CBSE Class 12 Mathematics Question Paper 2020 with Solution PDF-Set 65-1-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper comprises **four** Sections A, B, C and D. This question paper carries **36** questions. **All** questions are compulsory.

(ii) **Section A** – Questions no. **1 to 20** comprises of **20** questions of **1** mark each.

(iii) **Section B** – Questions no. **21 to 26** comprises of **6** questions of **2** marks each.

(iv) **Section C** – Questions no. **27 to 32** comprises of **6** questions of **4** marks each.

(v) **Section D** – Questions no. **33 to 36** comprises of **4** questions of **6** marks each.

(vi) There is no overall choice in the question paper. However, an internal choice has been provided in 3 questions of one mark, 2 questions of two marks, 2 questions of four marks and 2 questions of six marks. Only one of the choices in such questions have to be attempted.

(vii) In addition to this, separate instructions are given with each section and question, wherever necessary.

(viii) Use of calculators is not permitted.

Section A (20 marks)

Question numbers 1 to 20 carry 1 mark each. Question numbers 1 to 10 are multiple choice type questions. Select the correct option.

1. The matrix $\begin{bmatrix} 2 & -1 & 3 \\ \lambda & 0 & 7 \\ -1 & 1 & 4 \end{bmatrix}$ **is not invertible for**

(A) $\lambda = -1$

(B) $\lambda = 0$

(C) $\lambda = 1$

(D) $\lambda \in \mathbb{R} - \{1\}$ [1 mark]

Solution:

Concept:

- A square matrix is not invertible precisely when its determinant is zero.
- Expand a determinant along a row using alternating signs $+, -, +$.

Step 1: Expand along the first row Let the given matrix be M. Its determinant is

$$|M| = 2(0 \cdot 4 - 7 \cdot 1) - (-1)(4\lambda + 7) + 3(\lambda \cdot 1 - 0)$$
$$= -14 + 4\lambda + 7 + 3\lambda = 7\lambda - 7.$$

Step 2: Apply the singularity condition For non-invertibility, $7\lambda - 7 = 0$, so $\lambda = 1$. At $\lambda = -1$ and $\lambda = 0$, the determinant is respectively -14 and -7, so (A) and (B) fail. Every value other than 1 gives a nonzero determinant, contrary to (D).

Step 3: Conclusion The required value is $\boxed{\lambda = 1}$, option (C).

Quick Tip to Solve: Expand along the row containing zero to reduce the number of nonzero minors.

2. The number of arbitrary constants in the particular solution of a differential equation of second order is (are)

(A) 0

(B) 1

(C) 2

(D) 3 [1 mark]

Solution:

Concept:

- The general solution of a second-order differential equation contains two independent arbitrary constants.
- A particular solution is obtained by assigning fixed values to the arbitrary constants.

Quick Tip to Solve: Check whether the question says general solution or particular solution before counting constants.

Step 1: Distinguish general and particular solutions Two integrations introduce two arbitrary constants into a general solution of a second-order equation. Their values remain unspecified until conditions selecting a particular solution are imposed.

Step 2: Count the remaining arbitrary constants In a particular solution, those constants have definite values. They may still appear as fixed numbers in the expression, but they are no longer arbitrary. Thus (C) describes the general solution; (B) would leave a free constant, and (D) is also incorrect.

Step 3: Conclusion There are $\boxed{0}$ arbitrary constants in a particular solution, option (A).

Quick Tip to Solve: Check whether the question says general solution or particular solution before counting constants.

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3. The value of $\tan^{-1}\left(\tan\dfrac{7\pi}{6}\right)$ is

(A) $\dfrac{\pi}{6}$

(B) $\dfrac{\pi}{2}$

(C) $\dfrac{\pi}{3}$

(D) $\dfrac{7\pi}{6}$

[1 mark]

Solution:

Concept:

- The principal range of $\tan^{-1}x$ is $(-\pi/2, \pi/2)$.
- Tangent has period π: $\tan(\theta+\pi) = \tan\theta$.

Step 1: Reduce the angle using periodicity Since $7\pi/6 = \pi + \pi/6$,

$$\tan\frac{7\pi}{6} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}.$$

The angle $\pi/6$ lies inside the principal range of inverse tangent.

Step 2: Select the principal value The unique angle in that range with tangent $1/\sqrt{3}$ is $\pi/6$. Option (B) is a point where tangent is undefined; (C) has tangent $\sqrt{3}$. Although (D) has the same tangent as the given angle, it is outside the principal range.

Step 3: Conclusion The value is $\boxed{\pi/6}$, option (A).

Quick Tip to Solve: Subtract a suitable multiple of pi before applying inverse tangent to a tangent value.

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4. The corner points of the feasible region determined by the system of linear inequalities are $(0,0)$, $(4,0)$, $(2,4)$ and $(0,5)$. If the maximum value of $z = ax + by$, where $a, b > 0$ occurs at both $(2,4)$ and $(4,0)$, then

(A) $a = 2b$

(B) $2a = b$

(C) $a = b$

(D) $3a = b$ [1 mark]

Solution:

Concept:

- For a bounded polygonal feasible region, a linear objective attains its maximum at a corner point.
- If two points both attain the maximum, their objective values must be equal.

Step 1: Equate the two maximum values At $(2,4)$, the objective value is $2a + 4b$, while at $(4,0)$ it is $4a$. Therefore

$$2a + 4b = 4a \quad \Longrightarrow \quad a = 2b.$$

Step 2: Check all corners and alternatives With $a = 2b$, the four corner values are $0, 8b, 8b, 5b$. Since $b > 0$, the common value $8b$ is indeed the maximum. Options (B), (C) and (D) give respectively $a = b/2, b, b/3$; none satisfies $2a + 4b = 4a$ for positive b.

Step 3: Conclusion The required relation is $\boxed{a = 2b}$, option (A).

Quick Tip to Solve: When an objective is optimal at two vertices, first equate its values at those vertices.

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5. If A and B are two independent events with $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$, then $P(B' \mid A)$ is equal to

(A) $\frac{1}{4}$

(B) $\frac{1}{3}$

(C) $\frac{3}{4}$

(D) 1 [1 mark]

Solution:

Concept:

- Independence gives $P(A \cap B) = P(A)P(B)$.
- For $P(A) > 0$, $P(B \mid A) = P(A \cap B)/P(A)$, and $P(B' \mid A) = 1 - P(B \mid A)$.

Step 1: Find the conditional probability of B Using the given probabilities,

$$P(A \cap B) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}, \qquad P(B \mid A) = \frac{1/12}{1/3} = \frac{1}{4}.$$

Step 2: Take the complement within A Within event A, the events B and B' are complementary.

Quick Tip to Solve: Independence makes conditioning on A leave the probability of B, and its complement, unchanged.

Hence $P(B' \mid A) = 1 - 1/4 = 3/4$. Option (A) gives the probability of B rather than its complement; (B) merely repeats $P(A)$, and (D) incorrectly treats B' as certain.

Step 3: Conclusion The probability is $\boxed{3/4}$, option (C).

Quick Tip to Solve: Independence makes conditioning on A leave the probability of B, and its complement, unchanged.

6. If A is a square matrix such that $A^2 = A$, then $(I - A)^3 + A$ is equal to

(A) I

(B) 0

(C) $I - A$

(D) $I + A$ [1 mark]

Solution:

Concept:

- A matrix satisfying $A^2 = A$ is idempotent, so $A^3 = A^2 A = A$.
- The identity matrix I commutes with every square matrix of the same order, allowing the binomial expansion.

Step 1: Expand the cube Using commutation with I,

$$(I - A)^3 = I - 3A + 3A^2 - A^3.$$

Substitute $A^2 = A$ and $A^3 = A$ to obtain $(I - A)^3 = I - A$.

Step 2: Add A and assess the options Thus $(I - A)^3 + A = (I - A) + A = I$. Option (C) omits the final addition of A. Options (B) and (D) do not follow from the simplification. In particular, for the permitted choice $A = I$, the expression equals I, ruling out both 0 and $I + A = 2I$.

Step 3: Conclusion The expression equals $\boxed{I}$, option (A).

Quick Tip to Solve: Reduce every positive power of an idempotent matrix to A before collecting terms.

7. $\int_1^e \frac{\log x}{x}\,dx$ **is equal to**

(A) $\frac{e^2}{2}$

(B) 1

(C) $\frac{1}{2}$

(D) $-\infty$ [1 mark]

Solution:

Concept:

- Use substitution when a factor is the derivative of another expression: set $u = \log x$, giving $du = dx/x$.
- In calculus, $\log x$ denotes the natural logarithm, so $\log 1 = 0$ and $\log e = 1$.

Step 1: Transform both limits For $u = \log x$, the lower limit $x = 1$ becomes $u = 0$, and the upper limit $x = e$ becomes $u = 1$. Consequently,

$$\int_1^e \frac{\log x}{x}\,dx = \int_0^1 u\,du.$$

Step 2: Evaluate and compare the choices The transformed integral is $[u^2/2]_0^1 = 1/2$. Option (A) fails to account for the logarithmic substitution; (B) omits the factor $1/2$. The integrand is continuous and nonnegative on $[1, e]$, so the integral is finite and cannot be the negative infinite value in (D).

Step 3: Conclusion The integral is $\boxed{1/2}$, option (C).

Quick Tip to Solve: For log x divided by x, substitute log x and change the limits immediately.

8. A point P lies on the line segment joining the points $(-1, 3, 2)$ and $(5, 0, 6)$. If x-coordinate of P is 2, then its z-coordinate is

(A) -1

(B) 4

(C) $\frac{3}{2}$

(D) 8 [1 mark]

Solution:

Concept:

- A point on the segment from A to B has coordinates $A + t(B - A)$, where $0 \le t \le 1$.
- The same parameter t must be used in all three coordinates.

Step 1: Determine the position on the segment For $A = (-1, 3, 2)$ and $B = (5, 0, 6)$, the coordinates of P are $(-1 + 6t, 3 - 3t, 2 + 4t)$. Since its given x-coordinate is 2, we obtain $-1 + 6t = 2$, hence $t = 1/2$. Thus P is the midpoint of the segment.

Step 2: Calculate and compare the options The corresponding coordinate is $z = 2 + 4(1/2) = 4$. The

Quick Tip to Solve: When the given coordinate is halfway between its endpoint values, every coordinate is the midpoint average.

midpoint also has $y = 3/2$; that value is a different coordinate, so option (C) is a tempting distractor. Options (A) and (D) are outside the endpoint range $2 \leq z \leq 6$ and cannot describe a point on this segment.

Step 3: Conclusion The z-coordinate is $\boxed{4}$. Hence, the correct option is (B).

