CBSE Class 12 Mathematics Question Paper 2020 with Solution PDF-Set 65-2-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper comprises **four** Sections A, B, C and D. This question paper carries **36** questions. **All** questions are compulsory.

(ii) **Section A** – Questions no. **1 to 20** comprises of **20** questions of **1** mark each.

(iii) **Section B** – Questions no. **21 to 26** comprises of **6** questions of **2** marks each.

(iv) **Section C** – Questions no. **27 to 32** comprises of **6** questions of **4** marks each.

(v) **Section D** – Questions no. **33 to 36** comprises of **4** questions of **6** marks each.

(vi) There is no overall choice in the question paper. However, an internal choice has been provided in 3 questions of one mark, 2 questions of two marks, 2 questions of four marks and 2 questions of six marks. Only one of the choices in such questions have to be attempted.

(vii) In addition to this, separate instructions are given with each section and question, wherever necessary.

(viii) Use of calculators is **not** permitted.

Section A (20 marks)

Question numbers 1 to 20 carry 1 mark each.

Question numbers 1 to 10 are multiple choice type questions. Select the correct option.

1. The area of a triangle formed by vertices O, A and B, where $\overrightarrow{OA} = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\overrightarrow{OB} = -3\hat{i} - 2\hat{j} + \hat{k}$ is

(A) $3\sqrt{5}$ **sq. units**

(B) $5\sqrt{5}$ **sq. units**

(C) $6\sqrt{5}$ **sq. units**

(D) 4 **sq. units** [1 mark]

Solution:

Concept:

- The triangle area is half the magnitude of the cross product of its two sides from the same vertex.
- For vectors $\mathbf{a}$ and $\mathbf{b}$, area $= \frac{1}{2}|\mathbf{a} \times \mathbf{b}|$.

Step 1: Compute the cross product Using the two sides from O,

$$\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & -2 & 1 \end{vmatrix} = 8\hat{i} - 10\hat{j} + 4\hat{k}.$$

The signs follow the alternating cofactors of the determinant.

Step 2: Find the triangle area and compare options The magnitude is $\sqrt{8^2 + (-10)^2 + 4^2} = \sqrt{180} = 6\sqrt{5}$. This gives the parallelogram area; the triangle has half that area. Options (B), (C) and (D) do not give this half-magnitude.

Step 3: Conclusion The area is $3\sqrt{5}$ square units, so the correct option is (A).

Quick Tip to Solve: Always halve the cross-product magnitude for a triangle, even when one vertex is the origin.

2. The domain of the function $f(x) = \sin^{-1}(2x)$ is

(A) $[0, 1]$

(B) $[-1, 1]$

(C) $\left[-\frac{1}{2}, \frac{1}{2}\right]$

(D) $[-2, 2]$ [1 mark]

Solution:

Concept:

- The real function $\sin^{-1} u$ requires $-1 \leq u \leq 1$.
- The domain consists of every real input for which the function is defined.

Quick Tip to Solve: Apply the inverse-sine restriction to its entire argument before solving for x.

Step 1: Restrict the argument Here the argument is $2x$. Therefore the defining condition is $-1 \leq 2x \leq 1$. The restriction applies to the whole argument, not just to x.

Step 2: Solve the inequality and check endpoints Dividing throughout by the positive number 2 gives $-\frac{1}{2} \leq x \leq \frac{1}{2}$. Both endpoints are included because inverse sine is defined at -1 and 1. Option (A) omits negative admissible inputs and includes inadmissible ones; (B) and (D) extend beyond the allowed endpoints.

Step 3: Conclusion The domain is $\left[-\frac{1}{2}, \frac{1}{2}\right]$, so the correct option is (C).

Quick Tip to Solve: Apply the inverse-sine restriction to its entire argument before solving for x.

3. The interval in which the function f given by $f(x) = x^2e^{-x}$ is strictly increasing, is

(A) $(-\infty, \infty)$

(B) $(-\infty, 0)$

(C) $(2, \infty)$

(D) $(0, 2)$ [1 mark]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval where its derivative is positive.
- The exponential factor e^{-x} is positive for every real x.

Step 1: Differentiate by the product rule For $f(x) = x^2e^{-x}$,

$$f\prime(x) = 2xe^{-x} - x^2e^{-x} = xe^{-x}(2 - x).$$

Thus the sign of the derivative is the sign of $x(2 - x)$.

