CBSE Class 12 Mathematics Question Paper 2019 with Solution PDF-Set 65-3-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 100**

General Instructions

(i) All questions are compulsory.

(ii) The question paper consists of **29** questions divided into four sections : A, B, C and D. Section A comprises of **4** questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 1 question of Section A, 3 questions of Section B, 3 questions of Section C and 3 questions of Section D. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

Section A (4 marks)

Questions number 1 to 4 carry 1 mark each.

1. Find the integrating factor of the differential equation $x\dfrac{dy}{dx} - 2y = 2x^2$. [1 mark]

Solution:

Concept:

- A first-order linear equation has standard form $y\prime + P(x)y = Q(x)$, where P is the coefficient of y after the coefficient of $y\prime$ is made unity.
- An integrating factor is $e^{\int P(x)\,dx}$; any nonzero constant multiple is equally valid.

Step 1: Write the standard form For $x \neq 0$, divide every term by x:

$$\frac{dy}{dx} - \frac{2}{x}y = 2x.$$

Hence the coefficient to integrate is $P(x) = -2/x$, rather than the unnormalised coefficient -2.

Step 2: Integrate the coefficient On an interval not containing zero,

$$\int P(x)\,dx = -2\ln|x|.$$

Exponentiating gives $e^{-2\ln|x|} = |x|^{-2} = x^{-2}$. The absolute value allows the calculation on either side of zero.

Step 3: Conclusion The integrating factor is $\boxed{\dfrac{1}{x^2}}$, for $x \neq 0$.

Quick Tip to Solve: First divide by the coefficient of dy/dx; here that changes the coefficient of y from -2 to $-2/x$.

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2. Find $\dfrac{dy}{dx}$, if $xy^2 - x^2 = 4$. [1 mark]

Solution:

Concept:

- In implicit differentiation, treat y as a function of x, so $d(y^2)/dx = 2y\,dy/dx$.
- The product rule is $d(uv)/dx = u\,dv/dx + v\,du/dx$.

Step 1: Differentiate both sides Apply the product rule to xy^2, the power rule to x^2, and note that the derivative of the constant 4 is zero:

$$y^2 + 2xy\frac{dy}{dx} - 2x = 0.$$

The term y^2 arises by differentiating the factor x.

Quick Tip to Solve: In xy^2, differentiate both factors: forgetting the product-rule term y^2 changes the numerator.

Step 2: Isolate the derivative Move the terms without dy/dx to the other side:

$$2xy\frac{dy}{dx} = 2x - y^2.$$

The original relation excludes both $x = 0$ and real points with $y = 0$, so division by $2xy$ is valid on the real curve.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = \frac{2x - y^2}{2xy}}.$$

Quick Tip to Solve: In xy^2, differentiate both factors: forgetting the product-rule term y^2 changes the numerator.

3. If A is a square matrix of order 3 with $|A| = 4$, then write the value of $|-2A|$. [1 mark]

Solution:

Concept:

- For a square matrix A of order n and scalar k, $|kA| = k^n|A|$.
- The exponent is the matrix order because multiplying the matrix by k multiplies every row of its determinant by k.

Step 1: Identify the order and scalar The matrix has three rows and three columns, so $n = 3$. Here the scalar multiplying all entries is $k = -2$, and the given determinant is $|A| = 4$. The determinant therefore acquires three factors of -2.

Step 2: Apply the determinant property Substitute the data into the scaling rule:

$$|-2A| = (-2)^3|A| = (-8) \times 4.$$

Since three is odd, the cube retains the negative sign; a single factor of -2 would account for only one scaled row.

Step 3: Conclusion

$$\boxed{|-2A| = -32}.$$

Quick Tip to Solve: Raise the scalar to the matrix order before multiplying by the original determinant; an odd order preserves a negative scalar sign.

4(a). If a line has the direction ratios $-18, 12, -4$, then what are its direction cosines? [1 mark]

Solution:

Concept:

- If (a, b, c) are direction ratios, one corresponding set of direction cosines is $(l, m, n) = (a, b, c)/\sqrt{a^2 + b^2 + c^2}$.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$; reversing the orientation reverses all three signs together.

Step 1: Find the normalising length For the direction ratios $(-18, 12, -4)$, the positive normalising factor is

$$\sqrt{(-18)^2 + 12^2 + (-4)^2} = \sqrt{324 + 144 + 16} = 22.$$

Squaring each ratio makes its contribution to the length positive.

Step 2: Normalise the direction ratios Divide all three ratios by the same length:

$$(l, m, n) = \left(\frac{-18}{22}, \frac{12}{22}, \frac{-4}{22}\right).$$

As a check, the squared simplified numerators add to $81 + 36 + 4 = 121$, the squared denominator.

Step 3: Conclusion One set is $\boxed{\left(-\frac{9}{11}, \frac{6}{11}, -\frac{2}{11}\right)}$. For the opposite orientation, all three signs are reversed.

Quick Tip to Solve: Normalise the ratios by their common Euclidean length; do not discard the individual signs when taking the square root.

OR

4(b). Find the cartesian equation of the line which passes through the point $(-2, 4, -5)$ and is parallel to the line $\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}$. [1 mark]

Solution:

Concept:

- A line through (x_1, y_1, z_1) with direction ratios (a, b, c) has symmetric equation $(x - x_1)/a = (y - y_1)/b = (z - z_1)/c$.
- Parallel lines have proportional direction ratios; numerators must be expressed in the same coordinate-minus-constant convention.

Step 1: Read the direction ratios Rewrite the middle fraction with y first:

$$\frac{x+3}{3} = \frac{y-4}{-5} = \frac{z+8}{6}.$$

Thus suitable direction ratios are $(3, -5, 6)$. In particular, the coefficient along y is negative because the printed numerator is $4 - y$.

Step 2: Insert the required point The required line passes through $(-2, 4, -5)$ and has the same

Quick Tip to Solve: Change $4 - y$ to $y - 4$ only if the denominator also changes from 5 to -5.

direction ratios. Its three numerators are therefore $x-(-2)=x+2$, $y-4$, and $z-(-5)=z+5$, respectively.

Step 3: Conclusion The required cartesian equation is

$$\boxed{\frac{x+2}{3}=\frac{y-4}{-5}=\frac{z+5}{6}}.$$

Quick Tip to Solve: Change $4-y$ to $y-4$ only if the denominator also changes from 5 to -5.

Section B (16 marks)

Questions number 5 to 12 carry 2 marks each.

5. Let $*$ be a binary operation on $\mathbb{R}-\{-1\}$ defined by $a*b=\frac{a}{b+1}$, for all $a,b\in\mathbb{R}-\{-1\}$. Show that $*$ is neither commutative nor associative in $\mathbb{R}-\{-1\}$. [2 marks]

Solution:

Concept:

- Commutativity requires $a*b=b*a$ for every admissible pair. A single counterexample disproves it.
- Associativity requires $(a*b)*c=a*(b*c)$ for every admissible triple. Choose values for which both intermediate results remain in the stated set.

Step 1: Test commutativity Take $a=1$ and $b=2$, both in $\mathbb{R}\setminus\{-1\}$. Applying the printed rule gives

$$1*2=\frac{1}{2+1}=\frac{1}{3},\qquad 2*1=\frac{2}{1+1}=1.$$

The results differ, so the order of the two inputs matters.

Step 2: Test associativity Take $a=b=c=1$. First $1*1=1/2$, an admissible intermediate value. Evaluate the two bracketings separately:

$$(1*1)*1=\frac{1}{2}*1=\frac{1/2}{1+1}=\frac{1}{4},$$
$$1*(1*1)=1*\frac{1}{2}=\frac{1}{1/2+1}=\frac{2}{3}.$$

Thus the bracketings give unequal results, even though every operation in this calculation is defined.

Step 3: Check the stated domain Strictly, the formula is not closed on the stated set: $(-2)*1=-1$, which lies outside it. Thus its description as a binary operation on that set has a closure defect. Nevertheless, the admissible counterexamples above directly establish both requested failures for the given rule.

Step 4: Conclusion The rule is neither commutative nor associative: $1*2\neq 2*1$ and $(1*1)*1\neq 1*(1*1)$.

Quick Tip to Solve: Use $(1,2)$ for commutativity and $(1,1,1)$ for associativity; their intermediate values avoid the excluded input -1.

6. If $A = \begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix}$, then show that $A^3 = A$. [2 marks]

Solution:

Concept:

- Matrix multiplication is computed by taking the dot product of a row of the first matrix with a column of the second.
- For a square matrix, $A^3 = A^2A$. If $A^2 = A$, the matrix is idempotent and multiplying once more reproduces it.

Step 1: Compute the square Use ordinary matrix multiplication, keeping the negative signs in the first column:

$$\begin{aligned} A^2 &= \begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 9-12 & -18+24 \\ 6-8 & -12+16 \end{bmatrix} \\ &= \begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix} = A. \end{aligned}$$

For example, the top-left entry is $(-3)(-3) + 6(-2)$, not the square of the top-left entry alone.

Step 2: Use the square to find the cube Multiply the identity $A^2 = A$ on the right by A. By associativity of matrix multiplication,

$$A^3 = A^2A = AA = A^2.$$

The square has already been evaluated, so no new four-entry multiplication is needed. This argument does not require cancelling A or assuming that it is invertible.

Step 3: Conclusion

$$\boxed{A^3 = A = \begin{bmatrix} -3 & 6 \\ -2 & 4 \end{bmatrix}}.$$

Hence the required identity is proved.

Quick Tip to Solve: Calculate A^2 first; once it equals A, replace A^2 by A in $A^3 = A^2A$.

7. Form the differential equation representing the family of curves $y^2 = m(a^2 - x^2)$ by eliminating the arbitrary constants m and a. [2 marks]

Solution:

Concept:

- To form a differential equation for a family with arbitrary constants, differentiate and eliminate those constants from the resulting relations.
- Write $y\prime = dy/dx$ and $y\prime\prime = d^2y/dx^2$. The product rule gives $d(yy\prime)/dx = (y\prime)^2 + yy\prime\prime$.

Quick Tip to Solve: After the first derivative removes a, differentiate $yy\prime$ with the product rule and eliminate m by multiplying the second relation by x.

Step 1: Differentiate to eliminate the constant term In $y^2 = m(a^2 - x^2)$, both m and a are constants. Differentiating with respect to x gives

$$2yy\prime = -2mx, \qquad yy\prime = -mx.$$

The constant term ma^2 disappears, so the first differentiated equation no longer contains a.

Step 2: Differentiate again and eliminate m Differentiate $yy\prime = -mx$ once more:

$$(y\prime)^2 + yy\prime\prime = -m.$$

Multiply this identity by x, then use $-mx = yy\prime$ from the preceding step:

$$x\big((y\prime)^2 + yy\prime\prime\big) = yy\prime.$$

This elimination avoids dividing by x, and neither arbitrary constant remains.

Step 3: Conclusion The required second-order differential equation is

$$\boxed{x\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] - y\frac{dy}{dx} = 0}.$$

Quick Tip to Solve: After the first derivative removes a, differentiate $yy\prime$ with the product rule and eliminate m by multiplying the second relation by x.

8. Find: $\displaystyle\int \frac{\sin x - \cos x}{\sqrt{1 + \sin 2x}}\,dx,\ 0 < x < \pi/2.$ [2 marks]

Solution:

Concept:

- The identity $1 + \sin 2x = (\sin x + \cos x)^2$ converts the radical into an absolute value.
- For a nonzero differentiable function $u(x)$, $\int u'(x)/u(x)\,dx = \log|u(x)| + C$, where C is an arbitrary constant.

