

CBSE Class 12 Mathematics Question Paper 2020 with Solution PDF-Set 65-1-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper comprises **four** Sections A, B, C and D. This question paper carries **36** questions. **All** questions are compulsory.
- (ii) **Section A** – Questions no. **1 to 20** comprises of **20** questions of **1** mark each.
- (iii) **Section B** – Questions no. **21 to 26** comprises of **6** questions of **2** marks each.
- (iv) **Section C** – Questions no. **27 to 32** comprises of **6** questions of **4** marks each.
- (v) **Section D** – Questions no. **33 to 36** comprises of **4** questions of **6** marks each.
- (vi) There is no overall choice in the question paper. However, an internal choice has been provided in 3 questions of one mark, 2 questions of two marks, 2 questions of four marks and 2 questions of six marks. Only one of the choices in such questions have to be attempted.
- (vii) In addition to this, separate instructions are given with each section and question, wherever necessary.
- (viii) Use of calculators **is not** permitted.

Section A (20 marks)

Question numbers 1 to 20 carry 1 mark each.

Question numbers 1 to 10 are multiple choice type questions. Select the correct option.

1. If A is a 3×3 matrix and $|A| = -2$, then the value of $|A \cdot (\text{adj } A)|$ is

- (A) -2
- (B) 2
- (C) -8
- (D) 8

[1 mark]

Solution:

Concept:

- For an order- n square matrix, $A \text{ adj } A = |A|I_n$, where I_n is the identity matrix.
- The determinant of cI_n is c^n , since every diagonal entry equals c .

Step 1: Apply the adjugate identity Here A has order three and its determinant is -2 . Therefore the matrix product is

$$A \text{ adj } A = -2I_3.$$

The vertical bars in the question ask for the determinant of this entire product, rather than its scalar factor.

Step 2: Evaluate the determinant The resulting diagonal matrix has three diagonal entries equal to -2 . Hence

$$|A \text{ adj } A| = (-2)(-2)(-2) = (-2)^3.$$

Options (A) and (B) forget the order of the matrix; option (D) loses the negative sign of an odd power.

Step 3: Conclusion The required determinant is -8 . Hence, the correct option is (C).

Quick Tip to Solve: Replace $A \text{ adj } A$ by $|A|I_n$ first, then raise the scalar determinant to the matrix order.

2. The number of arbitrary constants in the particular solution of a differential equation of second order is (are)

- (A) 0
- (B) 1
- (C) 2
- (D) 3

[1 mark]

Solution:

Concept:

- A general solution of a second-order differential equation normally contains two arbitrary constants.
- A particular solution is obtained by assigning specific values to those constants using suitable conditions.

Quick Tip to Solve: Underline the word "particular": the order counts constants in the general solution, while a particular solution fixes them.

Step 1: Distinguish the two kinds of solution The order tells us the number of independent arbitrary constants in the general solution. Thus a second-order equation normally gives a family written as $y = F(x, C_1, C_2)$, with two free constants. The question, however, asks about a particular solution.

Step 2: Fix the arbitrary constants In a particular solution, the constants have been assigned fixed values, say $C_1 = c_1$ and $C_2 = c_2$. The resulting function $y = F(x, c_1, c_2)$ contains no arbitrary constant. A fixed numerical constant is not an arbitrary one. Options (B), (C) and (D) still leave free choices and do not describe a particular solution.

Step 3: Conclusion The number of arbitrary constants is $\boxed{0}$. Hence, the correct option is (A).

Quick Tip to Solve: Underline the word "particular": the order counts constants in the general solution, while a particular solution fixes them.

3. The principal value of $\cos^{-1}\left(\cos \frac{13\pi}{6}\right)$ is

- (A) $\frac{13\pi}{6}$
- (B) $\frac{\pi}{2}$
- (C) $\frac{\pi}{3}$
- (D) $\frac{\pi}{6}$

[1 mark]

Solution:

Concept:

- The principal range of $\cos^{-1} t$ is $[0, \pi]$, for $-1 \leq t \leq 1$.
- Cosine is periodic: $\cos(2\pi + \theta) = \cos \theta$.

Step 1: Reduce the angle Separate one complete revolution from the angle:

$$\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}.$$

Therefore

$$\cos \frac{13\pi}{6} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Periodicity applies to the cosine before we take its inverse.

Step 2: Choose the principal representative The angle $\pi/6$ lies in $[0, \pi]$ and has cosine $\sqrt{3}/2$, so it is the unique principal inverse-cosine value. Option (A) lies outside the principal range. Options (B) and (C) have cosines 0 and $1/2$, respectively, and therefore cannot give the stated input.

Step 3: Conclusion The principal value is $\boxed{\pi/6}$. Hence, the correct option is (D).

Quick Tip to Solve: For $\cos^{-1}(\cos \theta)$, reduce the cosine first and then choose its angle in $[0, \pi]$.

4. The corner points of the feasible region determined by the system of linear inequalities are $(0, 0)$, $(4, 0)$, $(2, 4)$ and $(0, 5)$. If the maximum value of $z = ax + by$, where $a, b > 0$ occurs at both $(2, 4)$ and $(4, 0)$, then

- (A) $a = 2b$
- (B) $2a = b$
- (C) $a = b$
- (D) $3a = b$

[1 mark]

Solution:

Concept:

- The objective function is $z = ax + by$, where a, b are positive constants.
- If two feasible vertices both attain the maximum, their objective-function values must be equal.

Step 1: Equate the two maximum values At $(2, 4)$, the value is $2a + 4b$; at $(4, 0)$, it is $4a$. Since both points attain the same maximum,

$$2a + 4b = 4a \implies 4b = 2a.$$

This equation follows from equality of objective values, not equality of the coordinates.

Step 2: Check the other vertices The equality gives $a = 2b$. With this relation, both specified vertices have value $8b$, whereas $(0, 0)$ has value 0 and $(0, 5)$ has value $5b$. Since $b > 0$, neither exceeds $8b$. The relations in (B), (C) and (D) do not satisfy $2a + 4b = 4a$.

Step 3: Conclusion The relation is $a = 2b$. Hence, the correct option is (A).

Quick Tip to Solve: When two vertices maximize the same linear objective, equate their objective values before checking the remaining vertices.

5. If A and B are two independent events with $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, then $P(B' | A)$ is equal to

- (A) $\frac{1}{4}$
- (B) $\frac{1}{3}$
- (C) $\frac{3}{4}$
- (D) 1

[1 mark]

Solution:

Concept:

- For independent events, $P(A \cap B) = P(A)P(B)$.
- Conditional probability is $P(B | A) = P(A \cap B)/P(A)$, and $P(B' | A) = 1 - P(B | A)$.

Step 1: Use independence under the condition Because $P(A) = 1/3 > 0$, conditioning on A is defined.

Quick Tip to Solve: Independence removes the effect of conditioning on A ; take the complement only after identifying $P(B) = 1/4$.

Independence gives

$$P(B | A) = \frac{P(A)P(B)}{P(A)} = P(B) = \frac{1}{4}.$$

Thus learning that A occurred does not change the probability of B .

Step 2: Take the conditional complement Within the event A , the alternatives B and B' are complementary. Consequently

$$P(B' | A) = 1 - \frac{1}{4} = \frac{3}{4}.$$

Option (A) gives $P(B | A)$ instead of its complement; (B) confuses the answer with $P(A)$; (D) incorrectly treats B' as certain.

Step 3: Conclusion The probability is $\boxed{3/4}$. Hence, the correct option is (C).

Quick Tip to Solve: Independence removes the effect of conditioning on A ; take the complement only after identifying $P(B) = 1/4$.

6. If A is a square matrix such that $A^2 = A$, then $(I - A)^3 + A$ is equal to

- (A) I
- (B) 0
- (C) $I - A$
- (D) $I + A$

[1 mark]

Solution:

Concept:

- A matrix satisfying $A^2 = A$ is idempotent; multiplying once more gives $A^3 = A$.
- The identity matrix I commutes with A , so the ordinary binomial expansion applies to $I - A$.

Step 1: Expand the matrix expression Because the identity commutes with every square matrix of the same order,

$$(I - A)^3 = I - 3A + 3A^2 - A^3.$$

This expansion retains the signs of the alternating binomial terms.

Step 2: Use idempotence and add the last term Substitute $A^2 = A$ and $A^3 = A$:

$$(I - A)^3 = I - 3A + 3A - A = I - A.$$

Adding the separate A in the question now cancels the last negative term. Option (C) stops before this addition. Options (B) and (D) do not result from the simplification; the identity term remains throughout.

Step 3: Conclusion The expression equals $\boxed{I}$. Hence, the correct option is (A).

Quick Tip to Solve: First observe that $I - A$ is also idempotent; its cube is $I - A$, and the final added A cancels.

7. $\int_{-\pi/2}^{\pi/2} \frac{1}{x^2} \sin\left(\frac{1}{x}\right) dx$, where $x \neq 0$, is equal to

(A) -2

(B) 0

(C) 1

(D) π

[1 mark]

Solution:

Concept:

- The function $f(x) = x^{-2} \sin(1/x)$ is odd for $x \neq 0$. Odd-function cancellation applies to convergent symmetric integrals.
- An integral with an interior singularity requires both one-sided improper integrals to converge. A Cauchy principal value instead uses a common symmetric cutoff.

Step 1: Find the exact one-sided integrals Put $a = \pi/2$. On each side of zero, an antiderivative is $\cos(1/x)$, since its derivative is $x^{-2} \sin(1/x)$. For $0 < \varepsilon < a$,

$$\int_{\varepsilon}^a f(x) dx = \cos(2/\pi) - \cos(1/\varepsilon).$$

The last cosine has no limit as $\varepsilon \rightarrow 0^+$: cutoffs $1/(2n\pi)$ and $1/((2n+1)\pi)$ give values 1 and -1 . Thus the right-hand improper integral does not converge; the left-hand integral also fails.

Step 2: Distinguish symmetric cancellation from convergence For the same cutoff on both sides, oddness gives

$$\int_{-a}^{-\varepsilon} f(x) dx + \int_{\varepsilon}^a f(x) dx = 0.$$

Hence its Cauchy principal value is 0 , the value in option (B). However, the ordinary improper integral printed in the question does not exist. Excluding $x = 0$ does not ensure convergence. Options (A), (C) and (D) are neither the principal value nor values of a convergent integral.

Step 3: Conclusion As an ordinary improper integral, the expression does not exist and no printed option is correct. If symmetric Cauchy principal value is intended, the value is $\boxed{0}$, corresponding to (B).

Quick Tip to Solve: Before using oddness across zero, check the one-sided limits; equal cutoffs establish a principal value but cannot establish convergence.