Step 2: Determine the sign on each interval If $x < 0$, then $x < 0$ and $2 - x > 0$, so the derivative is negative. For $0 < x < 2$, both factors are positive. For $x > 2$, the second factor is negative. Hence options (B) and (C) describe decreasing intervals, and (A) crosses both sign changes.

Step 3: Conclusion The function is strictly increasing on $(0, 2)$, so the correct option is (D).

Quick Tip to Solve: Factor the derivative and discard the always-positive exponential when studying its sign.

4. The value of k so that f defined by $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$ is

(A) 0
(B) $\frac{1}{2}$
(C) 1
(D) 2

[1 mark]

Solution:

Concept:

- Continuity at zero requires $\lim_{x \to 0} f(x) = f(0) = k$.
- The bound $|\sin t| \leq 1$ holds for every real t.

Step 1: Bound the oscillating expression For every nonzero x, $-x^2 \leq x^2 \sin(1/x) \leq x^2$. Equivalently, $|x^2 \sin(1/x)| \leq x^2$. Although $\sin(1/x)$ oscillates near zero, this bound controls the product.

Step 2: Apply the squeeze theorem and continuity Both bounding functions tend to zero as $x \to 0$, so the squeeze theorem gives $\lim_{x \to 0} x^2 \sin(1/x) = 0$. The assigned value at zero must equal this limit. The nonzero choices $\frac{1}{2}, 1, 2$ create a discontinuity at zero.

Step 3: Conclusion Continuity requires $k = 0$, so the correct option is (A).

Quick Tip to Solve: For a bounded oscillating factor multiplied by a vanishing power, use the squeeze theorem.

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5. The derivative of $\log x$ with respect to $\frac{1}{x}$ is

(A) $-\frac{1}{x^3}$
(B) $-\frac{1}{x}$
(C) $-x$
(D) $\frac{1}{x}$

[1 mark]

Solution:

Concept:

- If $y = y(x)$ and $u = u(x)$, then $\frac{dy}{du} = \frac{dy/dx}{du/dx}$, provided $du/dx \neq 0$.
- For the real logarithm here, $x > 0$, and $\frac{d}{dx} \log x = \frac{1}{x}$.

Step 1: Differentiate both quantities with respect to x Put $y = \log x$ and $u = 1/x = x^{-1}$. Their derivatives are $dy/dx = 1/x$ and $du/dx = -x^{-2}$. The latter is nonzero throughout the domain.

Step 2: Form the derivative ratio Therefore

$$\frac{dy}{du} = \frac{1/x}{-1/x^2} = -x.$$

Quick Tip to Solve: Differentiate both expressions with respect to x, then divide in the requested order.

The reciprocal must be inverted when dividing: neither $-1/x^3$ nor $-1/x$ is the resulting expression. The positive value $1/x$ is only the derivative with respect to x, not with respect to $1/x$.

Step 3: Conclusion The requested derivative is $-x$, so the correct option is (C).

Quick Tip to Solve: Differentiate both expressions with respect to x, then divide in the requested order.

6. The coordinates of the foot of the perpendicular drawn from the point $(2, -3, 4)$ on the y-axis is

(A) $(2, 3, 4)$

(B) $(-2, -3, -4)$

(C) $(0, -3, 0)$

(D) $(2, 0, 4)$ [1 mark]

Solution:

Concept:

- Every point on the y-axis has coordinates $(0, t, 0)$.
- A perpendicular to the y-axis has zero component along its direction $(0, 1, 0)$.

Step 1: Write a general point on the axis Let $P = (2, -3, 4)$ and let the foot be $H = (0, t, 0)$. Then $\overrightarrow{PH} = (-2, t + 3, -4)$. The unknown coordinate is only the coordinate along the axis.

Step 2: Impose perpendicularity and compare choices The dot product with $(0, 1, 0)$ must vanish, so $t + 3 = 0$, giving $t = -3$. Options (A), (B) and (D) have nonzero x or z coordinates and therefore do not even lie on the y-axis. The original y coordinate is preserved in this projection.

Step 3: Conclusion The foot is $(0, -3, 0)$, so the correct option is (C).

Quick Tip to Solve: Projection onto a coordinate axis preserves that coordinate and sets the other two to zero.

7. The relation R in the set $\{1, 2, 3\}$ given by $R = \{(1, 2), (2, 1), (1, 1)\}$ is

(A) symmetric and transitive, but not reflexive

(B) reflexive and symmetric, but not transitive

(C) symmetric, but neither reflexive nor transitive

(D) an equivalence relation [1 mark]

Solution:

Concept:

- Reflexivity requires every (a, a); symmetry requires (b, a) whenever (a, b) occurs.
- Transitivity requires (a, c) whenever both (a, b) and (b, c) occur.