Step 1: Simplify the denominator using the given interval Since $0 < x < \pi/2$, both $\sin x$ and $\cos x$ are positive. Therefore the positive square root is

$$\sqrt{1 + \sin 2x} = \sqrt{(\sin x + \cos x)^2} = \sin x + \cos x.$$

The domain restriction is essential: a square root of a square is generally an absolute value, not the signed expression itself.

Step 2: Substitute the sum of sine and cosine Put $u = \sin x + \cos x$. Then $du = (\cos x - \sin x)\,dx$,

Quick Tip to Solve: Turn $1 + \sin 2x$ into a perfect square first, and use the interval to choose the positive sign before substituting.

which is the negative of the numerator times dx. Thus

$$I = \int \frac{\sin x - \cos x}{\sin x + \cos x}\, dx$$
$$= -\int \frac{du}{u} = -\log u + C.$$

The logarithm needs no absolute-value signs here because $u > 0$. Differentiating the obtained expression gives $-(\cos x - \sin x)/(\sin x + \cos x)$, the required integrand.

Step 3: Conclusion The required integral is $\boxed{-\log(\sin x + \cos x) + C}$ for $0 < x < \pi/2$.

Quick Tip to Solve: Turn $1 + \sin 2x$ into a perfect square first, and use the interval to choose the positive sign before substituting.

9(a). Find: $\int \frac{\sin(x-a)}{\sin(x+a)}\, dx$. [2 marks]

Solution:

Concept:

- The sine difference formula is $\sin(u - v) = \sin u \cos v - \cos u \sin v$; the parameter a is constant.
- Since $\cot t = \cos t / \sin t$, its antiderivative is $\log|\sin t|$ on any interval where $\sin t \neq 0$.

Step 1: Express the numerator in terms of the denominator's angle To introduce $x + a$ into the numerator, write $x - a = (x + a) - 2a$. Applying the sine difference formula gives

$$\sin(x - a) = \sin(x + a)\cos 2a - \cos(x + a)\sin 2a.$$

Divide by $\sin(x + a)$, which must be nonzero on the interval of integration:

$$\frac{\sin(x-a)}{\sin(x+a)} = \cos 2a - \sin 2a \cot(x + a).$$

Step 2: Integrate the constant and cotangent terms Both $\cos 2a$ and $\sin 2a$ are constants with respect to x. Also $d(x + a)/dx = 1$, so there is no additional substitution factor. Therefore

$$I = x\cos 2a - \sin 2a \log|\sin(x + a)| + C.$$

As a check, differentiating gives $\cos 2a - \sin 2a \cot(x + a)$, which is exactly the rewritten integrand.

Step 3: Conclusion $\boxed{\int \frac{\sin(x-a)}{\sin(x+a)}\, dx = x\cos 2a - \sin 2a \log|\sin(x + a)| + C}$, on intervals where $\sin(x + a) \neq 0$.

Quick Tip to Solve: Rewrite $x - a$ as $(x + a) - 2a$; this makes the denominator cancel in one term and leaves an integrable cotangent in the other.

OR

9(b). Find: $\int (\log x)^2\,dx$. [2 marks]

Solution:

Concept:

- Integration by parts gives $\int u\,dv = uv - \int v\,du$; select the logarithmic factor as u to lower its power on differentiation.
- Here $\log x$ denotes the natural logarithm, with $d(\log x)/dx = 1/x$ and real domain $x > 0$.

Step 1: Apply integration by parts to the squared logarithm Set $u = (\log x)^2$ and $dv = dx$. Then $du = 2\log x\,dx/x$ and $v = x$. Hence

$$I = x(\log x)^2 - \int x\frac{2\log x}{x}\,dx$$
$$= x(\log x)^2 - 2\int \log x\,dx.$$

The remaining integral has a lower power of the logarithm, so the same method now completes the calculation.

Step 2: Evaluate the remaining logarithmic integral For $J = \int \log x\,dx$, take $u = \log x$ and $dv = dx$ to obtain

$$J = x\log x - \int x\frac{1}{x}\,dx = x\log x - x.$$

Substituting this into the first expression gives

$$I = x(\log x)^2 - 2x\log x + 2x + C.$$

Differentiation checks the result: the extra terms $2\log x$, $-2\log x - 2$ and 2 cancel, leaving only $(\log x)^2$. A single arbitrary constant C includes all constants of integration.

Step 3: Conclusion $\boxed{\int (\log x)^2\,dx = x\left[(\log x)^2 - 2\log x + 2\right] + C}$, for $x > 0$.

Quick Tip to Solve: Use $(\log x)^2$ as the first factor in integration by parts, then repeat once for $\log x$; the power drops at each application.

10. Mother, father and son line up at random for a family photo. If A and B are two events given by A = Son on one end, B = Father in the middle, find $P(B/A)$. [2 marks]

Solution:

Concept:

- For events A and B with $P(A) > 0$, the conditional probability is $P(B \mid A) = P(A \cap B)/P(A)$.
- A random ordering of three distinct people has $3! = 6$ equally likely outcomes. Conditioning restricts the sample space to the outcomes in A.

Quick Tip to Solve: Once the son is fixed at either end, the two remaining positions are equally likely to place the father in the middle or at the other end.

Step 1: List the arrangements in the conditioning event Denote mother, father and son by M, F, S, respectively, with each ordering read from left to right. The six possible arrangements are

$$\Omega = \{MFS, MSF, FMS, FSM, SMF, SFM\}.$$

The son is at one end in four of these arrangements:

$$A = \{MFS, FMS, SMF, SFM\}, \qquad P(A) = \frac{4}{6}.$$

These four arrangements remain equally likely after we are told that A occurred.

Step 2: Count the arrangements with the father in the middle The father occupies the middle position only in MFS and SFM. Both also have the son at an end, so

$$B = \{MFS, SFM\}, \qquad A \cap B = B, \qquad P(A \cap B) = \frac{2}{6}.$$

Apply the conditional probability formula:

$$P(B \mid A) = \frac{2/6}{4/6} = \frac{2}{4}.$$

The denominator counts the four arrangements compatible with the given information, rather than all six unrestricted arrangements.

Step 3: Conclusion The required probability is $\boxed{P(B/A) = \frac{1}{2}}$.

Quick Tip to Solve: Once the son is fixed at either end, the two remaining positions are equally likely to place the father in the middle or at the other end.

11(a). **Let X be a random variable which assumes values x_1, x_2, x_3, x_4 such that $2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4)$. Find the probability distribution of X.** [2 marks]

Solution:

Concept:

- A discrete probability distribution assigns a nonnegative probability to every possible value of X, and these probabilities sum to 1.
- When multiples of probabilities are equal, set their common value to k; the probabilities are then inversely proportional to their respective multipliers.

Step 1: Express all four probabilities using a common value Write $p_i = P(X = x_i)$, for $i = 1, 2, 3, 4$. The given relation is $2p_1 = 3p_2 = p_3 = 5p_4$. Let this common value be k. Dividing each equality by its multiplier gives

$$p_1 = \frac{k}{2}, \qquad p_2 = \frac{k}{3}, \qquad p_3 = k, \qquad p_4 = \frac{k}{5}.$$

In particular, the probabilities are not in the ratio $2 : 3 : 1 : 5$; the multipliers occur in the denominators.

Step 2: Determine the common value by normalisation Since X assumes exactly these four values,

Quick Tip to Solve: Take the reciprocal ratio $1/2 : 1/3 : 1 : 1/5 = 15 : 10 : 30 : 6$, then divide each entry by the sum 61.

their probabilities must add to one. Therefore

$$\frac{k}{2}+\frac{k}{3}+k+\frac{k}{5}=1,$$
$$\frac{15k+10k+30k+6k}{30}=1,$$
$$\frac{61k}{30}=1, \qquad k=\frac{30}{61}.$$

Substituting into the four expressions yields numerators $15, 10, 30, 6$ over the common denominator 61. All are positive and their sum is $61/61 = 1$. Moreover, each prescribed multiple equals $30/61$, so the given relation is satisfied.

Step 3: Conclusion The probability distribution is

X	x_1	x_2	x_3	x_4
$P(X)$	$15/61$	$10/61$	$30/61$	$6/61$

.

Quick Tip to Solve: Take the reciprocal ratio $1/2 : 1/3 : 1 : 1/5 = 15 : 10 : 30 : 6$, then divide each entry by the sum 61.

OR

11(b). A coin is tossed 5 times. Find the probability of getting [2 marks]

11(b)(i). at least 4 heads, and

Solution:

Concept:

- For five independent tosses of a fair coin, the number of heads X has a binomial distribution with $n = 5$ and $p = \frac{1}{2}$.
- The probability of exactly r heads is $P(X = r) = \binom{5}{r}(\frac{1}{2})^r(\frac{1}{2})^{5-r} = \binom{5}{r}/32$.

Step 1: Identify the required outcomes At least four heads means either exactly four heads or exactly five heads. These events are mutually exclusive, so their probabilities are added:

$$P(X \geq 4) = P(X = 4) + P(X = 5).$$

Step 2: Apply the binomial formula There are $\binom{5}{4} = 5$ ways to choose the four head positions and $\binom{5}{5} = 1$ way to obtain five heads. Each particular sequence of five tosses has probability $(\frac{1}{2})^5$. Therefore,

$$P(X \geq 4) = \frac{\binom{5}{4}+\binom{5}{5}}{32} = \frac{5+1}{32} = \frac{6}{32}.$$

Step 3: Conclusion The probability of getting at least four heads is $\boxed{\frac{3}{16}}$.

Quick Tip to Solve: Translate "at least four" into the two counts 4 and 5 before adding their binomial probabilities.

11(b)(ii). at most 4 heads.

Solution:

Concept:

- Let X denote the number of heads in five independent tosses of a fair coin. Its possible values are $0, 1, 2, 3, 4, 5$.
- The complement rule states that $P(E) = 1 - P(E^c)$. The probability of a specified sequence of five tosses is $(\frac{1}{2})^5$.

Step 1: Express the event using its complement At most four heads includes zero, one, two, three or four heads. The only excluded possibility is five heads. Thus the complement of $X \leq 4$ is $X = 5$, and

$$P(X \leq 4) = 1 - P(X = 5).$$

Step 2: Calculate the excluded probability Five heads requires a head on every toss, giving the single sequence HHHHH. Independence allows the five probabilities to be multiplied:

$$P(X = 5) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

Substitution in the complement rule gives $P(X \leq 4) = 1 - \frac{1}{32}$.

Step 3: Conclusion The probability of getting at most four heads is $\boxed{\frac{31}{32}}$.

Quick Tip to Solve: For at most four heads in five tosses, subtract the single all-heads outcome from the total probability.

12(a). Find a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$, where $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$. [2 marks]

Solution:

Concept:

- The cross product $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$, provided this cross product is nonzero.
- A nonzero vector $\vec{v}$ is made a unit vector by dividing it by its magnitude: $\hat{n} = \vec{v}/|\vec{v}|$. Its negative is also a unit vector in the opposite direction.

Step 1: Calculate a vector perpendicular to both Use the components in the printed order to expand

Quick Tip to Solve: Both given vectors have opposite $\hat{j}$ and $\hat{k}$ coefficients, so try $\hat{j} + \hat{k}$ as a perpendicular direction, then divide by $\sqrt{2}$.

the determinant:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$
$$= \hat{i}(-14+14) - \hat{j}(2-21) + \hat{k}(-2+21)$$
$$= 19\hat{j} + 19\hat{k}.$$

The cross product is nonzero, so the two vectors are not parallel and it gives the required perpendicular direction.