8. The image of the point $(2, -1, 5)$ in the plane $\vec{r} \cdot \hat{i} = 0$ is

- (A) $(-2, -1, 5)$
- (B) $(2, 1, -5)$
- (C) $(-2, 1, -5)$
- (D) $(2, 0, 0)$

[1 mark]

Solution:

Concept:

- The plane $\vec{r} \cdot \hat{i} = 0$ has Cartesian equation $x = 0$, the yz -plane.
- Reflection in $x = 0$ changes the sign of the perpendicular coordinate x , while preserving y and z .

Step 1: Identify the reflecting plane The given plane has normal vector $\hat{i}$, so the perpendicular from $P(2, -1, 5)$ meets it at $H(0, -1, 5)$. The image must lie on this same perpendicular, at an equal distance on the other side of the plane.

Step 2: Reflect and compare the choices Using $P' = 2H - P$, we obtain $P' = (-2, -1, 5)$. Option (B) changes the wrong coordinates; (C) changes all three signs; (D) does not preserve the coordinates parallel to the plane. Only (A) gives the required reflection.

Step 3: Conclusion The image is $(-2, -1, 5)$, option (A).

Quick Tip to Solve: For reflection in a coordinate plane, reverse only the coordinate normal to that plane.

9. If the projection of $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ on $\vec{b} = 2\hat{i} + \lambda\hat{k}$ is zero, then the value of λ is

- (A) 0
- (B) 1
- (C) $-\frac{2}{3}$
- (D) $-\frac{3}{2}$

[1 mark]

Solution:

Concept:

- The scalar projection of $\vec{a}$ on a nonzero vector $\vec{b}$ is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$.
- A zero projection therefore means $\vec{a} \cdot \vec{b} = 0$, or that the vectors are perpendicular.

Step 1: Use the zero-projection condition Here $\vec{b} = 2\hat{i} + \lambda\hat{k}$ is nonzero for every real λ , since its $\hat{i}$ component is 2. Hence its magnitude cannot make the projection undefined, and the numerator must vanish.

Step 2: Evaluate the dot product and check the options

$$\vec{a} \cdot \vec{b} = 1(2) + (-2)(0) + 3\lambda = 2 + 3\lambda.$$

Thus $2 + 3\lambda = 0$ gives $\lambda = -2/3$. For (A), (B), and (D), the dot products are respectively 2, 5, and $-5/2$, all nonzero. Option (C) alone makes the projection zero.

Quick Tip to Solve: For zero projection, set the dot product to zero; there is no need to calculate the magnitude.

Step 3: Conclusion $\lambda = -\frac{2}{3}$, option (C).

Quick Tip to Solve: For zero projection, set the dot product to zero; there is no need to calculate the magnitude.

10. The vector equation of the line passing through the point $(-1, 5, 4)$ and perpendicular to the plane $z = 0$ is

- (A) $\vec{r} = -\hat{i} + 5\hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j})$
(B) $\vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}$
(C) $\vec{r} = \hat{i} - 5\hat{j} - 4\hat{k} + \lambda\hat{k}$
(D) $\vec{r} = \lambda\hat{k}$

[1 mark]

Solution:

Concept:

- A line through position vector $\vec{a}$, with direction vector $\vec{d}$, has equation $\vec{r} = \vec{a} + \lambda\vec{d}$, $\lambda \in \mathbb{R}$.
- A line perpendicular to a plane is parallel to the normal vector of that plane.

Step 1: Choose the point and direction The position vector of the given point is $-\hat{i} + 5\hat{j} + 4\hat{k}$. The plane $z = 0$ has normal vector $\hat{k}$, so this is the required direction vector.

Step 2: Form the equation and test the choices The equation is $\vec{r} = -\hat{i} + 5\hat{j} + 4\hat{k} + \lambda\hat{k}$. Combining the last two terms gives option (B). Option (A) has direction $\hat{i} + \hat{j}$, parallel to the plane; (C) has fixed coordinates $x = 1, y = -5$; and (D) has $x = y = 0$. Neither (C) nor (D) passes through the given point.

Step 3: Conclusion $\vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}$, option (B).

Quick Tip to Solve: For a line normal to a coordinate plane, keep the other two coordinates fixed and vary the normal coordinate.

11. The position vectors of two points A and B are $\vec{OA} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{OB} = 2\hat{i} - \hat{j} + 2\hat{k}$, respectively. The position vector of a point P which divides the line segment joining A and B in the ratio $2 : 1$ is _____.

[1 mark]

Solution:

Concept:

- If P divides AB internally in the ratio $m : n$, its position vector is $\frac{m\vec{b} + n\vec{a}}{m + n}$.
- The ratio $2 : 1$ means $AP : PB = 2 : 1$, so the weight of B is 2 and that of A is 1.

Step 1: Apply the internal section formula With $\vec{a} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$, we have

$$\vec{OP} = \frac{\vec{a} + 2\vec{b}}{3}.$$

Quick Tip to Solve: In the internal section formula, multiply each endpoint vector by the ratio term belonging to the opposite segment.

The point is within the segment, so the denominator is the sum of the ratio terms.

Step 2: Simplify and verify the ratio

$$\overrightarrow{OP} = \frac{(2+4)\hat{i} + (-1-2)\hat{j} + (-1+4)\hat{k}}{3} = 2\hat{i} - \hat{j} + \hat{k}.$$

Both endpoints have the same first two coordinates. Their third coordinates are -1 and 2 , while P has third coordinate 1 ; hence $AP = 2$ and $PB = 1$, as required.

Step 3: Conclusion $\overrightarrow{OP} = 2\hat{i} - \hat{j} + \hat{k}$.

Quick Tip to Solve: In the internal section formula, multiply each endpoint vector by the ratio term belonging to the opposite segment.

12(a). The equation of the normal to the curve $y^2 = 8x$ at the origin is _____. [1 mark]

Solution:

Concept:

- The normal at a point is perpendicular to the tangent to the curve at that point.
- Writing x as a function of y identifies a vertical tangent without dividing by a zero derivative.

Step 1: Find the tangent direction at the origin Rewrite the parabola as $x = y^2/8$. Then $\frac{dx}{dy} = y/4$, which is zero at $y = 0$. Taking y as the parameter, the tangent direction is $(dx/dy, 1) = (0, 1)$ at the origin. Therefore the tangent is vertical, with equation $x = 0$.

Step 2: Find the perpendicular line through the origin A line perpendicular to this vertical tangent is horizontal. Since the required normal passes through $(0, 0)$, its equation is $y - 0 = 0(x - 0)$. This also agrees with the symmetry axis of the parabola, which is the normal at its vertex.

Step 3: Conclusion The equation of the normal is $y = 0$.

Quick Tip to Solve: At the vertex of a right-opening parabola, the tangent is vertical and the normal is its horizontal axis.

OR

12(b). The radius of a circle is increasing at the uniform rate of 3 cm/sec . At the instant when the radius of the circle is 2 cm , its area increases at the rate of _____ cm^2/s . [1 mark]

Solution:

Concept:

- The area of a circle of radius r is $A = \pi r^2$.
- When r changes with time t , the chain rule gives $\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$.

Step 1: Differentiate with respect to time The radius is increasing uniformly, so $dr/dt = 3 \text{ cm/s}$.

Quick Tip to Solve: Differentiate the area formula before substituting the radius; the chain-rule factor is the given radius rate.

Treating both radius and area as functions of time, differentiate $A = \pi r^2$ to obtain $dA/dt = 2\pi r(dr/dt)$. The extra factor dr/dt converts change per unit radius into change per unit time.

Step 2: Substitute the instantaneous radius At the specified instant $r = 2$ cm, so

$$\frac{dA}{dt} = 2\pi(2)(3) = 12\pi \text{ cm}^2/\text{s}.$$

The positive sign agrees with the increasing radius. Although the radius changes at a constant rate, the area rate depends on the radius and is evaluated at this particular instant.

Step 3: Conclusion The area increases at $12\pi \text{ cm}^2/\text{s}$.

Quick Tip to Solve: Differentiate the area formula before substituting the radius; the chain-rule factor is the given radius rate.

13(a). On applying elementary column operation $C_2 \rightarrow C_2 - 3C_1$ in the matrix equation $\begin{bmatrix} 4 & -2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$, **the RHS (Right Hand Side) of the equation becomes** _____. [1 mark]

Solution:

Concept:

- In a product AB , a column operation on the product is performed on the right factor B .
- The operation $C_2 \rightarrow C_2 - 3C_1$ leaves the first column unchanged and subtracts three times it from the second column.

Step 1: Apply the operation to the right factor Let $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$. Since columns of AB are A multiplied by the corresponding columns of B , the same column combination is obtained by changing only B . Its new second column is $\begin{bmatrix} -1 \\ 0 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$.

Step 2: Write and check the transformed RHS The transformed product is

$$\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 2 & -6 \end{bmatrix} = \begin{bmatrix} 4 & -14 \\ 5 & -12 \end{bmatrix}.$$

As a check, directly changing the second column of the original LHS gives $(-2, 3)^T - 3(4, 5)^T = (-14, -12)^T$, with the first column unchanged.

Step 3: Conclusion The RHS becomes $\begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 2 & -6 \end{bmatrix}$, equal to $\begin{bmatrix} 4 & -14 \\ 5 & -12 \end{bmatrix}$.

Quick Tip to Solve: Column operations on a matrix product act on its right factor; row operations act on its left factor.

13(b). A square matrix A is said to be symmetric if _____.

[1 mark]

Solution:

Concept:

- The transpose A^T is formed by interchanging rows and columns of a matrix A .
- A square matrix is symmetric when corresponding entries on opposite sides of its principal diagonal are equal.

Step 1: Apply the definition For a square matrix $A = (a_{ij})$, symmetry means $a_{ij} = a_{ji}$ for every pair of indices i, j . The entry in position (i, j) therefore equals the entry in position (j, i) .

Step 2: Express the condition using transpose The (i, j) -entry of A^T is a_{ji} . Thus equality of these entries is exactly the matrix equation $A^T = A$. This condition is stronger than merely requiring the matrix to be square.

Step 3: Conclusion The blank is $A^T = A$.

Quick Tip to Solve: Check equality of entries reflected across the principal diagonal to test symmetry.

14. A relation R in a set A is called _____, if $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

[1 mark]

Solution:

Concept:

- A relation R on A is a subset of $A \times A$; its elements are ordered pairs.
- Symmetry requires the reverse pair to belong to the relation whenever the original pair belongs to it.

Step 1: Identify the stated property The condition says that whenever $(a_1, a_2) \in R$, the pair (a_2, a_1) is also in R , for all choices of $a_1, a_2 \in A$. It therefore requires closure under reversal of an ordered pair.

Step 2: Match it with the definition This is precisely the definition of a symmetric relation. Reflexivity instead asks for every pair (a, a) , while transitivity involves two linked pairs and a third implied pair. Neither of those is the condition given here.

Step 3: Conclusion The blank is symmetric .

Quick Tip to Solve: If a relation condition reverses the order of each pair, identify it as symmetry.

15. The greatest integer function defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at $x =$ _____ . [1 mark]

Solution:

Concept:

- The greatest integer function $[x]$ equals the largest integer less than or equal to x .
- Differentiability at an interior point implies continuity there.

