos x} \left(\frac{\cos^2 x}{\sin x} - \sin x \ln(\sin x) \right).\end{aligned}$$

Quick Tip to Solve: For each variable power $f(x)^{g(x)}$, multiply the original term by $g' \ln f + g f' / f$; apply this separately to the two terms before adding.

15. Find: $\int \frac{2 \cos x}{(1 - \sin x)(2 - \cos^2 x)} dx.$

[4 marks]

Solution:

Concept:

- Use the identity $\cos^2 x = 1 - \sin^2 x$ to express the denominator entirely in terms of $\sin x$. The substitution $t = \sin x$ then gives $dt = \cos x dx$.
- The standard integrals needed are $\int \frac{du}{u} = \ln |u| + C$ and $\int \frac{dt}{1+t^2} = \tan^{-1} t + C$, where C is an arbitrary constant.

Step 1: Convert to a rational function Let the required integral be I . Since $2 - \cos^2 x = 1 + \sin^2 x$,

Quick Tip to Solve: Replace $2 - \cos^2 x$ by $1 + \sin^2 x$ before substituting. Keep the minus sign when integrating $1/(1-t)$.

take $t = \sin x$, so that $dt = \cos x \, dx$. Thus

$$I = \int \frac{2 \cos x}{(1 - \sin x)(1 + \sin^2 x)} dx = \int \frac{2}{(1 - t)(1 + t^2)} dt.$$

Step 2: Determine the partial fractions Write

$$\frac{2}{(1 - t)(1 + t^2)} = \frac{A}{1 - t} + \frac{Bt + D}{1 + t^2}.$$

Multiplication by the common denominator gives the polynomial identity

$$\begin{aligned} 2 &= A(1 + t^2) + (Bt + D)(1 - t) \\ &= (A + D) + (B - D)t + (A - B)t^2. \end{aligned}$$

Equating constant, linear and quadratic coefficients yields $A + D = 2$, $B - D = 0$ and $A - B = 0$. Therefore $A = B = D = 1$, and

$$\frac{2}{(1 - t)(1 + t^2)} = \frac{1}{1 - t} + \frac{t}{1 + t^2} + \frac{1}{1 + t^2}.$$

Step 3: Integrate the three terms Integrate term by term. In the first term, the derivative of $1 - t$ is -1 , which supplies a minus sign. In the second, the derivative of $1 + t^2$ is $2t$, which supplies the factor $\frac{1}{2}$. Hence

$$\begin{aligned} I &= \int \frac{dt}{1 - t} + \int \frac{t \, dt}{1 + t^2} + \int \frac{dt}{1 + t^2} \\ &= -\ln |1 - t| + \frac{1}{2} \ln(1 + t^2) + \tan^{-1} t + C. \end{aligned}$$

Step 4: Conclusion Substituting $t = \sin x$, the required antiderivative is

$$-\ln |1 - \sin x| + \frac{1}{2} \ln(1 + \sin^2 x) + \tan^{-1}(\sin x) + C.$$

This holds on any interval on which $\sin x \neq 1$.

Quick Tip to Solve: Replace $2 - \cos^2 x$ by $1 + \sin^2 x$ before substituting. Keep the minus sign when integrating $1/(1 - t)$.

16(a). Check whether the relation R defined on the set $A = \{1, 2, 3, 4, 5, 6\}$ as $R = \{(a, b) : b = a + 1\}$ is reflexive, symmetric or transitive. [4 marks]

Solution:

Concept:

- A relation R on A is reflexive if $(a, a) \in R$ for every $a \in A$.
- A relation is symmetric if $(a, b) \in R$ always implies $(b, a) \in R$, for $a, b \in A$.

Quick Tip to Solve: For the successor relation, use $(1, 1)$ to test reflexivity, reverse $(1, 2)$ for symmetry, and link $(1, 2), (2, 3)$ for transitivity.

- A relation is transitive if $(a, b) \in R$ and $(b, c) \in R$ together always imply $(a, c) \in R$, for $a, b, c \in A$. A single counterexample disproves any of these universal properties.

Step 1: List the ordered pairs in the relation The condition for membership is $b = a + 1$, with both entries belonging to $A = \{1, 2, 3, 4, 5, 6\}$. Taking $a = 1, 2, 3, 4, 5$ gives the pairs

$$R = \{(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)\}.$$

For $a = 6$, the rule would give $b = 7$, but $7 \notin A$, so no further pair is included. We now test each property separately against its definition.

Step 2: Test reflexivity Reflexivity requires every element to be related to itself. In particular, it would require $(1, 1) \in R$. However, for $a = 1$, the relation requires $b = 2$, whereas the pair $(1, 1)$ has $b = 1$. Therefore $(1, 1) \notin R$, disproving reflexivity. More generally, a diagonal pair would require $a = a + 1$, which is impossible.

Step 3: Test symmetry We have $(1, 2) \in R$, since $2 = 1 + 1$. Its reverse is $(2, 1)$. This reversed pair is not in R , because with first entry 2, the second entry must be 3, not 1. Thus

$$(1, 2) \in R \quad \text{but} \quad (2, 1) \notin R,$$

which disproves symmetry.

Step 4: Test transitivity Choose the two related pairs $(1, 2) \in R$ and $(2, 3) \in R$. They have the matching middle entry needed in the definition of transitivity. Transitivity would therefore require $(1, 3) \in R$. However, $3 \neq 1 + 1$, so $(1, 3) \notin R$. This is a valid counterexample with all three elements in A . In general, two successive applications of the relation give $c = b + 1 = a + 2$, whereas direct membership of (a, c) requires $c = a + 1$.

Step 5: Conclusion The relation R is neither reflexive, nor symmetric, nor transitive.

Quick Tip to Solve: For the successor relation, use $(1, 1)$ to test reflexivity, reverse $(1, 2)$ for symmetry, and link $(1, 2), (2, 3)$ for transitivity.

OR

16(b). Let $f : \mathbb{N} \rightarrow Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in \mathbb{N} : y = 4x + 3, \text{ for some } x \in \mathbb{N}\}$. Show that f is invertible. Find its inverse. [4 marks]

Solution:

Concept:

- A function has an inverse on its stated codomain precisely when it is bijective, that is, both one-one and onto.
- To prove one-one, show that equal images imply equal inputs. To prove onto, take an arbitrary element of the codomain and exhibit a preimage in the domain.
- The inverse reverses the original mapping: if $y = f(x)$, then $f^{-1}(y) = x$. Its domain is Y and its codomain is $\mathbb{N}$.

Quick Tip to Solve: Use the definition of Y to prove onto directly; then subtract 3 and divide by 4 to reverse the original function.

Step 1: Prove that the function is one-one Let $x_1, x_2 \in \mathbb{N}$ and suppose $f(x_1) = f(x_2)$. Using the given rule,

$$\begin{aligned}4x_1 + 3 &= 4x_2 + 3, \\4(x_1 - x_2) &= 0, \\x_1 &= x_2.\end{aligned}$$

Thus distinct natural numbers cannot have the same image, so f is one-one.

Step 2: Prove that the function is onto its codomain Take any $y \in Y$. By the definition of Y , there exists a natural number n for which $y = 4n + 3$. Therefore $f(n) = 4n + 3 = y$, so every element of the stated codomain has a preimage in $\mathbb{N}$. This proves that $f : \mathbb{N} \rightarrow Y$ is onto. Together with the one-one property, this establishes bijectivity and hence invertibility.

Step 3: Obtain the inverse rule Solve the equation $y = 4x + 3$ for the original input:

$$y - 3 = 4x, \quad x = \frac{y - 3}{4}.$$

Since $y \in Y$, this expression is a natural number by the definition of Y . Thus it defines a function $g : Y \rightarrow \mathbb{N}$ by $g(y) = (y - 3)/4$.

Step 4: Verify both compositions For every $x \in \mathbb{N}$,

$$g(f(x)) = \frac{(4x + 3) - 3}{4} = x.$$

For every $y \in Y$,

$$f(g(y)) = 4\left(\frac{y - 3}{4}\right) + 3 = y.$$

These identities verify that g reverses f on the full stated domain and codomain.