Step 2: Normalize and check the direction Its magnitude is

$$|\vec{a} \times \vec{b}| = \sqrt{0^2 + 19^2 + 19^2} = 19\sqrt{2}.$$

Dividing each component by $19\sqrt{2}$ gives components $0, 1/\sqrt{2}, 1/\sqrt{2}$. Their squared sum is $1/2+1/2 = 1$. The dot products with the original vectors are $(-7+7)/\sqrt{2} = 0$ and $(-2+2)/\sqrt{2} = 0$, confirming perpendicularity to both.

Step 3: Conclusion One required unit vector is $\boxed{(\hat{j}+\hat{k})/\sqrt{2}}$. Its negative, $-(\hat{j}+\hat{k})/\sqrt{2}$, is equally valid.

Quick Tip to Solve: Both given vectors have opposite $\hat{j}$ and $\hat{k}$ coefficients, so try $\hat{j}+\hat{k}$ as a perpendicular direction, then divide by $\sqrt{2}$.

OR

12(b). Show that the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\hat{i} - 3\hat{j} + 5\hat{k}$ are coplanar. [2 marks]

Solution:

Concept:

- Three vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar if and only if their scalar triple product $[\vec{a}\ \vec{b}\ \vec{c}]$ is zero.
- The scalar triple product equals the determinant formed by the three component rows. Its absolute value is the volume of the parallelepiped determined by the vectors.

Step 1: Form the scalar triple product Denote the given vectors in order by $\vec{a}, \vec{b}, \vec{c}$. Writing their components along $\hat{i}, \hat{j}, \hat{k}$ gives

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}.$$

We evaluate this determinant to test whether the three directions lie in one plane through the origin.

Step 2: Expand along the first row The cofactor signs in the first row are $+, -, +$; therefore the

Quick Tip to Solve: As a quick check, the third vector is $3\vec{a} + \vec{b}$; a linear combination of the first two must lie in their plane.

middle contribution is positive because its entry is -2:

$$\begin{aligned}[\vec{a}\ \vec{b}\ \vec{c}] &= 1(15-12)+2(-10+4)+3(6-3)\\ &= 3-12+9\\ &= 0.\end{aligned}$$

Thus the volume determined by the three vectors is zero, which is precisely the scalar-triple-product test for coplanarity.

Step 3: Conclusion Since $[\vec{a}\ \vec{b}\ \vec{c}]=0$, the three given vectors are $\boxed{\text{coplanar}}$.

Quick Tip to Solve: As a quick check, the third vector is $3\vec{a}+\vec{b}$; a linear combination of the first two must lie in their plane.

Section C (44 marks)

Questions number 13 to 23 carry 4 marks each.

13(a). Show that the relation R on the set $\mathbb{Z}$ of all integers, given by $R=\{(a,b) : 2$ divides $(a-b)\}$ is an equivalence relation. [4 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation when it is reflexive, symmetric and transitive. Each property must hold for arbitrary elements of that set.
- For integers a, b, the notation $2 \mid (a-b)$ means that $a-b=2k$ for some integer k. Thus aRb means precisely that their difference is an even integer.

Step 1: Prove reflexivity on the integers Take any $a\in\mathbb{Z}$. To prove reflexivity, we must verify $(a,a)\in R$. The difference of the two entries is

$$a-a=0=2\cdot 0.$$

Since $0\in\mathbb{Z}$, the integer 2 divides this difference. Hence $(a,a)\in R$ for every integer a, so R is reflexive. This uses the fact that zero is divisible by every nonzero integer.

Step 2: Prove symmetry by reversing the difference Let $a,b\in\mathbb{Z}$ and suppose $(a,b)\in R$. By the definition of R, there is an integer k such that $a-b=2k$. Reversing the order gives

$$b-a=-(a-b)=-2k=2(-k).$$

The number $-k$ is also an integer, so $2\mid(b-a)$, which means $(b,a)\in R$. Therefore R is symmetric. No assumption about whether a or b is larger is needed.

Step 3: Prove transitivity by adding the differences Let $a,b,c\in\mathbb{Z}$, with $(a,b)\in R$ and $(b,c)\in R$. Then there are integers m,n satisfying

$$a-b=2m,\qquad b-c=2n.$$

Quick Tip to Solve: For divisibility relations, use a zero difference for reflexivity, negate the difference for symmetry, and add two differences for transitivity.

Adding these equalities eliminates the common middle element b:

$$\begin{aligned} a - c &= (a - b) + (b - c) \\ &= 2m + 2n = 2(m + n). \end{aligned}$$

Since integers are closed under addition, $m + n \in \mathbb{Z}$. Consequently $2 \mid (a - c)$, so $(a, c) \in R$. This establishes transitivity for every such triple of integers.

Step 4: Conclusion The relation R is reflexive, symmetric and transitive on $\mathbb{Z}$. Hence $\boxed{R \text{ is an equivalence relation on } \mathbb{Z}}$.

Quick Tip to Solve: For divisibility relations, use a zero difference for reflexivity, negate the difference for symmetry, and add two differences for transitivity.

OR

13(b). If $f(x) = \dfrac{4x+3}{6x-4}$, $x \neq \dfrac{2}{3}$, show that $fof(x) = x$ for all $x \neq \dfrac{2}{3}$. Also, find the inverse of f. [4 marks]

Solution:

Concept:

- Composition means substituting $f(x)$ into f; both applications must be defined.
- A function satisfying $f \circ f = I$ on its domain is its own inverse, provided it maps that domain into itself. Here $I(x) = x$ is the identity function.

Step 1: Check the domain of the composition Put $D = \mathbb{R} \setminus \{2/3\}$. For $x \in D$, the denominator is nonzero. Check that $f(x)$ cannot equal the excluded input $2/3$:

$$\frac{4x+3}{6x-4} = \frac{2}{3} \quad \Longrightarrow \quad 12x + 9 = 12x - 8,$$

which is impossible. Thus $f(x) \in D$, so the second application of f is legitimate for every allowed x.

Step 2: Substitute the function into itself Replace each input in the rule by $f(x)$, and combine the numerator and denominator separately:

$$\begin{aligned} f(f(x)) &= \frac{4\left(\frac{4x+3}{6x-4}\right) + 3}{6\left(\frac{4x+3}{6x-4}\right) - 4} \\ &= \frac{\dfrac{16x + 12 + 18x - 12}{6x - 4}}{\dfrac{24x + 18 - 24x + 16}{6x - 4}} \\ &= \frac{34x/(6x-4)}{34/(6x-4)} = x. \end{aligned}$$

Both denominators are nonzero because $x \neq 2/3$.

Quick Tip to Solve: In this composition, simplify the outer numerator and denominator separately: they become $34x/(6x-4)$ and $34/(6x-4)$, making the cancellation immediate.

Step 3: Solve for the inverse and verify its domain To obtain the inverse explicitly, set $y = f(x)$ and isolate x:

$$y(6x-4) = 4x+3,$$
$$(6y-4)x = 4y+3,$$
$$x = \frac{4y+3}{6y-4}.$$

For any $y \in D$, taking $x = f(y)$ gives $f(x) = y$, proving surjectivity onto D. If $f(u) = f(v)$, applying f gives $u = v$, proving injectivity. Thus the inverse exists on D.

Step 4: Conclusion For every $x \neq 2/3$, $(f \circ f)(x) = x$. Moreover,

$$\boxed{f^{-1}(x) = \frac{4x+3}{6x-4} = f(x), \qquad x \neq \frac{2}{3}.}$$

Quick Tip to Solve: In this composition, simplify the outer numerator and denominator separately: they become $34x/(6x-4)$ and $34/(6x-4)$, making the cancellation immediate.

14(a). **If** $\sin y = x\sin(a+y)$**, prove that** $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$**.** [4 marks]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x, while a is a constant. Thus $\dfrac{d}{dx}\sin y = \cos y \dfrac{dy}{dx}$.
- The product rule gives $\dfrac{d}{dx}[x\sin(a+y)] = \sin(a+y) + x\cos(a+y)\dfrac{dy}{dx}$.
- Use the subtraction identity $\sin U\cos V - \cos U\sin V = \sin(U-V)$ to simplify the resulting denominator.

Step 1: Differentiate the given relation Differentiate $\sin y = x\sin(a+y)$ with respect to x, using the chain rule on the left and the product rule on the right:

$$\cos y \frac{dy}{dx} = \sin(a+y) + x\cos(a+y)\frac{dy}{dx}.$$

Collecting the terms containing the derivative on the left gives

$$[\cos y - x\cos(a+y)]\frac{dy}{dx} = \sin(a+y).$$

Step 2: Simplify the coefficient using the original equation The claimed expression is defined where $\sin a \neq 0$. Multiply the collected equation by $\sin(a+y)$. In the resulting coefficient, substitute

Quick Tip to Solve: After differentiating, multiply by $\sin(a+y)$ and replace $x\sin(a+y)$ by $\sin y$; the coefficient becomes $\sin((a+y)-y) = \sin a$.

$x\sin(a+y)=\sin y$:

$$[\cos y\sin(a+y)-x\cos(a+y)\sin(a+y)]\frac{dy}{dx}$$
$$=\sin^2(a+y),$$
$$[\cos y\sin(a+y)-\cos(a+y)\sin y]\frac{dy}{dx}$$
$$=\sin^2(a+y).$$

Step 3: Apply the angle subtraction identity The bracket is exactly a sine difference:

$$\cos y\sin(a+y)-\cos(a+y)\sin y=\sin a.$$

Hence the differentiated equation reduces to

$$\sin a\,\frac{dy}{dx}=\sin^2(a+y).$$

Divide by the nonzero constant $\sin a$ to obtain the claimed derivative.

Step 4: Conclusion Therefore, wherever the stated expression is defined,

$$\boxed{\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}},\qquad \sin a\neq 0.$$

Quick Tip to Solve: After differentiating, multiply by $\sin(a+y)$ and replace $x\sin(a+y)$ by $\sin y$; the coefficient becomes $\sin((a+y)-y)=\sin a$.

OR

14(b). If $(\sin x)^y=x+y$, find $\frac{dy}{dx}$. [4 marks]

Solution:

Concept:

- Logarithmic differentiation handles a variable base raised to a variable power: if $u(x)>0$, then $\log(u^y)=y\log u$, where $y=y(x)$.
- Implicit differentiation treats y as a function of x. Thus $\frac{d}{dx}\log(x+y)=\frac{1+y'}{x+y}$, where $y'=\frac{dy}{dx}$.
- The product and chain rules give $\frac{d}{dx}[y\log(\sin x)]=y'\log(\sin x)+y\cot x$. Here $\log$ denotes the natural logarithm.

Step 1: Take logarithms of the given relation The given relation contains the variable y both in the exponent and on the right-hand side. Taking logarithms first brings the exponent down as a product. On the real differentiable branch with $\sin x>0$, the original equation also gives $x+y=(\sin x)^y>0$,

Quick Tip to Solve: Take logarithms before differentiating $(\sin x)^y$; include y' both in the product rule and in the derivative of $\log(x+y)$.

so both logarithms are defined. Hence

$$(\sin x)^y = x + y,$$
$$y \log(\sin x) = \log(x + y).$$

Step 2: Differentiate both sides with respect to x Apply the product rule to the left side, remembering that the exponent y varies with x. On the right side, the derivative of $x + y$ is $1 + y'$, rather than just 1. Therefore

$$y' \log(\sin x) + y\frac{\cos x}{\sin x} = \frac{1 + y'}{x + y}.$$

Using $\cos x / \sin x = \cot x$ and multiplying by the positive quantity $x + y$, we obtain

$$(x + y)y' \log(\sin x) + y(x + y) \cot x = 1 + y'.$$

Step 3: Collect the terms containing the derivative Move the term y' on the right to the left, and move the term that does not contain the derivative to the right. Factor out the common y':

$$\big[(x + y) \log(\sin x) - 1\big]y' = 1 - y(x + y) \cot x.$$

For $0 < \sin x \leq 1$, we have $\log(\sin x) \leq 0$. Since $x + y > 0$, the coefficient in square brackets is at most -1, so division by this coefficient is valid on the branch under consideration.

