

CBSE Class 12 Mathematics Question Paper 2020 with Solution PDF-Set 65-1-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper comprises **four** Sections A, B, C and D. This question paper carries **36** questions. **All** questions are compulsory.
- (ii) **Section A** – Questions no. **1 to 20** comprises of **20** questions of **1** mark each.
- (iii) **Section B** – Questions no. **21 to 26** comprises of **6** questions of **2** marks each.
- (iv) **Section C** – Questions no. **27 to 32** comprises of **6** questions of **4** marks each.
- (v) **Section D** – Questions no. **33 to 36** comprises of **4** questions of **6** marks each.
- (vi) There is no overall choice in the question paper. However, an internal choice has been provided in 3 questions of one mark, 2 questions of two marks, 2 questions of four marks and 2 questions of six marks. Only one of the choices in such questions have to be attempted.
- (vii) In addition to this, separate instructions are given with each section and question, wherever necessary.
- (viii) Use of calculators **is not permitted**.

Section A (20 marks)

Question numbers 1 to 20 carry 1 mark each.

1. If A is a square matrix of order 3 and $|A| = 5$, then the value of $|2A'|$ is

- (A) -10
- (B) 10
- (C) -40
- (D) 40

[1 mark]

Solution:

Concept:

- Transposing a square matrix does not change its determinant: $|A'| = |A|$.
- For an order- n matrix, multiplying every entry by k gives $|kA| = k^n|A|$, since each of its n rows contributes one factor of k .

Step 1: Use the transpose property The symbol A' denotes the transpose. Hence $|A'| = 5$, and the order remains 3. No minus sign is introduced by transposition.

Step 2: Apply the scalar determinant rule All three rows of A' are multiplied by 2. Therefore

$$|2A'| = 2^3|A'| = 8 \times 5 = 40.$$

Multiplying the determinant by just 2 would give 10, which misses the other two row factors. Neither negative option is possible because both factors in this calculation are positive.

Step 3: Conclusion The required value is $\boxed{40}$. Hence, the correct option is (D).

Quick Tip to Solve: Count the rows before taking a scalar outside a determinant: order 3 means a cube.

2. If A is a square matrix such that $A^2 = A$, then $(I - A)^3 + A$ is equal to

- (A) I
- (B) 0
- (C) $I - A$
- (D) $I + A$

[1 mark]

Solution:

Concept:

- A matrix satisfying $A^2 = A$ is idempotent; repeated positive powers of A equal A .
- The identity matrix I satisfies $IA = AI = A$, so ordinary polynomial expansion in A is valid.

Step 1: Square the complementary matrix Using $A^2 = A$,

$$(I - A)^2 = I - 2A + A^2 = I - A.$$

Quick Tip to Solve: First check whether the complementary matrix is idempotent; this avoids expanding the cube.

Thus $I - A$ is itself idempotent; the cube also equals $I - A$.

Step 2: Evaluate the requested expression Multiply the preceding identity by $I - A$ and then add A :

$$(I - A)^3 + A = (I - A) + A = I.$$

The zero matrix, $I - A$, and $I + A$ do not equal the requested expression for a general idempotent matrix. In particular, adding the final A is essential; stopping at the cube alone would incorrectly select option (C).

Step 3: Conclusion The expression equals I . Hence, the correct option is (A).

Quick Tip to Solve: First check whether the complementary matrix is idempotent; this avoids expanding the cube.

3. The principal value of $\tan^{-1}\left(\tan \frac{3\pi}{5}\right)$ is

- (A) $\frac{2\pi}{5}$
- (B) $\frac{-2\pi}{5}$
- (C) $\frac{3\pi}{5}$
- (D) $\frac{-3\pi}{5}$

[1 mark]

Solution:

Concept:

- The principal-value range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$.
- Tangent has period π : $\tan(\theta - \pi) = \tan \theta$.

Step 1: Reduce the angle into the principal interval The angle $3\pi/5$ exceeds $\pi/2$, so it is outside the allowed range. Subtract one period:

$$\frac{3\pi}{5} - \pi = -\frac{2\pi}{5}.$$

This lies strictly between $-\pi/2$ and $\pi/2$.

Step 2: Check the candidate values Since $\tan(3\pi/5) = \tan(-2\pi/5)$, the inverse tangent returns $-2\pi/5$. The positive value $2\pi/5$ has positive tangent, whereas the given angle has negative tangent. The values $3\pi/5$ and $-3\pi/5$ both lie outside the principal interval and cannot be principal values.

Step 3: Conclusion The principal value is $-\frac{2\pi}{5}$. Hence, the correct option is (B).

Quick Tip to Solve: For an angle between $\pi/2$ and π , subtract π before applying the inverse tangent.

4. If the projection of $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ on $\vec{b} = 2\hat{i} + \lambda\hat{k}$ is zero, then the value of λ is

- (A) 0
(B) 1
(C) $-\frac{2}{3}$
(D) $-\frac{3}{2}$

[1 mark]

Solution:

Concept:

- The scalar projection of $\vec{a}$ on a nonzero vector $\vec{b}$ is $(\vec{a} \cdot \vec{b})/|\vec{b}|$.
- The dot product is the sum of products of corresponding components; zero projection therefore requires zero dot product.

Step 1: Set up the orthogonality condition Here $\vec{b} = (2, 0, \lambda)$, so $|\vec{b}| = \sqrt{4 + \lambda^2} > 0$ for every real λ . Consequently, its projection is zero precisely when

$$\vec{a} \cdot \vec{b} = 1(2) + (-2)(0) + 3\lambda = 0.$$

Step 2: Solve and check the options The equation is $2 + 3\lambda = 0$, giving $\lambda = -2/3$. Substitution yields $2 + 3(-2/3) = 0$. The other printed values 0, 1, and $-3/2$ give dot products 2, 5, and $-5/2$, respectively, so none produces zero projection.

Step 3: Conclusion The required value is $\lambda = -\frac{2}{3}$. Hence, the correct option is (C).

Quick Tip to Solve: For a zero projection, set the dot product to zero; the length denominator need not be evaluated.

5. The vector equation of the line passing through the point $(-1, 5, 4)$ and perpendicular to the plane $z = 0$ is

- (A) $\vec{r} = -\hat{i} + 5\hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j})$
(B) $\vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}$
(C) $\vec{r} = \hat{i} - 5\hat{j} - 4\hat{k} + \lambda\hat{k}$
(D) $\vec{r} = \lambda\hat{k}$

[1 mark]

Solution:

Concept:

- A line through position vector $\vec{a}$ with direction vector $\vec{d}$ has equation $\vec{r} = \vec{a} + \lambda\vec{d}$, where $\lambda \in \mathbb{R}$.
- A line perpendicular to a plane has the direction of the plane normal. The plane $z = 0$ has normal $\hat{k}$.

Step 1: Identify the point and direction The given point has position vector $\vec{a} = -\hat{i} + 5\hat{j} + 4\hat{k}$. The direction vector may be chosen as $\hat{k}$, since it is normal to the plane $z = 0$. Thus the line keeps $x = -1$ and $y = 5$, with only its z -coordinate varying.

Quick Tip to Solve: For a line perpendicular to $z=0$, hold x and y fixed and let only z vary.

Step 2: Write the line and compare the options The equation is

$$\vec{r} = -\hat{i} + 5\hat{j} + 4\hat{k} + \lambda\hat{k} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}.$$

At $\lambda = 0$, it passes through $(-1, 5, 4)$. Option (A) has direction $\hat{i} + \hat{j}$, which is parallel to the plane. Options (C) and (D) have fixed x, y coordinates different from those of the given point.

Step 3: Conclusion The line is $\vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}$, $\lambda \in \mathbb{R}$. Hence, the correct option is (B).

Quick Tip to Solve: For a line perpendicular to $z=0$, hold x and y fixed and let only z vary.

6. The number of arbitrary constants in the particular solution of a differential equation of second order is (are)

- (A) 0
- (B) 1
- (C) 2
- (D) 3

[1 mark]

Solution:

Concept:

- The general solution of a second-order differential equation normally contains two independent arbitrary constants.
- A particular solution is obtained by assigning definite values to all arbitrary constants, usually using the supplied conditions.

Step 1: Distinguish general and particular solutions The order tells us how many arbitrary constants occur in the general solution. Thus second order corresponds to two constants at that stage. The question, however, specifically asks about a particular solution, after those constants have been fixed.

Step 2: Count the constants still arbitrary For example, a general solution may be $y = C_1x + C_2$. Choosing $C_1 = 2$ and $C_2 = 3$ gives the particular solution $y = 2x + 3$: its numbers are fixed and neither is arbitrary. Therefore no arbitrary constant remains. Choosing 2 confuses the general solution with a particular one; 1 would still leave a family, while 3 has no basis in the given order.

Step 3: Conclusion There are 0 arbitrary constants in the particular solution. Hence, the correct option is (A).

Quick Tip to Solve: Underline the word particular: every arbitrary constant has already been assigned a value.

7. $\int_{-\pi/4}^{\pi/4} \sec^2 x \, dx$ is equal to

- (A) -1
- (B) 0
- (C) 1
- (D) 2

[1 mark]

Solution:

Concept:

- The antiderivative of $\sec^2 x$ is $\tan x$, because $d(\tan x)/dx = \sec^2 x$.
- A definite integral is evaluated as the antiderivative at the upper limit minus its value at the lower limit.

Step 1: Use the antiderivative The function $\sec^2 x$ is continuous throughout $[-\pi/4, \pi/4]$, so the fundamental theorem of calculus applies:

$$\int_{-\pi/4}^{\pi/4} \sec^2 x \, dx = [\tan x]_{-\pi/4}^{\pi/4}.$$

Step 2: Substitute both limits We have $\tan(\pi/4) = 1$ and $\tan(-\pi/4) = -1$. Hence the integral is $1 - (-1) = 2$. Symmetric limits do not make this integral zero, since $\sec^2 x$ is even and positive. The value 1 would omit the contribution from the negative half of the interval, and -1 is incompatible with a positive integrand and increasing limits.

Step 3: Conclusion The integral equals $\boxed{2}$. Hence, the correct option is (D).

Quick Tip to Solve: Subtract the lower-limit value with brackets; subtracting minus one adds one.