Step 1: Test reflexivity and symmetry The set is $\{1, 2, 3\}$. Since $(2, 2)$ and $(3, 3)$ are absent, R is not reflexive. It is symmetric: $(1, 2)$ and $(2, 1)$ reverse each other, while $(1, 1)$ reverses itself.

Step 2: Find a transitivity counterexample The pairs $(2, 1)$ and $(1, 2)$ belong to R, but their required

Quick Tip to Solve: For transitivity, compose the two reversed off-diagonal pairs to test the missing diagonal pair.

consequence $(2, 2)$ does not. Thus R is not transitive. Option (A) incorrectly asserts transitivity, (B) incorrectly asserts reflexivity, and (D) requires all three properties of an equivalence relation.

Step 3: Conclusion The relation is symmetric but neither reflexive nor transitive, so the correct option is (C).

Quick Tip to Solve: For transitivity, compose the two reversed off-diagonal pairs to test the missing diagonal pair.

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8. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{a} \times \vec{b}| = 6$, then the value of $\vec{a} \cdot \vec{b}$ is

(A) 12

(B) 6

(C) $3\sqrt{3}$

(D) $6\sqrt{3}$ [1 mark]

Solution:

Concept:

- The dot and cross products satisfy $(\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$.
- For the angle θ between the vectors, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.

Step 1: Use the identity Substituting the three given magnitudes gives

$$(\vec{a} \cdot \vec{b})^2 = 3^2 \times 4^2 - 6^2 = 144 - 36 = 108.$$

Thus the magnitude of the scalar product is $\sqrt{108} = 6\sqrt{3}$.

Step 2: Check the choices and the sign Choices (A), (B) and (C) have squares 144, 36 and 27, respectively, so none satisfies the identity. Choice (D) has square 108. Strictly, the data allow both $+6\sqrt{3}$ and $-6\sqrt{3}$, since the angle can be acute or obtuse; the listed positive value is the only defensible option.

Step 3: Conclusion The listed answer is $\boxed{6\sqrt{3}}$, option (D). Without specifying the angle, the scalar product can be $\pm 6\sqrt{3}$.

Quick Tip to Solve: Square the dot product using the magnitude identity, then check whether the angle fixes its sign.

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9. If A is a non-singular square matrix of order 3 such that $A^2 = 3A$, then the value of $|A|$ is

(A) -3

(B) 3

(C) 9

(D) 27 [1 mark]

Solution:

Concept:

- A non-singular matrix A has an inverse A^{-1}.
- For a square matrix of order n, $|kI_n| = k^n$, where I_n is the identity matrix.

Step 1: Use non-singularity Premultiply the given relation $A^2 = 3A$ by A^{-1}. Associativity gives

$$A^{-1}A^2 = 3A^{-1}A \quad \Longrightarrow \quad A = 3I_3.$$

The inverse exists because the question explicitly states that A is non-singular.

Step 2: Evaluate the determinant and compare The resulting matrix has diagonal entries $3, 3, 3$, with all other entries zero. Its determinant is their product:

$$|A| = |3I_3| = 3 \times 3 \times 3 = 27.$$

Option (A) has the wrong sign. Option (B) omits two factors of 3, and option (C) uses only two factors instead of the three required by the order.

Step 3: Conclusion $\boxed{|A| = 27}$, option (D).

Quick Tip to Solve: Use the inverse to reduce the matrix equation, and count one scalar factor for each matrix row.

10. If $|\vec{a}| = 4$ and $-3 \leq \lambda \leq 2$, then $|\lambda\vec{a}|$ lies in

(A) $[0, 12]$

(B) $[2, 3]$

(C) $[8, 12]$

(D) $[-12, 8]$ [1 mark]

Solution:

Concept:

- Scalar multiplication changes magnitude by the absolute value: $|\lambda\vec{a}| = |\lambda|\,|\vec{a}|$.
- A magnitude is non-negative; a negative scalar reverses direction rather than making the magnitude negative.

Step 1: Find the scalar magnitude range The interval $[-3, 2]$ includes zero, so the minimum of $|\lambda|$ is 0. Its largest value is 3, attained when $\lambda = -3$. Every value between 0 and 3 occurs within this interval. Hence $0 \leq |\lambda| \leq 3$.

Quick Tip to Solve: When the scalar interval crosses zero, include zero and use the endpoint with greatest absolute value.