Step 1: Locate the jump in the given interval For $0 < x < 1$, $f(x) = 0$, whereas for $1 \leq x < 2$, $f(x) = 1$. Hence $\lim_{x \rightarrow 1^-} f(x) = 0$ and $\lim_{x \rightarrow 1^+} f(x) = 1$. The unequal one-sided limits show a jump at $x = 1$.

Step 2: Check differentiability elsewhere The jump makes f discontinuous at 1, so it cannot be differentiable there. On each open interval $(0, 1)$ and $(1, 2)$, the function is constant and has derivative zero. The endpoints 0, 2 are excluded from the domain.

Step 3: Conclusion The required value is $\boxed{x = 1}$.

Quick Tip to Solve: Only integers inside the stated open interval can be jump points of the floor function.

16. If A is a non-singular square matrix of order 3 and $A^2 = 2A$, then find the value of $|A|$. [1 mark]

Solution:

Concept:

- A non-singular matrix A has an inverse A^{-1} , with $A^{-1}A = I$.
- For an identity matrix of order n , $\det(kI_n) = k^n$.

Step 1: Use non-singularity The given equality is $A^2 = 2A$. Multiplying both sides on the left by A^{-1} gives

$$A^{-1}A^2 = 2A^{-1}A \implies A = 2I_3.$$

Cancellation is valid because the matrix is non-singular.

Step 2: Evaluate the determinant The matrix $2I_3$ is diagonal with all three diagonal entries equal to 2. Its determinant is the product of those entries, so

$$|A| = |2I_3| = 2 \cdot 2 \cdot 2 = 2^3.$$

The order three determines the exponent.

Step 3: Conclusion Therefore $\boxed{|A| = 8}$.

Quick Tip to Solve: Cancel an invertible matrix first, then use its order when taking a scalar determinant.

17. Two cards are drawn at random and one-by-one without replacement from a well-shuffled pack of 52 playing cards. Find the probability that one card is red and the other is black. [1 mark]

Solution:

Concept:

- A standard pack contains 26 red cards and 26 black cards.
- Without replacement, the second draw is from 51 cards; disjoint colour orders have probabilities that add.

Step 1: Calculate each possible order The required event consists of red then black, or black then red. If the first card is red, all 26 black cards remain among 51 cards. Thus

$$P(RB) = \frac{26}{52} \cdot \frac{26}{51}.$$

Similarly, $P(BR) = \frac{26}{52} \cdot \frac{26}{51}$.

Step 2: Add the disjoint events The orders RB and BR cannot both occur in the same draw. Therefore

$$P(\text{one of each colour}) = P(RB) + P(BR) = 2 \cdot \frac{1}{2} \cdot \frac{26}{51}.$$

Including both orders is essential because the question does not prescribe which colour comes first.

Step 3: Conclusion The probability is $\boxed{\frac{26}{51}}$.

Quick Tip to Solve: For one card of each colour without replacement, count both colour orders and use 51 on draw two.

18. Evaluate: $\int_1^3 |2x - 1| dx$

[1 mark]

Solution:

Concept:

- If $u \geq 0$, then $|u| = u$; determine the sign over the entire integration interval.
- The fundamental theorem evaluates $\int_a^b g(x) dx$ as $G(b) - G(a)$, where G is an antiderivative.

Step 1: Remove the absolute value correctly On $[1, 3]$, $2x - 1 \geq 2(1) - 1 = 1 > 0$. Hence $|2x - 1| = 2x - 1$ throughout the interval. The zero $x = \frac{1}{2}$ lies outside the limits, so no split of the integral is needed.

Step 2: Integrate and substitute the limits An antiderivative of $2x - 1$ is $x^2 - x$. Thus

$$\int_1^3 |2x - 1| dx = [x^2 - x]_1^3 = (9 - 3) - (1 - 1) = 6.$$

Subtracting the lower-limit value completes the definite integration.

Step 3: Conclusion The value of the integral is $\boxed{6}$.

Quick Tip to Solve: Locate the zero of the expression inside the modulus before deciding whether to split the integral.

19. Find: $\int \frac{dx}{\sqrt{9-4x^2}}$

[1 mark]

Solution:

Concept:

- The standard result is $\int du/\sqrt{1-u^2} = \sin^{-1} u + C$ for $|u| < 1$.
- A linear substitution must change both the expression under the radical and the differential.

Step 1: Reduce to the standard form Put $u = \frac{2x}{3}$. Then $dx = \frac{3}{2} du$ and $\sqrt{9-4x^2} = 3\sqrt{1-u^2}$. Consequently

$$\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2} \int \frac{du}{\sqrt{1-u^2}}.$$

The real integrand is defined for $-\frac{3}{2} < x < \frac{3}{2}$.

Step 2: Integrate and return to the original variable Using the inverse-sine formula gives $\frac{1}{2} \sin^{-1} u + C$. Substituting $u = \frac{2x}{3}$ yields the required antiderivative. Differentiation checks it:

$$\frac{d}{dx} \left[\frac{1}{2} \sin^{-1} \left(\frac{2x}{3} \right) \right] = \frac{1/3}{\sqrt{1-4x^2/9}} = \frac{1}{\sqrt{9-4x^2}}.$$

Step 3: Conclusion $\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2} \sin^{-1} \left(\frac{2x}{3} \right) + C.$

Quick Tip to Solve: In the inverse-sine formula, the coefficient of x contributes an extra factor of one-half here.

20(a). Find: $\int x^4 \log x \, dx$

[1 mark]

Solution:

Concept:

- Integration by parts: $\int u \, dv = uv - \int v \, du$. Choose the logarithmic factor as u , because its derivative simplifies.
- The power rule is $\int x^n \, dx = x^{n+1}/(n+1) + C$, for $n \neq -1$. Here $\log x$ denotes the natural logarithm and $x > 0$.

Step 1: Choose the two factors Put $u = \log x$ and $dv = x^4 \, dx$. Then $du = dx/x$ and $v = x^5/5$. This choice removes the logarithm from the integral that remains.

Quick Tip to Solve: In a product of a power and a logarithm, differentiate the logarithm and integrate the power.

Step 2: Apply integration by parts

$$\begin{aligned}\int x^4 \log x \, dx &= \frac{x^5}{5} \log x - \int \frac{x^5}{5} \frac{dx}{x} \\&= \frac{x^5}{5} \log x - \frac{1}{5} \int x^4 \, dx \\&= \frac{x^5}{5} \log x - \frac{x^5}{25} + C.\end{aligned}$$

Differentiating the two variable terms gives $x^4 \log x + x^4/5 - x^4/5$, which checks the integrand.

Step 3: Conclusion $\int x^4 \log x \, dx = \frac{x^5}{5} \log x - \frac{x^5}{25} + C$, where C is an arbitrary constant.

Quick Tip to Solve: In a product of a power and a logarithm, differentiate the logarithm and integrate the power.

OR

20(b). Find: $\int \frac{2x}{\sqrt[3]{x^2+1}} \, dx$

[1 mark]

Solution:

Concept:

- Substitution replaces a composite expression by one variable when its derivative is present in the integrand.
- The power rule gives $\int t^n \, dt = t^{n+1}/(n+1) + C$, provided $n \neq -1$. A cube root in the denominator is the power $t^{-1/3}$.

Step 1: Identify a suitable substitution Let $t = x^2 + 1$. Then $dt = 2x \, dx$, exactly the numerator including the differential. Also $t > 0$ for every real x , so the fractional powers here are well-defined.

Step 2: Integrate and substitute back

$$\begin{aligned}\int \frac{2x}{\sqrt[3]{x^2+1}} \, dx &= \int t^{-1/3} \, dt \\&= \frac{t^{2/3}}{2/3} + C \\&= \frac{3}{2}(x^2+1)^{2/3} + C.\end{aligned}$$

The chain rule checks the coefficient: $(3/2)(2/3)(x^2+1)^{-1/3}(2x)$ equals the original integrand.

Step 3: Conclusion $\int \frac{2x}{\sqrt[3]{x^2+1}} \, dx = \frac{3}{2}(x^2+1)^{2/3} + C$.

Quick Tip to Solve: Check whether the numerator is the derivative of the expression inside the root before expanding anything.

Section B (12 marks)

Question numbers 21 to 26 carry 2 marks each.

21(a). Find a unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$ where $\vec{a} = 5\hat{i} + 6\hat{j} - 2\hat{k}$ and $\vec{b} = 7\hat{i} + 6\hat{j} + 2\hat{k}$. [2 marks]

Solution:

Concept:

- For nonparallel vectors $\vec{a}$ and $\vec{b}$, their cross product $\vec{a} \times \vec{b}$ is perpendicular to both.
- A nonzero vector $\vec{n}$ becomes a unit vector on division by its magnitude: $\hat{n} = \vec{n}/|\vec{n}|$. Its negative is also a valid unit normal.

Step 1: Calculate a common perpendicular Using the given components,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 6 & -2 \\ 7 & 6 & 2 \end{vmatrix} \\ &= (12 + 12)\hat{i} - (10 + 14)\hat{j} + (30 - 42)\hat{k} \\ &= 24\hat{i} - 24\hat{j} - 12\hat{k}.\end{aligned}$$

In particular this is nonzero, so the two vectors are not parallel.

Step 2: Normalise and verify The magnitude is

$$|\vec{a} \times \vec{b}| = \sqrt{24^2 + (-24)^2 + (-12)^2} = 36.$$

Dividing by 36 gives $\hat{n} = (2\hat{i} - 2\hat{j} - \hat{k})/3$. Its magnitude is $\sqrt{4 + 4 + 1}/3 = 1$. Moreover,

$$\vec{a} \cdot \hat{n} = (10 - 12 + 2)/3 = 0,$$

$$\vec{b} \cdot \hat{n} = (14 - 12 - 2)/3 = 0.$$

These checks establish both the required perpendicularities.

Step 3: Conclusion One required unit vector is $\boxed{\frac{2\hat{i} - 2\hat{j} - \hat{k}}{3}}$. Its negative is equally valid.

Quick Tip to Solve: Cancel a common factor from the cross product before normalising; here use the simpler normal $(2, -2, -1)$.

21(b). Find the volume of the parallelepiped whose adjacent edges are represented by $2\vec{a}$, $-\vec{b}$ and $3\vec{c}$, where

$$\vec{a} = \hat{i} - \hat{j} + 2\hat{k}, \vec{b} = 3\hat{i} + 4\hat{j} - 5\hat{k}, \text{ and } \vec{c} = 2\hat{i} - \hat{j} + 3\hat{k}.$$

[2 marks]

Solution:

Concept:

- The volume determined by adjacent edge vectors $\vec{u}, \vec{v}, \vec{w}$ is the absolute scalar triple product $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$.
- The scalar triple product is linear in each vector. Scaling the three edges multiplies the signed product by the product of the three scale factors.