8. The length of the perpendicular drawn from the point $(4, -7, 3)$ on the y -axis is

- (A) 3 units
- (B) 4 units
- (C) 5 units
- (D) 7 units

[1 mark]

Solution:

Concept:

- A point on the y -axis has coordinates $(0, t, 0)$.
- The shortest distance is obtained by keeping the same y -coordinate and using the perpendicular components.

Step 1: Locate the perpendicular foot Let $P = (4, -7, 3)$. The foot on the y -axis is $Q = (0, -7, 0)$: the displacement $\vec{QP} = (4, 0, 3)$ has zero dot product with the axis direction $(0, 1, 0)$, so PQ is perpendicular to the axis.

Quick Tip to Solve: For distance from the y -axis, discard the y -coordinate and calculate the square root of the sum of squares of x and z .

Step 2: Calculate its length By the three-dimensional distance formula,

$$PQ = \sqrt{(4-0)^2 + (-7+7)^2 + (3-0)^2} = \sqrt{16+9} = 5.$$

Options (A) and (B) use only one perpendicular component. Option (D) uses the coordinate along the axis, which does not contribute to this distance.

Step 3: Conclusion The required length is 5 units, option (C).

Quick Tip to Solve: For distance from the y-axis, discard the y-coordinate and calculate the square root of the sum of squares of x and z.

9. If A and B are two independent events with $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, then $P(B' | A)$ is equal to

- (A) $\frac{1}{4}$
- (B) $\frac{1}{3}$
- (C) $\frac{3}{4}$
- (D) 1

[1 mark]

Solution:

Concept:

- Independence gives $P(A \cap B) = P(A)P(B)$.
- For $P(A) > 0$, $P(B | A) = P(A \cap B)/P(A)$; a conditional complement has probability $1 - P(B | A)$.

Step 1: Use independence Since $P(A) = 1/3 > 0$, conditioning on A is defined. Independence gives

$$P(B | A) = \frac{P(A)P(B)}{P(A)} = P(B) = \frac{1}{4}.$$

Thus knowledge that A occurred does not change the probability of B .

Step 2: Take the conditional complement Within the event A , the events B and B' are complementary. Hence

$$P(B' | A) = 1 - P(B | A) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Option (A) gives the probability of B , not its complement. Option (B) merely repeats $P(A)$, and option (D) would require B to be impossible given A .

Step 3: Conclusion $P(B' | A) = \frac{3}{4}$, option (C).

Quick Tip to Solve: When events are independent, first remove the conditioning; then take the complement if the requested event has a prime.

10. The corner points of the feasible region determined by the system of linear inequalities are $(0, 0)$, $(4, 0)$, $(2, 4)$ and $(0, 5)$. If the maximum value of $z = ax + by$, where $a, b > 0$ occurs at both $(2, 4)$ and $(4, 0)$, then

- (A) $a = 2b$
- (B) $2a = b$
- (C) $a = b$
- (D) $3a = b$

[1 mark]

Solution:

Concept:

- A linear objective has the same value at any two points at which its maximum is attained.
- At a vertex (x, y) , substitute its coordinates in $z = ax + by$.

Step 1: Equate the two maximum values At $(2, 4)$, the value is $2a + 4b$; at $(4, 0)$, it is $4a$. Since both attain the maximum,

$$2a + 4b = 4a \implies 4b = 2a \implies a = 2b.$$

Step 2: Check the remaining vertices and options With $a = 2b$ and $b > 0$, both specified vertices give $8b$. The values at $(0, 0)$ and $(0, 5)$ are 0 and $5b$, respectively, which are smaller. Thus the relation is consistent with a maximum at both stated points. Options (B), (C) and (D) impose different positive ratios of a to b , so none satisfies the equality just derived.

Step 3: Conclusion $a = 2b$, option (A).

Quick Tip to Solve: If two vertices attain the same optimum, equate their objective values before checking the other vertices.

11. A relation R in a set A is called _____, if $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

[1 mark]

Solution:

Concept:

- A relation is symmetric when reversing any ordered pair in the relation produces another pair in the relation.
- Reflexivity requires $(a, a) \in R$; transitivity concerns a chain of two related pairs.

Step 1: Compare the printed condition with the definition The condition says that whenever $(a_1, a_2) \in R$, the reversed pair (a_2, a_1) also belongs to R , for every choice of $a_1, a_2 \in A$. This is precisely the defining implication for a symmetric relation.

Step 2: Distinguish other relation properties The statement does not require every element to be related to itself, so it is not the definition of reflexivity. It does not state that (a_1, a_2) and (a_2, a_3) imply (a_1, a_3) , so it is not the definition of transitivity either. The relevant feature is that the order of the two entries can be reversed.

Step 3: Conclusion The blank is symmetric.

Quick Tip to Solve: Look for the reversal of an ordered pair: that is the direct signal for symmetry of a relation.

12. The greatest integer function defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at $x =$ _____ . [1 mark]

Solution:

Concept:

- The greatest integer function $[x]$ is the largest integer less than or equal to x .
- Differentiability implies continuity; a jump discontinuity therefore prevents differentiability.

Step 1: Describe the function inside its domain For $0 < x < 1$, $f(x) = 0$, while for $1 \leq x < 2$, $f(x) = 1$. Thus it is constant, and has derivative zero, on each open interval $(0, 1)$ and $(1, 2)$. The only integer strictly between the endpoints of the given domain is 1.

Step 2: Check the jump at the interior integer At $x = 1$,

$$\lim_{x \rightarrow 1^-} f(x) = 0, \quad f(1) = \lim_{x \rightarrow 1^+} f(x) = 1.$$

The unequal one-sided limits show that f is not continuous there and hence cannot be differentiable there. The points 0 and 2 are excluded by the strict inequalities in the domain.

Step 3: Conclusion The blank is $x = 1$.

Quick Tip to Solve: For the greatest integer function, check integer points inside the stated domain and exclude open endpoints.

13(a). If A is a matrix of order 3×2 , then the order of the matrix A' is _____. [1 mark]

Solution:

Concept:

- Transposing a matrix interchanges its rows and columns.
- If a matrix has order $m \times n$, its transpose has order $n \times m$.

Step 1: Interpret the given order The order 3×2 means that A has three rows and two columns. The first number always counts rows, and the second counts columns; the matrix has six entries in total.

Step 2: Interchange rows and columns In this question, A' denotes the transpose of A . Each of the two columns of A becomes a row of A' , giving two rows. Each of the three rows of A becomes a column of A' , giving three columns. The total number of entries stays six, which checks the resulting dimensions.

Step 3: Conclusion The order of A' is 2×3 .

Quick Tip to Solve: For a transpose, reverse the two numbers in the order; the number of entries remains unchanged.

13(b). A square matrix A is said to be skew-symmetric, if _____.

[1 mark]

Solution:

Concept:

- A square matrix A is skew-symmetric when its transpose equals its negative.
- Entrywise, this condition is $a_{ji} = -a_{ij}$ for all row and column indices.

Step 1: Apply the definition The transpose A' is obtained by interchanging rows and columns. For skew-symmetry, the entry in position (j, i) must be the negative of the entry in position (i, j) . Expressing that requirement as a matrix equation gives $A' = -A$.

Step 2: Check the distinguishing consequence Equivalently, $A + A' = O$, where O is the zero matrix of the same order. For a diagonal entry the condition gives $a_{ii} = -a_{ii}$, so each diagonal entry is zero for real matrices. In contrast, the equation $A' = A$ defines a symmetric matrix and would not fill this blank correctly.

Step 3: Conclusion The blank is $A' = -A$.

Quick Tip to Solve: Remember the minus sign: transpose equal to the negative of the matrix is the condition for skew-symmetry.

14(a). The equation of the normal to the curve $y^2 = 8x$ at the origin is _____.

[1 mark]

Solution:

Concept:

- A normal is perpendicular to the tangent at the same point of the curve.
- For $y^2 = 8x$, write $x = y^2/8$ to find the tangent direction at the origin without dividing by zero.

Step 1: Find the tangent direction Differentiating with respect to y , we get $dx/dy = y/4$. At $(0, 0)$, this derivative is zero. Thus the tangent has no first-order horizontal change as y varies and is the vertical line $x = 0$.

Step 2: Use perpendicularity The normal to a vertical tangent is horizontal. Since the point of contact is the origin, its horizontal line has ordinate zero throughout. Equivalently, the normal slope is $-dx/dy = 0$ at this point.

Step 3: Conclusion The equation of the normal is $y = 0$.

Quick Tip to Solve: At the vertex of a right-opening parabola, the tangent is vertical and the normal is its horizontal axis.

14(b). The radius of a circle is increasing at the uniform rate of 3 cm/sec. At the instant when the radius of the circle is 2 cm, its area increases at the rate of _____ cm²/s. [1 mark]

Solution:

Concept:

- The area of a circle of radius r is $A = \pi r^2$.
- The chain rule gives $dA/dt = 2\pi r dr/dt$, where t denotes time.

Step 1: Differentiate the area Both the radius and the area depend on time. Differentiating $A = \pi r^2$ with respect to t ,

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

The radius is increasing, so its derivative is positive.

Step 2: Substitute the instantaneous radius At the stated instant, $r = 2$ cm and $dr/dt = 3$ cm/s. Hence

$$\frac{dA}{dt} = 2\pi(2)(3) = 12\pi \text{ cm}^2/\text{s}.$$

The units are area per unit time, as required. This rate is instantaneous because the radius changes continuously with time.

Step 3: Conclusion The blank is 12π , and the area increases at $12\pi \text{ cm}^2/\text{s}$.

Quick Tip to Solve: Differentiate before substituting the radius: the area rate equals circumference times the radius rate.

15. The position vectors of two points A and B are $\vec{OA} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{OB} = 2\hat{i} - \hat{j} + 2\hat{k}$, respectively. The position vector of a point P which divides the line segment joining A and B in the ratio 2 : 1 is _____. [1 mark]

Solution:

Concept:

- For internal division $AP : PB = m : n$, the position vector is $\vec{p} = (n\vec{a} + m\vec{b})/(m + n)$.
- The coefficient of each endpoint vector is the ratio number attached to the opposite segment.

Step 1: Apply the internal section formula Here $\vec{a} = 2\hat{i} - \hat{j} - \hat{k}$, $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$, and $AP : PB = 2 : 1$. Therefore

$$\vec{p} = \frac{\vec{a} + 2\vec{b}}{3}.$$

Since the question specifies the line segment, internal division applies.