Quick Tip to Solve: When the given coordinate is halfway between its endpoint values, every coordinate is the midpoint average.

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9. If the projection of $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ on $\vec{b} = 2\hat{i} + \lambda\hat{k}$ is zero, then the value of λ is

(A) 0

(B) 1

(C) $\dfrac{-2}{3}$

(D) $\dfrac{-3}{2}$

[1 mark]

Solution:

Concept:

- The scalar projection of $\vec{a}$ on a nonzero vector $\vec{b}$ is $\vec{a} \cdot \vec{b}/|\vec{b}|$.
- Zero projection means the vectors are perpendicular, so their dot product is zero.

Step 1: Use the projection condition Here $\vec{b} = 2\hat{i} + \lambda\hat{k}$ has magnitude $\sqrt{4 + \lambda^2} > 0$ for every real λ. Thus the denominator cannot cause the projection to vanish. Using the corresponding components gives $\vec{a} \cdot \vec{b} = 1(2) + (-2)(0) + 3\lambda = 2 + 3\lambda$.

Step 2: Solve and check the choices The required condition is $2 + 3\lambda = 0$, which gives $\lambda = -2/3$. Substitution gives a dot product of zero. For options (A), (B) and (D), the dot products are respectively 2, 5 and $-5/2$; all are nonzero, so none has zero projection.

Step 3: Conclusion $\boxed{\lambda = -\dfrac{2}{3}}$. Hence, the correct option is (C).

Quick Tip to Solve: For zero projection, set the dot product to zero immediately; there is no need to calculate the magnitude.

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10. The vector equation of the line passing through the point $(-1, 5, 4)$ and perpendicular to the plane $z = 0$ is

(A) $\vec{r} = -\hat{i} + 5\hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j})$

(B) $\vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}$

(C) $\vec{r} = \hat{i} - 5\hat{j} - 4\hat{k} + \lambda\hat{k}$

(D) $\vec{r} = \lambda\hat{k}$ [1 mark]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$ and direction vector $\vec{d}$ has equation $\vec{r} = \vec{a} + \lambda\vec{d}$.
- A line perpendicular to a plane is parallel to the normal vector of that plane.

Step 1: Identify the direction and point The plane $z = 0$ is the xy-plane, whose normal vector is $\hat{k}$. Therefore the required line varies only in its z-coordinate. The fixed point has position vector $-\hat{i} + 5\hat{j} + 4\hat{k}$, giving $\vec{r} = -\hat{i} + 5\hat{j} + 4\hat{k} + \lambda\hat{k}$.

Step 2: Check the equations against both conditions Combining the $\hat{k}$ terms gives $-\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}$, exactly option (B). At $\lambda = 0$ it passes through the given point. Option (A) has a direction in the plane. Option (C) keeps $x = 1, y = -5$, and option (D) keeps $x = y = 0$, so neither passes through the required point.

Step 3: Conclusion $\boxed{\vec{r} = -\hat{i} + 5\hat{j} + (4 + \lambda)\hat{k}}$. Hence, the correct option is (B).

Quick Tip to Solve: For a line normal to the xy-plane, hold x and y fixed and let only z vary.

11. The position vectors of two points A and B are $\overrightarrow{OA} = 2\hat{i} - \hat{j} - \hat{k}$ and $\overrightarrow{OB} = 2\hat{i} - \hat{j} + 2\hat{k}$, respectively. The position vector of a point P which divides the line segment joining A and B in the ratio $2 : 1$ is _____. [1 mark]

Solution:

Concept:

- If $AP : PB = m : n$, internal division gives $\overrightarrow{OP} = (n\overrightarrow{OA} + m\overrightarrow{OB})/(m + n)$.
- The weight of an endpoint is the length of the opposite part of the segment.

Step 1: Substitute the endpoint vectors With $AP : PB = 2 : 1$, use $m = 2, n = 1$. Thus

$$\overrightarrow{OP} = \frac{(2\hat{i} - \hat{j} - \hat{k}) + 2(2\hat{i} - \hat{j} + 2\hat{k})}{3}.$$

The numerator simplifies to $6\hat{i} - 3\hat{j} + 3\hat{k}$.

Step 2: Simplify and verify the division Dividing each component by 3 gives $2\hat{i} - \hat{j} + \hat{k}$. Both endpoints have the same x and y coordinates, so these remain unchanged. Along the z-coordinate, the distances from -1 to 1 and from 1 to 2 are 2 and 1, confirming the prescribed internal ratio.

Step 3: Conclusion The required position vector is $\boxed{2\hat{i} - \hat{j} + \hat{k}}$.

Quick Tip to Solve: For internal division in the ratio 2:1, double the second endpoint vector and add the first, then divide by 3.

12(a). **The equation of the normal to the curve $y^2 = 8x$ at the origin is _____.** [1 mark]

Solution:

Concept:

- The normal at a point is perpendicular to the tangent at that point.
- For a curve written as $x = f(y)$, the derivative dx/dy identifies the tangent direction even when dy/dx is undefined.

Step 1: Find the tangent at the vertex Write the curve as $x = y^2/8$. Differentiation gives $dx/dy = y/4$, which is zero at the origin. A tangent direction is therefore $(dx/dy, 1) = (0, 1)$. This is a vertical direction, so the tangent at the origin is the line $x = 0$.

Step 2: Use perpendicularity at the origin The normal must be perpendicular to this vertical tangent and pass through $(0, 0)$. It is consequently horizontal. A horizontal line has constant y, and passing through the origin fixes that constant as zero. This avoids trying to substitute $y = 0$ into $dy/dx = 4/y$, where that derivative is undefined.

Step 3: Conclusion The normal at the origin is $\boxed{y = 0}$.

Quick Tip to Solve: At the vertex of a right-opening parabola, the tangent is vertical and the normal is the horizontal axis.

OR

12(b). **The radius of a circle is increasing at the uniform rate of 3 cm/sec. At the instant when the radius of the circle is 2 cm, its area increases at the rate of _____ cm^2/s.** [1 mark]

Solution:

Concept:

- The area of a circle of radius r is $A = \pi r^2$.
- The chain rule gives $dA/dt = 2\pi r\, dr/dt$, where t denotes time.

Step 1: Differentiate with respect to time Both the radius and area vary with time. Differentiating $A = \pi r^2$ with respect to t, rather than just to r, gives

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}.$$

The factor dr/dt accounts for the specified rate at which the radius changes.

Step 2: Evaluate at the given instant At this instant, $r = 2\,\text{cm}$ and $dr/dt = 3\,\text{cm/s}$. Substitution gives

$$\frac{dA}{dt} = 2\pi(2)(3) = 12\pi\ \text{cm}^2/\text{s}.$$

The result is positive because the radius is increasing. The units are area per time, as required, rather than the length per time used for the radius.

Step 3: Conclusion The area increases at $\boxed{12\pi\ \text{cm}^2/\text{s}}$.

Quick Tip to Solve: Multiply the circle circumference by the radial speed to obtain its instantaneous area growth rate.

13(a). **If A is a square matrix of order 3 and A_{ij} is the cofactor of the element a_{ij}, then value of $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13}$ is _____.** [1 mark]

Solution:

Concept:

- Expanding a determinant along row i gives $\sum_j a_{ij}A_{ij}$.
- The product sum of one row with the cofactors of a different row is zero, as follows from a determinant with equal rows.

Step 1: Interpret the given sum as a determinant Replace the first row of A by its second row, producing a matrix B. The cofactors along the first row of B are still A_{11}, A_{12}, A_{13}, because deleting the first row leaves exactly the same minors as in A. Therefore expansion of $\det B$ along its first row gives the stated sum.

Step 2: Use the repeated-row property The first and second rows of B are identical. A determinant with two equal rows is zero, so $\det B = 0$. Consequently $a_{21}A_{11} + a_{22}A_{12} + a_{23}A_{13} = 0$. This is not the ordinary second-row expansion of $\det A$, which would instead involve A_{21}, A_{22}, A_{23}.

Step 3: Conclusion The required value is $\boxed{0}$.

Quick Tip to Solve: Check the row indices: entries from one row multiplied by cofactors from another row always sum to zero.

OR

13(b). **If the matrix A is both symmetric and skew symmetric, then A is a _____.** [1 mark]

Solution:

Concept:

- A symmetric matrix satisfies $A^T = A$, where A^T denotes the transpose.
- A skew-symmetric matrix satisfies $A^T = -A$. A zero matrix has every entry equal to zero.

Step 1: Apply both given properties Since the same matrix is symmetric and skew symmetric, its transpose must obey both equations simultaneously. Thus

$$A = A^T = -A.$$

This condition concerns the whole matrix, including its off-diagonal entries; merely having zero diagonal entries is insufficient.

Step 2: Compare corresponding entries Adding A to both sides gives $2A = O$, where O is the zero matrix of the same order. Hence every entry satisfies $2a_{ij} = 0$, so $a_{ij} = 0$. Conversely, the zero matrix satisfies both transpose conditions.

Step 3: Conclusion A is a $\boxed{\text{zero matrix}}$.

Quick Tip to Solve: Equate the two expressions for the transpose; this forces every entry, not just the diagonal, to vanish.

14. A relation R in a set A is called _____, if $(a_1, a_2) \in R$ implies $(a_2, a_1) \in R$, for all $a_1, a_2 \in A$.

[1 mark]

Solution:

Concept:

- A relation on A is a subset of $A \times A$, so its members are ordered pairs.
- A relation is symmetric when each related pair remains related after reversing its order.

Step 1: Interpret the implication The hypothesis states that whenever (a_1, a_2) belongs to R, the reversed pair (a_2, a_1) also belongs to R. The assertion applies to every choice of the two elements in A, which is exactly the symmetry condition.

Step 2: Distinguish the other relation properties Reflexivity would require $(a, a) \in R$ for every $a \in A$; transitivity would concern two pairs (a, b) and (b, c) and require (a, c). Neither is what the stated implication describes. It specifically checks reversal of a single pair.

Step 3: Conclusion The blank is $\boxed{\text{symmetric}}$.

Quick Tip to Solve: When the condition reverses the two entries of a related ordered pair, identify symmetry immediately.

15. The greatest integer function defined by $f(x) = [x]$, $0 < x < 2$ is not differentiable at $x =$ _____.

[1 mark]

Solution:

Concept:

- The greatest integer function $[x]$ is the largest integer not exceeding x.
- Differentiability at an interior point implies continuity there; a jump discontinuity therefore prevents differentiability.

Step 1: Describe the function on the given interval For $0 < x < 1$, $[x] = 0$, whereas for $1 \leq x < 2$, $[x] = 1$. On each open interval $(0, 1)$ and $(1, 2)$, the function is constant and has derivative zero.