Step 1: Account for the scaled edge vectors The actual edges are $2\vec{a}$, $-\vec{b}$, $3\vec{c}$, so

$$V = |(2\vec{a}) \cdot ((-\vec{b}) \times (3\vec{c}))| = |-6\vec{a} \cdot (\vec{b} \times \vec{c})|.$$

The negative sign affects orientation; the absolute value ensures a positive volume.

Step 2: Evaluate the scalar triple product First compute

$$\begin{aligned} \vec{b} \times \vec{c} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix} \\ &= (12 - 5)\hat{i} - (9 + 10)\hat{j} + (-3 - 8)\hat{k} \\ &= 7\hat{i} - 19\hat{j} - 11\hat{k}. \end{aligned}$$

Since $\vec{a} = \hat{i} - \hat{j} + 2\hat{k}$,

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = 7 + 19 - 22 = 4.$$

Hence the signed product for the actual edges is $-6(4) = -24$, whose absolute value gives their enclosed volume.

Step 3: Conclusion The volume of the parallelepiped is 24 cubic units.

Quick Tip to Solve: Compute the triple product of the unscaled vectors first, then multiply its absolute value by $|2(-1)3| = 6$.

22. Examine the applicability of Rolle's theorem for the function $f(x) = \sin 2x$ in $[0, \pi]$. Hence find the points where the tangent is parallel to x -axis.

[2 marks]

Solution:

Concept:

- Rolle's theorem applies when f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$. It guarantees an interior point with $f'(c) = 0$.
- A tangent to $y = f(x)$ is parallel to the x -axis when its slope $f'(x)$ is zero.

Quick Tip to Solve: Solve for the angle $2x$ within $(0, 2\pi)$ first, then halve each solution and evaluate its ordinate.

Step 1: Check all hypotheses of Rolle's theorem The function $f(x) = \sin 2x$ is continuous on $[0, \pi]$ and differentiable on $(0, \pi)$, since sine and the linear inner function have these properties. Also

$$f(0) = \sin 0 = 0, \quad f(\pi) = \sin 2\pi = 0.$$

Thus the endpoint values agree, and all three conditions are satisfied.

Step 2: Find every interior point with a horizontal tangent Differentiate to get $f'(x) = 2 \cos 2x$. Thus

$$2 \cos 2x = 0 \iff 2x = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}.$$

Since $0 < 2x < 2\pi$, only $2x = \pi/2$ and $2x = 3\pi/2$ occur. The corresponding abscissae are $x = \pi/4$ and $x = 3\pi/4$. Their ordinates are $\sin(\pi/2) = 1$ and $\sin(3\pi/2) = -1$, respectively. The endpoints have derivative 2, so add no further horizontal tangents.

Step 3: Conclusion Rolle's theorem is applicable. The tangent is parallel to the x -axis at $(\pi/4, 1)$ and $(3\pi/4, -1)$.

Quick Tip to Solve: Solve for the angle $2x$ within $(0, 2\pi)$ first, then halve each solution and evaluate its ordinate.

23. Find the values of x for which the function $f(x) = 2 + 3x - x^3$ is decreasing.

[2 marks]

Solution:

Concept:

- A differentiable function is decreasing where its derivative is negative: $f'(x) < 0$.
- To solve a quadratic inequality, locate its zeroes and check the sign on the intervals they determine.

Step 1: Differentiate the given polynomial For $f(x) = 2 + 3x - x^3$, differentiate each term with respect to x . The constant contributes zero, the linear term contributes 3, and the cubic term contributes $-3x^2$. Therefore,

$$f'(x) = 3 - 3x^2 = 3(1 - x^2).$$

The derivative exists for every real number, so the sign test can be applied throughout the real line.

Step 2: Find where the derivative is negative The required inequality is

$$3(1 - x^2) < 0 \iff x^2 > 1.$$

Thus either $x < -1$ or $x > 1$. On the intermediate interval $-1 < x < 1$, the derivative is positive and the function increases instead. At $x = -1$ and $x = 1$, the derivative is zero; these stationary boundary points separate the intervals of increase and decrease.

Step 3: Conclusion The function is decreasing for $x \in (-\infty, -1) \cup (1, \infty)$.

Quick Tip to Solve: Factor the derivative first; the sign of $1 - x^2$ is negative outside its two roots.

24. The probability of finding a green signal on a busy crossing X is 30%. What is the probability of finding a green signal on X on two consecutive days out of three? [2 marks]

Solution:

Concept:

- For independent daily observations, multiply the probabilities of the specified outcomes.
- Write G for a green signal and N for a non-green signal; $P(G) = 0.30$ and $P(N) = 0.70$.

Step 1: List the consecutive-day outcomes Treat the observations on different days as independent trials with the same probability of a green signal. The event asks for green signals on two consecutive days out of three. The two admissible sequences are GGN and NGG : green on the first two days or on the last two days, with the remaining day non-green. The sequence GNG has two green signals but they are not consecutive, and GGG has three green signals rather than two.

Step 2: Add the probabilities of the disjoint sequences Each admissible sequence has two green outcomes and one non-green outcome. By independence,

$$P(GGN) = (0.3)^2(0.7) = 0.063, \quad P(NGG) = (0.7)(0.3)^2 = 0.063.$$

These sequences are mutually exclusive, so their probabilities add:

$$P(\text{two consecutive green days}) = 2(0.3)^2(0.7) = 0.126.$$

A binomial coefficient of three would also count the nonconsecutive sequence GNG , so it is not appropriate here.

Step 3: Conclusion The required probability is $\boxed{0.126}$, or $\boxed{12.6\%}$.

Quick Tip to Solve: List the three-day sequences explicitly: consecutiveness excludes the arrangement GNG .

25(a). Prove that $\sin^{-1}(2x\sqrt{1-x^2}) = 2\cos^{-1}x$, $\frac{1}{\sqrt{2}} \leq x \leq 1$. [2 marks]

Solution:

Concept:

- The principal range of $\cos^{-1}x$ is $[0, \pi]$, while that of $\sin^{-1}u$ is $[-\pi/2, \pi/2]$.
- The double-angle identity is $\sin 2\theta = 2\sin\theta\cos\theta$; $\sin^{-1}(\sin t) = t$ when t lies in the principal sine-inverse range.

Step 1: Choose an angle and establish its range Let $\theta = \cos^{-1}x$, so that $x = \cos\theta$. Since $1/\sqrt{2} \leq x \leq 1$ and cosine decreases on $[0, \pi]$, we have

$$0 \leq \theta \leq \frac{\pi}{4}, \quad 0 \leq 2\theta \leq \frac{\pi}{2}.$$

In particular, $\sin\theta \geq 0$, and hence $\sqrt{1-x^2} = \sqrt{1-\cos^2\theta} = \sin\theta$. This also covers both endpoints of the given interval.

Quick Tip to Solve: Before cancelling sine with inverse sine, verify that the doubled angle stays in $[0, \pi/2]$.

Step 2: Apply the double-angle identity and principal value Substitution now gives

$$2x\sqrt{1-x^2} = 2\cos\theta\sin\theta = \sin 2\theta.$$

Because $2\theta \in [0, \pi/2]$, it lies in the principal range of $\sin^{-1}$. Therefore

$$\sin^{-1}(2x\sqrt{1-x^2}) = \sin^{-1}(\sin 2\theta) = 2\theta.$$

Replacing θ by $\cos^{-1}x$ completes the identity; the range check is essential to justify undoing the sine.

Step 3: Conclusion Thus $\boxed{\sin^{-1}(2x\sqrt{1-x^2}) = 2\cos^{-1}x}$ for $1/\sqrt{2} \leq x \leq 1$.

Quick Tip to Solve: Before cancelling sine with inverse sine, verify that the doubled angle stays in $[0, \pi/2]$.

OR

25(b). Consider a bijective function $f : \mathbb{R}_+ \rightarrow (7, \infty)$ given by $f(x) = 16x^2 + 24x + 7$, where $\mathbb{R}_+$ is the set of all positive real numbers. Find the inverse function of f . [2 marks]

Solution:

Concept:

- To find an inverse function, set $y = f(x)$, solve for x , and use the original domain to choose the valid root.
- The domain and codomain interchange under inversion: $f^{-1} : (7, \infty) \rightarrow \mathbb{R}_+$.

Step 1: Complete the square and isolate the input Let $y = 16x^2 + 24x + 7$, where $x > 0$. Completing the square produces

$$y = (4x + 3)^2 - 2, \quad (4x + 3)^2 = y + 2.$$

Since $x > 0$, we have $4x + 3 > 3 > 0$; therefore only the positive square root is admissible. Thus

$$4x + 3 = \sqrt{y + 2}, \quad x = \frac{\sqrt{y + 2} - 3}{4}.$$

The negative root would give a negative input, outside $\mathbb{R}_+$.

Step 2: State the inverse and check its range Interchanging the input notation gives

$$f^{-1}(x) = \frac{\sqrt{x + 2} - 3}{4}, \quad x > 7.$$

For every $x > 7$, $\sqrt{x + 2} > 3$, so this expression is positive as required. Also, if $g(x) = (\sqrt{x + 2} - 3)/4$, then

$$f(g(x)) = (4g(x) + 3)^2 - 2 = x.$$

Conversely, for $t > 0$, $\sqrt{f(t) + 2} = 4t + 3$, and hence $g(f(t)) = t$. Both compositions confirm the inverse on the stated domains.

Quick Tip to Solve: Complete the square as $(4x + 3)^2 - 2$; positivity of the original input selects the square root.

Step 3: Conclusion $f^{-1} : (7, \infty) \rightarrow \mathbb{R}_+, \quad f^{-1}(x) = \frac{\sqrt{x+2}-3}{4}.$

Quick Tip to Solve: Complete the square as $(4x+3)^2 - 2$; positivity of the original input selects the square root.

26. Find the value of k so that the lines $x = -y = kz$ and $x - 2 = 2y + 1 = -z + 1$ are perpendicular to each other. [2 marks]

Solution:

Concept:

- Two perpendicular line directions have zero scalar product: $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$.
- Direction ratios are the coefficients of a parameter in parametric line equations; multiplying all three by a nonzero common factor preserves the direction.

Step 1: Extract direction ratios using parameters For the first line, write its direction in the proportional form $(k, -k, 1)$. For nonzero k , this follows by setting the common value $x = -y = kz$ equal to kt : $x = kt, y = -kt, z = t$. For the second line let $x - 2 = 2y + 1 = -z + 1 = s$. Then

$$x = 2 + s, \quad y = \frac{s-1}{2}, \quad z = 1 - s.$$

Its direction ratios are $(1, 1/2, -1)$, or equivalently $(2, 1, -2)$. The additive constants specify positions, not directions.

Step 2: Impose perpendicularity Take the scalar product of $(k, -k, 1)$ and $(2, 1, -2)$. The perpendicularity condition gives

$$2k - k - 2 = 0 \implies k = 2.$$

This value is nonzero, so the parameterisation used above is valid. Substituting it back yields directions $(2, -2, 1)$ and $(2, 1, -2)$; their dot product is $4 - 2 - 2 = 0$, confirming the right angle between the directions.