Step 5: Conclusion The function f is invertible, with

$$f^{-1} : Y \rightarrow \mathbb{N}, \quad f^{-1}(y) = \frac{y - 3}{4} \quad (y \in Y).$$

Quick Tip to Solve: Use the definition of Y to prove onto directly; then subtract 3 and divide by 4 to reverse the original function.

17. Using properties of determinants, show that

$$\begin{vmatrix} 3a & -a+b & -a+c \\ -b+a & 3b & -b+c \\ -c+a & -c+b & 3c \end{vmatrix} = 3(a+b+c)(ab+bc+ca).$$

[4 marks]

Solution:

Concept:

- Adding a multiple of one column to another column leaves a determinant unchanged. The corresponding property also holds for rows.
- A common factor in one column can be taken outside the determinant. A column with two zero entries makes cofactor expansion particularly simple.

Step 1: Add the columns to expose a common factor Denote the given determinant by Δ , and write $s = a + b + c$. The sum of the entries in each row is s . Apply $C_1 \leftarrow C_1 + C_2 + C_3$, keeping the other columns unchanged:

$$\Delta = \begin{vmatrix} s & b-a & c-a \\ s & 3b & c-b \\ s & b-c & 3c \end{vmatrix} = s \begin{vmatrix} 1 & b-a & c-a \\ 1 & 3b & c-b \\ 1 & b-c & 3c \end{vmatrix}.$$

This factorization is valid even if $s = 0$; no division is used.

Step 2: Create zeros below the first entry Perform $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$, keeping the first row unchanged. For example, $3b - (b - a) = a + 2b$. Thus

$$\Delta = s \begin{vmatrix} 1 & b-a & c-a \\ 0 & a+2b & a-b \\ 0 & a-c & a+2c \end{vmatrix}.$$

These row operations leave the determinant unchanged.

Step 3: Expand and simplify the remaining minor Expand along the first column. The cofactor sign at position $(1, 1)$ is positive, giving

$$\begin{aligned} \Delta &= s[(a+2b)(a+2c) - (a-b)(a-c)] \\ &= s[a^2 + 2ab + 2ac + 4bc - a^2 + ab + ac - bc] \\ &= s(3ab + 3ac + 3bc) \\ &= 3s(ab + bc + ca). \end{aligned}$$

Finally substitute the row sum $s = a + b + c$ to express the answer in the original variables.

Step 4: Conclusion Hence, using properties of determinants,

$$\boxed{\Delta = 3(a+b+c)(ab+bc+ca).}$$

Quick Tip to Solve: Equal row sums suggest adding all columns into one column. After extracting the common sum, subtract the first row from the other two rows.

18. If $(x - a)^2 + (y - b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}, \text{ is a constant independent of } a \text{ and } b.$$

[4 marks]

Solution:

Concept:

- Implicit differentiation treats y as a function of x ; write $y' = dy/dx$ and $y'' = d^2y/dx^2$.
- The given equation is a circle with centre (a, b) and radius $c > 0$. Its upper and lower open semicircles are differentiable branches $y = y(x)$, on which $y - b$ has a fixed sign.
- For real $u \neq 0$, $(u^2)^{3/2} = |u|^3$. Retaining this absolute value distinguishes the signs of the constant on the two branches.

Step 1: Differentiate the circle equation once Since a, b, c are constants, differentiation with respect to x gives

$$2(x - a) + 2(y - b)y' = 0.$$

At any point where the branch is differentiable, $y - b \neq 0$, so

$$y' = -\frac{x - a}{y - b}.$$

Squaring and using the original circle equation, we obtain

$$1 + (y')^2 = \frac{(y - b)^2 + (x - a)^2}{(y - b)^2} = \frac{c^2}{(y - b)^2}.$$

Step 2: Differentiate again to find the second derivative Differentiate $(x - a) + (y - b)y' = 0$, using the product rule on its second term:

$$1 + (y')^2 + (y - b)y'' = 0.$$

Therefore,

$$y'' = -\frac{1 + (y')^2}{y - b} = -\frac{c^2}{(y - b)^3}.$$

In particular, $y'' \neq 0$ on either differentiable branch, so the requested quotient is defined there.

Step 3: Simplify the quotient and track its sign Because $c > 0$, the positive real power in the numerator is

$$[1 + (y')^2]^{3/2} = \left(\frac{c^2}{(y - b)^2}\right)^{3/2} = \frac{c^3}{|y - b|^3}.$$

Substituting the second derivative now yields

$$\frac{[1 + (y')^2]^{3/2}}{y''} = -c \frac{(y - b)^3}{|y - b|^3}.$$

The sign of $y - b$ cannot change on a single differentiable branch. At the two points $y = b$, the circle

Quick Tip to Solve: Differentiate the implicit equation twice before substituting; use $|y - b|^3$, not $(y - b)^3$, when taking the power $3/2$.

has a vertical tangent and the displayed derivatives with respect to x are not defined.

Step 4: Conclusion The quotient is $\boxed{-c}$ on the upper semicircle ($y > b$), and $\boxed{c}$ on the lower semicircle ($y < b$). Thus it is a constant on each differentiable branch, independent of a and b , with magnitude c .

Quick Tip to Solve: Differentiate the implicit equation twice before substituting; use $|y - b|^3$, not $(y - b)^3$, when taking the power $3/2$.

19. Find the equation of the normal to the curve $x^2 = 4y$ which passes through the point $(-1, 4)$.

[4 marks]

Solution:

Concept:

- Every point of the parabola $x^2 = 4y$ can be written as $P(2t, t^2)$, where t is real.
- Differentiating the curve gives the tangent slope $dy/dx = x/2$. At P , this is t , so the normal slope is $-1/t$ for $t \neq 0$.
- A line of slope m through (x_1, y_1) has equation $y - y_1 = m(x - x_1)$. Requiring a normal to pass through a fixed point determines its point of contact.

Step 1: Write the normal at a general point Implicitly differentiating $x^2 = 4y$, we get

$$2x = 4 \frac{dy}{dx}, \quad \frac{dy}{dx} = \frac{x}{2}.$$

At $P(2t, t^2)$, the tangent slope is t . For $t \neq 0$, the normal is therefore

$$y - t^2 = -\frac{1}{t}(x - 2t), \quad x + ty - 2t - t^3 = 0.$$

At $t = 0$, the point is the vertex: its tangent is horizontal and its normal is $x = 0$. That normal does not pass through $(-1, 4)$, so it supplies no additional answer.

Step 2: Impose the condition of passing through the given point Substitute $x = -1$ and $y = 4$ into the normal equation:

$$-1 + 4t - 2t - t^3 = 0, \quad t^3 - 2t + 1 = 0.$$

Since $t = 1$ is a root, factor the cubic completely over the reals:

$$(t - 1)(t^2 + t - 1) = 0.$$

Thus there are three admissible, nonzero parameter values,

$$t = 1, \quad t = \frac{-1 + \sqrt{5}}{2}, \quad t = \frac{-1 - \sqrt{5}}{2}.$$

Every real root corresponds to a point of the parabola, so all three normals must be retained.

Step 3: Substitute the three parameter values For each root, $t^3 = 2t - 1$. The general normal equation

Quick Tip to Solve: Parameterise this parabola by $(2t, t^2)$; factor the resulting cubic and retain every real root, since a point can lie on three normals.

simplifies to

$$x + ty - 4t + 1 = 0, \quad x + 1 + t(y - 4) = 0.$$

The value $t = 1$ gives $x + y - 3 = 0$. For the two other values, multiplying by 2 gives

$$2(x + 1) + (-1 + \sqrt{5})(y - 4) = 0,$$

$$2(x + 1) + (-1 - \sqrt{5})(y - 4) = 0.$$

Each equation visibly passes through $(-1, 4)$. Their slopes are $-1/t$, the negative reciprocals of the tangent slopes at their respective points of contact, verifying normality.