Step 2: Examine the only interior integer At $x = 1$,

$$\lim_{x \to 1^-} f(x) = 0, \qquad \lim_{x \to 1^+} f(x) = 1.$$

The unequal one-sided limits establish a jump, so differentiability fails there. The endpoints 0 and 2 do not belong to the specified domain.

Step 3: Conclusion The function is not differentiable at $\boxed{x = 1}$.

Quick Tip to Solve: Look for integers strictly inside the given domain: the floor function jumps exactly at those points.

16. If A is a square matrix of order 3 and $|A| = 2$, then find the value of $|-AA'|$. [1 mark]

Solution:

Concept:

- For square matrices, $|BC| = |B|\,|C|$, and transposition preserves the determinant: $|A^T| = |A|$.
- For an order-n matrix, $|kA| = k^n|A|$, because all n rows are scaled by k.

Step 1: Separate the negative factor and product Here A has order 3, and $A\prime$ denotes its transpose. Therefore

$$|-AA\prime| = (-1)^3|AA\prime| = -|A|\,|A\prime|.$$

The negative sign acts on a matrix of odd order, so it changes the determinant sign.

Step 2: Substitute the given determinant Transposition gives $|A\prime| = |A| = 2$. Thus the product determinant is $|AA\prime| = 2 \times 2 = 4$, and multiplying the whole order-three matrix by -1 supplies a factor $(-1)^3$.

Step 3: Conclusion

$$\boxed{|-AA\prime| = -4}.$$

Quick Tip to Solve: Before substituting the determinant, use the matrix order to determine the sign introduced by a scalar factor.

17. Two cards are drawn at random and one-by-one without replacement from a well-shuffled pack of 52 playing cards. Find the probability that one card is red and the other is black. [1 mark]

Solution:

Concept:

- A standard pack contains 26 red cards and 26 black cards.
- Without replacement, the second draw is from 51 cards. The two colour orders are mutually exclusive, so their probabilities add.

Step 1: Compute one colour order Let RB denote drawing red first and black second. The first draw has probability $26/52$. After a red card is removed, all 26 black cards remain among 51 cards, so

$$P(RB) = \frac{26}{52}\frac{26}{51} = \frac{13}{51}.$$

Step 2: Include the reverse order Likewise, black followed by red has probability $P(BR) = (26/52)(26/51) = 13/51$. Either order meets the condition, and they cannot occur together in a single pair of draws. Hence add these two probabilities rather than retaining only the first order.

Step 3: Conclusion

$$\boxed{P(\text{one red and one black}) = \frac{26}{51}}.$$

Quick Tip to Solve: For different-colour draws, count both possible orders and reduce the second-draw denominator by one.

18. Evaluate: $\int_1^3 |2x-1|\,dx$ [1 mark]

Solution:

Concept:

- An absolute value satisfies $|u| = u$ when $u \geq 0$, and $|u| = -u$ when $u < 0$.
- The fundamental theorem gives $\int_a^b g(x)\,dx = G(b) - G(a)$, where $G'(x) = g(x)$.

Step 1: Check the sign throughout the interval The expression $2x-1$ vanishes at $x = 1/2$, which lies outside $[1, 3]$. For every point of the integration interval, $2x - 1 \geq 1 > 0$. Consequently, the modulus can be removed without splitting the integral.

Step 2: Integrate and apply the limits An antiderivative of $2x-1$ is $x^2 - x$. Thus

$$\int_1^3 |2x-1|\,dx = \left[x^2 - x\right]_1^3 = (9-3) - (1-1).$$

The lower endpoint contributes zero, and the upper endpoint contributes six.

Step 3: Conclusion

$$\boxed{\int_1^3 |2x-1|\,dx = 6}.$$

Quick Tip to Solve: Find the zero of the expression inside the modulus first; split only if it lies within the integration limits.

19. Find: $\int \dfrac{dx}{\sqrt{9-4x^2}}$ [1 mark]

Solution:

Concept:

- The standard integral is $\int du/\sqrt{1-u^2} = \sin^{-1} u + C$ for $|u| < 1$.
- A linear substitution must include the corresponding differential factor; C denotes an arbitrary constant.

Step 1: Reduce to the standard form Set $u = 2x/3$. Then $dx = (3/2)\,du$ and $\sqrt{9-4x^2} = 3\sqrt{1-u^2}$. Hence

$$\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}}.$$

Step 2: Integrate and verify the factor The standard formula gives $\frac{1}{2}\sin^{-1} u + C$. Substituting back, its derivative is

$$\frac{1}{2}\,\frac{2/3}{\sqrt{1-4x^2/9}} = \frac{1}{\sqrt{9-4x^2}},$$

as required. The real-valued integrand is defined on $-3/2 < x < 3/2$.

Quick Tip to Solve: Use the coefficient of x inside the square as a substitution and retain its reciprocal differential factor.

Step 3: Conclusion

$$\boxed{\int \frac{dx}{\sqrt{9-4x^2}} = \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C}.$$

Quick Tip to Solve: Use the coefficient of x inside the square as a substitution and retain its reciprocal differential factor.

20(a). Find: $\int x^4 \log x \, dx$ [1 mark]

Solution:

Concept:

- Integration by parts: $\int u\,dv = uv - \int v\,du$. Choose the logarithm as u, since its derivative is simpler.
- For $x > 0$, $d(\log x)/dx = 1/x$; the power rule gives $\int x^n dx = x^{n+1}/(n+1) + C$ for $n \neq -1$.

Step 1: Choose the factors Take $u = \log x$ and $dv = x^4 dx$. Then $du = dx/x$ and $v = x^5/5$. This choice turns the remaining integrand into a single power of x.

Step 2: Apply integration by parts

$$\int x^4 \log x \, dx = \frac{x^5}{5}\log x - \frac{1}{5}\int x^4 dx = \frac{x^5}{5}\log x - \frac{x^5}{25} + C.$$

Differentiating the expression gives $x^4 \log x + x^4/5 - x^4/5$, confirming the original integrand.

Step 3: Conclusion

$$\boxed{\int x^4 \log x \, dx = \frac{x^5}{5}\log x - \frac{x^5}{25} + C}, \qquad x > 0.$$

Quick Tip to Solve: Choose the logarithm for differentiation in integration by parts; its derivative cancels one power of x.

OR

20(b). Find: $\int \frac{2x}{\sqrt[3]{x^2+1}}\,dx$ [1 mark]

Solution:

Concept:

- Use substitution when the numerator is the derivative of the expression inside a root: here $d(x^2+1) = 2x\,dx$.
- The power rule is $\int t^n dt = t^{n+1}/(n+1) + C$, where $n \neq -1$.

Step 1: Substitute the inner expression Put $t = x^2 + 1$, so $dt = 2x\,dx$. The cube root in the

Quick Tip to Solve: Replace the whole expression inside the cube root by t; the numerator becomes dt without any extra factor.

denominator becomes $t^{1/3}$. Thus the integral is $\int t^{-1/3}dt$. Since $x^2+1>0$, this substitution introduces no real-domain restriction.

Step 2: Integrate and return to x

$$\int t^{-1/3}dt = \frac{t^{2/3}}{2/3} + C = \frac{3}{2}t^{2/3} + C.$$

Substitute back $t = x^2+1$. Differentiation gives $\frac{3}{2}\frac{2}{3}(x^2+1)^{-1/3}(2x)$, which is exactly the given integrand.

Step 3: Conclusion

$$\boxed{\int \frac{2x}{\sqrt[3]{x^2+1}}dx = \frac{3}{2}(x^2+1)^{2/3} + C}.$$

Quick Tip to Solve: Replace the whole expression inside the cube root by t; the numerator becomes dt without any extra factor.

Section B (12 marks)

Question numbers 21 to 26 carry 2 marks each.

21(a). Find a unit vector perpendicular to each of the vectors $\vec{a}$ and $\vec{b}$ where $\vec{a} = 5\hat{i}+6\hat{j}-2\hat{k}$ and $\vec{b} = 7\hat{i}+6\hat{j}+2\hat{k}$. [2 marks]

Solution:

Concept:

- The cross product $\vec{a}\times\vec{b}$ is perpendicular to both nonparallel vectors $\vec{a}$ and $\vec{b}$.
- A unit vector in the direction of a nonzero vector $\vec{v}$ is $\vec{v}/|\vec{v}|$, where $|\vec{v}|$ is its magnitude. The opposite direction also qualifies.

Step 1: Compute a perpendicular vector

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 6 & -2 \\ 7 & 6 & 2 \end{vmatrix}$$
$$= (12+12)\hat{i} - (10+14)\hat{j} + (30-42)\hat{k}$$
$$= 24\hat{i} - 24\hat{j} - 12\hat{k}.$$

This vector is nonzero, so the two given vectors are not parallel and can be used to define a perpendicular direction.

Step 2: Normalize and check Its magnitude is $\sqrt{24^2+(-24)^2+(-12)^2} = \sqrt{1296} = 36$. Dividing each component by 36 gives $\vec{u} = (2\hat{i}-2\hat{j}-\hat{k})/3$. In fact, $\vec{u}\cdot\vec{a} = (10-12+2)/3 = 0$ and $\vec{u}\cdot\vec{b} = (14-12-2)/3 = 0$. Also $|\vec{u}| = \sqrt{4+4+1}/3 = 1$, verifying both requirements.

Quick Tip to Solve: Compute the cross product first, then divide by its magnitude; either of the two opposite unit vectors is valid.

Step 3: Conclusion One required unit vector is $\boxed{\dfrac{2\hat{i}-2\hat{j}-\hat{k}}{3}}$. Its negative is also a valid answer.

Quick Tip to Solve: Compute the cross product first, then divide by its magnitude; either of the two opposite unit vectors is valid.

OR

21(b). Find the volume of the parallelopiped whose adjacent edges are represented by $2\vec{a}$, $-\vec{b}$ and $3\vec{c}$, where

$\vec{a}=\hat{i}-\hat{j}+2\hat{k}$, $\vec{b}=3\hat{i}+4\hat{j}-5\hat{k}$, and $\vec{c}=2\hat{i}-\hat{j}+3\hat{k}$. [2 marks]

Solution:

Concept:

- The volume of a parallelepiped with adjacent edge vectors $\vec{p},\vec{q},\vec{r}$ is $|\vec{p}\cdot(\vec{q}\times\vec{r})|$. The absolute value makes the geometric volume nonnegative.
- A scalar triple product is linear in each vector: $[\alpha\vec{a},\beta\vec{b},\gamma\vec{c}]=\alpha\beta\gamma[\vec{a},\vec{b},\vec{c}]$.