Step 4: Conclusion The required derivative is

$$\boxed{\frac{dy}{dx} = \frac{1 - y(x + y) \cot x}{(x + y) \log(\sin x) - 1}}.$$

Quick Tip to Solve: Take logarithms before differentiating $(\sin x)^y$; include y' both in the product rule and in the derivative of $\log(x + y)$.

15. If $\sin^{-1}\left(\frac{3}{x}\right) + \sin^{-1}\left(\frac{4}{x}\right) = \frac{\pi}{2}$, then find the value of x. [4 marks]

Solution:

Concept:

- The principal value $\sin^{-1} t$ exists for $-1 \leq t \leq 1$ and lies in $[-\pi/2, \pi/2]$. Its sign is the same as the sign of t.
- If two acute angles A and B satisfy $A + B = \pi/2$, then $\sin B = \cos A$, with $\cos A = \sqrt{1 - \sin^2 A}$.
- After squaring an equation, check the sign restrictions and substitute the candidate into the original inverse-trigonometric relation to exclude extraneous roots.

Step 1: Determine the permitted values of x Both inverse sine expressions must be real. Their

Quick Tip to Solve: First reject negative x using the signs of the principal inverse sine values. Complementary angles then give $3^2 + 4^2 = x^2$, with the positive root required.

arguments therefore satisfy

$$\left|\frac{3}{x}\right| \le 1, \qquad \left|\frac{4}{x}\right| \le 1,$$

giving $|x| \ge 4$. If $x \le -4$, both arguments are negative, so both principal inverse sine values are negative. Their sum cannot equal the positive number $\pi/2$. Hence any solution must have $x \ge 4$. At $x = 4$, the second angle is already $\pi/2$ and the first is positive, so equality is impossible there; any solution has $x > 4$.

Step 2: Use the complementary-angle relation Put $A = \sin^{-1}(3/x)$ and $B = \sin^{-1}(4/x)$. Because $x > 4$, both angles are acute. The given equation states $B = \pi/2 - A$, so taking sine gives

$$\frac{4}{x} = \sin(\pi/2 - A) = \cos A,$$
$$\frac{4}{x} = \sqrt{1 - \frac{9}{x^2}}.$$

The positive square root is required because A is acute. Both sides are nonnegative, so we may square and multiply by x^2:

$$16 = x^2 - 9, \qquad x^2 = 25.$$

The restriction $x > 4$ leaves the positive candidate 5; the negative algebraic root is inadmissible.

Step 3: Verify the candidate in the principal ranges For the candidate, let $A = \sin^{-1}(3/5)$. This is an acute angle and

$$\cos A = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}.$$

Therefore $\sin(\pi/2 - A) = 4/5$. The angle $\pi/2 - A$ is also acute and belongs to the principal range of inverse sine, which gives

$$\sin^{-1}\left(\frac{4}{5}\right) = \frac{\pi}{2} - A.$$

Adding $A = \sin^{-1}(3/5)$ verifies exactly the required sum. Thus the candidate satisfies the original equation, not merely its squared form.

Step 4: Conclusion The unique value satisfying the given equation is $\boxed{x = 5}$.

Quick Tip to Solve: First reject negative x using the signs of the principal inverse sine values. Complementary angles then give $3^2 + 4^2 = x^2$, with the positive root required.

16. Using properties of determinants, prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$. [4 marks]

Solution:

Concept:

- A determinant is linear in each column separately: a column written as a sum splits the determinant into a sum, while a scalar factor in one column can be taken outside.

Quick Tip to Solve: Write the columns as identity columns plus multiples of $(a, b, c)^T$. Every expansion term containing two copies of that vector vanishes immediately.

- A determinant with two proportional columns is zero. The determinant of the identity matrix is 1.

Step 1: Express each column as a sum Let the given determinant be D. Introduce the column vectors $v = (a, b, c)^T$, $e_1 = (1, 0, 0)^T$, $e_2 = (0, 1, 0)^T$ and $e_3 = (0, 0, 1)^T$. The three columns of the given determinant are respectively $e_1 + av$, $e_2 + bv$ and $e_3 + cv$. Using vertical bars to denote the determinant formed from these columns,

$$D = |e_1 + av,\ e_2 + bv,\ e_3 + cv|.$$

Step 2: Split the columns using linearity Expanding the sum in each column gives eight determinants. Four of these contain at least two columns chosen from av, bv, cv; those columns are proportional to the same vector v, so all four determinants vanish. Thus only the identity determinant and the terms with exactly one such column remain:

$$\begin{aligned} D = {} & |e_1, e_2, e_3| + a|v, e_2, e_3| \\ & + b|e_1, v, e_3| + c|e_1, e_2, v|. \end{aligned}$$

For example, the term $ab|v, v, e_3|$ is zero because its first two columns are identical; the other discarded terms vanish for the same reason.

Step 3: Evaluate the surviving determinants The first determinant is 1. In each of the other three, expand along the two identity columns. Only the indicated component of v survives, with a positive cofactor sign:

$$\begin{vmatrix} a & 0 & 0 \\ b & 1 & 0 \\ c & 0 & 1 \end{vmatrix} = a, \qquad \begin{vmatrix} 1 & a & 0 \\ 0 & b & 0 \\ 0 & c & 1 \end{vmatrix} = b, \qquad \begin{vmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & c \end{vmatrix} = c.$$

The outside factors a, b, c multiply these three values respectively. No division by these variables was used, so zero values are also included.

Step 4: Conclusion Therefore, using the properties of determinants,

$$D = 1 + a(a) + b(b) + c(c) = \boxed{1 + a^2 + b^2 + c^2}.$$

Hence the required identity is proved for all real a, b, c.

Quick Tip to Solve: Write the columns as identity columns plus multiples of $(a, b, c)^T$. Every expansion term containing two copies of that vector vanishes immediately.

17. If $y = (\cot^{-1} x)^2$, show that $(x^2 + 1)^2 \dfrac{d^2y}{dx^2} + 2x(x^2 + 1) \dfrac{dy}{dx} = 2$. [4 marks]

Solution:

Concept:

- For the principal inverse cotangent, $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1 + x^2}$, valid for every real x.
- The chain rule differentiates a composite function, while the product rule gives $\dfrac{d}{dx}(uv) = u'v + uv'$.

Quick Tip to Solve: After the first differentiation, multiply by $1 + x^2$. Differentiating this product produces both required derivative terms directly.

Here y' and y'' denote the first and second derivatives of y.

Step 1: Differentiate the given square Start with $y = (\cot^{-1} x)^2$. Treat the inverse cotangent as the inner function and the square as the outer function. By the chain rule,

$$\begin{aligned} y' &= 2\cot^{-1} x \, \frac{d}{dx}(\cot^{-1} x) \\ &= 2\cot^{-1} x \left(-\frac{1}{1+x^2}\right) \\ &= -\frac{2\cot^{-1} x}{1+x^2}. \end{aligned}$$

The negative sign is essential because inverse cotangent is a decreasing function. Also, $1 + x^2 > 0$ for every real x, so this derivative has no real singularity.

Step 2: Remove the denominator before differentiating again Multiply the first derivative equation by $1 + x^2$, obtaining

$$(1+x^2)y' = -2\cot^{-1} x.$$

Now differentiate both sides with respect to x. The left side is a product of two functions, so it requires both product-rule terms:

$$\begin{aligned} \frac{d}{dx}\left[(1+x^2)y'\right] &= (1+x^2)y'' + 2xy', \\ \frac{d}{dx}(-2\cot^{-1} x) &= -2\left(-\frac{1}{1+x^2}\right). \end{aligned}$$

Equating the two derivatives gives

$$(1+x^2)y'' + 2xy' = \frac{2}{1+x^2}.$$

Multiplication by $1 + x^2$ now produces exactly the coefficients of the derivatives specified in the question, without needing a lengthy quotient-rule expansion for y''.

Step 3: Conclusion Multiplying the last identity by $1 + x^2$, we obtain

$$\boxed{(x^2+1)^2\frac{d^2y}{dx^2} + 2x(x^2+1)\frac{dy}{dx} = 2}.$$

Hence the required differential identity holds for every real x.

Quick Tip to Solve: After the first differentiation, multiply by $1 + x^2$. Differentiating this product produces both required derivative terms directly.

18. Find: $\displaystyle\int \frac{\sin 2x}{(\sin^2 x+1)(\sin^2 x+3)}\,dx$. [4 marks]

Solution:

Concept:

- Choose a substitution whose differential occurs in the numerator: $\frac{d}{dx}(\sin^2 x) = 2\sin x\cos x = \sin 2x$.
- A rational expression with distinct linear factors can be separated into partial fractions, and $\int \frac{du}{u+k} = \ln|u+k| + C$.

Step 1: Substitute for the repeated expression Denote the given integral by I. Both denominator factors contain $\sin^2 x$, while its derivative is exactly the numerator. Put

$$t = \sin^2 x, \qquad dt = \sin 2x\,dx.$$

The integral therefore becomes the rational integral

$$I = \int \frac{dt}{(t+1)(t+3)}.$$

No additional factor of two is needed because it is already included in $\sin 2x$.

Step 2: Determine the partial fractions Write

$$\frac{1}{(t+1)(t+3)} = \frac{A}{t+1} + \frac{B}{t+3}.$$

Multiplying by $(t+1)(t+3)$ gives the polynomial identity

$$1 = A(t+3) + B(t+1) = (A+B)t + 3A + B.$$

Comparing coefficients yields $A+B=0$ and $3A+B=1$. Subtracting these equations gives $2A=1$, so $A=\frac{1}{2}$ and $B=-\frac{1}{2}$.

Step 3: Integrate the two simple terms Substitute these constants and integrate term by term:

$$\begin{aligned} I &= \frac{1}{2}\int\left(\frac{1}{t+1} - \frac{1}{t+3}\right)dt \\ &= \frac{1}{2}\ln|t+1| - \frac{1}{2}\ln|t+3| + C. \end{aligned}$$

Since $t = \sin^2 x \geq 0$, both logarithm arguments are positive for every real x. Thus the absolute-value signs may be omitted when restoring x.

Step 4: Conclusion

$$\boxed{I = \frac{1}{2}\ln\left(\frac{\sin^2 x+1}{\sin^2 x+3}\right) + C.}$$

Here C is an arbitrary constant of integration.

Quick Tip to Solve: When the numerator is $\sin 2x$ and the denominator depends on $\sin^2 x$, try $t = \sin^2 x$ first; the denominator constants differ by 2, giving coefficients $1/2$ and $-1/2$.

19. Prove that $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ and hence evaluate $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$. [4 marks]

Solution:

Concept:

- Substitution in a definite integral changes both its integrand and limits. Reversing the limits changes its sign.
- Complementary angles satisfy $\tan(\pi/2 - x) = \cot x$. For $\tan x > 0$, $\sqrt{\cot x} = 1/\sqrt{\tan x}$.

Step 1: Prove the reflection property Suppose f is integrable on $[a, b]$. Put $u = a + b - x$, so $x = a + b - u$ and $dx = -du$. The limits $x = a, b$ become $u = b, a$, respectively. Thus

$$\begin{aligned}\int_a^b f(x)\,dx &= -\int_b^a f(a+b-u)\,du \\ &= \int_a^b f(a+b-u)\,du.\end{aligned}$$

Replacing the dummy variable u by x proves the stated identity: the integral equals $\int_a^b f(a+b-x)\,dx$.