Step 3: Conclusion The required value is $k = 2$.

Quick Tip to Solve: Convert each common-value equation to parametric form; the coefficient of y is not its direction ratio.

Section C (24 marks)

Question numbers 27 to 32 carry 4 marks each.

27. A furniture trader deals in only two items – chairs and tables. He has ₹ 50,000 to invest and a space to store at most 35 items. A chair costs him ₹ 1,000 and a table costs him ₹ 2,000. The trader earns a profit of ₹ 150 and ₹ 250 on a chair and table, respectively. Formulate the above problem as an LPP to maximise the profit and solve it graphically. [4 marks]

Solution:

Concept:

- Let x denote the number of chairs and y the number of tables. Both variables are non-negative since they represent furniture items.
- The investment and storage limits define linear inequalities. Their common region in the first quadrant is the feasible region.
- A linear objective function attains its maximum over a bounded feasible polygon at a corner point. Evaluate profit at all vertices.

Step 1: Formulate the linear programming problem The total investment is $1000x + 2000y$ rupees and must not exceed 50000 rupees. Thus $1000x + 2000y \leq 50000$, which simplifies to $x + 2y \leq 50$. The storage restriction gives $x + y \leq 35$, since each chair or table counts as one item. The profit in rupees is $Z = 150x + 250y$. The required LPP is therefore

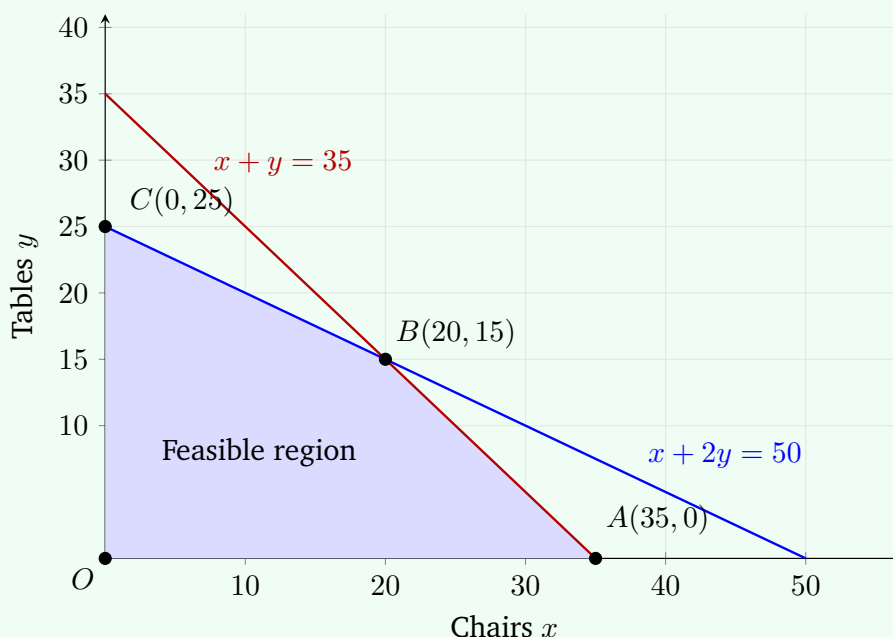
$$\text{Maximise } Z = 150x + 250y$$

subject to

$$x + 2y \leq 50, \quad x + y \leq 35, \quad x \geq 0, \quad y \geq 0.$$

Step 2: Plot the boundaries and identify the feasible region The line $x + 2y = 50$ has intercepts $(50, 0)$ and $(0, 25)$; the line $x + y = 35$ has intercepts $(35, 0)$ and $(0, 35)$. Draw both lines. The origin satisfies both inequalities, so for each line choose the side containing the origin. Their common portion in the first quadrant is the shaded region. At the intersection, subtract $x + y = 35$ from $x + 2y = 50$ to obtain $y = 15$. Substituting back gives $x = 20$. Hence the vertices of the feasible polygon are $O(0, 0)$, $A(35, 0)$, $B(20, 15)$ and $C(0, 25)$.

Quick Tip to Solve: Divide the investment inequality by 1000, then subtract the two boundary equations to locate their intersection quickly.



Shaded feasible polygon for the investment and storage constraints.

Step 3: Compare profit at every vertex Since the feasible polygon is bounded, apply the corner-point method:

Vertex	Z (rupees)
$O(0, 0)$	0
$A(35, 0)$	$150(35) + 250(0) = 5250$
$B(20, 15)$	$150(20) + 250(15) = 6750$
$C(0, 25)$	$150(0) + 250(25) = 6250$

The greatest profit occurs at $B(20, 15)$. For this choice the total number of items is $20 + 15 = 35$, and the investment is $1000(20) + 2000(15) = 50000$ rupees. Thus both limits are satisfied exactly. The optimal coordinates are whole numbers, so the solution also represents a physically possible purchase of furniture.

Step 4: Conclusion Buy 20 chairs and 15 tables. The maximum profit is ₹ 6750.

Quick Tip to Solve: Divide the investment inequality by 1000, then subtract the two boundary equations to locate their intersection quickly.

28. If $x = a \sec^3 \theta$, $y = a \tan^3 \theta$, then find $\frac{d^2 y}{dx^2}$ at $\theta = \frac{\pi}{4}$.

[4 marks]

Solution:

Concept:

- For coordinates depending on a parameter θ , the derivative is $dy/dx = (dy/d\theta)/(dx/d\theta)$, wherever $dx/d\theta \neq 0$.
- The second derivative is $d^2 y/dx^2 = [d(dy/dx)/d\theta]/(dx/d\theta)$. A derivative with respect to the

Quick Tip to Solve: Simplify the first derivative to $\sin \theta$, then divide its parameter derivative by $dx/d\theta$ before substituting $\pi/4$.

parameter must therefore be divided by $dx/d\theta$ to become a derivative with respect to x .

Step 1: Differentiate the coordinates with respect to the parameter With a a nonzero constant, applying the chain rule to the cubes gives

$$\begin{aligned}\frac{dx}{d\theta} &= 3a \sec^2 \theta (\sec \theta \tan \theta) = 3a \sec^3 \theta \tan \theta, \\ \frac{dy}{d\theta} &= 3a \tan^2 \theta \sec^2 \theta.\end{aligned}$$

At the specified angle $\pi/4$, the derivative of x is nonzero, so the parametric derivative formula can be used in a neighbourhood of this point.

Step 2: Simplify the first derivative Taking the ratio of the derivatives, we obtain

$$\frac{dy}{dx} = \frac{3a \tan^2 \theta \sec^2 \theta}{3a \sec^3 \theta \tan \theta} = \frac{\tan \theta}{\sec \theta} = \sin \theta.$$

The common constant and trigonometric factors cancel. This simple expression for the slope makes the next differentiation direct: $\frac{d}{d\theta}(dy/dx) = \cos \theta$.

Step 3: Find the second derivative and substitute the angle Divide the derivative of the slope by the derivative of x :

$$\frac{d^2y}{dx^2} = \frac{\cos \theta}{3a \sec^3 \theta \tan \theta}.$$

The exact trigonometric values are $\cos(\pi/4) = 1/\sqrt{2}$, $\sec(\pi/4) = \sqrt{2}$, and $\tan(\pi/4) = 1$. Substitution gives

$$\left. \frac{d^2y}{dx^2} \right|_{\theta=\pi/4} = \frac{1/\sqrt{2}}{3a(\sqrt{2})^3(1)} = \frac{1/\sqrt{2}}{6a\sqrt{2}}.$$

Step 4: Conclusion Hence $\left. \frac{d^2y}{dx^2} \right|_{\theta=\pi/4} = \boxed{\frac{1}{12a}}$, for $a \neq 0$.

Quick Tip to Solve: Simplify the first derivative to $\sin \theta$, then divide its parameter derivative by $dx/d\theta$ before substituting $\pi/4$.

29. Find: $\int \frac{2x+1}{\sqrt{3+2x-x^2}} dx$

[4 marks]

Solution:

Concept:

- Split a numerator into a multiple of the derivative of the expression under the square root and a constant. The derivative term can then be integrated by substitution.
- Complete the square and use $\int \frac{du}{\sqrt{b^2 - u^2}} = \sin^{-1}(u/b) + C$, for $|u| < b$, with $b > 0$.

Step 1: Decompose the numerator using the derivative of the radicand Let $f(x) = 3 + 2x - x^2$. Then

Quick Tip to Solve: Write $2x + 1 = -(2 - 2x) + 3$ first; the derivative term gives a square root and completing the square gives the inverse-sine term.

$f'(x) = 2 - 2x$. Since

$$2x + 1 = -(2 - 2x) + 3,$$

the requested integral I becomes

$$I = - \int \frac{2 - 2x}{\sqrt{3 + 2x - x^2}} dx + 3 \int \frac{dx}{\sqrt{3 + 2x - x^2}}.$$

The first term is suited to substitution; the remaining constant gives a standard inverse-trigonometric integral.

Step 2: Integrate the derivative term by substitution Put $t = 3 + 2x - x^2$, so $dt = (2 - 2x) dx$. The first term is

$$- \int t^{-1/2} dt = -2\sqrt{t} = -2\sqrt{3 + 2x - x^2}.$$

The negative sign comes from the numerator decomposition, and the factor two follows from the power rule. Combine all constants into one final constant.

Step 3: Complete the square for the remaining term The quadratic under the radical can be written as

$$3 + 2x - x^2 = 4 - (x - 1)^2.$$

Setting $u = x - 1$ gives $du = dx$, and consequently

$$3 \int \frac{dx}{\sqrt{4 - (x - 1)^2}} = 3 \sin^{-1} \left(\frac{x - 1}{2} \right).$$

The shift by one centres the quadratic at its maximum. The real integrand is defined on $-1 < x < 3$, where the radicand is positive.

Step 4: Conclusion Therefore,

$$\int \frac{2x + 1}{\sqrt{3 + 2x - x^2}} dx = -2\sqrt{3 + 2x - x^2} + 3 \sin^{-1} \left(\frac{x - 1}{2} \right) + C,$$

where C is an arbitrary constant and $-1 < x < 3$.

Quick Tip to Solve: Write $2x + 1 = -(2 - 2x) + 3$ first; the derivative term gives a square root and completing the square gives the inverse-sine term.

30(a). There are two bags, I and II. Bag I contains 3 red and 5 black balls and Bag II contains 4 red and 3 black balls. One ball is transferred randomly from Bag I to Bag II and then a ball is drawn randomly from Bag II. If the ball so drawn is found to be black in colour, then find the probability that the transferred ball is also black. [4 marks]

Solution:

Concept:

- Conditional probability uses known information: $P(E | F) = P(E \cap F) / P(F)$, where $P(F) > 0$.