Step 1: Evaluate the basic scalar triple product Using the given components,

$$\begin{aligned}[\vec{a},\vec{b},\vec{c}] &= \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix} \\ &= 1(12-5)-(-1)(9+10)+2(-3-8) \\ &= 7+19-22=4.\end{aligned}$$

Equivalently, $\vec{b}\times\vec{c}=7\hat{i}-19\hat{j}-11\hat{k}$, whose dot product with $\vec{a}$ is 4.

Step 2: Include all edge multipliers The actual edges are $2\vec{a},-\vec{b},3\vec{c}$, so their scalar triple product is $2(-1)(3)[\vec{a},\vec{b},\vec{c}]=-6(4)=-24$. The negative sign records orientation, not a negative volume. Therefore take the absolute value, giving $|-24|=24$. Each of the three edge multipliers must be included; using only the unscaled determinant would give the wrong volume.

Step 3: Conclusion The volume is $\boxed{24 \text{ cubic units}}$.

Quick Tip to Solve: Multiply the basic determinant by all three edge scale factors, then take its absolute value for the volume.

22. If $f(x) = \sqrt{\tan\sqrt{x}}$, then find $f'\left(\frac{\pi^2}{16}\right)$. [2 marks]

Solution:

Concept:

- For a composite function, apply the chain rule at every layer: $d\sqrt{u}/dx = u\prime/(2\sqrt{u})$.
- The inner derivatives are $d(\tan v)/dx = \sec^2 v\, v\prime$ and $d\sqrt{x}/dx = 1/(2\sqrt{x})$ for $x > 0$.

Step 1: Differentiate all three layers The layers, from outside to inside, are the square root, tangent and $\sqrt{x}$. Thus

$$f\prime(x) = \frac{1}{2\sqrt{\tan\sqrt{x}}}\sec^2\sqrt{x}\,\frac{1}{2\sqrt{x}}$$
$$= \frac{\sec^2\sqrt{x}}{4\sqrt{x}\sqrt{\tan\sqrt{x}}}.$$

Each factor is necessary; differentiating the tangent alone would omit both square-root derivatives.

Step 2: Evaluate at the specified argument At $x = \pi^2/16$, the positive square root is $\sqrt{x} = \pi/4$. Therefore $\tan(\pi/4) = 1$, $\sqrt{\tan(\pi/4)} = 1$, and $\sec^2(\pi/4) = 2$. The denominator is $4(\pi/4)(1) = \pi$. The function is real and differentiable near this point because $x > 0$ and $\tan\sqrt{x} > 0$ there.

Step 3: Conclusion

$$\boxed{f\prime\left(\frac{\pi^2}{16}\right) = \frac{2}{\pi}}.$$

Quick Tip to Solve: Write one derivative factor for each of the three layers, then substitute sqrt(x) = pi/4 only after differentiating.

23. Using differentials, find the approximate value of $\sqrt{25.3}$ up to two places of decimals. [2 marks]

Solution:

Concept:

- For a differentiable function $y = f(x)$, a small change h produces the differential $dy = f\prime(x)h$. Thus $f(x+h) \approx f(x) + dy$.
- For $f(x) = \sqrt{x}$, the derivative is $f\prime(x) = 1/(2\sqrt{x})$, with $x > 0$.

Step 1: Choose a nearby perfect square Take $x = 25$ and $h = 25.3 - 25 = 0.3$. Then $f(25) = 5$. The change is small compared with 25, so using the tangent approximation is appropriate. A perfect square gives both the original function value and its derivative exactly.

Step 2: Calculate the differential At $x = 25$,

$$dy = \frac{0.3}{2\sqrt{25}} = \frac{0.3}{10} = 0.03.$$

Therefore the first-order estimate is $f(25) + dy = 5 + 0.03$. Since the derivative is positive, the estimate must be slightly larger than 5; this also checks the sign of the correction.

Quick Tip to Solve: Use 25 as the base value: add the increment 0.3 divided by twice its square root.

Step 3: Conclusion Using differentials, $\boxed{\sqrt{25.3} \approx 5.03}$, correct to two decimal places.

Quick Tip to Solve: Use 25 as the base value: add the increment 0.3 divided by twice its square root.

24. The probability of finding a green signal on a busy crossing X is 30%. What is the probability of finding a green signal on X on two consecutive days out of three? [2 marks]

Solution:

Concept:

- On each day, let G denote a green signal and N a non-green signal. Then $P(G) = 0.30$ and $P(N) = 0.70$.
- For independent daily trials, multiply probabilities within a sequence and add probabilities of mutually exclusive sequences.

Step 1: List the consecutive-day possibilities Two green-signal days out of three means exactly two green days. For these to be consecutive, the allowed sequences are GGN and NGG. The sequence GNG has two green days, but they are not consecutive. The sequence GGG has three green days, so it is not included.

Step 2: Add the probabilities Treating the signals on different days as independent trials with the same green probability,

$$P(GGN) = (0.30)^2(0.70) = 0.063,$$
$$P(NGG) = (0.70)(0.30)^2 = 0.063.$$

These two sequences cannot occur together in one three-day observation, so their probabilities add.

Step 3: Conclusion The required probability is $\boxed{0.063 + 0.063 = 0.126}$, or 12.6%.

Quick Tip to Solve: List the sequences first: consecutiveness allows GGN and NGG, but excludes GNG.

25(a). Prove that $\sin^{-1}(2x\sqrt{1-x^2}) = 2\cos^{-1} x$, $\dfrac{1}{\sqrt{2}} \le x \le 1$. [2 marks]

Solution:

Concept:

- The principal value of $\cos^{-1} x$ lies in $[0, \pi]$, and that of $\sin^{-1} t$ lies in $[-\pi/2, \pi/2]$.
- The double-angle identity is $\sin 2\theta = 2\sin\theta\cos\theta$. Also, $\sin^{-1}(\sin u) = u$ when $u \in [-\pi/2, \pi/2]$.

Step 1: Use the domain to choose an angle Put $\theta = \cos^{-1} x$, so $x = \cos\theta$. Since $1/\sqrt{2} \le x \le 1$, the decreasing cosine function on its principal interval gives $0 \le \theta \le \pi/4$. Hence $\sin\theta \ge 0$, and $\sqrt{1-x^2} = \sqrt{1-\cos^2\theta} = \sin\theta$.

Step 2: Apply the double-angle identity within the principal range The argument on the left becomes

$$2x\sqrt{1-x^2} = 2\cos\theta\sin\theta = \sin 2\theta.$$

Quick Tip to Solve: Before cancelling inverse sine with sine, check that twice the angle lies between 0 and pi/2.

As $0 \le 2\theta \le \pi/2$, inverse sine returns the angle 2θ itself. The range restriction is essential: inverse sine does not return every angle unchanged.

Step 3: Conclusion Thus $\boxed{\sin^{-1}(2x\sqrt{1-x^2}) = 2\theta = 2\cos^{-1} x}$ for the given interval.

Quick Tip to Solve: Before cancelling inverse sine with sine, check that twice the angle lies between 0 and pi/2.

OR

25(b). Consider a bijective function $f : \mathbb{R}_+ \to (7, \infty)$ given by $f(x) = 16x^2 + 24x + 7$, where $\mathbb{R}_+$ is the set of all positive real numbers. Find the inverse function of f. [2 marks]

Solution:

Concept:

- To find an inverse, write $y = f(x)$, solve for x in terms of y, and keep only the root in the original domain.
- The domain of f^{-1} is the range $(7, \infty)$ of f; its values lie in $\mathbb{R}_+$.

Step 1: Solve the quadratic by completing the square Let $y = 16x^2 + 24x + 7$, where $x > 0$. Adding 2 to both sides gives

$$y + 2 = 16x^2 + 24x + 9 = (4x + 3)^2.$$

As $x > 0$, we have $4x + 3 > 3 > 0$, so the positive square root must be used: $4x + 3 = \sqrt{y + 2}$.

Step 2: Identify and check the inverse mapping Rearranging gives $x = (\sqrt{y + 2} - 3)/4$. For $y > 7$, $\sqrt{y + 2} > 3$, so this value is positive as required. Substituting it into $f(x) = (4x + 3)^2 - 2$ gives $(\sqrt{y + 2})^2 - 2 = y$, confirming the inverse expression. The negative square-root branch would give a negative original input, and is excluded by the domain.

Step 3: Conclusion $\boxed{f^{-1} : (7, \infty) \to \mathbb{R}_+, \quad f^{-1}(x) = \dfrac{\sqrt{x + 2} - 3}{4}}$.

Quick Tip to Solve: Complete the square as $f(x) + 2 = (4x + 3)^2$, then use positivity to choose the square root.

26. Find the value of k so that the lines $x = -y = kz$ and $x - 2 = 2y + 1 = -z + 1$ are perpendicular to each other. [2 marks]

Solution:

Concept:

- Two lines have perpendicular directions when the scalar product of their direction vectors is zero.
- Direction ratios can be read from a parameterization: the coefficients of the same parameter in x, y, z form a direction vector.

Step 1: Find direction vectors without dividing by k For the first line, use $z = t$. Since $x = -y = kz$,

Quick Tip to Solve: Parameterize the common expressions first; coefficients of y and z must be inverted correctly.

its parameterization is $x = kt,\ y = -kt,\ z = t$, with direction vector $\mathbf{a} = (k, -k, 1)$. This choice remains valid even when $k = 0$. For the second line, put $x - 2 = 2y + 1 = -z + 1 = s$. Then $x = s + 2,\ y = (s - 1)/2,\ z = 1 - s$, giving $\mathbf{b} = (1, 1/2, -1)$.

Step 2: Apply the perpendicularity condition The constant terms specify positions and do not affect direction ratios. Perpendicular directions require

$$\mathbf{a} \cdot \mathbf{b} = k - \frac{k}{2} - 1 = 0.$$

Thus $k/2 = 1$. As a check, at this value the vectors are $(2, -2, 1)$ and $(1, 1/2, -1)$, whose scalar product is $2 - 1 - 1 = 0$.

Step 3: Conclusion The required value is $\boxed{k = 2}$.

Quick Tip to Solve: Parameterize the common expressions first; coefficients of y and z must be inverted correctly.

Section C (24 marks)

Question numbers 27 to 32 carry 4 marks each.

27. A furniture trader deals in only two items – chairs and tables. He has ₹50,000 to invest and a space to store at most 35 items. A chair costs him ₹1,000 and a table costs him ₹2,000. The trader earns a profit of ₹150 and ₹250 on a chair and table, respectively. Formulate the above problem as an LPP to maximise the profit and solve it graphically. [4 marks]

Solution:

Concept:

- Let x and y be the numbers of chairs and tables. Both variables are non-negative.
- The common region satisfying all resource inequalities is the feasible region. A linear objective on a bounded feasible polygon attains its maximum at a vertex.