Step 2: Reflect the required integrand Denote the required integral by I. The sum of its limits is $\pi/6 + \pi/3 = \pi/2$, so the proved property gives

$$\begin{aligned}I &= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan(\pi/2-x)}} \\ &= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}} \\ &= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\,dx.\end{aligned}$$

All angles in the interval are in the first quadrant. Hence $\tan x > 0$, and multiplying by $\sqrt{\tan x}$ in the last step is valid; the denominators never vanish.

Step 3: Add the original and reflected forms Add the last integral to the original expression for I. The limits are identical, and the two numerators sum to their common denominator:

$$\begin{aligned}2I &= \int_{\pi/6}^{\pi/3} \frac{1+\sqrt{\tan x}}{1+\sqrt{\tan x}}\,dx \\ &= \int_{\pi/6}^{\pi/3} 1\,dx = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}.\end{aligned}$$

Step 4: Conclusion The identity is proved, and division by 2 gives

$$\boxed{\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \frac{\pi}{12}.}$$

Quick Tip to Solve: Add the limits first: their sum $\pi/2$ turns tangent into cotangent, making the original and reflected integrands add to 1.

20(a). Solve the differential equation: $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$. [4 marks]

Solution:

Concept:

- A first-order equation $dy/dx = F(y/x)$ is homogeneous. On an interval where $x \neq 0$, put $y = vx$, where v is a function of x, so that $dy/dx = v + x\,dv/dx$.
- After separation, use $\int dv/(1+v^2) = \tan^{-1} v$ and $\int v\,dv/(1+v^2) = \frac{1}{2}\log(1+v^2)$. The original equation is defined only when $x - y \neq 0$.

Step 1: Recognise the homogeneous form Both numerator and denominator have degree one. Dividing by x gives

$$\frac{dy}{dx} = \frac{1 + y/x}{1 - y/x}.$$

Substitute $y = vx$ and apply the product rule:

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v}.$$

Step 2: Separate the variables Move v to the right and combine over the common denominator:

$$\begin{aligned} x\frac{dv}{dx} &= \frac{1 + v - v(1-v)}{1-v} \\ &= \frac{1+v^2}{1-v}. \end{aligned}$$

Thus the equation becomes

$$\frac{1-v}{1+v^2}\,dv = \frac{dx}{x}.$$

Dividing by $1 + v^2 > 0$ loses no real constant-ratio solution. The value $v = 1$ is already excluded by the original denominator.

Step 3: Integrate and substitute back Splitting the numerator into its two terms and integrating both sides gives

$$\tan^{-1} v - \frac{1}{2}\log(1+v^2) = \log|x| + C.$$

Now replace v by y/x. The logarithms combine using

$$\log|x| + \frac{1}{2}\log\left(1 + \frac{y^2}{x^2}\right) = \frac{1}{2}\log(x^2+y^2).$$

Step 4: Conclusion The general solution is

$$\boxed{\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\log(x^2+y^2) = C.}$$

This applies locally with $x \neq 0$ and $x - y \neq 0$, extending across $x = 0$ by a continuous angular branch where the original equation is defined.

Quick Tip to Solve: After setting $y = vx$, simplify $(1+v)/(1-v) - v$ carefully: its numerator is $1 + v^2$, so the separated integral splits directly into inverse-tangent and logarithmic terms.

OR

20(b). Solve the differential equation: $(1+x^2)\,dy + 2xy\,dx = \cot x\,dx$. [4 marks]

Solution:

Concept:

- A linear differential equation has the form $dy/dx + P(x)y = Q(x)$. Its integrating factor is $I(x) = e^{\int P(x)\,dx}$, which turns the left side into the derivative of $I(x)y$.
- Use $\int \cot x\,dx = \log|\sin x| + C$, valid on an interval where $\sin x \neq 0$.

Step 1: Put the equation in linear form Divide the given equation by dx, and then by $1+x^2$, which is positive for every real x. This gives

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cot x}{1+x^2}.$$

Here $P(x) = 2x/(1+x^2)$ and $Q(x) = \cot x/(1+x^2)$. Work between successive multiples of π, where $\cot x$ is defined.

Step 2: Find the integrating factor The numerator of $P(x)$ is the derivative of its denominator, so the substitution $u = 1+x^2$, $du = 2x\,dx$, gives

$$\int \frac{2x}{1+x^2}\,dx = \log(1+x^2).$$

We may therefore choose

$$I(x) = e^{\log(1+x^2)} = 1+x^2.$$

Multiplying the linear equation by this factor makes the left side an exact product derivative:

$$\frac{d}{dx}[(1+x^2)y] = \cot x.$$

Step 3: Integrate the product derivative Integrate both sides with respect to x. For the right side, write $\cot x = \cos x/\sin x$ and use $u = \sin x$, $du = \cos x\,dx$. Thus

$$(1+x^2)y = \int \frac{\cos x}{\sin x}\,dx = \log|\sin x| + C.$$

The absolute value is needed because sine may be negative. Differentiating this relation gives $(1+x^2)y' + 2xy = \cot x$, recovering the given equation.

Step 4: Conclusion The general solution is

$$\boxed{y = \frac{\log|\sin x| + C}{1+x^2},}$$

on any interval $(k\pi, (k+1)\pi)$, where k is an integer and C is an arbitrary real constant.

Quick Tip to Solve: Spot $(1+x^2)\,dy + 2xy\,dx$ as $d[(1+x^2)y]$; recognising this product derivative immediately reduces the problem to integrating $\cot x$.

21. Find the value of λ for which the following lines are perpendicular to each other: $\frac{x-5}{5\lambda+2} = \frac{2-y}{5} = \frac{1-z}{-1}$; $\frac{x}{1} = \frac{y+\frac{1}{2}}{2\lambda} = \frac{z-1}{3}$. Hence, find whether the lines intersect or not. [4 marks]

Solution:

Concept:

- Read direction ratios after writing numerators as $x - x_0$, $y - y_0$, $z - z_0$.
- Two line directions are perpendicular when their direction vectors have zero dot product: $a_1a_2 + b_1b_2 + c_1c_2 = 0$.
- Lines intersect only if one point satisfies all three coordinate equations simultaneously. Orthogonal directions alone do not establish intersection in three dimensions.

Step 1: Read the direction ratios with the correct signs For the first line, $(2-y)/5 = (y-2)/(-5)$ and $(1-z)/(-1) = z-1$. Thus its standard symmetric form is

$$\frac{x-5}{5\lambda+2} = \frac{y-2}{-5} = \frac{z-1}{1}.$$

We may therefore take direction vectors

$$\vec{d_1} = (5\lambda+2, -5, 1), \qquad \vec{d_2} = (1, 2\lambda, 3).$$

Step 2: Apply the perpendicularity condition Set the dot product of these two direction vectors equal to zero:

$$\begin{aligned}(5\lambda+2)(1) + (-5)(2\lambda) + (1)(3) &= 0,\\ 5\lambda + 2 - 10\lambda + 3 &= 0,\\ -5\lambda + 5 &= 0.\end{aligned}$$

Hence $\lambda = 1$. At this value the directions are $(7, -5, 1)$ and $(1, 2, 3)$, and their dot product is $7 - 10 + 3 = 0$, as required.

Step 3: Test all three coordinates for intersection Introduce independent parameters t and s for the two lines at $\lambda = 1$:

$$\begin{aligned}L_1: &\quad (x, y, z) = (5 + 7t,\ 2 - 5t,\ 1 + t),\\ L_2: &\quad (x, y, z) = (s,\ -\tfrac{1}{2} + 2s,\ 1 + 3s).\end{aligned}$$

Equality of the z-coordinates requires $t = 3s$. Equality of the x-coordinates then gives

$$5 + 7(3s) = s \quad \Longrightarrow \quad s = -\frac{1}{4}, \qquad t = -\frac{3}{4}.$$

At these parameters, however,

$$y_{L_1} = 2 - 5\left(-\frac{3}{4}\right) = \frac{23}{4}, \qquad y_{L_2} = -\frac{1}{2} + 2\left(-\frac{1}{4}\right) = -1.$$

The unequal y-coordinates rule out a common point.

Step 4: Conclusion $\boxed{\lambda = 1}$. The lines have perpendicular directions but $\boxed{\text{do not intersect}}$; they are skew lines.

Quick Tip to Solve: Rewrite $2 - y$ and $1 - z$ before taking the dot product. For intersection, solve the simpler x- and z-equations first, then check the unused y-equation.

22. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $|\vec{a}| = 1,\ |\vec{b}| = 2$ and $|\vec{c}| = 3$. If the projection of $\vec{b}$ along $\vec{a}$ is equal to the projection of $\vec{c}$ along $\vec{a}$; and $\vec{b}, \vec{c}$ are perpendicular to each other, then find $|3\vec{a} - 2\vec{b} + 2\vec{c}|$.

[4 marks]

Solution:

Concept:

- The scalar projection of a vector $\vec{u}$ along a nonzero vector $\vec{a}$ is $(\vec{u} \cdot \vec{a})/|\vec{a}|$. Equal projections along the same vector give equal dot products with that vector.
- Perpendicular vectors have zero dot product. Also, $|\vec{u}|^2 = \vec{u} \cdot \vec{u}$, so expanding a dot product evaluates the magnitude of a linear combination.
- For any vectors $\vec{u}, \vec{v}$, $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v}$. The cross term disappears when the vectors are perpendicular.

Step 1: Translate the projection condition The projections of $\vec{b}$ and $\vec{c}$ along $\vec{a}$ are equal, so

$$\frac{\vec{b} \cdot \vec{a}}{|\vec{a}|} = \frac{\vec{c} \cdot \vec{a}}{|\vec{a}|}.$$

Since $|\vec{a}| = 1 \neq 0$, the common denominator may be cancelled. By symmetry of the dot product,

$$\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}, \qquad \vec{a} \cdot (\vec{c} - \vec{b}) = 0.$$

Thus the difference $\vec{c} - \vec{b}$ is perpendicular to $\vec{a}$. Notice that neither individual projection has to be zero; their equality is sufficient.

Step 2: Find the squared magnitude of the difference The given perpendicularity of $\vec{b}$ and $\vec{c}$ gives $\vec{b} \cdot \vec{c} = 0$. Expanding the square of their difference,

$$\begin{aligned} |\vec{c} - \vec{b}|^2 &= (\vec{c} - \vec{b}) \cdot (\vec{c} - \vec{b}) \\ &= |\vec{c}|^2 + |\vec{b}|^2 - 2\vec{b} \cdot \vec{c} \\ &= 3^2 + 2^2 - 0 = 13. \end{aligned}$$

Step 3: Expand the required squared magnitude Write the required vector as $3\vec{a} + 2(\vec{c} - \vec{b})$. Its two terms are perpendicular by Step 1, so

$$\begin{aligned} |3\vec{a} - 2\vec{b} + 2\vec{c}|^2 &= 9|\vec{a}|^2 + 4|\vec{c} - \vec{b}|^2 + 12\vec{a} \cdot (\vec{c} - \vec{b}) \\ &= 9(1)^2 + 4(13) + 0 \\ &= 61. \end{aligned}$$

Step 4: Conclusion A magnitude is nonnegative. Taking the positive square root gives

$$\boxed{|3\vec{a} - 2\vec{b} + 2\vec{c}| = \sqrt{61}}.$$

Quick Tip to Solve: Equal projections along $\vec{a}$ make $\vec{c} - \vec{b}$ perpendicular to $\vec{a}$. Group the expression as $3\vec{a} + 2(\vec{c} - \vec{b})$ so the mixed term vanishes immediately.