Step 4: Conclusion The required normals are $x + y - 3 = 0$ and the two lines

$$2(x + 1) + (-1 \pm \sqrt{5})(y - 4) = 0.$$

There are three distinct normals through the given point.

Quick Tip to Solve: Parameterise this parabola by $(2t, t^2)$; factor the resulting cubic and retain every real root, since a point can lie on three normals.

20. Prove that $\int_0^a f(x) dx = \int_0^a f(a - x) dx$ **and hence evaluate** $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$. [4 marks]

Solution:

Concept:

- A change of variable in a definite integral changes both the integrand and the limits; reversing the limits removes a minus sign.
- Reflection about the midpoint gives $\int_0^a f(x) dx = \int_0^a f(a - x) dx$, for an integrable function f .
- Use $\sin x + \cos x = \sqrt{2} \cos(x - \pi/4)$ and $\int \sec t dt = \log |\sec t + \tan t| + C$.

Step 1: Prove the reflection property In $K = \int_0^a f(x) dx$, put $t = a - x$, so $dx = -dt$. When $x = 0$, $t = a$; when $x = a$, $t = 0$. Therefore

$$K = - \int_a^0 f(a - t) dt = \int_0^a f(a - t) dt.$$

Since the variable of integration is a dummy variable, we may rename t as x . This proves the required identity.

Step 2: Reflect the given integral and add Let $I = \int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$. The denominator is positive throughout this interval, so the integral exists. Applying the proved property with $a = \pi/2$, and using the complementary-angle identities, gives

$$I = \int_0^{\pi/2} \frac{\pi/2 - x}{\cos x + \sin x} dx.$$

Quick Tip to Solve: When the denominator is unchanged by $x \mapsto \pi/2 - x$, add the reflected integral to the original one to replace the numerator x by the constant $\pi/2$.

Add this to the original expression. The terms involving x in the numerator cancel:

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}.$$

Thus it remains to evaluate the simpler integral $J = \int_0^{\pi/2} (\sin x + \cos x)^{-1} dx$, with $I = \pi J/4$.

Step 3: Evaluate the remaining trigonometric integral Put $t = x - \pi/4$. The new limits are $-\pi/4$ and $\pi/4$. Hence

$$\begin{aligned} J &= \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} \sec t \, dt \\ &= \frac{1}{\sqrt{2}} [\log |\sec t + \tan t|]_{-\pi/4}^{\pi/4} \\ &= \frac{1}{\sqrt{2}} \log \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right). \end{aligned}$$

Both endpoint arguments are positive. Since $(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$, the logarithm equals $2 \log(1 + \sqrt{2})$, so $J = \sqrt{2} \log(1 + \sqrt{2})$.

Step 4: Conclusion The reflection identity has been proved, and substitution in $I = \pi J/4$ yields

$$\boxed{\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx = \frac{\pi}{2\sqrt{2}} \log(1 + \sqrt{2})}.$$

Quick Tip to Solve: When the denominator is unchanged by $x \mapsto \pi/2 - x$, add the reflected integral to the original one to replace the numerator x by the constant $\pi/2$.

21(a). Solve the differential equation: $x \frac{dy}{dx} = y - x \tan \left(\frac{y}{x} \right).$

[4 marks]

Solution:

Concept:

- A first-order equation $dy/dx = F(y/x)$ is homogeneous. The substitution $v = y/x$, or $y = vx$, gives $dy/dx = v + x dv/dx$.
- Once the variables are separated, integrate each side: $\int \cot v \, dv = \log |\sin v| + C$ and $\int dx/x = \log |x| + C$.
- Dividing by a function can exclude constant solutions; check any excluded values in the original differential equation.

Step 1: Recognise the homogeneous form The given equation is defined for $x \neq 0$ and $\cos(y/x) \neq 0$. Divide by x to obtain

$$\frac{dy}{dx} = \frac{y}{x} - \tan \left(\frac{y}{x} \right).$$

Its right-hand side depends only on y/x , so put $y = vx$, where v is a function of x . By the product

Quick Tip to Solve: Substitute $y = vx$ as soon as the ratio y/x appears; the two vx terms cancel. Check $\tan v = 0$ separately before dividing by it.

rule,

$$\frac{dy}{dx} = v + x \frac{dv}{dx}.$$

Step 2: Separate the variables and integrate Substitute the expressions for y and dy/dx into the given equation:

$$x \left(v + x \frac{dv}{dx} \right) = vx - x \tan v.$$

Cancel xv and divide by the nonzero x , giving

$$x \frac{dv}{dx} = -\tan v.$$

On a nonconstant solution interval with $\sin v \neq 0$, separate and integrate:

$$\begin{aligned} \cot v \, dv &= -\frac{dx}{x}, \\ \log |\sin v| &= -\log |x| + C_1, \\ \log |x \sin v| &= C_1. \end{aligned}$$

Absorbing the fixed sign and the positive exponential of C_1 into an arbitrary nonzero constant gives $x \sin v = C$.

Step 3: Restore the original variables and check excluded solutions Replace v by y/x to obtain $x \sin(y/x) = C$. The division by $\tan v$ used in separation excluded $\sin v = 0$, that is, $v = n\pi$ for an integer n . These give $y = n\pi x$, and direct substitution gives $xn\pi = n\pi x - x \tan(n\pi)$; thus each is a solution. They are included by allowing $C = 0$. To verify the implicit family, differentiate:

$$\sin(y/x) + \frac{\cos(y/x)}{x} \left(x \frac{dy}{dx} - y \right) = 0.$$

Multiplication by $x/\cos(y/x)$ recovers the original equation exactly on its domain.

Step 4: Conclusion The general solution is

$$\boxed{x \sin \left(\frac{y}{x} \right) = C},$$

on intervals where $x \neq 0$ and $\cos(y/x) \neq 0$. Here C is an arbitrary real constant; $C = 0$ includes the straight-line solutions $y = n\pi x$.

Quick Tip to Solve: Substitute $y = vx$ as soon as the ratio y/x appears; the two vx terms cancel. Check $\tan v = 0$ separately before dividing by it.

21(b). Solve the differential equation: $\frac{dy}{dx} = -\left[\frac{x + y \cos x}{1 + \sin x}\right]$.

[4 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $y' + P(x)y = Q(x)$, where $y' = dy/dx$. Its integrating factor is $IF = e^{\int P(x) dx}$.
- Multiplying by the integrating factor makes the left-hand side an exact product derivative: $\frac{d}{dx}(y IF) = Q(x) IF$. Integrate once to obtain the general solution.

Step 1: Put the equation in linear form Move the term containing y to the left. The given equation becomes

$$\frac{dy}{dx} + \frac{\cos x}{1 + \sin x} y = -\frac{x}{1 + \sin x}.$$

Thus $P(x) = \cos x/(1 + \sin x)$ and $Q(x) = -x/(1 + \sin x)$. Work on an interval where $1 + \sin x \neq 0$, because the original differential equation is undefined when this denominator vanishes.

Step 2: Calculate the integrating factor The numerator $\cos x$ is the derivative of the denominator $1 + \sin x$. Therefore,

$$\int P(x) dx = \log |1 + \sin x|.$$

On every allowed interval $1 + \sin x > 0$, so we may take

$$IF = e^{\log(1 + \sin x)} = 1 + \sin x.$$

Step 3: Multiply and integrate the product derivative Multiplying the linear equation by $1 + \sin x$ gives

$$(1 + \sin x) \frac{dy}{dx} + y \cos x = -x.$$

By the product rule this is

$$\frac{d}{dx} [y(1 + \sin x)] = -x.$$

Integrating with respect to x , with arbitrary constant C , yields

$$y(1 + \sin x) = -\frac{x^2}{2} + C.$$

As a check, differentiating this relation gives $(1 + \sin x)y' + y \cos x = -x$; division by the nonzero denominator recovers the printed differential equation exactly.