Step 1: Formulate the LPP The total investment is $1000x + 2000y$ rupees and the total number of items is $x + y$. The investment cannot exceed 50000 rupees and the storage cannot exceed 35 items. The profit is $150x + 250y$ rupees. Hence the required LPP is

$$\begin{aligned} \text{Maximise} \quad & Z = 150x + 250y, \\ \text{subject to} \quad & 1000x + 2000y \le 50000, \\ & x + y \le 35, \\ & x \ge 0, \quad y \ge 0. \end{aligned}$$

Dividing the investment inequality by 1000 gives the convenient form $x + 2y \le 50$.

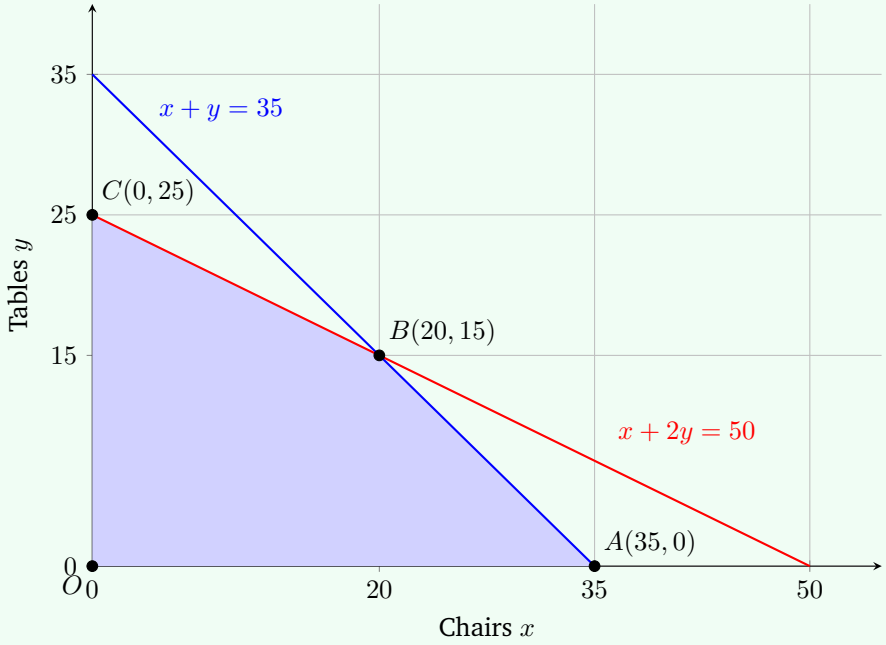
Step 2: Construct the feasible polygon The line $x + y = 35$ passes through $(35, 0)$ and $(0, 35)$. The line $x + 2y = 50$ passes through $(50, 0)$ and $(0, 25)$. For each inequality the origin satisfies the condition, so shade the common part on the origin side of both lines, within the first quadrant. This gives polygon

Quick Tip to Solve: Find the intersection of the two resource lines, then compare its profit with every feasible vertex on the axes.

$OABC$ in the graph. At the intersection B, solve

$$x + y = 35,$$
$$x + 2y = 50.$$

Subtracting the first equation from the second gives $y = 15$, and then $x = 20$. Thus the feasible vertices are $O(0, 0)$, $A(35, 0)$, $B(20, 15)$ and $C(0, 25)$. Although $(50, 0)$ and $(0, 35)$ are useful for drawing the lines, they violate the storage and investment limits, respectively, and are not feasible vertices.



The shaded polygon OABC is the feasible region.

Step 3: Evaluate the profit at the vertices Since the feasible region is bounded, compare the objective at every vertex:

(x, y)	$Z = 150x + 250y$ (rupees)
$(0, 0)$	0
$(35, 0)$	5250
$(20, 15)$	$3000 + 3750 = 6750$
$(0, 25)$	6250

The largest value occurs at B. This point represents whole numbers of furniture items, so it is also a permissible purchase. Check its resource use: $20(1000) + 15(2000) = 50000$ rupees, and $20 + 15 = 35$ items. Both constraints are satisfied exactly. Compared with the all-table feasible vertex, this combination earns 500 rupees more; compared with the all-chair vertex, it earns 1500 rupees more.

Step 4: Conclusion Purchase $\boxed{20 \text{ chairs and } 15 \text{ tables}}$ for a maximum profit of $\boxed{₹\,6750}$.

Quick Tip to Solve: Find the intersection of the two resource lines, then compare its profit with every feasible vertex on the axes.

28. **If $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $a > 0$, then find $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{3}$.** [4 marks]

Solution:

Concept:

- For parametric coordinates, $dy/dx = (dy/d\theta)/(dx/d\theta)$, provided $dx/d\theta \neq 0$.
- The second derivative is $d^2y/dx^2 = [d(dy/dx)/d\theta]/(dx/d\theta)$. The parameter derivative of the slope alone is not the second derivative with respect to x.

Step 1: Differentiate the parametric coordinates Since a is constant, differentiating both given expressions with respect to θ gives

$$\frac{dx}{d\theta} = a(1 - \cos\theta), \qquad \frac{dy}{d\theta} = a\sin\theta.$$

At the required value $\theta = \pi/3$, we have $dx/d\theta = a/2 \neq 0$, since $a > 0$. Hence the parametric formula for the slope is valid here. Dividing the coordinate derivatives, we find

$$\frac{dy}{dx} = \frac{\sin\theta}{1 - \cos\theta}.$$

The parameter must remain variable while differentiating this slope; substituting its given value first would discard the information needed to compute the second derivative.

Step 2: Differentiate the slope using the quotient rule Differentiate the slope with respect to θ. The numerator derivative is $\cos\theta$, while the denominator derivative is $\sin\theta$. Therefore,

$$\begin{aligned}\frac{d}{d\theta}\left(\frac{dy}{dx}\right) &= \frac{\cos\theta(1 - \cos\theta) - \sin^2\theta}{(1 - \cos\theta)^2} \\ &= \frac{\cos\theta - (\cos^2\theta + \sin^2\theta)}{(1 - \cos\theta)^2} \\ &= -\frac{1}{1 - \cos\theta}.\end{aligned}$$

We used the identity $\cos^2\theta + \sin^2\theta = 1$ to simplify the numerator. Now apply the chain rule to convert this parameter derivative into the derivative with respect to x:

$$\frac{d^2y}{dx^2} = \frac{-1/(1 - \cos\theta)}{a(1 - \cos\theta)} = -\frac{1}{a(1 - \cos\theta)^2}.$$

Step 3: Substitute the required angle The printed angle is $\theta = \pi/3$. Using $\cos(\pi/3) = 1/2$ in the general expression,

$$\left.\frac{d^2y}{dx^2}\right|_{\theta=\pi/3} = -\frac{1}{a(1 - 1/2)^2} = -\frac{1}{a/4}.$$

The positive denominator confirms that the curvature expressed by this second derivative is negative at the specified point.

Step 4: Conclusion Thus, $\left.\dfrac{d^2y}{dx^2}\right|_{\theta=\pi/3} = -\dfrac{4}{a}$.

Quick Tip to Solve: Differentiate the slope with respect to the parameter, divide by $dx/d\theta$ again, and only then substitute the angle.

29. Evaluate $\int_1^3 e^x\,dx$ as limit of the sums. [4 marks]

Solution:

Concept:

- For a continuous function f on $[a,b]$, divide the interval into n equal parts of width $h=(b-a)/n$. Then its definite integral is the limit $\lim_{n\to\infty} h\sum_{r=1}^{n} f(a+rh)$.
- A finite geometric progression with first term t and common ratio $q\neq 1$ has sum $t(q^n-1)/(q-1)$. Also $\lim_{u\to 0}(e^u-1)/u=1$.

Step 1: Express the integral as a sum over equal subintervals The function e^x is continuous on $[1,3]$, so its integral equals the limit of Riemann sums. Divide this interval into n equal subintervals. Their common width and right endpoints are

$$h=\frac{3-1}{n}=\frac{2}{n},\qquad x_r=1+\frac{2r}{n},\quad r=1,2,\ldots,n.$$

The right-endpoint sum therefore has the form

$$\begin{aligned} S_n &= \frac{2}{n}\sum_{r=1}^{n} e^{1+2r/n} \\ &= \frac{2e}{n}\left(e^{2/n}+e^{4/n}+\cdots+e^{2n/n}\right). \end{aligned}$$

Step 2: Sum the geometric progression exactly The bracket contains n terms, with first term $e^{2/n}$ and common ratio $e^{2/n}$. Since this ratio is greater than one for every positive integer n, the geometric-sum formula applies without a zero denominator. It gives

$$\sum_{r=1}^{n} e^{2r/n}=\frac{e^{2/n}\left((e^{2/n})^n-1\right)}{e^{2/n}-1}=\frac{e^{2/n}(e^2-1)}{e^{2/n}-1}.$$

Consequently,

$$S_n=e(e^2-1)e^{2/n}\frac{2/n}{e^{2/n}-1}.$$

Step 3: Take the limit of the sums Set $u=2/n$. As $n\to\infty$, we have $u\to 0^+$, $e^u\to 1$, and

$$\frac{u}{e^u-1}\longrightarrow 1,$$

by the standard limit for the exponential function. Hence

$$\lim_{n\to\infty} S_n=e(e^2-1)\cdot 1\cdot 1.$$

Step 4: Conclusion Therefore, by the limit of sums, $\int_1^3 e^x\,dx=e(e^2-1)=e^3-e$.

Quick Tip to Solve: Factor out e, sum the remaining exponential terms as a geometric progression, and arrange the result to use $(e^u-1)/u\to 1$.

30(a). There are two bags, I and II. Bag I contains 3 red and 5 black balls and Bag II contains 4 red and 3 black balls. One ball is transferred randomly from Bag I to Bag II and then a ball is drawn randomly from Bag II. If the ball so drawn is found to be black in colour, then find the probability that the transferred ball is also black. [4 marks]

Solution:

Concept:

- Let T_B and T_R denote transfer of a black and a red ball respectively, and let D_B denote drawing a black ball from Bag II. The required probability is the conditional probability $P(T_B \mid D_B)$.
- For mutually exclusive, exhaustive transfer events, total probability gives $P(D_B) = P(T_B)P(D_B \mid T_B) + P(T_R)P(D_B \mid T_R)$. Bayes' theorem then gives $P(T_B \mid D_B) = P(T_B \cap D_B)/P(D_B)$.