23. Find the equations of the tangent and the normal to the curve $y = \dfrac{x-7}{(x-2)(x-3)}$ at the point where it cuts the x-axis. [4 marks]

Solution:

Concept:

- At a point on the x-axis, $y = 0$. A rational expression is zero when its numerator is zero and its denominator is nonzero.
- The slope of the tangent to $y = f(x)$ at (x_0, y_0) is $m_t = f'(x_0)$. The perpendicular normal has slope $m_n = -1/m_t$, provided $m_t \neq 0$.
- A line of slope m through (x_0, y_0) has equation $y - y_0 = m(x - x_0)$.

Step 1: Locate the point of intersection The curve is defined for $x \neq 2, 3$. Setting $y = 0$ gives

$$\frac{x-7}{(x-2)(x-3)} = 0 \quad \Longrightarrow \quad x = 7.$$

At $x = 7$, the denominator is $(7-2)(7-3) = 20 \neq 0$, so the required point really lies on the curve. Thus both the tangent and the normal pass through $P = (7, 0)$.

Step 2: Differentiate and evaluate the tangent slope Write the denominator as $x^2 - 5x + 6$. Applying the quotient rule, with numerator derivative 1 and denominator derivative $2x - 5$, gives

$$\frac{dy}{dx} = \frac{x^2 - 5x + 6 - (x-7)(2x-5)}{(x^2 - 5x + 6)^2}.$$

Substitute $x = 7$ before expanding the numerator: the second term vanishes because it contains $x - 7$. Hence

$$m_t = \left.\frac{dy}{dx}\right|_{x=7} = \frac{20 - 0}{20^2} = \frac{1}{20}.$$

This finite, nonzero slope allows the usual negative-reciprocal formula for the normal.

Step 3: Form both line equations For the tangent, insert $P = (7, 0)$ and $m_t = 1/20$ into the point-slope equation:

$$y - 0 = \frac{1}{20}(x - 7).$$

For the normal, first calculate its slope and then use the same point:

$$m_n = -\frac{1}{m_t} = -20, \qquad y - 0 = -20(x - 7).$$

As checks, both equations give $y = 0$ at $x = 7$, and the slopes have product $(1/20)(-20) = -1$, as required for perpendicular lines.

Step 4: Conclusion At $(7, 0)$, the required equations are

$$\boxed{\text{Tangent: } x - 20y - 7 = 0}, \qquad \boxed{\text{Normal: } 20x + y - 140 = 0}.$$

Quick Tip to Solve: At the zero $x = 7$ of the numerator, substitute directly into the quotient-rule expression; its term containing $x - 7$ disappears.

Section D (36 marks)

Questions number 24 to 29 carry 6 marks each.

24. Find the local maxima and local minima, if any, of the following function. Also find the local maximum and the local minimum values, as the case may be: $f(x) = \sin x + \frac{1}{2}\cos 2x,\ 0 \le x \le \frac{\pi}{2}$.

[6 marks]

Solution:

Concept:

- A differentiable function has stationary points where $f'(x) = 0$. A change in f' from positive to negative gives a local maximum; a change from negative to positive gives a local minimum.
- At a stationary interior point c, $f''(c) < 0$ confirms a local maximum and $f''(c) > 0$ confirms a local minimum. At a boundary of a restricted interval, use the available one-sided neighbourhood instead.
- The identities $\sin 2x = 2\sin x \cos x$ and $\cos 2x = 1 - 2\sin^2 x$ simplify both the derivative and the extremum values.

Step 1: Differentiate and find the stationary points For $f(x) = \sin x + \frac{1}{2}\cos 2x$, apply the chain rule to the double-angle term:

$$\begin{aligned} f'(x) &= \cos x - \sin 2x \\ &= \cos x(1 - 2\sin x). \end{aligned}$$

Thus $f'(x) = 0$ when $\cos x = 0$ or $\sin x = 1/2$. In the given interval $0 \le x \le \pi/2$, the resulting points are

$$x = \frac{\pi}{2} \quad \text{and} \quad x = \frac{\pi}{6}.$$

The second point is interior; the first is the right endpoint. Keep this distinction when classifying them.

Step 2: Classify the interior stationary point On $(0, \pi/2)$, $\cos x > 0$. Also $\sin x < 1/2$ on $(0, \pi/6)$ and $\sin x > 1/2$ on $(\pi/6, \pi/2)$. Consequently,

$$f'(x) > 0 \text{ on } (0, \pi/6), \qquad f'(x) < 0 \text{ on } (\pi/6, \pi/2).$$

The function therefore increases up to $\pi/6$ and decreases after it, giving a local maximum there. As a second-derivative check,

$$f''(x) = -\sin x - 2\cos 2x, \qquad f''(\pi/6) = -\frac{1}{2} - 2\left(\frac{1}{2}\right) = -\frac{3}{2} < 0.$$

Step 3: Examine the interval endpoints The function decreases as x approaches $\pi/2$ from the left. Hence $\pi/2$ is a one-sided local minimum on the stated domain. Moreover, $f'(\pi/2) = 0$ and $f''(\pi/2) = 1 > 0$, so it is also a local minimum of the same trigonometric expression extended beyond the endpoint.

At the left endpoint $x = 0$, the function increases immediately to its right. Thus 0 is also a one-sided local minimum when neighbourhoods are restricted to $[0, \pi/2]$. It is not stationary, since $f'(0) = 1$. If

Quick Tip to Solve: Factor the derivative as $\cos x(1 - 2\sin x)$, then check endpoints separately. Completing the square in $\sin x$ quickly verifies both extremum values.

local extrema are defined only at interior points, neither endpoint is counted and there is no interior local minimum.

Step 4: Calculate the extremum values Substituting the maximum point gives

$$f(\pi/6) = \sin(\pi/6) + \frac{1}{2}\cos(\pi/3) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}.$$

At the two endpoints,

$$f(0) = 0 + \frac{1}{2} = \frac{1}{2}, \qquad f(\pi/2) = 1 + \frac{1}{2}(-1) = \frac{1}{2}.$$

An independent value check follows by completing the square:

$$f(x) = \frac{1}{2} + \sin x - \sin^2 x = \frac{3}{4} - \left(\sin x - \frac{1}{2}\right)^2.$$

Since $0 \le \sin x \le 1$, the squared term lies between 0 and $1/4$. The calculated maximum and endpoint minima therefore also give the absolute bounds on the stated interval.

Step 5: Conclusion The local maximum occurs at $\boxed{x = \pi/6}$, with value $\boxed{3/4}$. The stationary local minimum is at $\boxed{x = \pi/2}$, with value $\boxed{1/2}$. On the restricted closed interval, $x = 0$ is additionally a one-sided local minimum with the same value $1/2$. There is no interior local minimum.

Quick Tip to Solve: Factor the derivative as $\cos x(1 - 2\sin x)$, then check endpoints separately. Completing the square in $\sin x$ quickly verifies both extremum values.

25(a). **If** $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, **find** A^2 **and show that** $A^2 = A^{-1}$**.** [6 marks]

Solution:

Concept:

- Matrix multiplication uses row-column products: if $B = AA$, then $b_{ij} = \sum_{k=1}^{3} a_{ik}a_{kj}$.
- The inverse of a square matrix A is the matrix B satisfying $AB = BA = I$, where I is the identity matrix of the same order.
- Matrix multiplication is associative, so $A(A^2) = (A^2)A = A^3$. Thus proving $A^3 = I$ identifies A^2 as the inverse directly.

Step 1: Multiply each row of the matrix by its columns Use row-column multiplication. Each entry uses all three terms, including zero terms; square the matrix as a product, not entry by entry. For the first row of A^2, the three products are

$$1 - 2 + 1 = 0, \qquad -1 + 1 + 0 = 0, \qquad 1 + 0 + 0 = 1.$$

Quick Tip to Solve: After computing A^2, test $A^3 = I$; this proves the inverse identity without calculating nine cofactors.

For the second row, they are

$$2-2+0=0, \qquad -2+1+0=-1, \qquad 2+0+0=2.$$

The third row $(1,0,0)$ selects the first row of A, giving the entries $1,-1,1$. Therefore

$$A^2=\begin{bmatrix}1&-1&1\\2&-1&0\\1&0&0\end{bmatrix}\begin{bmatrix}1&-1&1\\2&-1&0\\1&0&0\end{bmatrix}=\begin{bmatrix}0&0&1\\0&-1&2\\1&-1&1\end{bmatrix}.$$

Step 2: Multiply the square by the original matrix Let $B=A^2$. To test the inverse identity, calculate BA by row-column multiplication:

$$\begin{aligned}BA&=\begin{bmatrix}0&0&1\\0&-1&2\\1&-1&1\end{bmatrix}\begin{bmatrix}1&-1&1\\2&-1&0\\1&0&0\end{bmatrix}\\&=\begin{bmatrix}1&0&0\\-2+2&1&0\\1-2+1&-1+1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=I.\end{aligned}$$

Since $B=AA$, associativity also gives $AB=A(AA)=(AA)A=BA=I$. Thus the candidate satisfies both inverse identities; no assumption that A^{-1} already exists is needed.

Step 3: Conclusion

$$\boxed{A^2=\begin{bmatrix}0&0&1\\0&-1&2\\1&-1&1\end{bmatrix}=A^{-1}}.$$

Quick Tip to Solve: After computing A^2, test $A^3=I$; this proves the inverse identity without calculating nine cofactors.

OR

25(b). Using matrix method, solve the following system of equations: $2x-3y+5z=13,\ 3x+2y-4z=-2,\ x+y-2z=-2$. [6 marks]

Solution:

Concept:

- A system of linear equations can be written as $AX=B$, where A is the coefficient matrix, X is the column of unknowns and B is the column of constants.
- When $|A|\neq 0$, the unique solution is $X=A^{-1}B$, with $A^{-1}=\dfrac{\operatorname{adj}A}{|A|}$.
- The cofactor $C_{ij}=(-1)^{i+j}M_{ij}$, where M_{ij} is the minor obtained by deleting row i and column j. The adjugate is the transpose of the cofactor matrix.

Quick Tip to Solve: Here the first cofactor is zero and $|A|=-1$; transpose the cofactor matrix, then change every sign to obtain the inverse.

Step 1: Express the equations as a matrix equation Use the same order x, y, z in every row of the coefficient matrix, retaining the signs of the coefficients. The given system is $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 13 \\ -2 \\ -2 \end{bmatrix}.$$

To solve by inversion, first check that the determinant is nonzero.

Step 2: Calculate the determinant and cofactors Expanding along the first row,

$$\begin{aligned} |A| &= 2\{2(-2) - (-4)(1)\} - (-3)\{3(-2) - (-4)(1)\} \\ &\quad + 5\{3(1) - 2(1)\} \\ &= 2(0) + 3(-2) + 5(1) = -1. \end{aligned}$$

Thus A is invertible. The cofactors, with the alternating signs included, are

$$\begin{aligned} &C_{11} = -4 + 4 = 0, && C_{12} = -(-6 + 4) = 2, && C_{13} = 3 - 2 = 1, \\ &C_{21} = -(6 - 5) = -1, && C_{22} = -4 - 5 = -9, && C_{23} = -(2 + 3) = -5, \\ &C_{31} = 12 - 10 = 2, && C_{32} = -(-8 - 15) = 23, && C_{33} = 4 + 9 = 13. \end{aligned}$$

Transpose the cofactor matrix to obtain

$$\operatorname{adj} A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}, \qquad A^{-1} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}.$$

Step 3: Multiply the inverse by the constant column Premultiplying $AX = B$ by A^{-1} gives $X = A^{-1}B$. Hence

$$\begin{aligned} X &= \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 13 \\ -2 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} -2 + 4 \\ -26 - 18 + 46 \\ -13 - 10 + 26 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}. \end{aligned}$$

Check the result in each original equation:

$$\begin{aligned} 2(2) - 3(2) + 5(3) &= 4 - 6 + 15 = 13, \\ 3(2) + 2(2) - 4(3) &= 6 + 4 - 12 = -2, \\ 2 + 2 - 2(3) &= -2. \end{aligned}$$

Every constant is reproduced, which also checks the signs used in the matrix multiplication.