Quick Tip to Solve: Update Bag II to eight balls in both transfer cases, then divide the black-transfer contribution by the total probability of a black draw.

- Total probability sums the probabilities arising from mutually exclusive and exhaustive transfer events.
- Bayes' theorem gives $P(E | F) = P(E)P(F | E)/P(F)$, reversing the direction of conditioning.

Step 1: Define the events for the transfer and the observed draw Let T_B and T_R denote transferring a black and a red ball respectively, and let D_B denote drawing a black ball from Bag II after the transfer. Bag I initially contains $3 + 5 = 8$ balls. Hence

$$P(T_B) = \frac{5}{8}, \quad P(T_R) = \frac{3}{8}.$$

The two transfer events are mutually exclusive and exhaustive. The required probability is $P(T_B | D_B)$, since the colour of the later draw is known. It is different from the unconditional transfer probability.

Step 2: Compute the probability of a black draw in each case After a black transfer, Bag II has four red and four black balls. After a red transfer, it has five red and three black balls. Therefore

$$P(D_B | T_B) = \frac{4}{8}, \quad P(D_B | T_R) = \frac{3}{8}.$$

Both denominators are eight because the transfer increases the total in Bag II from seven to eight. Applying total probability gives

$$P(D_B) = \frac{5}{8} \cdot \frac{4}{8} + \frac{3}{8} \cdot \frac{3}{8} = \frac{20}{64} + \frac{9}{64} = \frac{29}{64}.$$

The first contribution corresponds to a black transfer followed by a black draw, and the second to a red transfer followed by a black draw.

Step 3: Condition on the observed black draw The joint probability of a black transfer and a black draw is

$$P(T_B \cap D_B) = P(T_B)P(D_B | T_B) = \frac{20}{64}.$$

Bayes' theorem now yields

$$P(T_B | D_B) = \frac{P(T_B \cap D_B)}{P(D_B)} = \frac{20/64}{29/64} = \frac{20}{29}.$$

As a check, this is larger than $5/8$: observing a black draw favours the transfer case that made black balls more numerous in Bag II.

Step 4: Conclusion The required probability is $\boxed{\frac{20}{29}}$.

Quick Tip to Solve: Update Bag II to eight balls in both transfer cases, then divide the black-transfer contribution by the total probability of a black draw.

30(b). An urn contains 5 red, 2 white and 3 black balls. Three balls are drawn, one-by-one, at random without replacement. Find the probability distribution of the number of white balls. Also, find the mean and the variance of the number of white balls drawn. [4 marks]

Solution:

Concept:

- A random variable assigns a numerical value to an outcome. Here X is the number of white balls in a sample of three balls drawn without replacement.
- There are $\binom{n}{r}$ ways to choose r objects from n . Each unordered selection of three distinct balls is equally likely.
- For a discrete distribution, $E(X) = \sum_x xP(X = x)$ and $\text{Var}(X) = E(X^2) - [E(X)]^2$.

Step 1: Identify the possible values and the sample space The urn contains ten balls: two white and eight non-white (five red and three black). Let X count the white balls among the three drawn. Since only two white balls are available, X can take only the values 0, 1, 2; the value three is impossible. There are $\binom{10}{3} = 120$ unordered selections. Although the balls are drawn successively, each selection corresponds to $3!$ ordered draws, so counting unordered selections gives the same probabilities.

Step 2: Calculate the probability distribution To obtain exactly x white balls, choose x of the two white balls and $3 - x$ of the eight non-white balls. Thus

$$P(X = x) = \frac{\binom{2}{x} \binom{8}{3-x}}{\binom{10}{3}}, \quad x = 0, 1, 2.$$

In particular,

$$\begin{aligned} P(X = 0) &= \frac{1 \cdot 56}{120} = \frac{7}{15}, \\ P(X = 1) &= \frac{2 \cdot 28}{120} = \frac{7}{15}, \\ P(X = 2) &= \frac{1 \cdot 8}{120} = \frac{1}{15}. \end{aligned}$$

These probabilities are non-negative and sum to $(7 + 7 + 1)/15 = 1$, verifying that the distribution is complete.

Step 3: Find the mean Multiply each possible value by its probability and add:

$$E(X) = 0 \cdot \frac{7}{15} + 1 \cdot \frac{7}{15} + 2 \cdot \frac{1}{15} = \frac{9}{15} = \frac{3}{5}.$$

This also agrees with three draws multiplied by the initial white-ball proportion $2/10$, which gives $3(2/10) = 3/5$.

Step 4: Find the second moment and variance The second moment is

$$E(X^2) = 0^2 \cdot \frac{7}{15} + 1^2 \cdot \frac{7}{15} + 2^2 \cdot \frac{1}{15} = \frac{11}{15}.$$

Quick Tip to Solve: Combine red and black balls into eight non-white balls; count selections without replacement and calculate the variance from the second moment.

Subtract the square of the mean:

$$\text{Var}(X) = \frac{11}{15} - \left(\frac{3}{5}\right)^2 = \frac{55}{75} - \frac{27}{75} = \frac{28}{75}.$$

Sampling is without replacement, so successive draws are dependent; an independent-trial binomial distribution would give a different variance.

Step 5: Conclusion The distribution is $P(X = 0) = 7/15$, $P(X = 1) = 7/15$, $P(X = 2) = 1/15$. The mean is $\boxed{3/5}$ and the variance is $\boxed{28/75}$.

Quick Tip to Solve: Combine red and black balls into eight non-white balls; count selections without replacement and calculate the variance from the second moment.

31(a). Find the general solution of the differential equation $y e^y dx = (y^3 + 2x e^y) dy$.

[4 marks]

Solution:

Concept:

- Treat x as a function of y . A linear equation $\frac{dx}{dy} + P(y)x = Q(y)$ has integrating factor $I(y) = e^{\int P(y) dy}$.
- Multiplication by the integrating factor makes the left side the derivative $\frac{d}{dy}[I(y)x]$, so the general solution contains one arbitrary constant.

Step 1: Express the equation in linear form The printed equation is

$$y e^y dx = (y^3 + 2x e^y) dy.$$

On an interval where $y \neq 0$, divide by $y e^y dy$ and rearrange:

$$\frac{dx}{dy} = y^2 e^{-y} + \frac{2x}{y}, \quad \frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y}.$$

This is linear in the dependent variable x . In particular, the exponential depends only on the independent variable y ; there is no need to classify the original equation as homogeneous.

Step 2: Find and apply the integrating factor Here $P(y) = -2/y$, and hence an integrating factor on either interval excluding zero is

$$I(y) = e^{\int -2/y dy} = e^{-2 \ln |y|} = \frac{1}{y^2}.$$

Multiplying the entire linear equation by $1/y^2$ gives

$$\frac{1}{y^2} \frac{dx}{dy} - \frac{2x}{y^3} = e^{-y}.$$

By the product rule, the left side equals $\frac{d}{dy}(x/y^2)$. Thus the equation can now be integrated directly

Quick Tip to Solve: When the coefficients naturally depend on y , switch to x as the dependent variable before choosing the integrating factor.

with respect to y .

Step 3: Integrate and check the family Integration gives

$$\frac{x}{y^2} = \int e^{-y} dy = -e^{-y} + C.$$

Equivalently, $x = y^2(C - e^{-y})$. Differentiating the latter expression yields

$$\frac{dx}{dy} = 2y(C - e^{-y}) + y^2 e^{-y} = \frac{2x}{y} + y^2 e^{-y},$$

which reproduces the required equation after multiplication by ye^y . The division by y restricts this derivation to $y \neq 0$. In the original differential equation, the constant solution $y = 0$ also satisfies both sides identically and is stated separately.

Step 4: Conclusion The general nonconstant solution is $\boxed{\frac{x}{y^2} + e^{-y} = C}$, or $\boxed{x = y^2(C - e^{-y})}$, on intervals with $y \neq 0$. The original equation also admits $y = 0$.

Quick Tip to Solve: When the coefficients naturally depend on y , switch to x as the dependent variable before choosing the integrating factor.

OR

31(b). Find the particular solution of the differential equation $x \frac{dy}{dx} = y - x \tan\left(\frac{y}{x}\right)$, given that $y = \frac{\pi}{4}$ at $x = 1$. [4 marks]

Solution:

Concept:

- A homogeneous first-order equation whose right side depends on y/x is reduced by $v = y/x$, so $y = vx$ and $dy/dx = v + x dv/dx$, for $x \neq 0$.
- After separation, use $\int \cot v dv = \ln |\sin v|$ and determine the integration constant from the prescribed point.

Step 1: Substitute the ratio y/x Dividing the given equation by x gives

$$\frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right).$$

Put $v = y/x$, or $y = vx$. The product rule gives $dy/dx = v + x dv/dx$. Substitute this into the differential equation:

$$v + x \frac{dv}{dx} = v - \tan v, \quad x \frac{dv}{dx} = -\tan v.$$

At the initial point $x = 1$, $v = \pi/4$, so both $\sin v$ and $\cos v$ are nonzero. Separation is therefore valid in a neighbourhood of the initial point.

Quick Tip to Solve: Substitute $y = vx$ before separating; then use the initial value of v to select the correct inverse-sine branch.

Step 2: Separate and integrate Rearranging gives

$$\cot v \, dv = -\frac{dx}{x}.$$

Integrating both sides,

$$\ln |\sin v| = -\ln |x| + C_1, \quad \ln |x \sin v| = C_1.$$

The sign is fixed on any connected solution interval through the initial point, so the constant can be written in the form

$$x \sin v = C, \quad x \sin\left(\frac{y}{x}\right) = C.$$

Taking the logarithm did not lose the requested solution, since its initial ratio is $\pi/4$, rather than a zero of $\sin v$.

Step 3: Apply the initial condition and verify At $x = 1$, $y = \pi/4$, and hence

$$C = 1 \cdot \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

To check the result, differentiate $x \sin v = 1/\sqrt{2}$:

$$\sin v + x \cos v \frac{dv}{dx} = 0.$$

Since $x \, dv/dx = dy/dx - y/x$, division by $\cos v$ recovers

$$\frac{dy}{dx} = \frac{y}{x} - \tan\left(\frac{y}{x}\right).$$

The branch through the prescribed point has v near $\pi/4$. It may also be expressed as $y = x \sin^{-1}(1/(\sqrt{2}x))$, with the principal inverse sine, on $x > 1/\sqrt{2}$. At the excluded endpoint the tangent is undefined, so the solution interval does not include it.

Step 4: Conclusion The particular solution through $(1, \pi/4)$ is $x \sin\left(\frac{y}{x}\right) = \frac{1}{\sqrt{2}}$, taking the branch $y = x \sin^{-1}(1/(\sqrt{2}x))$ for $x > 1/\sqrt{2}$.

Quick Tip to Solve: Substitute $y = vx$ before separating; then use the initial value of v to select the correct inverse-sine branch.