Step 1: Find the transfer probabilities Bag I contains $3 + 5 = 8$ balls. Since the transfer is random, each ball has the same chance of selection. Thus

$$P(T_B) = \frac{5}{8}, \qquad P(T_R) = \frac{3}{8}.$$

These two events account for all possible transfers, and their probabilities add to one. The contents of Bag II must now be updated separately for these two cases before computing the probability of its draw.

Step 2: Evaluate the black-draw probability in each branch If the transferred ball is black, Bag II has 4 red and 4 black balls, giving $P(D_B \mid T_B) = 4/8$. If the transferred ball is red, Bag II has 5 red and 3 black balls, giving $P(D_B \mid T_R) = 3/8$. In either case Bag II contains eight balls after transfer. Therefore

$$\begin{aligned} P(D_B) &= \frac{5}{8} \cdot \frac{4}{8} + \frac{3}{8} \cdot \frac{3}{8} \\ &= \frac{20 + 9}{64} = \frac{29}{64}. \end{aligned}$$

A black draw can follow either colour of transfer, so both branches belong in the denominator.

Step 3: Condition on the observed black draw The joint probability of a black transfer and a black draw is

$$P(T_B \cap D_B) = \frac{5}{8} \cdot \frac{4}{8} = \frac{20}{64}.$$

Hence Bayes' theorem yields

$$P(T_B \mid D_B) = \frac{20/64}{29/64} = \frac{20}{29}.$$

The observed colour makes a black transfer more likely than its original probability $5/8$, because that transfer increases the black fraction in Bag II. We do not require the drawn ball to be the very ball transferred; the question concerns its colour only.

Step 4: Conclusion The required probability is $\boxed{\frac{20}{29}}$.

Quick Tip to Solve: Update Bag II to eight balls in each branch, then divide the black-transfer-and-black-draw probability by the total black-draw probability.

OR

30(b). An urn contains 5 red, 2 white and 3 black balls. Three balls are drawn, one-by-one, at random without replacement. Find the probability distribution of the number of white balls. Also, find the mean and the variance of the number of white balls drawn. [4 marks]

Solution:

Concept:

- Let X be the number of white balls in the three draws. There are 2 white and $5+3=8$ non-white balls, so X can take only the values $0, 1, 2$.
- Without replacement, all unordered selections of three balls are equally likely. Thus $P(X=x) = \binom{2}{x}\binom{8}{3-x}/\binom{10}{3}$. Here $\binom{n}{r}$ counts selections of r objects from n.
- The mean is $E(X) = \sum xP(X=x)$, and the variance is $\text{Var}(X) = E(X^2) - [E(X)]^2$.

Step 1: Count the possible selections There are ten balls, so the number of unordered selections is $\binom{10}{3} = 120$. Each selection occurs in 3! orders, so counting unordered selections gives the same probabilities. Group the red and black balls as non-white.

Step 2: Obtain and check the probability distribution For $X = x$, choose x white balls and $3-x$ non-white balls. Hence

$$P(X=0) = \frac{\binom{2}{0}\binom{8}{3}}{120} = \frac{56}{120} = \frac{7}{15},$$
$$P(X=1) = \frac{\binom{2}{1}\binom{8}{2}}{120} = \frac{56}{120} = \frac{7}{15},$$
$$P(X=2) = \frac{\binom{2}{2}\binom{8}{1}}{120} = \frac{8}{120} = \frac{1}{15}.$$

The probabilities sum to $7/15 + 7/15 + 1/15 = 1$. Three white balls are impossible.

Step 3: Calculate the mean and the second moment Weight each value by its probability:

$$E(X) = 0\cdot\frac{7}{15} + 1\cdot\frac{7}{15} + 2\cdot\frac{1}{15} = \frac{3}{5},$$
$$E(X^2) = 0^2\cdot\frac{7}{15} + 1^2\cdot\frac{7}{15} + 2^2\cdot\frac{1}{15} = \frac{11}{15}.$$

Step 4: Find the variance Subtract the square of the mean from the second moment:

$$\text{Var}(X) = \frac{11}{15} - \left(\frac{3}{5}\right)^2 = \frac{55-27}{75} = \frac{28}{75}.$$

Step 5: Conclusion The distribution is

x	0	1	2
$P(X=x)$	$\frac{7}{15}$	$\frac{7}{15}$	$\frac{1}{15}$

with mean $\boxed{3/5}$ and variance $\boxed{28/75}$.

Quick Tip to Solve: Combine red and black into eight non-white balls, count selections without replacement, and use the second moment to calculate variance.

31(a). Find the general solution of the differential equation $y\,e^{y}\,dx = (y^3 + 2x\,e^{y})\,dy$. [4 marks]

Solution:

Concept:

- Treat x as dependent on y. For the linear equation $dx/dy + P(y)x = Q(y)$, the integrating factor is $e^{\int P(y)dy}$.
- Multiplication by the integrating factor converts the left side into the derivative of x times that factor, so integration supplies an arbitrary constant.

Step 1: Express the equation as a linear equation in x Starting from $ye^{y}dx = (y^3 + 2xe^{y})dy$, divide by ye^{y} on an interval where $y \neq 0$. This gives

$$\frac{dx}{dy} = y^2e^{-y} + \frac{2x}{y}, \qquad \frac{dx}{dy} - \frac{2}{y}x = y^2e^{-y}.$$

The equation is linear in x, with coefficient $P(y) = -2/y$ and right side $Q(y) = y^2e^{-y}$. Choosing y as independent variable avoids introducing an unnecessary homogeneous substitution.

Step 2: Find the integrating factor and integrate The integrating factor is

$$\text{IF} = e^{\int(-2/y)dy} = e^{-2\log|y|} = y^{-2}.$$

Multiplying by this factor, and recognizing the product derivative, gives

$$\frac{1}{y^2}\frac{dx}{dy} - \frac{2x}{y^3} = e^{-y} \quad \Longrightarrow \quad \frac{d}{dy}\left(\frac{x}{y^2}\right) = e^{-y}.$$

Integration with respect to y now yields $x/y^2 = -e^{-y} + C$, where C is arbitrary. The negative sign arises because the derivative of $-e^{-y}$ is e^{-y}.

Step 3: Verify the family in the original equation Rearranging gives $x = y^2(C - e^{-y})$. Its derivative is

$$\frac{dx}{dy} = 2y(C - e^{-y}) + y^2e^{-y} = \frac{2x}{y} + y^2e^{-y}.$$

Multiplying by ye^{y} recovers precisely the printed equation. Thus the integrating factor and exponential sign are verified. The division by y excludes $y = 0$; viewed as a constant curve, $y = 0$ also satisfies the original differential relation since $dy = 0$.

Step 4: Conclusion The general solution is $\boxed{\frac{x}{y^2} + e^{-y} = C}$, for $y \neq 0$, equivalently $x = y^2(C - e^{-y})$. The original equation also admits the constant solution $y = 0$.

Quick Tip to Solve: Choose x as the dependent variable here: the integrating factor y^{-2} immediately cancels the y^2 on the right side.

OR

31(b). Find the particular solution of the differential equation $x\dfrac{dy}{dx} = y - x\tan\left(\dfrac{y}{x}\right)$, given that $y = \dfrac{\pi}{4}$ at $x = 1$. [4 marks]

Solution:

Concept:

- A homogeneous first-order equation whose right side depends on y/x is reduced by setting $v = y/x$, so $y = xv$ and $dy/dx = v + x\,dv/dx$.
- Separate the variables and use $\int \cot v\,dv = \log|\sin v| + C$. The initial condition determines the constant and selects the branch of the solution.

Step 1: Substitute the ratio y/x Dividing the equation by x, where $x \neq 0$, gives $dy/dx = y/x - \tan(y/x)$. Put $v = y/x$, hence $y = xv$ and $dy/dx = v + x\,dv/dx$. Substitution in the printed equation gives

$$x\left(v + x\frac{dv}{dx}\right) = xv - x\tan v, \qquad x\frac{dv}{dx} = -\tan v.$$

The terms involving v cancel. At the given initial point $x = 1$, $v = \pi/4$, so both $\sin v$ and $\cos v$ are nonzero in a sufficiently small interval about that point.

Step 2: Separate and integrate Multiplying the reduced equation by $\cot v$ and dividing by x gives

$$\cot v\,dv = -\frac{dx}{x}.$$

Integrating both sides yields $\log|\sin v| = -\log|x| + C$, or $\log|x\sin v| = C$. Thus the integrated relation can be written as $x\sin v = K$, with a constant K. Returning to the original variables gives $x\sin(y/x) = K$.

Step 3: Apply the initial condition and verify Substituting $x = 1$ and $y = \pi/4$ determines $K = \sin(\pi/4) = 1/\sqrt{2}$. To verify the resulting relation, differentiate its left side:

$$\sin v + x\cos v\frac{dv}{dx} = 0.$$

Where $\cos v \neq 0$, this gives $x\,dv/dx = -\tan v$, which reverses the substitution and reproduces the original equation. The branch through the initial point has $0 < v < \pi/2$. It can therefore be written explicitly using the principal inverse sine as $y = x\sin^{-1}(1/(\sqrt{2}x))$, on $x > 1/\sqrt{2}$. At the lower endpoint the tangent is undefined, so that endpoint is excluded.

Step 4: Conclusion The required particular solution is $\boxed{x\sin\left(\dfrac{y}{x}\right) = \dfrac{1}{\sqrt{2}}}$, on the branch through $(1, \pi/4)$. Equivalently, $y = x\sin^{-1}(1/(\sqrt{2}x))$ for $x > 1/\sqrt{2}$.

Quick Tip to Solve: After setting v=y/x, cancel the repeated xv terms first; the remaining tangent separates directly, and the initial point fixes x sin(v).

32. Let N be the set of natural numbers and R be the relation on $N \times N$ defined by $(a, b)R(c, d)$ iff $ad = bc$ for all $a, b, c, d \in N$. Show that R is an equivalence relation. [4 marks]

Solution:

Concept:

- An equivalence relation is reflexive, symmetric and transitive on its underlying set. Here elements of that set are ordered pairs $(a, b) \in N \times N$.
- The rule $(a, b)R(c, d) \iff ad = bc$ compares cross-products. Natural numbers are positive, so every coordinate is nonzero and may be cancelled from an equality.

Step 1: Prove reflexivity Choose any $(a, b) \in N \times N$. For this pair to be related to itself, the defining equality must hold with $(c, d) = (a, b)$. It becomes

$$ab = ba,$$

which follows from commutativity of multiplication. Therefore $(a, b)R(a, b)$ for every element of the underlying set. This proves reflexivity.