Step 2: Combine the components The numerator is

$$\vec{a} + 2\vec{b} = 6\hat{i} - 3\hat{j} + 3\hat{k}.$$

Quick Tip to Solve: For a point twice as far from A as from B, weight B twice as heavily in the internal section formula.

Division by 3 leaves the common first two coordinates unchanged. The third coordinate is $(-1 + 2 \cdot 2)/3 = 1$, lying between the endpoint values -1 and 2 .

Step 3: Conclusion The required position vector is $\boxed{2\hat{i} - \hat{j} + \hat{k}}$.

Quick Tip to Solve: For a point twice as far from A as from B, weight B twice as heavily in the internal section formula.

16. If $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 2 & 3 \\ 3 & 3 & 5 \end{bmatrix}$, then find $A \cdot (\text{adj } A)$.

[1 mark]

Solution:

Concept:

- For any square matrix A , $A \text{ adj } A = (\det A)I$.
- Expand the determinant along a row with zeros to minimise the calculation; I is the identity matrix of the same order.

Step 1: Calculate the determinant The first row has only one nonzero entry. Its cofactor sign is positive, so

$$\det A = 2 \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 2(2 \cdot 5 - 3 \cdot 3) = 2.$$

The entries below the first entry do not enter this expansion.

Step 2: Use the adjoint identity The given matrix has order 3, so use I_3 . Multiplication by its adjoint is therefore the scalar determinant times I_3 . Thus every diagonal entry of the product is 2 and every off-diagonal entry is zero; computing the nine cofactors separately is unnecessary.

Step 3: Conclusion

$$A \text{ adj } A = 2I_3 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Quick Tip to Solve: Before computing an adjoint explicitly, check whether the requested product is immediately given by the determinant identity.

17(a). Find: $\int x^4 \log x \, dx$

[1 mark]

Solution:

Concept:

- Integration by parts states $\int u \, dv = uv - \int v \, du$.

Quick Tip to Solve: In a polynomial times logarithm integral, differentiate the logarithm and integrate the polynomial.

- Choose $u = \log x$, whose derivative is $1/x$, and integrate the polynomial factor; the real logarithm requires $x > 0$.

Step 1: Choose the factors Set $u = \log x$ and $dv = x^4 dx$. Then $du = dx/x$ and $v = x^5/5$. The logarithmic factor becomes simpler on differentiation.

Step 2: Apply integration by parts Substitution into the formula gives

$$\int x^4 \log x \, dx = \frac{x^5}{5} \log x - \frac{1}{5} \int x^4 \, dx = \frac{x^5}{5} \log x - \frac{x^5}{25} + C.$$

Differentiating the first term produces an extra $x^4/5$, which is exactly cancelled by differentiating the second term.

Step 3: Conclusion

$$\boxed{\int x^4 \log x \, dx = \frac{x^5}{5} \log x - \frac{x^5}{25} + C}, \quad x > 0.$$

Quick Tip to Solve: In a polynomial times logarithm integral, differentiate the logarithm and integrate the polynomial.

OR

17(b). Find: $\int \frac{2x}{\sqrt[3]{x^2+1}} \, dx$

[1 mark]

Solution:

Concept:

- Use substitution when the numerator is the derivative of the expression inside a root.
- The power rule is $\int u^n \, du = u^{n+1}/(n+1) + C$ for $n \neq -1$.

Step 1: Substitute the inner expression Let $u = x^2 + 1$. Then $du = 2x \, dx$, which matches the full numerator. Since $x^2 + 1 > 0$ for real x , the cube-root denominator is nonzero and we may write it as $u^{1/3}$.

Step 2: Integrate the resulting power The integral becomes

$$\int u^{-1/3} \, du = \frac{u^{2/3}}{2/3} + C = \frac{3}{2} u^{2/3} + C.$$

Restoring $u = x^2 + 1$ completes the substitution. Differentiation gives $(3/2)(2/3)(x^2 + 1)^{-1/3}(2x)$, recovering the integrand.

Step 3: Conclusion

$$\boxed{\int \frac{2x}{\sqrt[3]{x^2+1}} \, dx = \frac{3}{2} (x^2 + 1)^{2/3} + C}.$$

Quick Tip to Solve: Absorb the entire factor $2x \, dx$ into du ; then the cube root becomes an ordinary fractional power.

18. Evaluate: $\int_1^3 |2x - 1| dx$

[1 mark]

Solution:

Concept:

- For $z \geq 0$, $|z| = z$; for $z < 0$, $|z| = -z$.
- Check the sign throughout the integration interval before removing an absolute value.

Step 1: Determine the sign on the interval The expression $2x - 1$ vanishes at $x = 1/2$, which is outside $[1, 3]$. For every x in this interval, $2x - 1 \geq 1 > 0$. Thus $|2x - 1| = 2x - 1$ throughout and no splitting of the integral is necessary.

Step 2: Evaluate at the limits An antiderivative of $2x - 1$ is $x^2 - x$. Therefore

$$\int_1^3 |2x - 1| dx = [x^2 - x]_1^3 = (9 - 3) - (1 - 1) = 6.$$

The positive result agrees with the positive integrand on the full interval.

Step 3: Conclusion The value of the definite integral is $\boxed{6}$.

Quick Tip to Solve: Locate the zero of the expression inside the absolute value first; split only if that zero lies inside the limits.

19. Two cards are drawn at random and one-by-one without replacement from a well-shuffled pack of 52 playing cards. Find the probability that one card is red and the other is black.

[1 mark]

Solution:

Concept:

- There are 26 red cards and 26 black cards in a standard pack of 52 cards.
- For successive draws without replacement, multiply the first-draw probability by the conditional probability for the second draw.

Step 1: Count the two possible orders The required event consists of two disjoint possibilities: red followed by black, or black followed by red. If the first card is red, all 26 black cards remain among the 51 remaining cards, and similarly for the reverse order.

Step 2: Add their probabilities By the multiplication rule and addition of disjoint events,

$$P(\text{one red and one black}) = \frac{26}{52} \frac{26}{51} + \frac{26}{52} \frac{26}{51} = \frac{26}{51}.$$

Each order must be counted because the question does not prescribe which colour is drawn first.

Step 3: Conclusion The required probability is $\boxed{\frac{26}{51}}$.

Quick Tip to Solve: Include both colour orders; after one draw the denominator is 51, not 52.

20. Find: $\int \frac{dx}{\sqrt{9-4x^2}}$

[1 mark]

Solution:

Concept:

- The standard inverse-sine integral is $\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$, for $-1 < u < 1$.
- A linear substitution must also transform the differential; here set $u = 2x/3$.

Step 1: Reduce to the standard form Let $u = \frac{2x}{3}$. Then $dx = \frac{3}{2} du$ and $\sqrt{9-4x^2} = 3\sqrt{1-u^2}$. Thus

$$\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}}.$$

Step 2: Integrate and substitute back Applying the standard formula gives

$$\frac{1}{2} \sin^{-1} u + C = \frac{1}{2} \sin^{-1} \left(\frac{2x}{3} \right) + C.$$

This real antiderivative applies on $-3/2 < x < 3/2$, where the given integrand is defined.

Step 3: Conclusion $\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2} \sin^{-1} \left(\frac{2x}{3} \right) + C$.

Quick Tip to Solve: The coefficient of x inside the square root contributes a factor of one-half.

Section B (12 marks)

Question numbers 21 to 26 carry 2 marks each.

21(a). Prove that $\sin^{-1}(2x\sqrt{1-x^2}) = 2 \cos^{-1} x$, $\frac{1}{\sqrt{2}} \leq x \leq 1$.

[2 marks]

Solution:

Concept:

- The principal range of $\cos^{-1} x$ is $[0, \pi]$, and that of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$.
- The double-angle identity is $\sin 2\theta = 2 \sin \theta \cos \theta$. Inverting sine is valid directly only within its principal interval.

Step 1: Use the given interval Put $\theta = \cos^{-1} x$, so $x = \cos \theta$. Since $1/\sqrt{2} \leq x \leq 1$ and cosine decreases on $[0, \pi]$, we have

$$0 \leq \theta \leq \frac{\pi}{4}, \quad 0 \leq 2\theta \leq \frac{\pi}{2}.$$

Also $\sin \theta \geq 0$, hence $\sqrt{1-x^2} = \sqrt{1-\cos^2 \theta} = \sin \theta$.

Step 2: Apply the identity within the principal range Consequently,

$$2x\sqrt{1-x^2} = 2 \cos \theta \sin \theta = \sin 2\theta.$$

Quick Tip to Solve: Check the range of the doubled angle before using inverse sine to cancel sine.

Because $2\theta \in [0, \pi/2]$, which lies in the principal range of inverse sine,

$$\sin^{-1}(2x\sqrt{1-x^2}) = \sin^{-1}(\sin 2\theta) = 2\theta.$$

The interval check also includes both endpoints of the stated domain.

Step 3: Conclusion Therefore $\sin^{-1}(2x\sqrt{1-x^2}) = 2\cos^{-1}x$ for $1/\sqrt{2} \leq x \leq 1$, as required.

Quick Tip to Solve: Check the range of the doubled angle before using inverse sine to cancel sine.

OR

21(b). Consider a bijective function $f : \mathbb{R}_+ \rightarrow (7, \infty)$ given by $f(x) = 16x^2 + 24x + 7$, where $\mathbb{R}_+$ is the set of all positive real numbers. Find the inverse function of f . [2 marks]

Solution:

Concept:

- To obtain an inverse, write $y = f(x)$ and solve for the original input x in terms of the output y .
- The inverse has domain $(7, \infty)$ and codomain $\mathbb{R}_+$. The positive-input condition selects the admissible square-root branch.

Step 1: Complete the square Let $y = 16x^2 + 24x + 7$, with $x > 0$. Completing the square gives

$$y = (4x + 3)^2 - 2, \quad (4x + 3)^2 = y + 2.$$

Since $x > 0$, we have $4x + 3 > 3$; therefore $4x + 3 = \sqrt{y + 2}$, rather than the negative square root.