Step 4: Conclusion The unique solution of the given system is $\boxed{x = 2,\ y = 2,\ z = 3}$.

Quick Tip to Solve: Here the first cofactor is zero and $|A| = -1$; transpose the cofactor matrix, then change every sign to obtain the inverse.

26. There are two boxes I and II. Box I contains 3 red and 6 black balls. Box II contains 5 red and n black balls. One of the two boxes, box I and box II is selected at random and a ball is drawn at random. The ball drawn is found to be red. If the probability that this red ball comes out from box II is $\frac{3}{5}$, find the value of n. [6 marks]

Solution:

Concept:

- Let E_1 and E_2 denote selection of boxes I and II, respectively, and let R denote drawing a red ball. Random selection of one of two boxes gives $P(E_1) = P(E_2) = \frac{1}{2}$.
- The total probability of a red ball is $P(R) = P(E_1)P(R \mid E_1) + P(E_2)P(R \mid E_2)$, since exactly one box is selected.
- Bayes' theorem reverses the conditioning: $P(E_2 \mid R) = \dfrac{P(E_2)P(R \mid E_2)}{P(R)}$. Here the observed information is that the drawn ball is red.

Step 1: Find the red-ball probability for each box Box I contains $3 + 6 = 9$ balls, of which three are red. Box II contains $5 + n$ balls, of which five are red. Therefore,

$$P(R \mid E_1) = \frac{3}{9} = \frac{1}{3}, \qquad P(R \mid E_2) = \frac{5}{5+n}.$$

The unknown n is a non-negative integer. These conditional probabilities describe drawing red after a particular box has been selected; they are distinct from the probability of the box after a red ball is observed.

Step 2: Express the total probability of drawing red A red ball can arise from either of two mutually exclusive routes: select box I and draw red, or select box II and draw red. Adding these route probabilities gives

$$\begin{aligned} P(R) &= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{5}{5+n} \\ &= \frac{1}{6} + \frac{5}{2(5+n)} \\ &= \frac{n+20}{6(n+5)}. \end{aligned}$$

The box choices are equally likely, although the probabilities of obtaining a red ball from them need not initially be equal.

Step 3: Apply the given posterior probability The statement that the observed red ball comes from box II with probability $\frac{3}{5}$ means $P(E_2 \mid R) = \frac{3}{5}$. Bayes' theorem yields

$$\begin{aligned} \frac{3}{5} &= \frac{\frac{1}{2} \cdot \frac{5}{n+5}}{\frac{n+20}{6(n+5)}} \\ &= \frac{15}{n+20}. \end{aligned}$$

Since $n + 20 > 0$, cross-multiplication is valid and gives

$$3(n+20) = 75, \qquad 3n + 60 = 75, \qquad 3n = 15.$$

Quick Tip to Solve: With equal box-selection probabilities, cancel the common factor $\frac{1}{2}$ in Bayes' formula; compare the red-ball fractions $\frac{1}{3}$ and $\frac{5}{n+5}$.

Step 4: Check the ball count in the original probability The candidate value is $n = 5$, which is an admissible whole number of black balls. For this count, box II has ten balls and its red-ball probability is $\frac{1}{2}$. The joint probabilities of drawing red from boxes I and II become $\frac{1}{6}$ and $\frac{1}{4}$, respectively. Thus the conditional probability required by the question is

$$\frac{\frac{1}{4}}{\frac{1}{6}+\frac{1}{4}} = \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{3}{5},$$

agreeing exactly with the given information.

Step 5: Conclusion Hence, $\boxed{n = 5}$: box II contains five black balls.

Quick Tip to Solve: With equal box-selection probabilities, cancel the common factor $\frac{1}{2}$ in Bayes' formula; compare the red-ball fractions $\frac{1}{3}$ and $\frac{5}{n+5}$.

27(a). Find the vector and cartesian equations of the plane passing through the points $(2, 5, -3)$, $(-2, -3, 5)$ and $(5, 3, -3)$. Also, find the point of intersection of this plane with the line passing through points $(3, 1, 5)$ and $(-1, -3, -1)$. [6 marks]

Solution:

Concept:

- For three non-collinear points A, B, C, a normal to their plane is $\overrightarrow{AB} \times \overrightarrow{AC}$, perpendicular to both directions within the plane.
- If $\vec{a}$ is a position vector of a point on the plane and $\vec{n}$ is a normal, its vector equation is $(\vec{r}-\vec{a})\cdot\vec{n} = 0$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
- The line through points with position vectors $\vec{p}, \vec{q}$ is $\vec{r} = \vec{p} + t(\vec{q} - \vec{p})$, where $t \in \mathbb{R}$. Substituting these coordinates into the plane equation determines their intersection.

Step 1: Construct two directions in the plane Denote the given points by $A = (2, 5, -3)$, $B = (-2, -3, 5)$ and $C = (5, 3, -3)$. Subtracting coordinates gives

$$\overrightarrow{AB} = (-4, -8, 8),$$
$$\overrightarrow{AC} = (3, -2, 0).$$

Both vectors lie in the required plane. Their cross product supplies a normal vector:

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}$$
$$= 16\hat{i} + 24\hat{j} + 32\hat{k}$$
$$= 8(2\hat{i} + 3\hat{j} + 4\hat{k}).$$

This is nonzero, so the three points are non-collinear and determine a unique plane. Use the simpler proportional normal $\vec{n} = 2\hat{i} + 3\hat{j} + 4\hat{k}$.

Quick Tip to Solve: Use two vectors with the same starting point to find the normal; here divide the cross product by eight to obtain $(2, 3, 4)$ before substituting the line coordinates.

Step 2: Form the vector and cartesian equations Using point A and this normal, the plane is

$$[\vec{r} - (2\hat{i} + 5\hat{j} - 3\hat{k})] \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 0.$$

The dot product of the position vector of A with the normal is $4 + 15 - 12 = 7$. Hence the vector form simplifies to

$$\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 7.$$

Writing $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, the corresponding cartesian form is

$$2x + 3y + 4z = 7.$$

As a check, substitution of B gives $-4 - 9 + 20 = 7$, and substitution of C gives $10 + 9 - 12 = 7$. Thus both remaining points satisfy the equation.

Step 3: Parametrize the given line Let $P = (3, 1, 5)$ and $Q = (-1, -3, -1)$. Their difference is $Q - P = (-4, -4, -6)$, so any point on the line can be written as

$$(x, y, z) = (3, 1, 5) + t(-4, -4, -6), \qquad t \in \mathbb{R}.$$

Equivalently,

$$x = 3 - 4t, \qquad y = 1 - 4t, \qquad z = 5 - 6t.$$

The direction has dot product $2(-4) + 3(-4) + 4(-6) = -44$ with the plane normal. Since this is nonzero, the line meets the plane in exactly one point.

Step 4: Determine and verify the intersection At an intersection, the line coordinates must also satisfy the plane equation. Substituting them gives

$$\begin{aligned} 2(3 - 4t) + 3(1 - 4t) + 4(5 - 6t) &= 7, \\ 29 - 44t &= 7, \\ t &= \frac{1}{2}. \end{aligned}$$

Inserting this parameter in the line gives

$$(x, y, z) = (3 - 2,\ 1 - 2,\ 5 - 3) = (1, -1, 2).$$

This point is on the line by construction, and the plane check is $2(1) + 3(-1) + 4(2) = 7$.

Step 5: Conclusion The required vector and cartesian equations are

$$\boxed{\vec{r} \cdot (2\hat{i} + 3\hat{j} + 4\hat{k}) = 7}, \qquad \boxed{2x + 3y + 4z = 7}.$$

The line intersects this plane at $\boxed{(1, -1, 2)}$.

Quick Tip to Solve: Use two vectors with the same starting point to find the normal; here divide the cross product by eight to obtain $(2, 3, 4)$ before substituting the line coordinates.

OR

27(b). Find the equation of the plane passing through the intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and parallel to x-axis. Hence, find the distance of the plane from x-axis. [6 marks]

Solution:

Concept:

- A plane through the line of intersection of two non-parallel planes $P_1 = 0$ and $P_2 = 0$ can be written as $P_1 + \lambda P_2 = 0$, where λ is a real parameter (with $P_2 = 0$ included separately).
- A plane is parallel to a direction vector when that vector is perpendicular to its normal. The direction of the x-axis is $\hat{i}$, so the coefficient of x must vanish.
- The perpendicular distance from (x_0, y_0, z_0) to $ax + by + cz + d = 0$ is $\dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$. Every point on a line parallel to a plane has the same distance from it.

Step 1: Express the given planes in Cartesian form Put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Expanding the two scalar products gives

$$P_1 = x + y + z - 1 = 0, \qquad P_2 = 2x + 3y - z + 4 = 0.$$

Their normals $(1, 1, 1)$ and $(2, 3, -1)$ are not proportional, so the two planes intersect in a line. Any point on this line satisfies both equations, and therefore also satisfies every linear combination of them.

Step 2: Impose the condition of parallelism Write the required member of the family as

$$(x + y + z - 1) + \lambda(2x + 3y - z + 4) = 0.$$

Collecting coefficients, this becomes

$$(1 + 2\lambda)x + (1 + 3\lambda)y + (1 - \lambda)z - 1 + 4\lambda = 0.$$

Its normal is perpendicular to $\hat{i}$ precisely when

$$1 + 2\lambda = 0 \quad \Longrightarrow \quad \lambda = -\frac{1}{2}.$$

Substitution gives

$$-\frac{1}{2}y + \frac{3}{2}z - 3 = 0.$$

Multiplication by -2 simplifies the equation to $y - 3z + 6 = 0$. Its normal $(0, 1, -3)$ has zero x-component, confirming the required parallelism.

Step 3: Calculate the distance from the x-axis An arbitrary point of the x-axis is $A = (t, 0, 0)$, where t is real. Applying the point-to-plane distance formula gives

$$d = \frac{|0 \cdot t + 1 \cdot 0 - 3 \cdot 0 + 6|}{\sqrt{0^2 + 1^2 + (-3)^2}} = \frac{6}{\sqrt{10}}.$$

The expression is independent of t, as it must be for a line parallel to a plane. Also, no point of the

Quick Tip to Solve: To make the plane parallel to the x-axis, eliminate x directly by forming $P_2 - 2P_1$. Then use the origin, which lies on the x-axis, to calculate the distance.

x-axis satisfies the plane equation, since substitution gives $6 \neq 0$; the distance is therefore positive.

Step 4: Conclusion The required plane is $\boxed{y - 3z + 6 = 0}$, or in vector form $\boxed{\vec{r} \cdot (\hat{j} - 3\hat{k}) = -6}$. Its distance from the x-axis is $\boxed{\dfrac{6}{\sqrt{10}} = \dfrac{3\sqrt{10}}{5} \text{ units}}$.

Quick Tip to Solve: To make the plane parallel to the x-axis, eliminate x directly by forming $P_2 - 2P_1$. Then use the origin, which lies on the x-axis, to calculate the distance.

28(a). Find the area of the triangle whose vertices are $(-1, 1)$, $(0, 5)$ and $(3, 2)$, using integration.

[6 marks]

Solution:

Concept:

- The straight line through (x_1, y_1) and (x_2, y_2), with $x_1 \neq x_2$, has slope $m = (y_2 - y_1)/(x_2 - x_1)$ and equation $y - y_1 = m(x - x_1)$.
- The area between an upper curve $y = u(x)$ and a lower curve $y = l(x)$ on $[a, b]$ is $\int_a^b [u(x) - l(x)]\, dx$. Split the interval wherever the upper or lower boundary changes.