Step 2: Prove symmetry Suppose $(a, b)R(c, d)$ for arbitrary pairs in $N \times N$. Then the defining rule gives $ad = bc$. Reversing this equality and commuting the factors gives

$$cb = da.$$

The left side is the first coordinate of (c, d) times the second coordinate of (a, b); the right side is the second coordinate of (c, d) times the first coordinate of (a, b). Thus this is exactly the condition $(c, d)R(a, b)$, proving symmetry.

Step 3: Prove transitivity Let $(a, b), (c, d), (e, f)$ be arbitrary elements of $N \times N$. Suppose $(a, b)R(c, d)$ and $(c, d)R(e, f)$. We must prove $(a, b)R(e, f)$, or equivalently $af = be$. The hypotheses give

$$ad = bc, \qquad cf = de.$$

Multiply the first equality by f and the second by b. The resulting equalities have the common expression bcf, so

$$adf = bcf = bde.$$

Rearrange the two outer expressions as $d(af) = d(be)$. Since d is a natural number, $d > 0$. Dividing by d therefore gives $af = be$, which is the defining equality for $(a, b)R(e, f)$. This establishes transitivity.

Step 4: Conclusion The relation R is reflexive, symmetric and transitive on $N \times N$. Hence R is an equivalence relation.

Quick Tip to Solve: For transitivity, multiply $ad = bc$ by f and $cf = de$ by b; then cancel the positive common coordinate d.

Section D (24 marks)

Question numbers 33 to 36 carry 6 marks each.

33. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone. [6 marks]

Solution:

Concept:

- The volume of a right circular cylinder of radius ρ and height x is $V = \pi\rho^2 x$. The cone has fixed positive height h and radius r.
- For an inscribed cylinder with its axis along the cone's axis and its lower base on the cone's base, similar triangles express its radius in terms of its height. Differentiate the resulting volume function and check the stationary point and the endpoints.

Step 1: Express the cylinder's radius using similarity Let the cylinder have height x and radius ρ, where $0 \le x \le h$. Its upper circular edge touches the cone at a distance $h - x$ from the vertex. In an axial section, the small triangle above the cylinder and the full triangle of the cone are similar. Their corresponding base radii and heights therefore satisfy

$$\frac{\rho}{r} = \frac{h-x}{h}, \qquad \rho = \frac{r}{h}(h-x).$$

For each fixed height, taking this largest permitted radius maximises the cylinder's volume, since volume increases with ρ^2.

Step 2: Form and differentiate the volume function Substitute the radius in $V = \pi\rho^2 x$:

$$V(x) = \frac{\pi r^2}{h^2} x(h-x)^2.$$

The factors outside $x(h-x)^2$ are positive constants. Differentiating with respect to x gives

$$\begin{aligned} V'(x) &= \frac{\pi r^2}{h^2}\left[(h-x)^2 - 2x(h-x)\right] \\ &= \frac{\pi r^2}{h^2}(h-x)(h-3x). \end{aligned}$$

Thus the stationary values satisfy $x = h$ or $x = h/3$. The value $x = h$ is an endpoint with zero cylinder radius; the interior candidate is $x = h/3$.

Step 3: Verify the greatest volume For $0 < x < h/3$, both $h - x$ and $h - 3x$ are positive, so $V'(x) > 0$. For $h/3 < x < h$, the first factor is positive and the second is negative, so $V'(x) < 0$. Hence volume increases up to $h/3$ and decreases thereafter. Also, $V(0) = V(h) = 0$, whereas $V(h/3) > 0$. This proves that the interior stationary point gives the absolute greatest volume on the allowed interval, not merely a stationary value.

Step 4: Calculate the maximum volume and compare with the cone At $x = h/3$, the corresponding

Quick Tip to Solve: Eliminate the cylinder's radius by similar triangles first; maximise $x(h-x)^2$ and check the endpoint volumes as well as the stationary point.

cylinder radius is

$$\rho = \frac{r}{h}\left(h - \frac{h}{3}\right) = \frac{2r}{3}.$$

Consequently,

$$V_{\max} = \pi\left(\frac{2r}{3}\right)^2 \frac{h}{3} = \frac{4\pi r^2 h}{27}.$$

The cone's volume is $V_{\text{cone}} = \pi r^2 h/3$. Therefore

$$\frac{V_{\max}}{V_{\text{cone}}} = \frac{4\pi r^2 h}{27}\frac{3}{\pi r^2 h} = \frac{4}{9}.$$

Step 5: Conclusion The cylinder of greatest volume has height $\boxed{h/3}$, one-third of the cone's height, and its greatest volume is $\boxed{4\pi r^2 h/27 = (4/9)V_{\text{cone}}}$.

Quick Tip to Solve: Eliminate the cylinder's radius by similar triangles first; maximise $x(h-x)^2$ and check the endpoint volumes as well as the stationary point.

34. Using integration, find the area of the region enclosed by the parabola $y = 3x^2$ and the line $3x - y + 6 = 0$. [6 marks]

Solution:

Concept:

- For two curves written as functions of x, the enclosed area is $\int_a^b (y_{\text{upper}} - y_{\text{lower}})\,dx$, where a and b are the abscissae of their intersection points.
- The intersection limits must first be found by equating the two ordinates. Checking their difference between these limits determines which curve is above the other.

Step 1: Express the line and find the intersection points The given straight line is $3x - y + 6 = 0$, so its ordinate is $y = 3x + 6$. At an intersection with the parabola $y = 3x^2$, both ordinates are equal. Thus

$$3x^2 = 3x + 6,$$
$$x^2 - x - 2 = 0,$$
$$(x-2)(x+1) = 0.$$

Hence the limiting abscissae are $x = -1$ and $x = 2$. Substituting in either curve gives the corresponding intersection points $(-1, 3)$ and $(2, 12)$. These are the endpoints of the enclosed region; no other intersections occur because the quadratic has exactly these two roots.

Step 2: Identify the upper boundary throughout the interval The difference between the line's ordinate and the parabola's ordinate is

$$(3x + 6) - 3x^2 = 3(2 - x)(x + 1).$$

For $-1 \le x \le 2$, both factors $2 - x$ and $x + 1$ are nonnegative. Therefore the line lies above the

Quick Tip to Solve: Factor the ordinate difference as $3(2-x)(x+1)$: this immediately gives both intersection limits and the correct upper-minus-lower order.

parabola over the whole interval, with equality only at the endpoints. For example, at $x = 0$, the line has ordinate 6, whereas the parabola has ordinate 0. A vertical strip consequently has height $3x + 6 - 3x^2$ and width dx. Integrating these nonnegative strip areas gives the desired enclosed area without needing to split the interval.

Step 3: Set up and evaluate the definite integral Denote the required area by A. Using the limits and boundary order established above,

$$\begin{aligned} A &= \int_{-1}^{2} (3x + 6 - 3x^2)\, dx \\ &= \left[\frac{3x^2}{2} + 6x - x^3\right]_{-1}^{2} \\ &= (6 + 12 - 8) - \left(\frac{3}{2} - 6 + 1\right) \\ &= 10 - \left(-\frac{7}{2}\right) = \frac{27}{2}. \end{aligned}$$

The lower-limit value must be subtracted as a whole: its negative sign adds $7/2$ to the upper-limit value. The positive result also agrees with the nonnegative strip height found above.

Step 4: Conclusion The area enclosed by $y = 3x^2$ and $3x - y + 6 = 0$ is $\boxed{\frac{27}{2} \text{ square units}}$.

Quick Tip to Solve: Factor the ordinate difference as $3(2 - x)(x + 1)$: this immediately gives both intersection limits and the correct upper-minus-lower order.

35(a). Find the equation of the plane that contains the point $A(2, 1, -1)$ and is perpendicular to the line of intersection of the planes $2x + y - z = 3$ and $x + 2y + z = 2$. Also find the angle between the plane thus obtained and the y-axis. [6 marks]

Solution:

Concept:

- If two intersecting planes have normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, their line of intersection has direction $\mathbf{n}_1 \times \mathbf{n}_2$, perpendicular to both normals.
- A plane perpendicular to a line has a normal vector parallel to that line's direction. A plane through position vector $\mathbf{a}$ with normal $\mathbf{n}$ satisfies $\mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0$.
- For the acute angle θ between a line with direction $\mathbf{d}$ and a plane with normal $\mathbf{n}$, $\sin\theta = \dfrac{|\mathbf{n} \cdot \mathbf{d}|}{|\mathbf{n}|\,|\mathbf{d}|}$.

Step 1: Obtain the direction of the intersection line From the equations $2x + y - z = 3$ and $x + 2y + z = 2$, their respective normal vectors are $\mathbf{n}_1 = (2, 1, -1)$ and $\mathbf{n}_2 = (1, 2, 1)$. A direction

Quick Tip to Solve: Cross the two given plane normals to get the new plane's normal, then use sine, rather than cosine, for the angle between an axis and the plane.

vector of their intersection is

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix} = 3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}.$$

We may divide this nonzero vector by 3 and use $\mathbf{n} = (1, -1, 1)$. Indeed, $(2, 1, -1) \cdot (1, -1, 1) = 0$ and $(1, 2, 1) \cdot (1, -1, 1) = 0$, confirming that this direction lies parallel to both original planes. The cross product is nonzero, so the intersection has a well-defined direction.

Step 2: Construct the required plane through the specified point Since the required plane is perpendicular to the intersection line, its normal must be parallel to $(1, -1, 1)$. Through $A(2, 1, -1)$, its point-normal equation is

$$\begin{aligned} (1, -1, 1) \cdot (x - 2, y - 1, z + 1) &= 0, \\ (x - 2) - (y - 1) + (z + 1) &= 0, \\ x - y + z &= 0. \end{aligned}$$

To check the point condition, substitution of A gives $2 - 1 - 1 = 0$. Its normal is the direction of the intersection line found above, so the perpendicularity condition is also satisfied. A prescribed normal direction together with one point fixes this plane uniquely.

Step 3: Calculate the angle with the y-axis The y-axis has unit direction vector $\mathbf{d} = (0, 1, 0)$. For the acute angle θ between this axis and the required plane, use the line-plane formula:

$$\begin{aligned} \sin\theta &= \frac{|(1, -1, 1) \cdot (0, 1, 0)|}{\sqrt{1^2 + (-1)^2 + 1^2}\sqrt{0^2 + 1^2 + 0^2}} \\ &= \frac{|-1|}{\sqrt{3}} = \frac{1}{\sqrt{3}}. \end{aligned}$$

The absolute value is needed because the angle with an axis does not depend on which direction along that axis is chosen. This sine formula gives the angle with the plane; a cosine formula with the normal would instead give its complementary acute angle.