Step 2: Solve for the input and verify its domain It follows that

$$x = \frac{\sqrt{y + 2} - 3}{4}.$$

For every $y > 7$, the square root exceeds 3, so this expression is positive. Conversely, substitution into $(4x + 3)^2 - 2$ gives exactly y . Also, for $x > 0$, $\sqrt{(4x + 3)^2} = 4x + 3$, so the reverse composition returns x . Thus the chosen branch gives the inverse on precisely the specified sets.

Step 3: Conclusion The inverse is $f^{-1} : (7, \infty) \rightarrow \mathbb{R}_+, \quad f^{-1}(x) = \frac{\sqrt{x + 2} - 3}{4}.$

Quick Tip to Solve: Complete the square first, and use the original positive domain to choose the root.

22. If $x = at^2$, $y = 2at$, then find $\frac{d^2y}{dx^2}$.

[2 marks]

Solution:

Concept:

- For a parametric curve, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ whenever $dx/dt \neq 0$.
- The second derivative is $\frac{d^2y}{dx^2} = \frac{d}{dt}(dy/dx) / \frac{dx}{dt}$; another division by dx/dt is essential.

Step 1: Find the first derivative Differentiating $x = at^2$ and $y = 2at$ with respect to the parameter t , treating a as a constant, yields

$$\frac{dx}{dt} = 2at, \quad \frac{dy}{dt} = 2a.$$

Thus at a regular point, where $2at \neq 0$,

$$\frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}.$$

Step 2: Differentiate with respect to x again First differentiate the first derivative with respect to t :

$$\frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{1}{t} \right) = -\frac{1}{t^2}.$$

Then apply the chain rule to convert this derivative to one with respect to x :

$$\frac{d^2y}{dx^2} = \frac{-1/t^2}{2at} = -\frac{1}{2at^3}.$$

The formula holds when $a \neq 0$ and $t \neq 0$; if $a = 0$, the parametrisation reduces to a single point and does not define this derivative.

Step 3: Conclusion $\frac{d^2y}{dx^2} = -\frac{1}{2at^3}$, at regular points of the given parametrisation.

Quick Tip to Solve: After differentiating dy/dx with respect to t , divide by dx/dt once more.

23. Find the points on the curve $y = x^3 - 3x^2 - 4x$ at which the tangent lines are parallel to the line $4x + y - 3 = 0$.

[2 marks]

Solution:

Concept:

- The slope of the tangent to $y = f(x)$ at an abscissa x is $f'(x)$. Parallel nonvertical lines have equal slopes.
- A line in the form $y = mx + c$ has slope m ; substitute each admissible abscissa into the original curve to find its ordinate.

Step 1: Find the required tangent slope The given line is $4x + y - 3 = 0$, or $y = -4x + 3$. Its slope is

Quick Tip to Solve: Read the slope as -4 by isolating y , then use the derivative only to find x ; use the original curve for y .

therefore -4 . Differentiating the curve with respect to x gives

$$\frac{dy}{dx} = 3x^2 - 6x - 4.$$

The derivative must equal the slope of the given line, not its intercept.

Step 2: Find the points on the curve Setting the tangent slope equal to -4 , we obtain

$$3x^2 - 6x - 4 = -4 \implies 3x(x - 2) = 0.$$

Thus the possible abscissae are $x = 0$ and $x = 2$. In the original equation, $y(0) = 0$, while $y(2) = 8 - 12 - 8 = -12$. Both abscissae give derivative -4 , so both points meet the parallelism condition.

Step 3: Conclusion The required points are $(0, 0)$ and $(2, -12)$.

Quick Tip to Solve: Read the slope as -4 by isolating y , then use the derivative only to find x ; use the original curve for y .

24(a). Find a unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$ where $\vec{a} = 5\hat{i} + 6\hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 6\hat{j} + 2\hat{k}$. [2 marks]

Solution:

Concept:

- The cross product $\vec{a} \times \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.
- For a nonzero vector $\vec{v}$, the unit vector in its direction is $\vec{v}/|\vec{v}|$, where $|\vec{v}|$ is its magnitude.

Step 1: Calculate a common perpendicular Using the determinant expansion with its alternating signs,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 6 & -2 \\ 7 & 6 & 2 \end{vmatrix} \\ &= (12 + 12)\hat{i} - (10 + 14)\hat{j} + (30 - 42)\hat{k} \\ &= 24\hat{i} - 24\hat{j} - 12\hat{k} \\ &= 12(2\hat{i} - 2\hat{j} - \hat{k}).\end{aligned}$$

This vector is nonzero, so it can be normalised.

Step 2: Normalise and verify The magnitude of $2\hat{i} - 2\hat{j} - \hat{k}$ is $\sqrt{4 + 4 + 1} = 3$. Dividing by this magnitude gives $\vec{u} = (2\hat{i} - 2\hat{j} - \hat{k})/3$. Its dot products with the given vectors are $(10 - 12 + 2)/3 = 0$ and $(14 - 12 - 2)/3 = 0$, respectively. Also $|\vec{u}| = 1$, confirming both required perpendicularities and the unit length.

Step 3: Conclusion One required unit vector is $\frac{2\hat{i} - 2\hat{j} - \hat{k}}{3}$. Its negative is also a valid answer.

Quick Tip to Solve: Remove the common factor 12 from the cross product before computing its magnitude; retain the middle-component minus sign.

24(b). Find the volume of the parallelepiped whose adjacent edges are represented by $2\vec{a}$, $-\vec{b}$ and $3\vec{c}$, where

$$\vec{a} = \hat{i} - \hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}, \text{ and } \vec{c} = 2\hat{i} - \hat{j} + 3\hat{k}.$$

[2 marks]

Solution:**Concept:**

- The volume of a parallelepiped with adjacent edge vectors $\vec{u}, \vec{v}, \vec{w}$ is $|\vec{u} \cdot (\vec{v} \times \vec{w})|$.
- The scalar triple product is linear in each vector, so edge multipliers multiply together; the absolute value makes the volume nonnegative.

Step 1: Account for the edge multipliers The actual edges are $2\vec{a}, -\vec{b}, 3\vec{c}$, rather than $\vec{a}, \vec{b}, \vec{c}$. Hence

$$V = |(2\vec{a}) \cdot ((-\vec{b}) \times (3\vec{c}))| = |-6 \vec{a} \cdot (\vec{b} \times \vec{c})|.$$

We first calculate the scalar triple product of the three unscaled vectors.

Step 2: Evaluate the scalar triple product The cross product is

$$\begin{aligned} \vec{b} \times \vec{c} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix} \\ &= (12 - 5)\hat{i} - (9 + 10)\hat{j} + (-3 - 8)\hat{k} \\ &= 7\hat{i} - 19\hat{j} - 11\hat{k}. \end{aligned}$$

Therefore $\vec{a} \cdot (\vec{b} \times \vec{c}) = 7 + 19 - 22 = 4$. Substitution gives $V = |-6 \times 4| = 24$. The negative edge multiplier changes the orientation, but volume remains positive.

Step 3: Conclusion The volume of the parallelepiped is 24 cubic units.

Quick Tip to Solve: Factor out the three edge multipliers first, then take the absolute value only after evaluating the scalar triple product.

25. Find the value of k so that the lines $x = -y = kz$ and $x - 2 = 2y + 1 = -z + 1$ are perpendicular to each other.

[2 marks]

Solution:**Concept:**

- Two line directions are perpendicular when the dot product of their direction vectors is zero.
- To read a direction vector safely, set all expressions in a symmetric line equation equal to a parameter and take the coefficients of that parameter.

Step 1: Extract the direction vectors For the first line, take $x = -y = kz$. Using $z = t$ gives $(x, y, z) = (kt, -kt, t)$, so a direction vector is $(k, -k, 1)$. This avoids dividing by an unknown k . For

Quick Tip to Solve: Parameterise before taking the dot product: the coefficient 2 in $2y+1$ gives direction component $1/2$, not 2.

the second line set $x - 2 = 2y + 1 = -z + 1 = s$. Then

$$x = 2 + s, \quad y = \frac{s - 1}{2}, \quad z = 1 - s.$$

Its direction vector is $(1, 1/2, -1)$.

Step 2: Apply perpendicularity The constants in the parametric equations locate points on the lines and do not affect their directions. Thus the required condition is

$$(k, -k, 1) \cdot (1, 1/2, -1) = k - \frac{k}{2} - 1 = 0.$$

It follows that $k/2 = 1$. Substituting the resulting value gives directions $(2, -2, 1)$ and $(1, 1/2, -1)$, whose dot product is $2 - 1 - 1 = 0$, verifying perpendicularity.

Step 3: Conclusion The required value is $k = 2$.

Quick Tip to Solve: Parameterise before taking the dot product: the coefficient 2 in $2y+1$ gives direction component $1/2$, not 2.

26. The probability of finding a green signal on a busy crossing X is 30%. What is the probability of finding a green signal on X on two consecutive days out of three? [2 marks]

Solution:

Concept:

- Use the usual independent-day trial model: each day has green-signal probability $p = 0.30$ and non-green probability $q = 1 - p = 0.70$.
- For independent trials, multiply the probabilities within one sequence and add the probabilities of mutually exclusive sequences. Consecutive days must be adjacent.

Step 1: Identify the favourable sequences Let G denote a green signal and N a non-green signal. To have green on two consecutive days out of the three days, with exactly two green days, the possible sequences are GGN and NGG . The sequence GNG has two green signals, but they are not consecutive; it must therefore be excluded. The sequence GGG has green on all three days, rather than two out of three.

Step 2: Calculate and add the probabilities Under the independent-day model,

$$P(GGN) = 0.30 \times 0.30 \times 0.70 = 0.063,$$

$$P(NGG) = 0.70 \times 0.30 \times 0.30 = 0.063.$$

These events cannot occur together, so their probabilities add:

$$P = 0.063 + 0.063 = 2(0.30)^2(0.70) = 0.126.$$

Using the binomial factor $\binom{3}{2} = 3$ would also count GNG and would ignore the consecutive-days requirement.

Quick Tip to Solve: List GGN and NGG explicitly; a binomial count of three arrangements incorrectly includes the nonconsecutive sequence GNG .

Step 3: Conclusion The probability of green signals on exactly two consecutive days out of three is $0.126 = 12.6\%$.

Quick Tip to Solve: List GGN and NGG explicitly; a binomial count of three arrangements incorrectly includes the nonconsecutive sequence GNG.