32. Let N be the set of natural numbers and R be the relation on $N \times N$ defined by $(a, b)R(c, d)$ iff $ad = bc$ for all $a, b, c, d \in N$. Show that R is an equivalence relation. [4 marks]

Solution:

Concept:

- An equivalence relation is reflexive, symmetric and transitive. The elements here are ordered pairs belonging to $N \times N$.
- The defining condition is $(a, b)R(c, d)$ if and only if $ad = bc$. All natural numbers are positive, so a natural-number factor can be cancelled.

Quick Tip to Solve: For transitivity, multiply $ad = bc$ by f , substitute $cf = de$, and cancel the positive factor d .

Step 1: Verify reflexivity Let (a, b) be any element of $N \times N$. Applying the defining condition to this ordered pair and itself requires $ab = ba$. This equality holds by commutativity of multiplication. Therefore $(a, b)R(a, b)$ for every element of the underlying set, which proves that R is reflexive. Notice that it is an ordered pair, rather than either individual coordinate, that is an element of the set on which the relation is defined.

Step 2: Verify symmetry Let $(a, b), (c, d) \in N \times N$, and assume $(a, b)R(c, d)$. By definition, $ad = bc$. Reversing the equality and using commutativity gives $cb = da$. This is precisely the equality required by the defining condition for $(c, d)R(a, b)$. Thus whenever one ordered pair is related to another, the relation also holds in the reverse direction, proving symmetry.

Step 3: Verify transitivity Choose any three ordered pairs $(a, b), (c, d), (e, f) \in N \times N$. Suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$. The two assumptions mean

$$ad = bc, \quad cf = de.$$

Multiply the first equality by f and use the second equality to substitute for cf :

$$adf = bcf = bde.$$

Consequently, $d(af) = d(be)$. Since d is a natural number, $d > 0$; cancelling this nonzero factor gives $af = be$. By the defining condition, this is exactly $(a, b)R(e, f)$. The implication holds for arbitrary three ordered pairs, so R is transitive.

Step 4: Conclusion Since R is reflexive, symmetric and transitive on $N \times N$, it is an equivalence relation.

Quick Tip to Solve: For transitivity, multiply $ad = bc$ by f , substitute $cf = de$, and cancel the positive factor d .

Section D (24 marks)

Question numbers 33 to 36 carry 6 marks each.

33. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone. [6 marks]

Solution:

Concept:

- The volume of a right circular cylinder of radius x and height y is $V = \pi x^2 y$. For a cylinder inscribed coaxially in a cone, similar axial triangles connect its radius and height.
- At an interior maximum of a differentiable volume function, its first derivative is zero. A negative second derivative confirms a local maximum; checking the endpoints establishes the greatest volume.

Step 1: Express the cylinder radius in terms of its height Let the cone have base radius $r > 0$ and

Quick Tip to Solve: Use similarity first to get $x = r(1 - y/h)$; the derivative of $y(h - y)^2$ factors immediately as $(h - y)(h - 3y)$.

height $h > 0$. Let x and y be the radius and height of the inscribed cylinder, respectively, with $0 \leq y \leq h$. Its base lies in the base plane of the cone, its axis coincides with the cone's axis, and its upper circular rim touches the conical surface. Above the upper rim is a smaller cone of height $h - y$ and radius x . In an axial section, the smaller and original cones give similar triangles. Their corresponding radii and heights therefore satisfy

$$\frac{x}{r} = \frac{h - y}{h}, \quad x = \frac{r(h - y)}{h}.$$

Thus raising the top of the cylinder reduces the available radius. This geometric constraint lets us express its volume using the single variable y .

Step 2: Form and differentiate the volume function Substitute the radius into the cylinder-volume formula:

$$V(y) = \pi x^2 y = \frac{\pi r^2}{h^2} y(h - y)^2.$$

The constants r and h are fixed while y varies. Differentiating by the product rule gives

$$\begin{aligned} V'(y) &= \frac{\pi r^2}{h^2} [(h - y)^2 - 2y(h - y)] \\ &= \frac{\pi r^2}{h^2} (h - y)(h - 3y). \end{aligned}$$

Setting the derivative equal to zero gives $y = h$ or $y = h/3$. The value $y = h$ is a degenerate endpoint where the cylinder radius vanishes. The interior critical height is $h/3$.

Step 3: Establish the greatest volume Differentiating once more,

$$V''(y) = \frac{\pi r^2}{h^2} (6y - 4h).$$

At the interior critical point,

$$V''\left(\frac{h}{3}\right) = -\frac{2\pi r^2}{h} < 0.$$

Hence the volume has a local maximum there. Moreover, $V(0) = V(h) = 0$, while the volume at $h/3$ is positive. There is only one interior critical point. Equivalently, $V'(y) > 0$ for $0 < y < h/3$, and $V'(y) < 0$ for $h/3 < y < h$. Thus this value is the absolute maximum over all allowable cylinder heights.

Step 4: Calculate the maximum volume and compare with the cone At $y = h/3$, the similarity relation gives

$$x = r \left(1 - \frac{1}{3}\right) = \frac{2r}{3}.$$

Therefore the greatest volume of the cylinder is

$$V_{\max} = \pi \left(\frac{2r}{3}\right)^2 \frac{h}{3} = \frac{4\pi r^2 h}{27}.$$

Quick Tip to Solve: Use similarity first to get $x = r(1 - y/h)$; the derivative of $y(h - y)^2$ factors immediately as $(h - y)(h - 3y)$.

The cone has volume $V_{\text{cone}} = \pi r^2 h/3$. Dividing the greatest cylinder volume by the cone volume gives

$$\frac{V_{\text{max}}}{V_{\text{cone}}} = \frac{4\pi r^2 h}{27} \frac{3}{\pi r^2 h} = \frac{4}{9}.$$

Step 5: Conclusion The cylinder of greatest volume has height $\boxed{h/3}$, one-third of the cone's height, and greatest volume $\boxed{4\pi r^2 h/27 = \frac{4}{9} V_{\text{cone}}}$, as required.

Quick Tip to Solve: Use similarity first to get $x = r(1 - y/h)$; the derivative of $y(h - y)^2$ factors immediately as $(h - y)(h - 3y)$.

34. Using integration, find the area of the region $\{(x, y) : x^2 + y^2 \leq 9, x + y \geq 3\}$.

[6 marks]

Solution:

Concept:

- The inequality $x^2 + y^2 \leq 9$ describes the disk of radius 3, centred at the origin. The inequality $x + y \geq 3$ selects the side of the line $y = 3 - x$ away from the origin.
- An area between an upper curve $y = f(x)$ and a lower curve $y = g(x)$, over $a \leq x \leq b$, is $\int_a^b [f(x) - g(x)] dx$. Here the upper boundary is the positive semicircle.
- For $a > 0$, $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}(x/a) + C$, with the inverse sine taken in radians.

Step 1: Find the intersection points of the boundaries Put $y = 3 - x$ in the circle equation. Then

$$\begin{aligned} x^2 + (3 - x)^2 &= 9, \\ 2x^2 - 6x &= 0, \\ 2x(x - 3) &= 0. \end{aligned}$$

Thus the boundaries meet at $(0, 3)$ and $(3, 0)$. These give the limits $x = 0$ and $x = 3$ for the desired circular segment.

Step 2: Identify the upper and lower boundaries The required region is inside the circle and above $y = 3 - x$. It lies entirely in the first quadrant: if $x < 0$, then $y \geq 3 - x > 3$, which is impossible inside the disk; similarly $y < 0$ is impossible. For $0 \leq x \leq 3$, the circle's upper boundary is $y = \sqrt{9 - x^2}$. Also,

$$(9 - x^2) - (3 - x)^2 = 2x(3 - x) \geq 0.$$

Both heights are non-negative, so the semicircle is above the line throughout this interval. Hence each vertical strip has height $\sqrt{9 - x^2} - (3 - x)$.

Quick Tip to Solve: Solve the line-circle intersection first; then integrate the upper semicircle minus the line between the two intersection abscissae.

Step 3: Integrate the strip height Denote the required area by A . Using integration,

$$\begin{aligned} A &= \int_0^3 \left[\sqrt{9-x^2} - (3-x) \right] dx \\ &= \left[\frac{x}{2} \sqrt{9-x^2} + \frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) - 3x + \frac{x^2}{2} \right]_0^3. \end{aligned}$$

At the upper limit the square-root term vanishes and $\sin^{-1}(1) = \pi/2$; at the lower limit every term vanishes. Consequently the integral evaluates to $9\pi/4 - 9 + 9/2$. As a check, the curved part contributes the first-quadrant area $9\pi/4$, while the integral under the line contributes $9/2$.

Step 4: Conclusion The required area is $\boxed{\frac{9\pi}{4} - \frac{9}{2} = \frac{9}{4}(\pi - 2)}$ square units.

Quick Tip to Solve: Solve the line-circle intersection first; then integrate the upper semicircle minus the line between the two intersection abscissae.

35(a). Find the equation of the plane that contains the point $A(2, 1, -1)$ and is perpendicular to the line of intersection of the planes $2x + y - z = 3$ and $x + 2y + z = 2$. Also find the angle between the plane thus obtained and the y -axis. [6 marks]

Solution:

Concept:

- A plane $ax + by + cz = d$ has normal vector (a, b, c) . A vector parallel to the intersection of two non-parallel planes is perpendicular to both their normal vectors.
- The cross product of the two plane normals gives a direction vector of their intersection line. A plane perpendicular to that line has a normal parallel to this direction vector.
- If θ is the acute angle between a line with direction vector $\mathbf{d}$ and a plane with normal $\mathbf{n}$, then $\sin \theta = |\mathbf{n} \cdot \mathbf{d}| / (|\mathbf{n}| |\mathbf{d}|)$.

Step 1: Find a direction vector of the intersection line The normals of the given planes are $\mathbf{n}_1 = (2, 1, -1)$ and $\mathbf{n}_2 = (1, 2, 1)$. If $\mathbf{v}$ is parallel to their line of intersection, then it is perpendicular to each normal. Calculate their cross product:

$$\begin{aligned} \mathbf{v} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} \\ &= 3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}. \end{aligned}$$

We may divide by 3 and use $(1, -1, 1)$. Indeed, its dot products with the two original normals are $2 - 1 - 1 = 0$ and $1 - 2 + 1 = 0$, verifying the direction independently.

Step 2: Use the given point to determine the plane The required plane is perpendicular to the intersection line, so take its normal as $\mathbf{n} = (1, -1, 1)$. The point-normal equation through $A(2, 1, -1)$

Quick Tip to Solve: Cross the two given normals to obtain the new plane's normal; use sine, rather than cosine, for the angle between an axis and a plane.

is

$$\begin{aligned}\mathbf{n} \cdot ((x, y, z) - (2, 1, -1)) &= 0, \\ (x - 2) - (y - 1) + (z + 1) &= 0.\end{aligned}$$

On simplifying, the constant terms cancel and the equation is $x - y + z = 0$. Substitution of A gives $2 - 1 - 1 = 0$, so the plane contains the stated point. Since the normal direction is prescribed and the point fixes its constant, this determines the plane uniquely.