Step 4: Conclusion The general solution is

$$\boxed{y(1 + \sin x) + \frac{x^2}{2} = C}, \quad \text{or} \quad \boxed{y = \frac{C - x^2/2}{1 + \sin x}}.$$

It is valid on intervals excluding $x = -\pi/2 + 2k\pi$, where k is any integer.

Quick Tip to Solve: Here the numerator of $P(x)$ is the derivative of its denominator, so the integrating factor is immediately $1 + \sin x$; multiplication then leaves only $-x$ to integrate.

22. If the lines $\frac{x-1}{-3} = \frac{y-2}{2\lambda} = \frac{z-3}{2}$ and $\frac{x-1}{3\lambda} = \frac{y-1}{2} = \frac{z-6}{-5}$ are perpendicular, find the value of λ . Hence find whether the lines are intersecting or not. [4 marks]

Solution:

Concept:

- The denominators of a line's symmetric equation are its direction ratios. Two perpendicular directions have zero scalar product: $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
- For two lines to intersect, one point must satisfy all three coordinate equations simultaneously. Use separate parameters for the two lines; orthogonal directions alone do not ensure intersection in three dimensions.

Step 1: Use perpendicular directions to determine the parameter The direction ratios of the two lines are $(-3, 2\lambda, 2)$ and $(3\lambda, 2, -5)$, respectively. Their scalar product must vanish:

$$\begin{aligned}(-3)(3\lambda) + (2\lambda)(2) + 2(-5) &= 0, \\ -9\lambda + 4\lambda - 10 &= 0, \\ -5\lambda &= 10.\end{aligned}$$

This fixes $\lambda = -2$. With this value their direction vectors are $(-3, -4, 2)$ and $(-6, 2, -5)$; indeed their scalar product is $18 - 8 - 10 = 0$.

Step 2: Write both lines in parametric form Let the common ratios in the first and second lines be t and s , respectively. After substituting $\lambda = -2$, the coordinate equations are

$$\begin{aligned}L_1 : \quad x &= 1 - 3t, \quad y = 2 - 4t, \quad z = 3 + 2t, \\ L_2 : \quad x &= 1 - 6s, \quad y = 1 + 2s, \quad z = 6 - 5s.\end{aligned}$$

The parameters need not be equal; a common point would require the three coordinates, rather than the parameter values, to agree.

Step 3: Check all three coordinates for a common point Equality of the x -coordinates gives $1 - 3t = 1 - 6s$, hence $t = 2s$. Equality of the y -coordinates then requires

$$2 - 4(2s) = 1 + 2s \implies s = \frac{1}{10}, \quad t = \frac{1}{5}.$$

These are the only parameter values that can make both x and y agree. However, their z -coordinates are

$$z_1 = 3 + \frac{2}{5} = \frac{17}{5}, \quad z_2 = 6 - \frac{5}{10} = \frac{11}{2}.$$

Since $17/5 \neq 11/2$, the third coordinate equation is inconsistent with the first two. Thus no pair of parameter values gives a common point.

Step 4: Conclusion $\lambda = -2$. The lines **do not intersect**. Their direction vectors are perpendicular, but the lines are nonparallel and nonintersecting, so they are skew lines.

Quick Tip to Solve: Use the zero dot product to find λ , then test intersection separately. Solve two coordinate equations and substitute the resulting parameter values into the third.

23. The scalar product of the vector $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ with a unit vector along the sum of the vectors $\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\vec{c} = \lambda\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to 1. Find the value of λ and hence find the unit vector along $\vec{b} + \vec{c}$. [4 marks]

Solution:

Concept:

- For a nonzero vector $\vec{v}$, the unit vector in its direction is $\hat{u} = \vec{v}/|\vec{v}|$, where $|\vec{v}|$ is its positive magnitude.
- The scalar product of vectors is the sum of products of corresponding Cartesian components:
 $\vec{a} \cdot \vec{u} = a_x u_x + a_y u_y + a_z u_z$.

Step 1: Add the given vectors and find the magnitude Adding corresponding components gives

$$\vec{b} + \vec{c} = (\lambda + 2)\hat{i} + 6\hat{j} - 2\hat{k}.$$

Consequently,

$$|\vec{b} + \vec{c}| = \sqrt{(\lambda + 2)^2 + 6^2 + (-2)^2} = \sqrt{(\lambda + 2)^2 + 40}.$$

This magnitude is always positive, since the component along $\hat{j}$ is nonzero. Thus the required direction is well defined for every real λ .

Step 2: Apply the prescribed scalar product Divide the vector sum by its magnitude to write

$$\hat{u} = \frac{(\lambda + 2)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(\lambda + 2)^2 + 40}}.$$

Using $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{a} \cdot \hat{u} = 1$, we obtain

$$\frac{\lambda + 2 + 6 - 2}{\sqrt{(\lambda + 2)^2 + 40}} = 1, \quad \lambda + 6 = \sqrt{(\lambda + 2)^2 + 40}.$$

The right side is positive, so a valid solution must satisfy $\lambda + 6 > 0$. Squaring and cancelling the quadratic terms gives

$$\begin{aligned} \lambda^2 + 12\lambda + 36 &= \lambda^2 + 4\lambda + 44, \\ 8\lambda &= 8, \quad \lambda = 1. \end{aligned}$$

Step 3: Normalize and check the direction At $\lambda = 1$, the sum has components $(3, 6, -2)$, so its magnitude is $\sqrt{9 + 36 + 4} = 7$. Dividing each component by 7 gives a vector whose squared magnitude is $(9 + 36 + 4)/49 = 1$. Its scalar product with $\vec{a}$ is $(3 + 6 - 2)/7 = 1$, verifying the original unsquared condition and excluding any extraneous root.

Step 4: Conclusion The required value and unit vector are

$$\boxed{\lambda = 1}, \quad \boxed{\hat{u} = \frac{3}{7}\hat{i} + \frac{6}{7}\hat{j} - \frac{2}{7}\hat{k}}.$$

Quick Tip to Solve: With $\vec{a} = (1, 1, 1)$, the dot product adds the components of the unit vector. Keep the positive norm in the denominator and check the unsquared equation after solving.

Section D (36 marks)

Question numbers 24 to 29 carry 6 marks each.

24(a). Show that for the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$, $A^3 - 6A^2 + 5A + 11I = 0$. Hence, find A^{-1} .

[6 marks]

Solution:

Concept:

- Matrix multiplication uses the row-column rule: $(AB)_{ij} = \sum_k a_{ik}b_{kj}$. Compute A^2 and A^3 to verify the given polynomial directly.
- For the identity matrix I , a matrix B is the inverse of A if $AB = BA = I$.

Step 1: Compute the square and cube of A Multiply each row of A by each column of A to obtain A^2 , then multiply A^2 by A to obtain A^3 :

$$A^2 = \begin{pmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{pmatrix}.$$

For example, $(A^2)_{23} = 1(1) + 2(-3) + (-3)(3) = -14$. Using row 2 of A^2 , we similarly get $(A^3)_{23} = (-3)(1) + 8(-3) + (-14)(3) = -69$.

Step 2: Verify the polynomial identity Combine the computed powers entry by entry and add $5A$:

$$A^3 - 6A^2 + 5A = \begin{pmatrix} -16 & -5 & -5 \\ -5 & -21 & 15 \\ -10 & 5 & -26 \end{pmatrix} + 5A = -11I.$$

All off-diagonal entries cancel, while the diagonal entries are $-16 + 5 = -11$, $-21 + 10 = -11$ and $-26 + 15 = -11$. Adding $11I$ proves the stated identity.

Step 3: Obtain the inverse from the verified identity Rearrange the verified polynomial identity and factor out A :

$$A(-A^2 + 6A - 5I) = 11I, \quad A^{-1} = \frac{-A^2 + 6A - 5I}{11}.$$

This proves invertibility: the numerator commutes with A because it is a polynomial in A . Dividing by 11 therefore gives both a left and a right inverse. Substitute the computed square into this formula; the numerator's first diagonal entry is $-4 + 6 - 5 = -3$, and its first-row second entry is $-2 + 6 = 4$. Obtain the other entries similarly, subtracting $5I$ only on the diagonal.