Section C (24 marks)

Question numbers 27 to 32 carry 4 marks each.

27. Let N be the set of natural numbers and R be the relation on $N \times N$ defined by $(a, b)R(c, d)$ iff $ad = bc$ for all $a, b, c, d \in N$. Show that R is an equivalence relation. [4 marks]

Solution:

Concept:

- A relation is an equivalence relation if it is reflexive, symmetric and transitive. The elements of the set here are ordered pairs in $N \times N$.
- The definition $(a, b)R(c, d)$ means $ad = bc$. Since natural numbers are positive, each coordinate is nonzero and can be cancelled from an equality.

Step 1: Establish reflexivity Choose any ordered pair $(a, b) \in N \times N$. For this pair to be related to itself, the defining condition becomes $ab = ba$, which holds by commutativity of multiplication. Hence $(a, b)R(a, b)$ for every pair in the underlying set, and the relation is reflexive.

Step 2: Establish symmetry Let $(a, b), (c, d) \in N \times N$ and suppose $(a, b)R(c, d)$. The defining equality is $ad = bc$. Reverse the equality and commute the factors to obtain $cb = da$. This is exactly the cross-product condition for $(c, d)R(a, b)$. Thus reversing the order of any two related pairs preserves the relation, and R is symmetric.

Step 3: Establish transitivity Choose three arbitrary pairs $(a, b), (c, d), (e, f) \in N \times N$. Suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$. Applying the definition separately gives

$$ad = bc, \quad cf = de.$$

Multiply the first equality by f and substitute the second equality into its right side:

$$adf = bcf = bde.$$

Rearranging products, this says $d(af) = d(be)$. Since d is a positive natural number, cancel d to get $af = be$. The last equality is the defining condition for $(a, b)R(e, f)$. Consequently, the two assumed relations imply the required third relation for all three pairs, proving transitivity.

Step 4: Conclusion Since R is reflexive, symmetric and transitive, it is an equivalence relation on $N \times N$.

Quick Tip to Solve: In transitivity, multiply the first cross-product equality by f , substitute $cf = de$, and cancel the positive middle coordinate d .

28. If $y = e^{x^2 \cos x} + (\cos x)^x$, then find $\frac{dy}{dx}$.

[4 marks]

Solution:

Concept:

- The sum rule allows differentiation of each term separately. The chain rule gives $\frac{d}{dx}e^{u(x)} = e^{u(x)}u'(x)$.
- For a variable base and exponent, logarithmic differentiation simplifies the power. Work on any open interval where $\cos x > 0$, so that $\log(\cos x)$ and the real-valued power are defined.

Step 1: Differentiate the exponential term Put $u = x^2 \cos x$. Apply the product rule to the polynomial and trigonometric factors:

$$u' = 2x \cos x - x^2 \sin x.$$

The negative sign comes from $\frac{d}{dx} \cos x = -\sin x$. Using the chain rule, the derivative of the first term is

$$\frac{d}{dx} (e^{x^2 \cos x}) = e^{x^2 \cos x} (2x \cos x - x^2 \sin x).$$

In particular, the square on x contributes both the factor $2x$ and the factor x^2 in the two product-rule terms.

Step 2: Use logarithmic differentiation for the power Let $v = (\cos x)^x$. Both the base and exponent depend on x , so the ordinary constant-exponent power rule is insufficient. On the stated domain, take natural logarithms:

$$\log v = x \log(\cos x).$$

Differentiate both sides. The left side uses the chain rule, and the right side uses the product rule:

$$\begin{aligned} \frac{v'}{v} &= \log(\cos x) + x \frac{-\sin x}{\cos x} \\ &= \log(\cos x) - x \tan x. \end{aligned}$$

Multiplying by v and restoring its expression gives

$$v' = (\cos x)^x [\log(\cos x) - x \tan x].$$

Each contribution is necessary: the logarithm arises from differentiating the exponent, while the tangent term arises from differentiating the base.

Step 3: Apply the sum rule The original function is the sum of the two terms just differentiated. Therefore add their derivatives, retaining the different exponential and power factors. No product rule between these two original terms is needed because they are joined by addition.

Step 4: Conclusion

$$\begin{aligned} \frac{dy}{dx} &= e^{x^2 \cos x} (2x \cos x - x^2 \sin x) \\ &\quad + (\cos x)^x [\log(\cos x) - x \tan x]. \end{aligned}$$

Quick Tip to Solve: Differentiate the exponent of e by the product rule, and take logarithms separately for the term whose base and exponent both vary.

29. Find: $\int \sec^3 x \, dx$

[4 marks]

Solution:

Concept:

- Integration by parts gives $\int u \, dv = uv - \int v \, du$. Split the odd power $\sec^3 x$ into $\sec x \sec^2 x$.
- Use $\tan^2 x = \sec^2 x - 1$ and $\int \sec x \, dx = \ln |\sec x + \tan x| + C$, where C is arbitrary.

Step 1: Apply integration by parts Write $I = \int \sec^3 x \, dx$. Choose $u = \sec x$ and $dv = \sec^2 x \, dx$. Then

$$du = \sec x \tan x \, dx, \quad v = \tan x.$$

This choice works because the second factor integrates directly, while differentiating the first introduces a tangent. Integration by parts now yields

$$I = \sec x \tan x - \int \sec x \tan^2 x \, dx.$$

Step 2: Recover the original integral Substitute $\tan^2 x = \sec^2 x - 1$ in the remaining integral:

$$\begin{aligned} I &= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx \\ &= \sec x \tan x - I + \int \sec x \, dx. \end{aligned}$$

The original integral appears again on the right, with a negative sign. Bring it to the left, obtaining

$$2I = \sec x \tan x + \ln |\sec x + \tan x| + C.$$

Now divide by two. The resulting constant is still arbitrary and can again be denoted by C . The absolute value in the logarithm ensures validity wherever secant is defined, including intervals where its argument is negative.

Step 3: Verify by differentiation On an interval where $\cos x \neq 0$, the logarithmic term differentiates to $\sec x$. The derivative of the proposed expression is therefore

$$\begin{aligned} \frac{1}{2} \{ \sec x \tan^2 x + \sec^3 x + \sec x \} &= \frac{1}{2} \{ \sec x (\sec^2 x - 1) + \sec^3 x + \sec x \} \\ &= \sec^3 x. \end{aligned}$$

This check confirms both the positive sign of the logarithmic term and the factor of one half.

Step 4: Conclusion

$$\boxed{\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln |\sec x + \tan x| + C.}$$

Quick Tip to Solve: Take $\sec x$ as the first factor and integrate $\sec^2 x$; move the reappearing original integral to the left before dividing by two.

30(a). Find the general solution of the differential equation $y e^y dx = (y^3 + 2x e^y) dy$.

[4 marks]

Solution:

Concept:

- A linear equation in $x(y)$ has the form $dx/dy + P(y)x = Q(y)$.
- Its integrating factor is $IF = e^{\int P(y) dy}$. Multiplication by this factor converts the left side into a product derivative.

Step 1: Write the equation in linear form Since the coefficient containing x occurs beside dy , regard x as a function of y . For $y \neq 0$, divide the given equation by $y e^y dy$:

$$\frac{dx}{dy} = \frac{y^3 + 2x e^y}{y e^y} = y^2 e^{-y} + \frac{2x}{y}.$$

Hence the required linear form is

$$\frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y}.$$

Here $P(y) = -2/y$ and $Q(y) = y^2 e^{-y}$.

Step 2: Calculate and apply the integrating factor On an interval not containing zero,

$$IF = e^{\int (-2/y) dy} = e^{-2 \ln |y|} = \frac{1}{y^2}.$$

Multiply every term of the linear equation by this factor:

$$\frac{1}{y^2} \frac{dx}{dy} - \frac{2x}{y^3} = e^{-y}.$$

The product rule shows that its left side equals the derivative of x/y^2 , so

$$\frac{d}{dy} \left(\frac{x}{y^2} \right) = e^{-y}.$$

Step 3: Integrate and check the family Integrating with respect to y gives

$$\frac{x}{y^2} = -e^{-y} + C.$$

Here C is an arbitrary constant, and the negative sign arises because the derivative of e^{-y} is $-e^{-y}$. Conversely, differentiating this relation gives the product-derivative equation above; multiplying it by y^2 , then by $y e^y$, recovers the original equation. The division excluded $y = 0$, which separately satisfies the original equation because $y = 0$ and $dy = 0$ make both sides zero.

Step 4: Conclusion The general solution on intervals where $y \neq 0$ is

$$\boxed{\frac{x}{y^2} + e^{-y} = C, \quad \text{or} \quad x = y^2(C - e^{-y}).}$$

The original equation also admits the separate solution $y = 0$.

Quick Tip to Solve: Treat x as the dependent variable here: dividing by $y e^y$ reveals a linear equation with integrating factor y^{-2} .

30(b). Find the particular solution of the differential equation $x \frac{dy}{dx} = y - x \tan\left(\frac{y}{x}\right)$, given that $y = \frac{\pi}{4}$ at $x = 1$. [4 marks]

Solution:

Concept:

- A homogeneous equation has its derivative expressed as a function of y/x . Substitute $v = y/x$, so that $y = xv$ and $dy/dx = v + x dv/dx$.
- Separation uses $1/\tan v = \cot v$ and $\int \cot v dv = \ln |\sin v| + C$. The initial values determine the integration constant.
- The original equation requires $x \neq 0$ and $\cos(y/x) \neq 0$. Choose the solution branch containing the prescribed initial point.

Step 1: Substitute the ratio variable Dividing the equation by x , near the initial point $x = 1$, gives

$$\frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right).$$

Its right-hand side depends only on y/x . Put $y = xv$ and differentiate using the product rule:

$$\frac{dy}{dx} = v + x \frac{dv}{dx}.$$

Substitution gives

$$v + x \frac{dv}{dx} = v - \tan v, \quad x \frac{dv}{dx} = -\tan v.$$

The ratio terms cancel. The minus sign before the tangent is retained when separating variables.