Step 2: Scale by the vector magnitude Using $|\vec{a}| = 4$,

$$0 \leq |\lambda\vec{a}| = 4|\lambda| \leq 12.$$

Option (B) does not multiply by 4; option (C) excludes the attainable value zero. Option (D) allows negative magnitudes and incorrectly treats the signed scalar as the magnitude.

Step 3: Conclusion $\boxed{|\lambda\vec{a}| \in [0, 12]}$, option (A).

Quick Tip to Solve: When the scalar interval crosses zero, include zero and use the endpoint with greatest absolute value.

11. If the radius of the circle is increasing at the rate of 0.5 cm/s, then the rate of increase of its circumference is _____. [1 mark]

Solution:

Concept:

- The circumference of a circle of radius r is $C = 2\pi r$.
- Time differentiation gives $\dfrac{dC}{dt} = 2\pi\dfrac{dr}{dt}$, with rates measured per unit time.

Step 1: Differentiate the circumference Let t be time in seconds and let r and C be measured in centimetres. Since 2π is constant, differentiating $C = 2\pi r$ with respect to time yields

$$\frac{dC}{dt} = 2\pi\frac{dr}{dt}.$$

The radius itself need not be known because circumference depends linearly on radius.

Step 2: Substitute the given rate The radius increases at $\dfrac{dr}{dt} = 0.5\,\text{cm/s}$. Therefore

$$\frac{dC}{dt} = 2\pi(0.5) = \pi\,\text{cm/s}.$$

The rate is positive, as expected for an increasing radius; its unit is length per time, since circumference is a length.

Step 3: Conclusion The rate of increase is $\boxed{\pi\,\text{cm/s}}$.

Quick Tip to Solve: Differentiate circumference with respect to time; the radius cancels from the rate formula.

12. If $\begin{bmatrix} 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ k \\ -5 \end{bmatrix} = O$, where O is the null matrix, then the value of k is _____. [1 mark]

Solution:

Concept:

- The product of a 1×3 row matrix and a 3×1 column matrix is a 1×1 matrix.
- Its entry is the sum of the products of corresponding entries; a null matrix has every entry zero.

Step 1: Multiply corresponding entries The row has entries $3, -2, 0$, while the column has entries $2, k, -5$. Thus their product is

$$\begin{bmatrix} 3 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ k \\ -5 \end{bmatrix} = \begin{bmatrix} 3(2) + (-2)k + 0(-5) \end{bmatrix} = \begin{bmatrix} 6 - 2k \end{bmatrix}.$$

The third entry contributes zero.

Step 2: Use the null matrix condition Equality with the null matrix means that the only entry vanishes:

$$6 - 2k = 0 \quad \Longrightarrow \quad 2k = 6 \quad \Longrightarrow \quad k = 3.$$

Substitution confirms that the product entry is $6 - 2(3) = 0$, as required.

Step 3: Conclusion $\boxed{k = 3}$.

Quick Tip to Solve: For a row times a column, sum paired products and set the single resulting entry equal to zero.

13. The corner points of the feasible region of an LPP are $(0, 0)$, $(0, 8)$, $(2, 7)$, $(5, 4)$ and $(6, 0)$. The maximum profit $P = 3x + 2y$ occurs at the point _____. [1 mark]

Solution:

Concept:

- A linear objective on a bounded polygonal feasible region attains a maximum at a corner point.
- For each corner (x, y), evaluate the profit function $P = 3x + 2y$ and compare the values.

Step 1: Evaluate profit at every corner The given feasible corner points produce

$$\begin{aligned} P(0, 0) &= 0, \\ P(0, 8) &= 16, \\ P(2, 7) &= 6 + 14 = 20, \\ P(5, 4) &= 15 + 8 = 23, \\ P(6, 0) &= 18. \end{aligned}$$

Both coordinates must be used; the corner with the greatest x or greatest y alone need not give

Quick Tip to Solve: List the profit at every supplied vertex; never choose the largest coordinate without evaluating the objective.

greatest profit.

Step 2: Compare the values The largest of $0, 16, 20, 23, 18$ is 23, obtained at $(5, 4)$. The adjacent listed corners have smaller objective values, so there is no tied edge of maximum profit. The corner-point principle therefore identifies this point as the maximizing point in the feasible region.

Step 3: Conclusion Maximum profit occurs at $\boxed{(5, 4)}$, with $P_{\max} = 23$.