Quick Tip to Solve: Factor out A from the verified polynomial and isolate I . This obtains the inverse directly without calculating nine separate cofactors.

Step 4: Conclusion Thus $A^3 - 6A^2 + 5A + 11I = 0$, and

$$A^{-1} = \frac{1}{11} \begin{pmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{pmatrix}.$$

Quick Tip to Solve: Factor out A from the verified polynomial and isolate I . This obtains the inverse directly without calculating nine separate cofactors.

OR

24(b). Using matrix method, solve the following system of equations: $3x - 2y + 3z = 8$, $2x + y - z = 1$, $4x - 3y + 2z = 4$. [6 marks]

Solution:

Concept:

- A system of three linear equations can be expressed as $AX = B$, where A contains the coefficients, X is the column of unknowns, and B is the column of constants.
- If $|A| \neq 0$, the coefficient matrix is invertible and the unique solution is $X = A^{-1}B$, with $A^{-1} = \frac{\text{adj } A}{|A|}$.
- The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$ uses the minor obtained by deleting row i and column j . The adjugate is the transpose of the cofactor matrix.

Step 1: Express the equations in matrix form Keep the unknowns in the order x, y, z in each equation. Then the given system becomes $AX = B$, where

$$A = \begin{pmatrix} 3 & -2 & 3 \\ 2 & 1 & -1 \\ 4 & -3 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 8 \\ 1 \\ 4 \end{pmatrix}.$$

We first check that the inverse method applies by calculating the determinant.

Step 2: Calculate the determinant and cofactors Expanding along the first row, including its alternating signs, gives

$$\begin{aligned} |A| &= 3(2 - 3) - (-2)(4 + 4) + 3(-6 - 4) \\ &= -3 + 16 - 30 = -17. \end{aligned}$$

Thus $|A| \neq 0$, so the system has exactly one solution. For clarity, the nine signed minors are

$$\begin{aligned} C_{11} &= 2 - 3 = -1, & C_{12} &= -(4 + 4) = -8, & C_{13} &= -6 - 4 = -10, \\ C_{21} &= -(-4 + 9) = -5, & C_{22} &= 6 - 12 = -6, & C_{23} &= -(-9 + 8) = 1, \\ C_{31} &= 2 - 3 = -1, & C_{32} &= -(-3 - 6) = 9, & C_{33} &= 3 + 4 = 7. \end{aligned}$$

Quick Tip to Solve: Keep the variable order fixed while forming A , transpose the cofactor matrix to get its adjugate, and check the resulting values in all three equations.

Step 3: Form the inverse matrix Transposing the cofactor matrix changes its rows into columns. Hence

$$\text{adj } A = \begin{pmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{pmatrix}.$$

Dividing this matrix by the determinant gives

$$A^{-1} = -\frac{1}{17} \begin{pmatrix} -1 & -5 & -1 \\ -8 & -6 & 9 \\ -10 & 1 & 7 \end{pmatrix}.$$

It is essential to transpose the cofactors before using the inverse formula.

Step 4: Multiply by the constants column and verify Multiplying $AX = B$ on the left by A^{-1} , we obtain

$$\begin{aligned} X &= A^{-1}B \\ &= -\frac{1}{17} \begin{pmatrix} -8 & -5 & -4 \\ -64 & -6 & 36 \\ -80 & 1 & 28 \end{pmatrix} = -\frac{1}{17} \begin{pmatrix} -17 \\ -34 \\ -51 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}. \end{aligned}$$

Substitution provides an independent check in all three original equations:

$$3(1) - 2(2) + 3(3) = 8,$$

$$2(1) + 2 - 3 = 1,$$

$$4(1) - 3(2) + 2(3) = 4.$$

Each left-hand side equals its prescribed constant, confirming the matrix calculation.

Step 5: Conclusion The unique solution of the given system is $x = 1, y = 2, z = 3$.

Quick Tip to Solve: Keep the variable order fixed while forming A , transpose the cofactor matrix to get its adjugate, and check the resulting values in all three equations.

25. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. [6 marks]

Solution:

Concept:

- For a cylinder of base radius r and height h , its volume is $V = \pi r^2 h$.
- An axial cross-section of a cylinder inscribed in a sphere is a rectangle inscribed in a circle of radius R . A right triangle from the centre to a corner gives $r^2 + (h/2)^2 = R^2$.
- To maximise a differentiable function on a closed interval, find its interior critical points and compare their values with the endpoints. A negative second derivative at a stationary point confirms a local maximum.

Quick Tip to Solve: Use half the cylinder's height in the sphere constraint: $r^2 + h^2/4 = R^2$. Eliminating r^2 makes the volume a simple cubic in h .

Step 1: Relate the cylinder dimensions to the sphere Let r be the radius of the cylinder and h its height. Assume $R > 0$. The cylinder is centred in the sphere, so the distance from the sphere's centre to either circular face is $h/2$. In an axial section, a corner of the rectangle lies on the circle representing the sphere. By Pythagoras,

$$r^2 + \frac{h^2}{4} = R^2, \quad r^2 = R^2 - \frac{h^2}{4}.$$

For a nondegenerate cylinder, $0 < h < 2R$; the limiting endpoints $h = 0$ and $h = 2R$ give zero volume. This relation allows us to eliminate r and optimise a function of height alone.

Step 2: Express the volume as a function of height Substituting the constraint into the volume formula gives

$$V(h) = \pi r^2 h = \pi \left(R^2 - \frac{h^2}{4} \right) h = \pi R^2 h - \frac{\pi h^3}{4}.$$

The sphere's radius R is constant while the cylinder dimensions vary. Differentiating with respect to h ,

$$V'(h) = \pi R^2 - \frac{3\pi h^2}{4}.$$

Step 3: Locate and establish the maximum At an interior maximum the derivative must vanish. Therefore

$$\begin{aligned} V'(h) = 0 &\implies \frac{3h^2}{4} = R^2 \\ &\implies h = \frac{2R}{\sqrt{3}}. \end{aligned}$$

We take only the positive root because h is a length. This value lies strictly between 0 and $2R$. Also,

$$V''(h) = -\frac{3\pi h}{2}, \quad V''\left(\frac{2R}{\sqrt{3}}\right) = -\sqrt{3}\pi R < 0.$$

There is only one critical point in the physical interval. Since the continuous volume function has zero value at both endpoints and positive value at this critical point, it gives the global maximum as well as a local maximum.

Step 4: Evaluate the largest volume At the optimal height, the squared base radius becomes

$$r^2 = R^2 - \frac{1}{4} \left(\frac{2R}{\sqrt{3}} \right)^2 = R^2 - \frac{R^2}{3} = \frac{2R^2}{3}.$$

Substituting both dimensions into $V = \pi r^2 h$,

$$V_{\max} = \pi \left(\frac{2R^2}{3} \right) \left(\frac{2R}{\sqrt{3}} \right) = \frac{4\pi R^3}{3\sqrt{3}}.$$

The result is proportional to R^3 , as required for a volume when the containing sphere is scaled.

Step 5: Conclusion The height of the cylinder of maximum volume is $\boxed{h = \frac{2R}{\sqrt{3}}}$, as required. Its

Quick Tip to Solve: Use half the cylinder's height in the sphere constraint: $r^2 + h^2/4 = R^2$. Eliminating r^2 makes the volume a simple cubic in h .

maximum volume is $\frac{4\pi R^3}{3\sqrt{3}}$ cubic units.

Quick Tip to Solve: Use half the cylinder's height in the sphere constraint: $r^2 + h^2/4 = R^2$. Eliminating r^2 makes the volume a simple cubic in h .