Step 1: Find the equations of the three sides Denote the vertices by $A = (-1, 1)$, $B = (0, 5)$ and $C = (3, 2)$. For side AB, the slope is $(5 - 1)/(0 + 1) = 4$, giving

$$AB: \quad y - 1 = 4(x + 1), \qquad y = 4x + 5.$$

For side BC, the slope is $(2 - 5)/(3 - 0) = -1$, giving

$$BC: \quad y - 5 = -x, \qquad y = 5 - x.$$

Finally, the slope of AC is $(2 - 1)/(3 + 1) = 1/4$, so

$$AC: \quad y - 1 = \frac{1}{4}(x + 1), \qquad y = \frac{x + 5}{4}.$$

Step 2: Identify the upper and lower boundaries The triangle extends from $x = -1$ to $x = 3$. Its lower boundary throughout this interval is AC. Its upper boundary is AB for $-1 \leq x \leq 0$, and BC for $0 \leq x \leq 3$; hence the area integral must be split at the x-coordinate of B. The two vertical strip heights are

$$(4x + 5) - \frac{x + 5}{4} = \frac{15}{4}(x + 1), \quad -1 \leq x \leq 0,$$
$$(5 - x) - \frac{x + 5}{4} = \frac{5}{4}(3 - x), \quad 0 \leq x \leq 3.$$

Both expressions are non-negative on their respective intervals, which confirms that the chosen upper and lower boundaries give positive areas.

Quick Tip to Solve: Split the integral at $x = 0$, the middle vertex's x-coordinate: the upper side changes there from AB to BC, while AC remains the lower side.

Step 3: Integrate the left portion The area between AB and AC, from $x = -1$ to $x = 0$, is

$$\begin{aligned} S_1 &= \frac{15}{4}\int_{-1}^{0}(x+1)\,dx \\ &= \frac{15}{4}\left[\frac{x^2}{2}+x\right]_{-1}^{0} \\ &= \frac{15}{4}\left[0-\left(\frac{1}{2}-1\right)\right] = \frac{15}{8}. \end{aligned}$$

Step 4: Integrate the right portion and combine The area between BC and AC, from $x = 0$ to $x = 3$, is

$$\begin{aligned} S_2 &= \frac{5}{4}\int_{0}^{3}(3-x)\,dx \\ &= \frac{5}{4}\left[3x-\frac{x^2}{2}\right]_{0}^{3} \\ &= \frac{5}{4}\left(9-\frac{9}{2}\right) = \frac{45}{8}. \end{aligned}$$

These portions cover the entire triangle and share only their boundary at $x = 0$. Therefore their areas add:

$$S = S_1 + S_2 = \frac{15}{8}+\frac{45}{8} = \frac{60}{8}.$$

Step 5: Conclusion Using integration, the area of the given triangle is $\boxed{\frac{15}{2}\text{ square units}}$.

Quick Tip to Solve: Split the integral at $x = 0$, the middle vertex's x-coordinate: the upper side changes there from AB to BC, while AC remains the lower side.

OR

28(b). Find the area of the region bounded by the curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$, using integration. [6 marks]

Solution:

Concept:

- The area between curves is the integral of the upper ordinate minus the lower ordinate over their common horizontal interval.
- Both circles have radius one, with centres $(0,0)$ and $(1,0)$. Their common region is symmetric about the x-axis and about $x = \frac{1}{2}$.
- Use $\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(x\sqrt{1-x^2}+\sin^{-1}x\right)+C$, with the inverse sine measured in radians.

Step 1: Locate the intersection points Expand the translated circle and subtract the other circle's equation:

$$(x-1)^2+y^2=1, \quad x^2+y^2=1 \quad \Longrightarrow \quad -2x+1=0.$$

Quick Tip to Solve: Subtract the circle equations first to find the common chord $x = \frac{1}{2}$, then use symmetry to reduce the lens area to four times one integral.

Hence $x = \frac{1}{2}$. Substituting in $x^2 + y^2 = 1$ gives

$$y^2 = 1 - \frac{1}{4} = \frac{3}{4}, \qquad y = \pm\frac{\sqrt{3}}{2}.$$

Thus the two boundary arcs meet at $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$. The common lens extends from $x = 0$ to $x = 1$.

Step 2: Express the common area as definite integrals On $0 \le x \le \frac{1}{2}$, the translated circle supplies the smaller positive ordinate, $\sqrt{1-(x-1)^2}$. On $\frac{1}{2} \le x \le 1$, the circle centred at the origin supplies the smaller positive ordinate, $\sqrt{1-x^2}$. Reflecting the upper half in the x-axis therefore gives

$$A = 2\left\{\int_0^{1/2} \sqrt{1-(x-1)^2}\,dx + \int_{1/2}^{1} \sqrt{1-x^2}\,dx\right\}.$$

In the first integral put $u = 1 - x$; its limits become 1 and $\frac{1}{2}$, and $dx = -du$. Consequently the two integrals are equal, so

$$A = 4\int_{1/2}^{1} \sqrt{1-x^2}\,dx.$$

The factor four accounts for reflection in both symmetry axes; it does not represent the areas of the two entire circles.

Step 3: Evaluate the integral at both limits Applying the standard integral and using $\sin^{-1} 1 = \frac{\pi}{2}$ and $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$, we obtain

$$\begin{aligned} A &= 2\left[x\sqrt{1-x^2} + \sin^{-1} x\right]_{1/2}^{1} \\ &= 2\left[\frac{\pi}{2} - \left(\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)\right] \\ &= 2\left(\frac{\pi}{3} - \frac{\sqrt{3}}{4}\right). \end{aligned}$$

Here the product $x\sqrt{1-x^2}$ vanishes at $x = 1$, while at $x = \frac{1}{2}$ it equals $\frac{\sqrt{3}}{4}$.

Step 4: Conclusion The required area enclosed by the two circular arcs is

$$\boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2} \text{ square units}}.$$

Quick Tip to Solve: Subtract the circle equations first to find the common chord $x = \frac{1}{2}$, then use symmetry to reduce the lens area to four times one integral.

29. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours and 20 minutes available for cutting and 4 hours available for assembling. The profit is Rs. 50 each for type A and Rs. 60 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximize profit? Formulate the above LPP and solve it graphically and also find the maximum profit. [6 marks]

Solution:

Concept:

- Let x and y denote the numbers of type A and type B souvenirs. Available processing times give inequalities because production cannot exceed either capacity.
- The objective is to maximize the linear profit function $Z = 50x + 60y$, measured in rupees, subject to both time constraints and $x, y \geq 0$.
- For a bounded feasible polygon, the maximum of a linear objective occurs at a vertex. Plot the constraints, identify the common feasible region and compare the objective at every vertex.

Step 1: Convert the available times and formulate the LPP The available cutting time is $3(60) + 20 = 200$ minutes, and the available assembling time is $4(60) = 240$ minutes. Cutting one unit of A takes five minutes and one unit of B takes eight minutes, whereas assembling requires ten and eight minutes respectively. Hence the complete LPP is

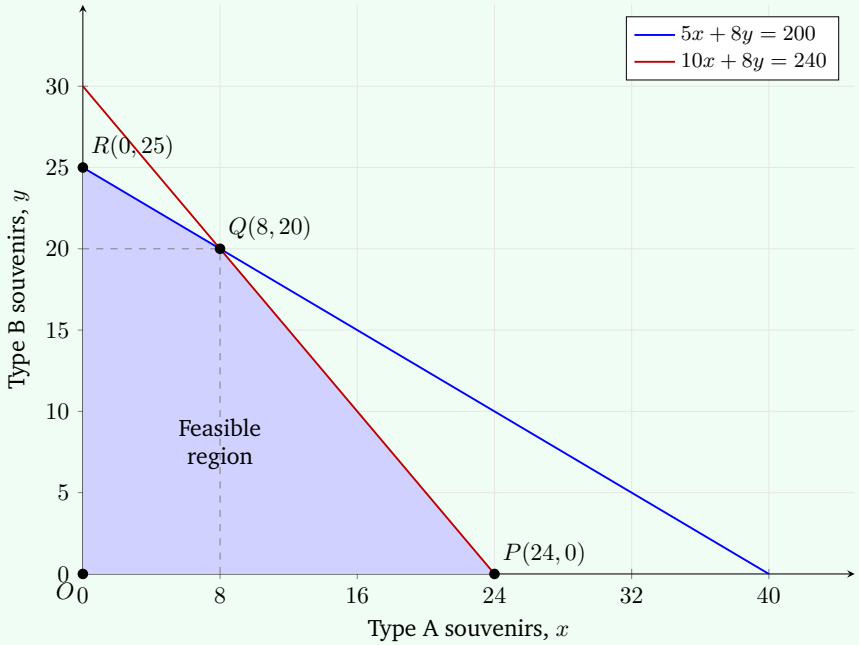
$$\begin{aligned} \text{Maximize} \quad & Z = 50x + 60y, \\ \text{subject to} \quad & 5x + 8y \leq 200, \\ & 10x + 8y \leq 240, \\ & x \geq 0, \quad y \geq 0. \end{aligned}$$

Production quantities are whole numbers in practice. We first solve the graphical LPP and then check that its maximizing point has integral coordinates.

Step 2: Draw the boundary lines and shade the feasible region The cutting boundary $5x + 8y = 200$ passes through $(40, 0)$ and $(0, 25)$. The assembling boundary $10x + 8y = 240$ passes through $(24, 0)$ and $(0, 30)$. Both inequalities include the origin, so their feasible half-planes lie on the origin's side of each line. Restricting to the first quadrant gives the shaded polygon $OPQR$ in the graph.

Its vertices on the coordinate axes are $O = (0, 0)$, $P = (24, 0)$ and $R = (0, 25)$. The stricter horizontal intercept is 24, and the stricter vertical intercept is 25; the other two intercepts are outside the common feasible region.

Quick Tip to Solve: The two time equations have the same coefficient of y; subtract them to obtain $x = 8$ immediately, then compare this intersection with both feasible axis vertices.



The shaded feasible polygon is $OPQR$; the maximum profit occurs at $Q = (8, 20)$.

Step 3: Calculate the intersection of the two resource boundaries At the remaining vertex Q, both available times are used completely. Solve the two boundary equations together:

$$5x + 8y = 200,$$
$$10x + 8y = 240.$$

Subtracting the first equation from the second gives $5x = 40$, so $x = 8$. The first equation then gives $40 + 8y = 200$, hence $y = 20$. Thus $Q = (8, 20)$, which lies in the first quadrant and has whole-number coordinates.

Step 4: Compare profits at all feasible vertices Evaluate $Z = 50x + 60y$ at each vertex of the bounded feasible polygon:

Vertex (x, y)	Z (rupees)
$(0, 0)$	0
$(24, 0)$	$50(24) = 1200$
$(8, 20)$	$50(8) + 60(20) = 1600$
$(0, 25)$	$60(25) = 1500$

The largest vertex value is attained only at Q. Moreover, producing eight A souvenirs and twenty B souvenirs uses $5(8) + 8(20) = 200$ cutting minutes and $10(8) + 8(20) = 240$ assembling minutes, exactly meeting both capacities. Since this graphical optimum already has whole-number coordinates, it is also attainable as an actual production plan.

Step 5: Conclusion The company should manufacture $\boxed{8}$ souvenirs of type A and $\boxed{20}$ souvenirs of type B. The maximum profit is $\boxed{\text{Rs. } 1600}$.

Quick Tip to Solve: The two time equations have the same coefficient of y; subtract them to obtain $x = 8$ immediately, then compare this intersection with both feasible axis vertices.