Step 3: Calculate the angle with the y-axis A direction vector of the y -axis is $\mathbf{d} = (0, 1, 0)$. The magnitude of the normal is $\sqrt{3}$, and $|\mathbf{n} \cdot \mathbf{d}| = |-1| = 1$. Therefore the acute angle θ between the plane and this axis satisfies

$$\sin \theta = \frac{1}{\sqrt{3}}, \quad \theta = \sin^{-1}\left(\frac{1}{\sqrt{3}}\right).$$

The absolute value is necessary because an axis is an unoriented line. Using cosine with this ratio would give the angle with the plane's normal; the requested line-plane angle is its complement.

Step 4: Conclusion The plane is $x - y + z = 0$, and its angle with the y -axis is $\sin^{-1}(1/\sqrt{3}) \approx 35.26^\circ$.

Quick Tip to Solve: Cross the two given normals to obtain the new plane's normal; use sine, rather than cosine, for the angle between an axis and a plane.

OR

35(b). Find the distance of the point $P(-2, -4, 7)$ from the point of intersection Q of the line $\vec{r} = (3\hat{i} - 2\hat{j} + 6\hat{k}) + \lambda(2\hat{i} - \hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 6$. Also write the vector equation of the line PQ . [6 marks]

Solution:

Concept:

- A point on a vector line is obtained by substituting a parameter into its three coordinates. At an intersection with a plane, those coordinates must also satisfy the plane equation.
- For points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, their distance is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

- A line through a point with position vector $\vec{p}$ and nonzero direction vector $\vec{d}$ has equation $\vec{r} = \vec{p} + t\vec{d}$, where $t \in \mathbb{R}$. Any nonzero multiple of the direction vector gives the same line.

Step 1: Express a general point of the given line Equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in the given vector equation gives

$$x = 3 + 2\lambda, \quad y = -2 - \lambda, \quad z = 6 + 2\lambda.$$

The plane equation is $x - y + z = 6$, since its normal vector is $\hat{i} - \hat{j} + \hat{k}$. We use this plane equation to find the particular parameter that corresponds to the intersection point Q .

Quick Tip to Solve: First substitute the parametric coordinates into the plane; use the resulting intersection point for both the distance and the direction of PQ .

Step 2: Locate the intersection point Substituting the line coordinates into the plane gives

$$\begin{aligned}(3 + 2\lambda) - (-2 - \lambda) + (6 + 2\lambda) &= 6, \\ 11 + 5\lambda &= 6, \\ \lambda &= -1.\end{aligned}$$

Therefore $Q = (3 - 2, -2 + 1, 6 - 2) = (1, -1, 4)$. This point lies on the plane because $1 - (-1) + 4 = 6$. The coefficient 5 is nonzero, so this is the unique intersection of the given line and plane.

Step 3: Calculate the distance from P to Q With $P = (-2, -4, 7)$, the displacement from P to Q is

$$\overrightarrow{PQ} = (1 + 2)\hat{i} + (-1 + 4)\hat{j} + (4 - 7)\hat{k} = 3\hat{i} + 3\hat{j} - 3\hat{k}.$$

Hence the square of the distance is $3^2 + 3^2 + (-3)^2 = 27$, and the distance is $\sqrt{27} = 3\sqrt{3}$. All three coordinate differences must be squared, including the negative difference in the third coordinate.

Step 4: Construct and verify the line PQ A convenient direction vector is $\hat{i} + \hat{j} - \hat{k}$, obtained by dividing $\overrightarrow{PQ}$ by 3. Thus a vector equation through P is

$$\vec{r} = -2\hat{i} - 4\hat{j} + 7\hat{k} + t(\hat{i} + \hat{j} - \hat{k}), \quad t \in \mathbb{R}.$$

At $t = 0$, it gives P ; at $t = 3$, it gives $(1, -1, 4) = Q$. These substitutions verify that the constructed line passes through both required points.

Step 5: Conclusion The required distance is $3\sqrt{3}$ units, and the line is

$$\vec{r} = -2\hat{i} - 4\hat{j} + 7\hat{k} + t(\hat{i} + \hat{j} - \hat{k}), \quad t \in \mathbb{R}.$$

Quick Tip to Solve: First substitute the parametric coordinates into the plane; use the resulting intersection point for both the distance and the direction of PQ.

36(a). If $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}$, then find A^{-1} and use it to solve the following system of equations:

$$x + 2y - 3z = 6$$

$$3x + 2y - 2z = 3$$

$$2x - y + z = 2$$

[6 marks]

Solution:

Concept:

- A square matrix A is invertible when $|A| \neq 0$; its inverse is $A^{-1} = \frac{\text{adj } A}{|A|}$.
- The cofactor C_{ij} is $(-1)^{i+j}$ times the minor obtained by deleting row i and column j . The adjugate is the transpose of the cofactor matrix.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant, and check the final values in all three original equations.

- A linear system with coefficient matrix A is written $AX = B$, where X is the column of unknowns and B is the column of constants. When A is invertible, $X = A^{-1}B$.

Step 1: Evaluate the determinant Expanding along the first row, with alternating signs, gives

$$\begin{aligned}|A| &= 1(2 - 2) - 2(3 + 4) + (-3)(-3 - 4) \\ &= 0 - 14 + 21 = 7.\end{aligned}$$

Thus A is nonsingular and has an inverse. In particular, the system will have a unique solution, which can be found by the inverse matrix method requested in the question.

Step 2: Calculate the cofactors Applying the cofactor sign pattern gives all nine entries:

$$\begin{aligned}C_{11} &= 2 - 2 = 0, & C_{12} &= -(3 + 4) = -7, \\ C_{13} &= -3 - 4 = -7, & C_{21} &= -(2 - 3) = 1, \\ C_{22} &= 1 + 6 = 7, & C_{23} &= -(-1 - 4) = 5, \\ C_{31} &= -4 + 6 = 2, & C_{32} &= -(-2 + 9) = -7, \\ C_{33} &= 2 - 6 = -4.\end{aligned}$$

Therefore the cofactor matrix is

$$C = \begin{bmatrix} 0 & -7 & -7 \\ 1 & 7 & 5 \\ 2 & -7 & -4 \end{bmatrix}.$$

The negative sign in the cofactors $C_{12}, C_{21}, C_{23}, C_{32}$ comes from their odd row-plus-column index.

Step 3: Form the adjugate and inverse Transposing the cofactor matrix, rather than using it directly, gives

$$\text{adj } A = C^T = \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}.$$

Consequently,

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}.$$

As a check, multiplication of A by this adjugate gives $7I$, consistent with the determinant calculated above.

Step 4: Use the inverse to solve the equations The given equations have matrix form

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant, and check the final values in all three original equations.

Multiplying on the left by A^{-1} , we obtain

$$\begin{aligned}\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \frac{1}{7} \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \\ &= \frac{1}{7} \begin{bmatrix} 3+4 \\ -42+21-14 \\ -42+15-8 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -5 \end{bmatrix}.\end{aligned}$$

Step 5: Verify in the original system Substitution into each original equation gives

$$\begin{aligned}1 + 2(-5) - 3(-5) &= 6, \\ 3(1) + 2(-5) - 2(-5) &= 3, \\ 2(1) - (-5) + (-5) &= 2.\end{aligned}$$

All three right-hand sides agree with the given constants, confirming the solution and its signs.

Step 6: Conclusion

$$A^{-1} = \frac{1}{7} \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}, \quad \boxed{x = 1, y = -5, z = -5}.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant, and check the final values in all three original equations.

OR

36(b). Using properties of determinants, prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2).$$

[6 marks]

Solution:

Concept:

- Adding a multiple of one column to another leaves a determinant unchanged. A common factor in a column can be taken outside the determinant.
- A determinant is linear in each column, and it vanishes when two columns are identical. Interchanging two columns reverses its sign.
- Write $s = a + b + c$, $p = ab + bc + ca$, and $t = a^2 + b^2 + c^2$. Then $bc = p - sa + a^2$, with analogous expressions for ca and ab .

Step 1: Make the first column constant Let the determinant on the left be D . Apply $C_1 \leftarrow C_1 + C_2 - 2C_3$.

Quick Tip to Solve: Start with $C_1 \leftarrow C_1 + C_2 - 2C_3$: it produces the same sum of squares in all three rows and exposes the required symmetric factor immediately.

In its first entry,

$$(b+c)^2 + a^2 - 2bc = a^2 + b^2 + c^2 = t.$$

The second and third entries also become t , since $(c+a)^2 + b^2 - 2ca = t$ and $(a+b)^2 + c^2 - 2ab = t$. Hence, taking the common factor out of the first column gives

$$D = t \begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}.$$

These operations are valid for all values of a, b, c ; no division by an expression in these variables is required.

Step 2: Reduce the third column using the symmetric sums In the determinant now obtained, apply $C_3 \leftarrow C_3 - C_2$. Since

$$\begin{aligned} bc - a^2 &= p - sa, \\ ca - b^2 &= p - sb, \\ ab - c^2 &= p - sc, \end{aligned}$$

we have

$$D = t \begin{vmatrix} 1 & a^2 & p - sa \\ 1 & b^2 & p - sb \\ 1 & c^2 & p - sc \end{vmatrix}.$$

Next apply $C_3 \leftarrow C_3 - pC_1$, so that the third column becomes $(-sa, -sb, -sc)^T$. Factor out $-s$, and then interchange columns two and three:

$$\begin{aligned} D &= -st \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix} \\ &= st \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}. \end{aligned}$$

The minus sign from the column interchange cancels the minus sign from the extracted factor.

Step 3: Evaluate the remaining determinant by row operations Denote the remaining determinant by V . Apply $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$. Factor the differences from the last two rows:

$$\begin{aligned} V &= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \\ &= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} \\ &= (b-a)(c-a)[(c+a) - (b+a)] \\ &= (b-a)(c-a)(c-b). \end{aligned}$$

Quick Tip to Solve: Start with $C_1 \leftarrow C_1 + C_2 - 2C_3$: it produces the same sum of squares in all three rows and exposes the required symmetric factor immediately.

Here expansion along the first column reduces the calculation to a 2×2 determinant. Finally, $b - a = -(a - b)$ and $c - b = -(b - c)$; these two sign changes cancel. Thus $V = (a - b)(b - c)(c - a)$. The factorisation also covers equal variables, for which both sides vanish.

Step 4: Conclusion Substituting V , s , and t into $D = stV$, we obtain

$$D = (a - b)(b - c)(c - a)(a + b + c)(a^2 + b^2 + c^2).$$

Hence the given identity is proved using properties of determinants.

Quick Tip to Solve: Start with $C_1 \leftarrow C_1 + C_2 - 2C_3$: it produces the same sum of squares in all three rows and exposes the required symmetric factor immediately.