26. A bag contains 5 red and 4 black balls, a second bag contains 3 red and 6 black balls. One of the two bags is selected at random and two balls are drawn at random (without replacement) both of which are found to be red. Find the probability that the balls are drawn from the second bag. [6 marks]

Solution:

Concept:

- Bayes' theorem reverses a conditional probability: after observing an event, the probability of a possible cause is proportional to its prior probability multiplied by the probability of the observation under that cause.
- For mutually exclusive and exhaustive events B_1, B_2 , the total probability of R is $P(R) = P(B_1)P(R | B_1) + P(B_2)P(R | B_2)$.
- In sampling without replacement, the second draw is made from one fewer ball. If the first ball is red, both the red-ball count and the total count decrease by one.

Step 1: Define the selection and observation events Let B_1 be the event that the first bag is selected, and B_2 the event that the second bag is selected. Let R denote the event that both drawn balls are red. Each bag initially contains nine balls. Since the bag is selected at random,

$$P(B_1) = P(B_2) = \frac{1}{2}.$$

The requested probability is $P(B_2 | R)$, since the observation of two red balls is already known. It is different from the probability of two red balls when the second bag has been selected.

Step 2: Calculate the likelihood of two red balls from each bag The first bag has five red balls. The probability of a red first draw is $5/9$; following that draw, four red balls remain among eight balls. Therefore,

$$P(R | B_1) = \frac{5}{9} \cdot \frac{4}{8} = \frac{5}{18}.$$

The second bag has three red balls. After a red first draw, it has two red balls among eight remaining balls. Thus,

$$P(R | B_2) = \frac{3}{9} \cdot \frac{2}{8} = \frac{1}{12}.$$

These products use conditional probabilities for the second draw, so they correctly account for drawing without replacement.

Step 3: Find the total probability of the observation The two possible bag choices are disjoint and

Quick Tip to Solve: Because the bag choices have equal prior probabilities and both bags have nine balls, compare the red-pair counts: $\binom{5}{2} : \binom{3}{2} = 10 : 3$. The second bag therefore contributes $3/(10 + 3)$.

cover every possible selection. Consequently,

$$\begin{aligned} P(R) &= P(B_1)P(R | B_1) + P(B_2)P(R | B_2) \\ &= \frac{1}{2} \cdot \frac{5}{18} + \frac{1}{2} \cdot \frac{1}{12} \\ &= \frac{5}{36} + \frac{1}{24} = \frac{13}{72}. \end{aligned}$$

The joint probability of selecting the second bag and observing two reds is $P(B_2 \cap R) = 1/24 = 3/72$.

Step 4: Apply Bayes' theorem to the second bag Restrict the sample space to outcomes in which both drawn balls are red. Bayes' theorem gives

$$\begin{aligned} P(B_2 | R) &= \frac{P(B_2)P(R | B_2)}{P(R)} \\ &= \frac{3/72}{13/72} = \frac{3}{13}. \end{aligned}$$

This posterior probability is below $1/2$, as expected: observing two reds favours the first bag, which contains more red balls.

Step 5: Conclusion The probability that the two red balls were drawn from the second bag is $\boxed{\frac{3}{13}}$.

Quick Tip to Solve: Because the bag choices have equal prior probabilities and both bags have nine balls, compare the red-pair counts: $\binom{5}{2} : \binom{3}{2} = 10 : 3$. The second bag therefore contributes $3/(10 + 3)$.

27(a). Using method of integration, find the area of the triangle whose vertices are $(1, 0)$, $(2, 2)$ and $(3, 1)$. [6 marks]

Solution:

Concept:

- The area between an upper boundary $y = f(x)$ and a lower boundary $y = g(x)$ over $[a, b]$ is $\int_a^b [f(x) - g(x)] dx$, provided $f(x) \geq g(x)$ throughout that interval.
- The straight line through (x_1, y_1) and (x_2, y_2) has equation $y - y_1 = m(x - x_1)$, where $m = (y_2 - y_1)/(x_2 - x_1)$.
- If the upper or lower boundary changes at an intermediate vertex, split the definite integral at that vertex's x -coordinate and add the areas.

Step 1: Find the equations of the three sides Denote the vertices by $A(1, 0)$, $B(2, 2)$ and $C(3, 1)$. The slopes of the sides are

$$m_{AB} = \frac{2 - 0}{2 - 1} = 2, \quad m_{BC} = \frac{1 - 2}{3 - 2} = -1, \quad m_{AC} = \frac{1 - 0}{3 - 1} = \frac{1}{2}.$$

Quick Tip to Solve: Split at $x = 2$: side AC stays below every vertical strip, while the upper side changes from AB to BC . Always integrate upper minus lower on each interval.

Substituting a vertex into the point-slope equation gives

$$AB : y = 2(x - 1) = 2x - 2,$$

$$BC : y - 2 = -(x - 2), \quad y = 4 - x,$$

$$AC : y = \frac{x - 1}{2}.$$

These equations also reproduce the coordinates at both endpoints of each side.

Step 2: Identify the boundaries for vertical strips The triangle extends from $x = 1$ to $x = 3$. Its lower boundary throughout this interval is side AC . The upper boundary is AB from $x = 1$ to $x = 2$, and BC from $x = 2$ to $x = 3$. The two strip heights are

$$\begin{aligned} (2x - 2) - \frac{x - 1}{2} &= \frac{3}{2}(x - 1), \quad 1 \leq x \leq 2, \\ (4 - x) - \frac{x - 1}{2} &= \frac{3}{2}(3 - x), \quad 2 \leq x \leq 3. \end{aligned}$$

Both heights are nonnegative on their respective intervals, so integrating them gives geometric area without sign cancellation.

Step 3: Integrate over the left portion Let S_1 be the area contributed by vertical strips for $1 \leq x \leq 2$. Then

$$\begin{aligned} S_1 &= \int_1^2 \left[(2x - 2) - \frac{x - 1}{2} \right] dx \\ &= \frac{3}{2} \int_1^2 (x - 1) dx \\ &= \left[\frac{3}{4}(x - 1)^2 \right]_1^2 = \frac{3}{4}. \end{aligned}$$

This integral ends at $x = 2$, where the upper boundary changes from AB to BC .

Step 4: Integrate over the right portion and add For $2 \leq x \leq 3$, the corresponding area is

$$\begin{aligned} S_2 &= \int_2^3 \left[(4 - x) - \frac{x - 1}{2} \right] dx \\ &= \frac{3}{2} \int_2^3 (3 - x) dx \\ &= \frac{3}{2} \left[3x - \frac{x^2}{2} \right]_2^3 \\ &= \frac{3}{2} \left(\frac{9}{2} - 4 \right) = \frac{3}{4}. \end{aligned}$$

The two portions meet only along the vertical line $x = 2$, which has zero area. Hence they cover the triangle without overlapping in area, and

$$S = S_1 + S_2 = \frac{3}{4} + \frac{3}{4} = \frac{3}{2}.$$

Quick Tip to Solve: Split at $x = 2$: side AC stays below every vertical strip, while the upper side changes from AB to BC . Always integrate upper minus lower on each interval.

Step 5: Conclusion Using integration, the area of the triangle is $\boxed{\frac{3}{2} \text{ square units}}$.

Quick Tip to Solve: Split at $x = 2$: side AC stays below every vertical strip, while the upper side changes from AB to BC . Always integrate upper minus lower on each interval.

OR

27(b). Using method of integration, find the area of the region enclosed between two circles $x^2 + y^2 = 4$ and $(x - 2)^2 + y^2 = 4$. [6 marks]

Solution:

Concept:

- The common region of two circles consists of the points satisfying both circle inequalities. For a vertical strip, its area is its height multiplied by dx .
- Reflection symmetry about the x -axis doubles the area above that axis. Here reflection about $x = 1$ also divides the common region into two equal parts.
- The standard antiderivative is $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C$, where $a > 0$ and inverse trigonometric angles are in radians.

Step 1: Identify the circles and their intersections The circle $x^2 + y^2 = 4$ has centre $(0, 0)$ and radius 2, while $(x - 2)^2 + y^2 = 4$ has centre $(2, 0)$ and radius 2. Subtracting their equations gives

$$(x - 2)^2 - x^2 = 0 \implies -4x + 4 = 0 \implies x = 1.$$

Substitution into the first circle gives $y^2 = 3$. Hence the intersection points are $(1, \sqrt{3})$ and $(1, -\sqrt{3})$. The enclosed common region extends from $x = 0$ to $x = 2$.