Step 2: Separate and integrate The initial value of v is $\pi/4$, where the tangent is nonzero. On this solution branch,

$$\cot v dv = -\frac{dx}{x}.$$

Integrating both sides yields

$$\ln |\sin v| = -\ln |x| + C.$$

Combine logarithms, exponentiate, and absorb the possible sign into the arbitrary constant:

$$x \sin v = K, \quad x \sin\left(\frac{y}{x}\right) = K.$$

Division by $\tan v$ omits constant-ratio solutions $v = n\pi$, but these do not satisfy the prescribed initial value. Thus the required particular solution has been retained.

Step 3: Determine the constant and verify Insert $x = 1$ and $y = \pi/4$ into the integrated relation:

$$K = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Quick Tip to Solve: Substitute $v = y/x$ first to cancel the ratio terms, and apply the initial values directly to $x \sin v = K$ after integration.

Differentiating this relation implicitly gives

$$\sin\left(\frac{y}{x}\right) + \cos\left(\frac{y}{x}\right) \frac{x \frac{dy}{dx} - y}{x} = 0.$$

Wherever the original differential equation is defined, rearrangement recovers

$$x \frac{dy}{dx} = y - x \tan\left(\frac{y}{x}\right).$$

The initial values satisfy the relation as well. For an explicit expression, select the principal inverse-sine branch through $y/x = \pi/4$. This branch exists for $x > 1/\sqrt{2}$; the endpoint is excluded because the tangent in the original equation becomes undefined there.

Step 4: Conclusion The particular solution is

$$x \sin\left(\frac{y}{x}\right) = \frac{1}{\sqrt{2}}.$$

Equivalently, the branch through the initial point is $y = x \sin^{-1}(1/(\sqrt{2}x))$, for $x > 1/\sqrt{2}$.

Quick Tip to Solve: Substitute $v = y/x$ first to cancel the ratio terms, and apply the initial values directly to $x \sin v = K$ after integration.

31. A furniture trader deals in only two items – chairs and tables. He has ₹ 50,000 to invest and a space to store at most 35 items. A chair costs him ₹ 1,000 and a table costs him ₹ 2,000. The trader earns a profit of ₹ 150 and ₹ 250 on a chair and table, respectively. Formulate the above problem as an LPP to maximise the profit and solve it graphically. [4 marks]

Solution:

Concept:

- In a linear programming problem, decision variables represent quantities to be chosen, constraints express the available resources, and a linear objective expresses the quantity to maximise.
- The feasible region is the common region satisfying all inequalities. For a bounded feasible polygon, the maximum of a linear objective occurs at a corner point.

Step 1: Define the variables and formulate the LPP Let x be the number of chairs and y the number of tables purchased. The quantities cannot be negative, so $x \geq 0$ and $y \geq 0$. The investment restriction is

$$1000x + 2000y \leq 50000 \iff x + 2y \leq 50.$$

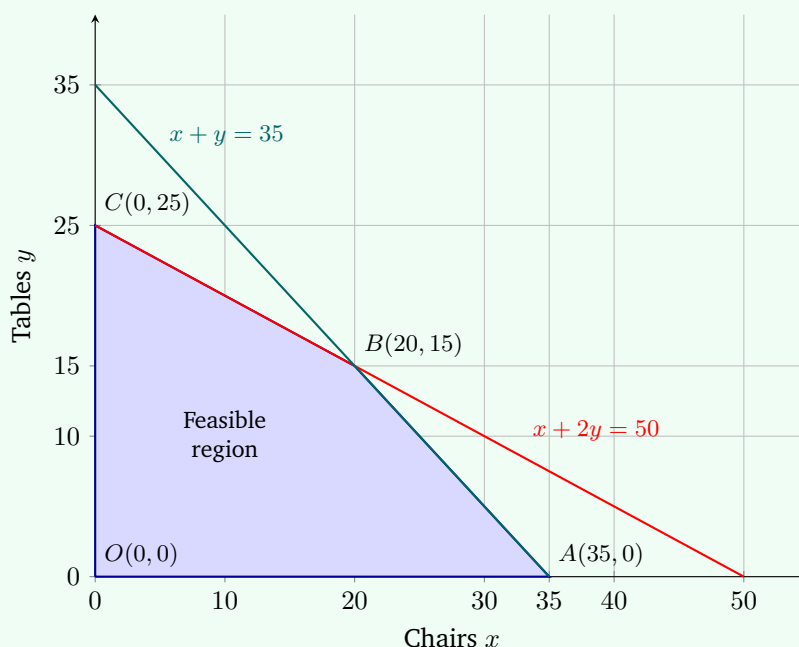
Since the trader can store at most 35 items, $x + y \leq 35$. The profit, in rupees, is $Z = 150x + 250y$. Thus the LPP is to maximise

$$Z = 150x + 250y$$

subject to $x + 2y \leq 50$, $x + y \leq 35$, $x \geq 0$, and $y \geq 0$. The resource inequalities allow unused money or storage; they do not require all resources to be used initially.

Quick Tip to Solve: Plot each resource line using its axis intercepts, shade the common side containing the origin, and evaluate profit only at the corners of that common region.

Step 2: Draw the boundary lines and identify the feasible region The investment boundary $x + 2y = 50$ passes through $(50, 0)$ and $(0, 25)$. The storage boundary $x + y = 35$ passes through $(35, 0)$ and $(0, 35)$. For both resource inequalities, the origin satisfies the inequality, so choose the side of each line containing the origin. Their common part in the first quadrant is the shaded quadrilateral $OABC$ shown below. At the intersection B , subtracting $x + y = 35$ from $x + 2y = 50$ gives $y = 15$, and then $x = 20$. Hence the feasible corner points are $O(0, 0)$, $A(35, 0)$, $B(20, 15)$, and $C(0, 25)$. The other axis intercepts of the boundary lines violate one of the resource restrictions and are not feasible corners.



Feasible region $OABC$ for the investment and storage restrictions.

Step 3: Evaluate the profit at each corner point Applying the corner-point method gives

Corner point (x, y)	$Z = 150x + 250y$ (rupees)
$(0, 0)$	0
$(35, 0)$	5250
$(20, 15)$	$3000 + 3750 = 6750$
$(0, 25)$	6250

The largest value in this table occurs at $B(20, 15)$. This point also represents whole numbers of items, so the graphical optimum is a valid purchase plan. It uses $20 + 15 = 35$ storage spaces and costs $1000(20) + 2000(15) = 50000$ rupees, satisfying both restrictions exactly.

Step 4: Conclusion The trader should purchase 20 chairs and 15 tables. The maximum profit is ₹ 6750.

Quick Tip to Solve: Plot each resource line using its axis intercepts, shade the common side containing the origin, and evaluate profit only at the corners of that common region.

32(a). There are two bags, I and II. Bag I contains 3 red and 5 black balls and Bag II contains 4 red and 3 black balls. One ball is transferred randomly from Bag I to Bag II and then a ball is drawn randomly from Bag II. If the ball so drawn is found to be black in colour, then find the probability that the transferred ball is also black. [4 marks]

Solution:

Concept:

- Conditional probability updates the probability after observing an event. Let T_B denote transfer of a black ball, T_R transfer of a red ball and D_B drawing a black ball from Bag II.
- The transfer events are mutually exclusive and exhaustive. Total probability gives $P(D_B) = P(T_B)P(D_B | T_B) + P(T_R)P(D_B | T_R)$.
- Bayes' theorem gives $P(T_B | D_B) = P(T_B \cap D_B)/P(D_B)$. After either transfer, Bag II contains one more ball than initially.

Step 1: Find the transfer probabilities Bag I has $3 + 5 = 8$ balls, each equally likely to be selected. Therefore,

$$P(T_B) = \frac{5}{8}, \quad P(T_R) = \frac{3}{8}.$$

These two events cover every transfer, since the bag contains only black and red balls.

Step 2: Count the balls in Bag II after each transfer After a black transfer, Bag II contains 4 red and 4 black balls. After a red transfer, it contains 5 red and 3 black balls. The total is 8 in both cases, so

$$P(D_B | T_B) = \frac{4}{8}, \quad P(D_B | T_R) = \frac{3}{8}.$$

Thus the joint probabilities of the transfer and the observed draw are

$$P(T_B \cap D_B) = \frac{5}{8} \cdot \frac{4}{8} = \frac{20}{64}, \quad P(T_R \cap D_B) = \frac{3}{8} \cdot \frac{3}{8} = \frac{9}{64}.$$

A black ball can be drawn even when the transferred ball was red, because Bag II already contained three black balls.

Step 3: Apply total probability and Bayes' theorem Since the two transfer events are disjoint, their black-draw probabilities add:

$$P(D_B) = \frac{20}{64} + \frac{9}{64} = \frac{29}{64}.$$

We are told that the draw is black, so the relevant sample space is restricted to this event. The required conditional probability is therefore

$$P(T_B | D_B) = \frac{P(T_B \cap D_B)}{P(D_B)} = \frac{20/64}{29/64}.$$

The numerator includes just the black-transfer route, while the denominator includes both routes that can produce the observed black draw.

Step 4: Conclusion The required probability that the transferred ball was also black is $\boxed{\frac{20}{29}}$.

Quick Tip to Solve: Count 8 balls in Bag II after either transfer, then compare the black-draw weights 20 and 9.

32(b). An urn contains 5 red, 2 white and 3 black balls. Three balls are drawn, one-by-one, at random without replacement. Find the probability distribution of the number of white balls. Also, find the mean and the variance of the number of white balls drawn. [4 marks]

Solution:

Concept:

- Let X count white balls in the three draws. Its possible values are 0, 1, 2.
- Without replacement, a sample of three balls has probability $P(X = x) = \frac{\binom{2}{x} \binom{8}{3-x}}{\binom{10}{3}}$, where the 8 non-white balls are the 5 red and 3 black balls together.
- For a discrete random variable, the mean is $E(X) = \sum xP(X = x)$ and the variance is $\text{Var}(X) = E(X^2) - [E(X)]^2$.

Step 1: Identify the possible outcomes and sample count The urn contains 10 balls: 2 white and 8 non-white. Three are drawn without replacement, so at most two can be white. Thus $X = 0, 1, 2$. The total number of equally likely unordered samples is $\binom{10}{3} = 120$.