Quick Tip to Solve: List the profit at every supplied vertex; never choose the largest coordinate without evaluating the objective.

14(a). The range of the principal value branch of the function $y = \sec^{-1} x$ is _____. [1 mark]

Solution:

Concept:

- The principal branch for the inverse secant is selected from $[0, \pi]$, with $\pi/2$ excluded.
- Secant is $\sec y = 1/\cos y$, so it is undefined when $\cos y = 0$.

Step 1: Choose the principal angle intervals For $x \geq 1$, the principal inverse secant angle lies in $[0, \pi/2)$. For $x \leq -1$, it lies in $(\pi/2, \pi]$. These two intervals supply the principal values for the domain $(-\infty, -1] \cup [1, \infty)$.

Step 2: Check endpoints and the exclusion At $x = 1$, the angle is 0, and at $x = -1$, it is π; both endpoints are included. The angle $\pi/2$ is excluded because $\cos(\pi/2) = 0$, making its secant undefined. Combining the two angle intervals gives all permissible output values of the principal branch.

Step 3: Conclusion The range is $\boxed{[0, \pi] \setminus \{\pi/2\}}$.

Quick Tip to Solve: For inverse secant, keep both endpoints of zero to pi and remove the angle where cosine vanishes.

OR

14(b). The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is _____. [1 mark]

Solution:

Concept:

- The principal branch of $\cos^{-1} t$ has values in $[0, \pi]$, so only an angle in this interval can be the required answer.
- For $\theta = \cos^{-1} t$, we require $\cos\theta = t$; the sign of cosine also determines the quadrant.

Step 1: Use the sign and reference angle We need an angle θ with $\cos\theta = -\frac{1}{2}$. Since $\cos(\pi/3) = \frac{1}{2}$, the reference angle is $\pi/3$. Cosine is negative in the second quadrant of the principal interval.

Step 2: Select the principal angle The required second-quadrant angle is $\theta = \pi - \pi/3 = 2\pi/3$. Indeed, $\cos(2\pi/3) = -1/2$, and $2\pi/3 \in [0, \pi]$. Other angles with the same cosine do not give additional

Quick Tip to Solve: For a negative cosine input, subtract its acute reference angle from π.

principal values.

Step 3: Conclusion The principal value is $\boxed{\frac{2\pi}{3}}$.

Quick Tip to Solve: For a negative cosine input, subtract its acute reference angle from π.

15(a). The distance between parallel planes $2x + y - 2z - 6 = 0$ and $4x + 2y - 4z = 0$ is _____ units. [1 mark]

Solution:

Concept:

- Parallel planes must first be written with identical coefficients of x, y, z before their constant terms are compared.
- For planes $ax+by+cz+d_1 = 0$ and $ax+by+cz+d_2 = 0$, their distance is $D = |d_1-d_2|/\sqrt{a^2+b^2+c^2}$.

Step 1: Normalize the second equation Dividing the second plane equation by 2 gives $2x + y - 2z = 0$. Thus both planes have the normal vector $(2, 1, -2)$, while their constant terms are -6 and 0.

Step 2: Apply the distance formula The length of the common normal vector is $\sqrt{2^2 + 1^2 + (-2)^2} = 3$. Therefore the perpendicular separation is

$$D = \frac{|-6-0|}{3} = 2.$$

Using the constants before normalizing the second equation would give an incorrect distance.

Step 3: Conclusion The distance between the planes is $\boxed{2 \text{ units}}$.

Quick Tip to Solve: Divide the second plane by 2 before subtracting the constant terms.

OR

15(b). If $P(1, 0, -3)$ is the foot of the perpendicular from the origin to the plane, then the cartesian equation of the plane is _____. [1 mark]

Solution:

Concept:

- The segment from a point to its perpendicular foot on a plane is normal to that plane.
- A plane through (x_0, y_0, z_0) with normal (a, b, c) is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Step 1: Identify a normal and a point Let the origin be $O = (0, 0, 0)$. Since $P = (1, 0, -3)$ is the perpendicular foot, $\overrightarrow{OP} = (1, 0, -3)$ is a normal vector. The foot P itself lies on the required plane.

Step 2: Write the point-normal equation Using normal $(1, 0, -3)$ and the point $(1, 0, -3)$, we obtain

$$(x - 1) + 0(y - 0) - 3(z + 3) = 0.$$

Quick Tip to Solve: Use the foot coordinates as both a point on the plane and its normal vector.

On expanding, the equation becomes $x - 3z - 10 = 0$. Substitution of P gives $1 + 9 - 10 = 0$, as required.