Step 2: Determine the boundary of a vertical strip Above the x -axis, the common region is below both semicircles. On $0 \leq x \leq 1$, its upper boundary is $y = \sqrt{4 - (x - 2)^2}$; on $1 \leq x \leq 2$, it is $y = \sqrt{4 - x^2}$. To see the switch, the difference of the squared heights is

$$[4 - (x - 2)^2] - [4 - x^2] = 4(x - 1).$$

Thus the smaller height changes at $x = 1$. Doubling for the equal region below the axis gives

$$A = 2 \left[\int_0^1 \sqrt{4 - (x - 2)^2} dx + \int_1^2 \sqrt{4 - x^2} dx \right].$$

Step 3: Use symmetry to reduce the two integrals In the first integral put $u = 2 - x$, so $dx = -du$. When $x = 0$, $u = 2$, and when $x = 1$, $u = 1$. Therefore

$$\int_0^1 \sqrt{4 - (x - 2)^2} dx = \int_2^1 \sqrt{4 - u^2} (-du) = \int_1^2 \sqrt{4 - u^2} du.$$

Quick Tip to Solve: First subtract the circle equations to get the common chord $x = 1$; symmetry then reduces the lens area to four times the single integral from 1 to 2.

Both contributions are equal, as also follows from the symmetry about $x = 1$. Consequently the required area can be evaluated from

$$A = 4 \int_1^2 \sqrt{4 - x^2} dx.$$

Step 4: Evaluate the definite integral Applying the standard formula with $a = 2$,

$$\begin{aligned} A &= 4 \left[\frac{x}{2} \sqrt{4 - x^2} + 2 \sin^{-1} \left(\frac{x}{2} \right) \right]_1^2 \\ &= 4 \left[\left(0 + 2 \cdot \frac{\pi}{2} \right) - \left(\frac{\sqrt{3}}{2} + 2 \cdot \frac{\pi}{6} \right) \right] \\ &= 4 \left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \right). \end{aligned}$$

The lower endpoint uses $\sin^{-1}(1/2) = \pi/6$; the upper endpoint uses $\sin^{-1}(1) = \pi/2$.

Step 5: Conclusion The area enclosed between the two circles is $\boxed{\frac{8\pi}{3} - 2\sqrt{3}}$ square units.

Quick Tip to Solve: First subtract the circle equations to get the common chord $x = 1$; symmetry then reduces the lens area to four times the single integral from 1 to 2.

28. A company produces two types of goods, A and B, that require gold and silver. Each unit of type A requires 3 g of silver and 1 g of gold while that of type B requires 1 g of silver and 2 g of gold. The company can use at the most 9 g of silver and 8 g of gold. If each unit of type A brings a profit of ₹ 40 and that of type B ₹ 50, find the number of units of each type that the company should produce to maximize profit. Formulate the above problem as an LPP and solve it graphically. Also find the maximum profit. [6 marks]

Solution:

Concept:

- Choose decision variables for the quantities produced. An “at most” resource limit gives a $\leq$ inequality, and quantities produced must be non-negative.
- The feasible region is the intersection of all constraint half-planes. Its points represent production plans obeying every resource limit.
- For a linear objective on a non-empty bounded polygonal feasible region, the maximum occurs at a corner point. Evaluate the profit at every vertex.

Step 1: Define the variables and formulate the objective Let x and y be the numbers of units of goods A and B respectively. Each A contributes a profit of ₹ 40, and each B contributes ₹ 50. Thus the total profit in rupees is

$$Z = 40x + 50y.$$

The company must maximize Z . Since production cannot be negative, $x \geq 0$ and $y \geq 0$.

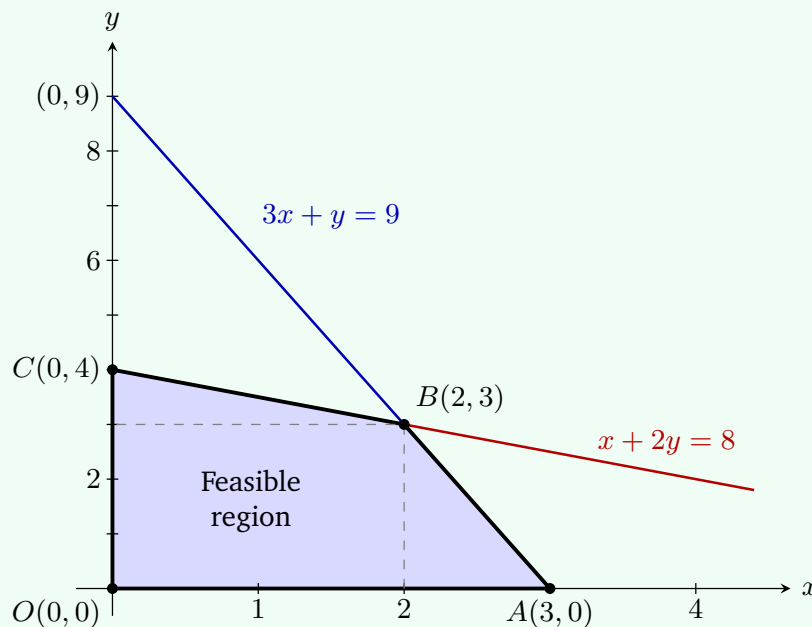
Quick Tip to Solve: Keep the two metals in separate constraint rows; the optimal corner (2, 3) uses all 9 g of silver and all 8 g of gold.

Step 2: Translate the available metals into constraints Each unit of A uses 3 g of silver and each unit of B uses 1 g. The silver used is therefore $3x + y$, which cannot exceed 9 g. Similarly the gold requirement is $x + 2y$, which cannot exceed 8 g. The complete LPP is

$$\begin{aligned} \text{Maximize } Z &= 40x + 50y, \\ \text{subject to } 3x + y &\leq 9, \quad x + 2y \leq 8, \\ x &\geq 0, \quad y \geq 0. \end{aligned}$$

The metal limits are upper bounds, so equality is required only on the corresponding boundary lines.

Step 3: Plot the constraint lines and shade the feasible region The silver boundary $3x + y = 9$ passes through $(3, 0)$ and $(0, 9)$. The gold boundary $x + 2y = 8$ passes through $(8, 0)$ and $(0, 4)$. For the part shown in the graph, $(4, 2)$ is another convenient point on the gold boundary. The origin satisfies both resource inequalities, so select the half-plane containing the origin for each line, and restrict to the first quadrant. The common shaded region is the bounded quadrilateral $OABC$. Its axis vertices are $O = (0, 0)$, $A = (3, 0)$, and $C = (0, 4)$. Point B is the intersection of the two resource boundaries.



Shaded feasible region $OABC$; x and y count units of A and B.

Step 4: Calculate the intersection vertex At B , both metal constraints are equalities:

$$3x + y = 9, \quad x + 2y = 8.$$

Multiplying the first equation by 2 and subtracting the second gives $5x = 10$, hence $x = 2$. Substituting into $3x + y = 9$ gives $y = 3$. Therefore $B = (2, 3)$, completing the list of four feasible corner points.

Quick Tip to Solve: Keep the two metals in separate constraint rows; the optimal corner $(2, 3)$ uses all 9 g of silver and all 8 g of gold.

Step 5: Compare the profit at every corner Apply the corner-point method:

Vertex	(x, y)	$Z = 40x + 50y$ (rupees)
O	$(0, 0)$	0
A	$(3, 0)$	120
B	$(2, 3)$	$80 + 150 = 230$
C	$(0, 4)$	200

The largest entry occurs at B . This plan uses $3(2) + 3 = 9$ g of silver and $2 + 2(3) = 8$ g of gold, so it respects both resource limits exactly. Both production quantities are whole numbers.