Step 2: Calculate the probability distribution To obtain exactly x white balls, choose x of the 2 white balls and $3 - x$ of the 8 other balls. Therefore,

$$\begin{aligned} P(X = 0) &= \frac{\binom{2}{0} \binom{8}{3}}{120} = \frac{56}{120} = \frac{7}{15}, \\ P(X = 1) &= \frac{\binom{2}{1} \binom{8}{2}}{120} = \frac{56}{120} = \frac{7}{15}, \\ P(X = 2) &= \frac{\binom{2}{2} \binom{8}{1}}{120} = \frac{8}{120} = \frac{1}{15}. \end{aligned}$$

The distribution is

x	0	1	2
$P(X = x)$	7/15	7/15	1/15

The probabilities sum to $(7 + 7 + 1)/15 = 1$.

Step 3: Find the mean and second moment Using the probability distribution,

$$\begin{aligned} E(X) &= 0 \cdot \frac{7}{15} + 1 \cdot \frac{7}{15} + 2 \cdot \frac{1}{15} = \frac{9}{15} = \frac{3}{5}, \\ E(X^2) &= 0^2 \cdot \frac{7}{15} + 1^2 \cdot \frac{7}{15} + 2^2 \cdot \frac{1}{15} = \frac{11}{15}. \end{aligned}$$

Step 4: Calculate the variance Subtract the square of the mean from the second moment:

$$\text{Var}(X) = \frac{11}{15} - \left(\frac{3}{5}\right)^2 = \frac{55}{75} - \frac{27}{75} = \frac{28}{75}.$$

Step 5: Conclusion The probability distribution is $P(X = 0) = 7/15$, $P(X = 1) = 7/15$, $P(X = 2) = 1/15$. The mean is $\boxed{3/5}$ and the variance is $\boxed{28/75}$.

Quick Tip to Solve: Combine red and black as 8 non-white balls; select from 2 white and 8 non-white balls without replacement.

Section D (24 marks)

Question numbers 33 to 36 carry 6 marks each.

33(a). If $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}$, then find A^{-1} and use it to solve the following system of equations:

$$x + 2y - 3z = 6$$

$$3x + 2y - 2z = 3$$

$$2x - y + z = 2$$

[6 marks]

Solution:

Concept:

- A square matrix is invertible when its determinant is non-zero. Its inverse is $A^{-1} = \frac{\text{adj } A}{|A|}$, where the adjugate is the transpose of the cofactor matrix.
- For a system written as $AX = B$, premultiplication by A^{-1} gives $X = A^{-1}B$. The order of multiplication matters.

Step 1: Calculate the determinant Expanding along the first row, with alternating signs, gives

$$\begin{aligned} |A| &= 1(2 - 2) - 2(3 + 4) + (-3)(-3 - 4) \\ &= 0 - 14 + 21 = 7. \end{aligned}$$

Since this is non-zero, the matrix has an inverse and the given system has a unique solution.

Step 2: Find the cofactors and the adjugate Each cofactor is $C_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the determinant left after deleting row i and column j . The nine cofactors are

$$\begin{aligned} C_{11} &= 2 - 2 = 0, & C_{12} &= -(3 + 4) = -7, \\ C_{13} &= -3 - 4 = -7, & C_{21} &= -(2 - 3) = 1, \\ C_{22} &= 1 + 6 = 7, & C_{23} &= -(-1 - 4) = 5, \\ C_{31} &= -4 + 6 = 2, & C_{32} &= -(-2 + 9) = -7, \\ C_{33} &= 2 - 6 = -4. \end{aligned}$$

Thus the cofactor matrix and its transpose are

$$C = \begin{bmatrix} 0 & -7 & -7 \\ 1 & 7 & 5 \\ 2 & -7 & -4 \end{bmatrix}, \quad \text{adj } A = \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}.$$

Step 3: Obtain the inverse and use the matrix equation Dividing the adjugate by the determinant gives

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}.$$

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the constant column; then check the three original equations to catch a sign or ordering error.

The coefficients in the three equations occur in precisely the rows of A . Therefore, with

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix},$$

the system is $AX = B$. Premultiplying gives

$$\begin{aligned} X &= A^{-1}B \\ &= \frac{1}{7} \begin{bmatrix} 0(6) + 1(3) + 2(2) \\ -7(6) + 7(3) - 7(2) \\ -7(6) + 5(3) - 4(2) \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 7 \\ -35 \\ -35 \end{bmatrix}. \end{aligned}$$

Step 4: Check in all three equations The resulting coordinates give $1 + 2(-5) - 3(-5) = 6$, $3(1) + 2(-5) - 2(-5) = 3$, and $2(1) - (-5) + (-5) = 2$. Thus every original equation is satisfied.

Step 5: Conclusion $A^{-1} = \frac{1}{7} \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}$, and the unique solution is $x = 1, y = -5, z = -5$.

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the constant column; then check the three original equations to catch a sign or ordering error.

OR

33(b). Using properties of determinants, prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2).$$

[6 marks]

Solution:

Concept:

- Adding a multiple of one column to another leaves a determinant unchanged. A factor common to a whole column can be taken outside the determinant.
- Interchanging two columns changes the sign. These properties allow a polynomial determinant to be reduced without dividing by expressions that might be zero.
- Write $s = a + b + c$, $p = ab + bc + ca$, and $t = a^2 + b^2 + c^2 = s^2 - 2p$. These are auxiliary expressions used to simplify the column entries.

Step 1: Rewrite the entries and subtract the second column Let the determinant on the left be D .

Quick Tip to Solve: Set s equal to the sum of the variables; subtract the square column from the other two, then use their linear terms to form a constant column.

Since $b+c = s-a$, and similarly for the other rows, the first column consists of $(s-a)^2, (s-b)^2, (s-c)^2$. Also,

$$bc = a^2 - sa + p, \quad ca = b^2 - sb + p, \quad ab = c^2 - sc + p.$$

Apply $C_1 \leftarrow C_1 - C_2$ and $C_3 \leftarrow C_3 - C_2$. Neither operation changes the determinant. We obtain

$$D = \begin{vmatrix} s^2 - 2sa & a^2 & p - sa \\ s^2 - 2sb & b^2 & p - sb \\ s^2 - 2sc & c^2 & p - sc \end{vmatrix}.$$

The same substitution is used in each row; in particular, the third column now has entries linear in the row variable.

Step 2: Make the first column constant Now apply $C_1 \leftarrow C_1 - 2C_3$. For example, its first entry becomes $s^2 - 2sa - 2(p - sa) = s^2 - 2p = t$. The other two entries are also t . Taking this common factor out gives

$$D = t \begin{vmatrix} 1 & a^2 & p - sa \\ 1 & b^2 & p - sb \\ 1 & c^2 & p - sc \end{vmatrix}.$$

In this last determinant, apply $C_3 \leftarrow C_3 - pC_1$, factor $-s$ from its third column, and interchange the last two columns. Hence

$$\begin{aligned} D &= -st \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} \\ &= st \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}. \end{aligned}$$

The two minus signs, one from the factor and one from the interchange, cancel.

Step 3: Factor the remaining determinant by row operations Denote the remaining determinant by V . Apply $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$. Since $b^2 - a^2 = (b-a)(b+a)$, and similarly for c ,

$$\begin{aligned} V &= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \\ &= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} \\ &= (b-a)(c-a)[(c+a) - (b+a)] \\ &= (b-a)(c-a)(c-b). \end{aligned}$$

Quick Tip to Solve: Set s equal to the sum of the variables; subtract the square column from the other two, then use their linear terms to form a constant column.

Here the last evaluation is expansion along the first column. Reversing both $b - a$ and $c - b$ introduces two minus signs, so $V = (a - b)(b - c)(c - a)$. All operations used are polynomial identities; no assumption that the factors are non-zero is required.

Step 4: Conclusion Substituting $s = a + b + c$ and $t = a^2 + b^2 + c^2$, we obtain

$$D = (a - b)(b - c)(c - a)(a + b + c)(a^2 + b^2 + c^2).$$

This proves the required identity for all real a, b, c .

Quick Tip to Solve: Set s equal to the sum of the variables; subtract the square column from the other two, then use their linear terms to form a constant column.

34. Using integration, find the area of the region bounded by the triangle whose vertices are $(2, -2)$, $(4, 5)$ and $(6, 2)$. [6 marks]

Solution:

Concept:

- The area between an upper curve $y = u(x)$ and a lower curve $y = l(x)$, from $x = a$ to $x = b$, is $\int_a^b [u(x) - l(x)] dx$. The integrand is a nonnegative vertical length.
- The line through (x_1, y_1) and (x_2, y_2) has equation $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$. Split an area integral wherever its bounding line changes.

Step 1: Find the equations of the three sides Write $A = (2, -2)$, $B = (4, 5)$ and $C = (6, 2)$. The slopes of AB , BC and AC are respectively

$$\frac{5 - (-2)}{4 - 2} = \frac{7}{2}, \quad \frac{2 - 5}{6 - 4} = -\frac{3}{2}, \quad \frac{2 - (-2)}{6 - 2} = 1.$$

Thus their equations are

$$AB: y + 2 = \frac{7}{2}(x - 2), \quad y = \frac{7}{2}x - 9,$$

$$BC: y - 5 = -\frac{3}{2}(x - 4), \quad y = 11 - \frac{3}{2}x,$$

$$AC: y + 2 = x - 2, \quad y = x - 4.$$

Each equation reproduces both vertices on its side, providing a check on the constants and signs.

Step 2: Identify the upper and lower boundaries The triangle extends from $x = 2$ to $x = 6$. The side AC is the lower boundary throughout. From $x = 2$ to $x = 4$, the upper boundary is AB ; from $x = 4$ to $x = 6$, it is BC . Indeed, the vertical differences are

$$y_{AB} - y_{AC} = \frac{5}{2}x - 5 = \frac{5}{2}(x - 2) \geq 0,$$

$$y_{BC} - y_{AC} = 15 - \frac{5}{2}x = \frac{5}{2}(6 - x) \geq 0$$

on the respective intervals. At $x = 4$, both upper sides give $y = 5$, whereas AC gives $y = 0$. Therefore

Quick Tip to Solve: Split at the middle vertex's x-coordinate, $x=4$; keep AC as the lower boundary in both integrals.

the vertical height there is 5, and the two integrals meet without a gap or overlap. Portions below the x -axis still contribute positive area because we integrate upper minus lower, not the upper ordinate alone.