Step 4: Conclusion The required plane is $\boxed{x - y + z = 0}$. Its angle with the y-axis is

$$\boxed{\theta = \sin^{-1}(1/\sqrt{3}) \approx 35.26^\circ}.$$

Quick Tip to Solve: Cross the two given plane normals to get the new plane's normal, then use sine, rather than cosine, for the angle between an axis and the plane.

OR

35(b). Find the distance of the point $P(-2,-4,7)$ from the point of intersection Q of the line $\vec{r} = (3\hat{i} - 2\hat{j} + 6\hat{k}) + \lambda(2\hat{i} - \hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 6$. Also write the vector equation of the line PQ. [6 marks]

Solution:

Concept:

- A point on a vector line $\vec{r} = \vec{a} + \lambda\vec{b}$ has coordinates obtained by comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$. The parameter λ is real.
- The intersection of a line and a plane is found by substituting the line coordinates into the plane equation. For points P and Q, their distance is $|\overrightarrow{PQ}|$, and their joining line has equation $\vec{r} = \overrightarrow{OP} + t\overrightarrow{PQ}$, $t \in \mathbb{R}$.

Step 1: Write a general point on the given line Comparing the components of the given vector equation, a point on the line has coordinates

$$x = 3 + 2\lambda, \qquad y = -2 - \lambda, \qquad z = 6 + 2\lambda.$$

The normal to the given plane is $\hat{i} - \hat{j} + \hat{k}$. Hence its Cartesian equation is $x - y + z = 6$. The point Q must satisfy both these descriptions.

Step 2: Determine the intersection parameter Substitute the line coordinates in the plane equation, taking care of the minus sign before y:

$$\begin{aligned}(3 + 2\lambda) - (-2 - \lambda) + (6 + 2\lambda) &= 6,\\ 11 + 5\lambda &= 6,\\ \lambda &= -1.\end{aligned}$$

The coefficient of λ is nonzero, so this substitution determines one intersection. Therefore

$$Q = (3 - 2, -2 + 1, 6 - 2) = (1, -1, 4).$$

As a check, $1 - (-1) + 4 = 6$, so this point lies on the plane.

Step 3: Calculate the displacement and its magnitude Using the coordinates of $P(-2,-4,7)$, the vector directed from P towards Q is

$$\overrightarrow{PQ} = (1 + 2)\hat{i} + (-1 + 4)\hat{j} + (4 - 7)\hat{k} = 3\hat{i} + 3\hat{j} - 3\hat{k}.$$

Its magnitude is the required distance:

$$PQ = \sqrt{3^2 + 3^2 + (-3)^2} = \sqrt{27} = 3\sqrt{3}.$$

Each coordinate difference is squared, so the negative third component contributes positively to the squared distance.

Step 4: Form the vector equation of the joining line Take P as the fixed point and $\overrightarrow{PQ}$ as the direction

Quick Tip to Solve: First substitute the parametric coordinates into $x - y + z = 6$; then use the displacement from P to the resulting intersection as the line direction.

vector. The vector equation is

$$\vec{r} = (-2\hat{i} - 4\hat{j} + 7\hat{k}) + t(3\hat{i} + 3\hat{j} - 3\hat{k}), \qquad t \in \mathbb{R}.$$

At $t = 0$ this gives P, and at $t = 1$ it gives Q, verifying that the line joins the required points. A nonzero scalar multiple of the direction vector defines the same line, so we may also use $\hat{i} + \hat{j} - \hat{k}$.

Step 5: Conclusion The required distance is $\boxed{3\sqrt{3} \text{ units}}$. A vector equation of PQ is

$$\boxed{\vec{r} = (-2\hat{i} - 4\hat{j} + 7\hat{k}) + s(\hat{i} + \hat{j} - \hat{k}), \quad s \in \mathbb{R}.}$$

Quick Tip to Solve: First substitute the parametric coordinates into $x - y + z = 6$; then use the displacement from P to the resulting intersection as the line direction.

36(a). If $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 2 & -2 \\ 2 & -1 & 1 \end{bmatrix}$, then find A^{-1} and use it to solve the following system of equations:

$x + 2y - 3z = 6$

$3x + 2y - 2z = 3$

$2x - y + z = 2$ [6 marks]

Solution:

Concept:

- For a square matrix A with $|A| \neq 0$, its inverse is $A^{-1} = \dfrac{1}{|A|} \operatorname{adj} A$. The adjugate is the transpose of the cofactor matrix, with cofactor $C_{ij} = (-1)^{i+j} M_{ij}$.
- A linear system can be written as $AX = B$, where X is the column of unknowns and B is the column of constants. Multiplication on the left by A^{-1} gives $X = A^{-1}B$.

Step 1: Evaluate the determinant Expand the determinant along the first row:

$$\begin{aligned} |A| &= 1(2 - 2) - 2(3 + 4) + (-3)(-3 - 4) \\ &= 0 - 14 + 21 = 7. \end{aligned}$$

Since the determinant is nonzero, the inverse exists and the system has a unique solution. The alternating signs in a first-row expansion are $+, -, +$; the third entry itself is -3.

Step 2: Calculate all the cofactors Delete the row and column of each entry and apply the cofactor

Quick Tip to Solve: Transpose the cofactor matrix before dividing by 7; multiply its rows by the constants column $(6, 3, 2)^T$, and check the resulting values in all three equations.

sign. This gives

$$\begin{aligned} C_{11} &= 2-2=0, & C_{12} &= -(3+4)=-7, \\ C_{13} &= -3-4=-7, & C_{21} &= -(2-3)=1, \\ C_{22} &= 1+6=7, & C_{23} &= -(-1-4)=5, \\ C_{31} &= -4+6=2, & C_{32} &= -(-2+9)=-7, \\ C_{33} &= 2-6=-4. \end{aligned}$$

Thus the cofactor matrix is

$$C = \begin{bmatrix} 0 & -7 & -7 \\ 1 & 7 & 5 \\ 2 & -7 & -4 \end{bmatrix}.$$

Step 3: Transpose the cofactor matrix to obtain the inverse The adjugate is C^T, not C. Consequently

$$\operatorname{adj} A = \begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}, \qquad A^{-1} = \frac{1}{7}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}.$$

This completes the inverse calculation required in the first part.

Step 4: Use the inverse to solve the system The coefficient matrix is precisely the given matrix, so the system becomes

$$AX = B, \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad B = \begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix}.$$

Multiplying by the computed inverse yields

$$\begin{aligned} X &= \frac{1}{7}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}\begin{bmatrix} 6 \\ 3 \\ 2 \end{bmatrix} \\ &= \frac{1}{7}\begin{bmatrix} 0+3+4 \\ -42+21-14 \\ -42+15-8 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ -5 \end{bmatrix}. \end{aligned}$$

To check all three equations, substitute these values:

$$\begin{aligned} 1+2(-5)-3(-5) &= 6, \\ 3(1)+2(-5)-2(-5) &= 3, \\ 2(1)-(-5)+(-5) &= 2. \end{aligned}$$

All the given constants are recovered.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by 7; multiply its rows by the constants column $(6, 3, 2)^T$, and check the resulting values in all three equations.

Step 5: Conclusion

$$\boxed{A^{-1} = \frac{1}{7}\begin{bmatrix} 0 & 1 & 2 \\ -7 & 7 & -7 \\ -7 & 5 & -4 \end{bmatrix}}$$

Using this inverse, the unique solution is $\boxed{x = 1,\ y = -5,\ z = -5}$.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by 7; multiply its rows by the constants column $(6, 3, 2)^T$, and check the resulting values in all three equations.

OR

36(b). Using properties of determinants, prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2).$$ [6 marks]

Solution:

Concept:

- Adding a multiple of one column to another column leaves a determinant unchanged. A common factor in a column can be taken outside the determinant.
- Interchanging two columns changes the sign of the determinant. After a row subtraction creates zeros, expansion along that row reduces the calculation to a determinant of order two.
- Write $s = a+b+c$, $T = a^2+b^2+c^2$ and $p = ab+bc+ca$. Then $bc = a^2 - sa + p$, with corresponding cyclic identities for ca and ab.

Step 1: Make the first column constant Let the determinant on the left-hand side be D. Apply the column operation $C_1 \leftarrow C_1 + C_2 - 2C_3$. Its first entry becomes

$$(b+c)^2 + a^2 - 2bc = a^2 + b^2 + c^2 = T.$$

The second and third entries similarly become T. Therefore, taking this common polynomial factor out of the first column,

$$D = T\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}.$$

This is a determinant identity and involves no division by T; it remains valid even when the common factor is zero.

Step 2: Turn the third column into a linear column In the determinant just obtained, use $C_3 \leftarrow C_3 - C_2 - pC_1$. To check its first entry explicitly,

$$bc - a^2 - p = -a^2 - ab - ac = -a(a+b+c) = -sa.$$

Quick Tip to Solve: Begin with $C_1 + C_2 - 2C_3$: the mixed products cancel and all three entries become $a^2+b^2+c^2$. Then use $bc = a^2 - a(a+b+c) + ab + bc + ca$ to obtain a linear column.

Likewise, the other entries are $-sb$ and $-sc$. Factoring $-s$ from this column gives

$$D = -sT \begin{vmatrix} 1 & a^2 & a \\ 1 & b^2 & b \\ 1 & c^2 & c \end{vmatrix}.$$

Now interchange the second and third columns. The resulting sign change cancels the preceding minus sign:

$$D = sT \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

Step 3: Factor the remaining determinant by row operations Denote the last determinant by V. Apply $R_2 \leftarrow R_2 - R_1$ and $R_3 \leftarrow R_3 - R_1$. Factor $b-a$ and $c-a$ from these rows, using the difference-of-squares identity:

$$\begin{aligned} V &= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{vmatrix} \\ &= (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix}. \end{aligned}$$

Expanding along the first column yields

$$\begin{aligned} V &= (b-a)(c-a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix} \\ &= (b-a)(c-a)(c-b) \\ &= (a-b)(b-c)(c-a). \end{aligned}$$

In the last line, reversing the signs of both $b-a$ and $c-b$ produces two minus signs, whose product is positive.

Step 4: Conclusion Substituting $s = a+b+c$, $T = a^2+b^2+c^2$ and the evaluated determinant into $D = sTV$, we obtain

$$\boxed{D = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)}.$$

Hence the required identity is proved for all values of a, b, c.

Quick Tip to Solve: Begin with $C_1 + C_2 - 2C_3$: the mixed products cancel and all three entries become $a^2+b^2+c^2$. Then use $bc = a^2 - a(a+b+c) + ab + bc + ca$ to obtain a linear column.