Step 6: Conclusion The company should produce 2 units of A and 3 units of B. The maximum profit is ₹ 230.

Quick Tip to Solve: Keep the two metals in separate constraint rows; the optimal corner $(2, 3)$ uses all 9 g of silver and all 8 g of gold.

29(a). Find the vector and cartesian equations of the plane passing through the points having position vectors $\hat{i} + \hat{j} - 2\hat{k}$, $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$. Write the equation of a plane passing through a point $(2, 3, 7)$ and parallel to the plane obtained above. Hence, find the distance between the two parallel planes. [6 marks]

Solution:

Concept:

- A plane through a point with position vector $\vec{a}$, perpendicular to a nonzero normal vector $\vec{n}$, has equation $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$, where $\vec{r}$ is the position vector of any point on the plane.
- For three noncollinear points A, B, C , the cross product $\vec{AB} \times \vec{AC}$ is normal to their plane. Parallel planes have proportional normal vectors.
- The perpendicular distance between $ax + by + cz = d_1$ and $ax + by + cz = d_2$ is $\frac{|d_2 - d_1|}{\sqrt{a^2 + b^2 + c^2}}$, provided their left-hand coefficients are identical.

Step 1: Find two vectors lying in the plane Write the given points as $A(1, 1, -2)$, $B(2, -1, 1)$ and $C(1, 2, 1)$. Subtract the coordinates of A from those of the other points:

$$\begin{aligned}\vec{AB} &= \hat{i} - 2\hat{j} + 3\hat{k}, \\ \vec{AC} &= \hat{j} + 3\hat{k}.\end{aligned}$$

Both vectors lie in the required plane. Their cross product will therefore be perpendicular to that plane.

Quick Tip to Solve: After finding the normal, retain exactly the same coefficients for the second plane; substitute its given point only to change the constant, then divide the constant difference by the normal's length.

Step 2: Obtain a normal and the first plane Expanding the cross product gives

$$\begin{aligned}\vec{AB} \times \vec{AC} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 0 & 1 & 3 \end{vmatrix} \\ &= -9\hat{i} - 3\hat{j} + \hat{k}.\end{aligned}$$

This is nonzero, so the three points are noncollinear. Choose the opposite normal $\vec{n} = 9\hat{i} + 3\hat{j} - \hat{k}$. Using A , the vector equation is

$$\begin{aligned}[\vec{r} - (\hat{i} + \hat{j} - 2\hat{k})] \cdot (9\hat{i} + 3\hat{j} - \hat{k}) &= 0, \\ \vec{r} \cdot (9\hat{i} + 3\hat{j} - \hat{k}) &= 14.\end{aligned}$$

With $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, the cartesian equation becomes $9x + 3y - z = 14$. As a check, the left side at B is $18 - 3 - 1 = 14$, and at C it is $9 + 6 - 1 = 14$.

Step 3: Form the parallel plane through the given point A parallel plane can use the same normal, so write it as $9x + 3y - z = d$. Since it passes through $P(2, 3, 7)$,

$$d = 9(2) + 3(3) - 7 = 20.$$

Thus its cartesian equation is $9x + 3y - z = 20$, and its vector equation is $\vec{r} \cdot (9\hat{i} + 3\hat{j} - \hat{k}) = 20$.

Step 4: Calculate the perpendicular separation The two equations now have identical coefficients of x, y, z . Their normal has length $\sqrt{9^2 + 3^2 + (-1)^2} = \sqrt{91}$. Hence the perpendicular separation is

$$D = \frac{|20 - 14|}{\sqrt{91}} = \frac{6}{\sqrt{91}}.$$

Step 5: Conclusion The required first plane is

$$\vec{r} \cdot (9\hat{i} + 3\hat{j} - \hat{k}) = 14, \quad 9x + 3y - z = 14.$$

The parallel plane through $(2, 3, 7)$ is $9x + 3y - z = 20$. Their distance is $\boxed{6/\sqrt{91} \text{ units}}$.

Quick Tip to Solve: After finding the normal, retain exactly the same coefficients for the second plane; substitute its given point only to change the constant, then divide the constant difference by the normal's length.

29(b). Find the equation of the line passing through $(2, -1, 2)$ and $(5, 3, 4)$, and of the plane passing through $(2, 0, 3)$, $(1, 1, 5)$ and $(3, 2, 4)$. Also find their point of intersection. [6 marks]

Solution:

Concept:

- The line through position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$, with real parameter t . The components of $\vec{b} - \vec{a}$ are direction ratios of the line.
- A plane through a point with position vector $\vec{p}$ and normal $\vec{n} \neq \vec{0}$ is $(\vec{r} - \vec{p}) \cdot \vec{n} = 0$. A cross product of two nonparallel vectors in the plane gives a normal.
- To intersect a line and a plane, substitute the line's parametric coordinates into the plane equation, solve for the parameter and substitute it back into the line.

Step 1: Write the line through the two points Let $A(2, -1, 2)$ and $B(5, 3, 4)$ be the points defining the line. Its direction vector is

$$\overrightarrow{AB} = (5 - 2)\hat{i} + [3 - (-1)]\hat{j} + (4 - 2)\hat{k} = 3\hat{i} + 4\hat{j} + 2\hat{k}.$$

The vector equation of the line is therefore

$$\vec{r} = (2\hat{i} - \hat{j} + 2\hat{k}) + t(3\hat{i} + 4\hat{j} + 2\hat{k}), \quad t \in \mathbb{R}.$$

Equivalently, its cartesian equation is

$$\frac{x - 2}{3} = \frac{y + 1}{4} = \frac{z - 2}{2}.$$

Setting each ratio equal to t gives $x = 2 + 3t$, $y = -1 + 4t$, $z = 2 + 2t$. These coordinates give A at $t = 0$ and B at $t = 1$.

Step 2: Calculate a normal to the required plane Denote the plane's three given points by $P(2, 0, 3)$, $Q(1, 1, 5)$, $R(3, 2, 4)$. Two vectors lying in the plane are

$$\overrightarrow{PQ} = -\hat{i} + \hat{j} + 2\hat{k}, \quad \overrightarrow{PR} = \hat{i} + 2\hat{j} + \hat{k}.$$

Their cross product is

$$\begin{aligned} \overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} \\ &= -3\hat{i} + 3\hat{j} - 3\hat{k}. \end{aligned}$$

It is nonzero, so the plane is uniquely determined. Dividing by -3 , choose the simpler normal $\vec{n} = \hat{i} - \hat{j} + \hat{k}$.

Quick Tip to Solve: After finding the plane, test the given endpoints of the line: here the first endpoint already satisfies the plane. Confirm uniqueness by checking that the line direction is not perpendicular to the plane normal.

Step 3: Use a point to obtain the plane equation Using $P(2, 0, 3)$, the point-normal equation becomes

$$(x - 2) - (y - 0) + (z - 3) = 0,$$
$$x - y + z = 5.$$

Its vector form is $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$. The other two points verify the equation: $1 - 1 + 5 = 5$ for Q , and $3 - 2 + 4 = 5$ for R .

Step 4: Substitute the line into the plane A common point must satisfy both equations. Substituting the parametric coordinates of the line gives

$$(2 + 3t) - (-1 + 4t) + (2 + 2t) = 5,$$
$$5 + t = 5,$$
$$t = 0.$$

Thus the common point is $(2, -1, 2)$. The parameter has a unique value because the line direction has nonzero scalar product with the plane normal: $(3, 4, 2) \cdot (1, -1, 1) = 1$. Direct substitution also checks $2 - (-1) + 2 = 5$.

Step 5: Conclusion The line and plane are respectively

$$\boxed{\frac{x - 2}{3} = \frac{y + 1}{4} = \frac{z - 2}{2}}, \quad \boxed{x - y + z = 5}.$$

Their unique point of intersection is $\boxed{(2, -1, 2)}$.

Quick Tip to Solve: After finding the plane, test the given endpoints of the line: here the first endpoint already satisfies the plane. Confirm uniqueness by checking that the line direction is not perpendicular to the plane normal.