Step 3: Integrate the two vertical lengths The required area is

$$\begin{aligned}\mathcal{A} &= \int_2^4 \left(\frac{5}{2}x - 5 \right) dx + \int_4^6 \left(15 - \frac{5}{2}x \right) dx \\ &= \left[\frac{5}{4}x^2 - 5x \right]_2^4 + \left[15x - \frac{5}{4}x^2 \right]_4^6 \\ &= [(20 - 20) - (5 - 10)] \\ &\quad + [(90 - 45) - (60 - 20)] \\ &= 5 + 5 = 10.\end{aligned}$$

Both contributions are positive, as required for geometric area. The calculation uses the boundary equations directly and includes the whole triangular region.

Step 4: Conclusion Hence, by integration, the area of the triangular region is 10 square units.

Quick Tip to Solve: Split at the middle vertex's x -coordinate, $x=4$; keep AC as the lower boundary in both integrals.

35. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone. [6 marks]

Solution:

Concept:

- For an inscribed coaxial cylinder, similar triangles in an axial section relate its radius to its height. If the cone has height $h > 0$ and base radius $r > 0$, its radius at height x above the base is $r(1 - x/h)$.
- A cylinder of radius R and height x has volume $V = \pi R^2 x$. An interior maximum is found from $V'(x) = 0$, then verified by a sign change or by $V''(x) < 0$ and comparison with endpoints.
- The volume of the given cone is $V_c = \frac{1}{3}\pi r^2 h$; this is the reference for the requested ratio of volumes.

Step 1: Relate the cylinder radius and height Let the inscribed right circular cylinder have height x and radius R , with $0 < x < h$. Its base lies in the cone's base plane and its axis coincides with the cone's axis. The top circular rim touches the conical surface. In an axial section, the small triangle above the cylinder is similar to the full triangular section of the cone: their heights are $h - x$ and h , while their corresponding half-base lengths are R and r . Hence

$$\frac{R}{r} = \frac{h - x}{h}, \quad R = \frac{r(h - x)}{h}.$$

Thus increasing the cylinder's height decreases its available radius. This dependence must be included when maximizing volume; height and radius cannot be varied independently.

Quick Tip to Solve: Use similar triangles first to eliminate the cylinder radius; maximize $x(h - x)^2$ over $0 < x < h$.

Step 2: Express and differentiate the volume Substituting the radius into $V = \pi R^2 x$ gives a function of height alone:

$$V(x) = \frac{\pi r^2}{h^2} x(h-x)^2.$$

The factor $\pi r^2/h^2$ is positive and constant. By the product rule,

$$\begin{aligned} V'(x) &= \frac{\pi r^2}{h^2} [(h-x)^2 - 2x(h-x)] \\ &= \frac{\pi r^2}{h^2} (h-x)(h-3x). \end{aligned}$$

Therefore $V'(x) = 0$ gives $x = h$ or $x = h/3$. The value $x = h$ makes the radius zero and lies at the degenerate endpoint; the only stationary height in $(0, h)$ is $h/3$.

Step 3: Verify that the stationary value is the greatest For $0 < x < h/3$, both $h-x$ and $h-3x$ are positive, so $V'(x) > 0$. For $h/3 < x < h$, the first factor stays positive but the second is negative, so $V'(x) < 0$. Thus the volume increases up to $h/3$ and decreases thereafter. Also, $V(0) = V(h) = 0$, so this interior maximum is the greatest volume over the entire permissible interval. As a further check,

$$V''(x) = \frac{\pi r^2}{h^2} (6x - 4h), \quad V''(h/3) = -\frac{2\pi r^2}{h} < 0.$$

Step 4: Evaluate the maximum volume and compare with the cone At $x = h/3$, similar triangles give $R = 2r/3$. Consequently,

$$\begin{aligned} V_{\max} &= \pi \left(\frac{2r}{3} \right)^2 \frac{h}{3} = \frac{4\pi r^2 h}{27}, \\ \frac{V_{\max}}{V_c} &= \frac{4\pi r^2 h/27}{\pi r^2 h/3} = \frac{4}{9}. \end{aligned}$$

The positive maximum is less than the cone's volume, as expected for an inscribed solid, and the ratio is independent of the dimensions r and h .

Step 5: Conclusion The cylinder of greatest volume has height $\boxed{h/3}$, and its greatest volume is

$$\boxed{4\pi r^2 h/27 = \frac{4}{9} V_c}, \text{ as required.}$$

Quick Tip to Solve: Use similar triangles first to eliminate the cylinder radius; maximize $x(h-x)^2$ over $0 < x < h$.

36(a). Find the equation of the plane that contains the point $A(2, 1, -1)$ and is perpendicular to the line of intersection of the planes $2x + y - z = 3$ and $x + 2y + z = 2$. Also find the angle between the plane thus obtained and the y -axis. [6 marks]

Solution:

Concept:

- For a plane $ax + by + cz = d$, the vector $a\hat{i} + b\hat{j} + c\hat{k}$ is normal to the plane. The direction of the intersection of two planes is perpendicular to both their normals and can therefore be found using their cross product.
- A plane through a point with position vector $\vec{a}$ and normal $\vec{n}$ has equation $(\vec{r} - \vec{a}) \cdot \vec{n} = 0$. If a line has direction $\vec{b}$, its acute angle θ with the plane satisfies $\sin \theta = |\vec{n} \cdot \vec{b}| / (|\vec{n}| |\vec{b}|)$.

Step 1: Find the direction of the intersection line The normals to the given planes are $\vec{n}_1 = 2\hat{i} + \hat{j} - \hat{k}$ and $\vec{n}_2 = \hat{i} + 2\hat{j} + \hat{k}$. A direction vector of their line of intersection is

$$\begin{aligned}\vec{n}_1 \times \vec{n}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} \\ &= 3\hat{i} - 3\hat{j} + 3\hat{k}.\end{aligned}$$

This nonzero vector is perpendicular to each given normal. Dividing by 3, we may use $\hat{i} - \hat{j} + \hat{k}$ as the direction of the intersection line.

Step 2: Form the plane through the prescribed point The required plane is perpendicular to the intersection line, so the direction of that line is a normal to this plane. Hence take $\vec{n} = \hat{i} - \hat{j} + \hat{k}$. Through $A(2, 1, -1)$, its point-normal equation is

$$(x - 2) - (y - 1) + (z + 1) = 0.$$

The constant terms cancel. This construction gives a plane containing A , whose normal is parallel to the intersection line, as required.

Step 3: Calculate the angle with the y -axis The y -axis has unit direction vector $\hat{j}$. If ϕ is the acute angle between this direction and the normal to the required plane, then $\cos \phi = 1/\sqrt{3}$. The acute angle θ between a line and a plane is complementary to its angle with the plane's normal. Equivalently,

$$\sin \theta = \frac{|(\hat{i} - \hat{j} + \hat{k}) \cdot \hat{j}|}{\sqrt{1^2 + (-1)^2 + 1^2}} = \frac{1}{\sqrt{3}}.$$

The absolute value ensures that the conventional acute line-plane angle is selected regardless of the orientation of the chosen normal.

Step 4: Conclusion The required plane is $x - y + z = 0$. Its angle with the y -axis is $\sin^{-1}(1/\sqrt{3})$, approximately 35.26° .

Quick Tip to Solve: Cross the two given plane normals, use that vector as the new plane's normal, and use sine for the angle between the plane and the y -axis.

36(b). Find the distance of the point $P(-2, -4, 7)$ from the point of intersection Q of the line $\vec{r} = (3\hat{i} - 2\hat{j} + 6\hat{k}) + \lambda(2\hat{i} - \hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 6$. Also write the vector equation of the line PQ . [6 marks]

Solution:

Concept:

- A line $\vec{r} = \vec{a} + \lambda\vec{b}$ has coordinates obtained by comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$. At its intersection with a plane $\vec{r} \cdot \vec{n} = d$, those coordinates must satisfy the plane equation.
- The distance between two points is the magnitude of their displacement: $PQ = |\overrightarrow{PQ}|$. Subtract corresponding coordinates to find this displacement, which also gives the direction of their joining line.

Step 1: Express the intersection point using the line parameter Comparing components of the given vector equation gives

$$x = 3 + 2\lambda, \quad y = -2 - \lambda, \quad z = 6 + 2\lambda.$$

The plane has normal $\hat{i} - \hat{j} + \hat{k}$, so its Cartesian equation is $x - y + z = 6$. For the intersection point Q , substitute the coordinates of the line into this equation:

$$\begin{aligned} (3 + 2\lambda) - (-2 - \lambda) + (6 + 2\lambda) &= 6, \\ 11 + 5\lambda &= 6, \\ \lambda &= -1. \end{aligned}$$

The coefficient 5 is nonzero, so the line meets the plane at a unique point.

Step 2: Determine Q and calculate the distance Substituting $\lambda = -1$ into the three line coordinates gives $Q(1, -1, 4)$. As a check, $1 - (-1) + 4 = 6$, so this point indeed lies in the plane. With $P(-2, -4, 7)$, the displacement from P to Q is

$$\overrightarrow{PQ} = (1 + 2)\hat{i} + (-1 + 4)\hat{j} + (4 - 7)\hat{k} = 3\hat{i} + 3\hat{j} - 3\hat{k}.$$

Consequently the required distance is

$$PQ = \sqrt{3^2 + 3^2 + (-3)^2} = \sqrt{27} = 3\sqrt{3}.$$

Step 3: Write a vector equation of the line PQ The line passes through P and has direction $\overrightarrow{PQ}$. Dividing this nonzero direction vector by 3 gives the simpler parallel vector $\hat{i} + \hat{j} - \hat{k}$. Use P as the fixed point and a new real parameter t . At $t = 0$ the point is P , while at $t = 3$ its coordinates become $(1, -1, 4)$, so the equation contains both required points.

Step 4: Conclusion The intersection is $Q(1, -1, 4)$, and $PQ = 3\sqrt{3}$ units. A vector equation of PQ is

$$\vec{r} = (-2\hat{i} - 4\hat{j} + 7\hat{k}) + t(\hat{i} + \hat{j} - \hat{k}), \quad t \in \mathbb{R}.$$

Quick Tip to Solve: Substitute all three line coordinates into $x - y + z = 6$ first; then subtract P from Q and simplify that displacement to obtain the direction of PQ .