Step 3: Conclusion The Cartesian equation is $\boxed{x - 3z - 10 = 0}$.

Quick Tip to Solve: Use the foot coordinates as both a point on the plane and its normal vector.

16. Evaluate: $\int_{-\pi/2}^{\pi/2} x \cos^2 x \, dx$. [1 mark]

Solution:

Concept:

- A function g is odd if $g(-x) = -g(x)$. Its definite integral over a symmetric interval $[-a, a]$ is zero.
- The cosine function is even; hence $\cos^2(-x) = \cos^2 x$. Multiplication by the odd function x makes the integrand odd.

Step 1: Check the symmetry of the integrand Put $g(x) = x \cos^2 x$. Then

$$g(-x) = (-x)\cos^2(-x) = -x\cos^2 x = -g(x).$$

Thus the integrand is odd and continuous over the interval of integration.

Step 2: Use the symmetric limits The limits $-\pi/2$ and $\pi/2$ are negatives of each other. The integral from $-\pi/2$ to 0 cancels the integral from 0 to $\pi/2$, by the substitution $x = -t$. Therefore the sum of these two integrals vanishes.

Step 3: Conclusion $\boxed{\int_{-\pi/2}^{\pi/2} x \cos^2 x \, dx = 0}$.

Quick Tip to Solve: Check oddness before expanding $\cos^2 x$ or integrating by parts.

17. Find the coordinates of the point where the line $\frac{x-1}{3} = \frac{y+4}{7} = \frac{z+4}{2}$ cuts the xy-plane. [1 mark]

Solution:

Concept:

- Every point in the xy-plane has $z = 0$; this is the condition for finding the intersection.
- Setting the three equal ratios of a line equal to a parameter t gives its coordinates as functions of t.

Step 1: Express the line parametrically Let

$$\frac{x-1}{3} = \frac{y+4}{7} = \frac{z+4}{2} = t.$$

Then $x = 1 + 3t$, $y = -4 + 7t$, and $z = -4 + 2t$. The constants in the numerators determine the starting point, while the denominators determine the direction.

Quick Tip to Solve: Set the coordinate absent from the plane name to zero: here set $z = 0$.

Step 2: Impose the plane equation At the intersection with the xy-plane, $z = 0$. Hence $-4 + 2t = 0$, so $t = 2$. Substitution gives $x = 1 + 6 = 7$ and $y = -4 + 14 = 10$. All three original ratios then equal 2, checking that the point lies on the line.

Step 3: Conclusion The required intersection point is $\boxed{(7, 10, 0)}$.

Quick Tip to Solve: Set the coordinate absent from the plane name to zero: here set $z = 0$.

18. Find the value of k, so that the function $f(x) = \begin{cases} kx^2 + 5 & \text{if } x \leq 1 \\ 2 & \text{if } x > 1 \end{cases}$ is continuous at $x = 1$. [1 mark]

Solution:

Concept:

- Continuity at $x = 1$ requires the left-hand limit, the right-hand limit, and $f(1)$ to be equal.
- Each branch is continuous on its own domain, so its limiting value at the boundary follows by direct substitution.

Step 1: Find the two one-sided limits For $x \leq 1$, the branch is $kx^2 + 5$. Therefore $\lim_{x \to 1^-} f(x) = k + 5$, and the equality sign in this branch also gives $f(1) = k + 5$. For $x > 1$, the function is constant, so $\lim_{x \to 1^+} f(x) = 2$.

Step 2: Match the boundary values Continuity requires $k + 5 = 2$, which gives $k = -3$. With this value, the left-hand limit is $-3 + 5 = 2$, the right-hand limit is 2, and $f(1) = 2$. All three values agree, which establishes continuity at the specified point.

Step 3: Conclusion The required value is $\boxed{k = -3}$.

Quick Tip to Solve: At a piecewise boundary, include the branch containing equality when computing $f(1)$.

19. Form the differential equation representing the family of curves $y = mx$, where m is an arbitrary constant. [1 mark]

Solution:

Concept:

- To form a differential equation from a family of curves, differentiate and eliminate its arbitrary constant.
- The family $y = mx$ contains one arbitrary constant m, so one differentiation is sufficient for a first-order equation.

Step 1: Differentiate the family For $y = mx$, the number m is constant along each member of the family. Differentiating with respect to x yields

$$\frac{dy}{dx} = m.$$

Quick Tip to Solve: Replace m by dy/dx in $y = mx$; avoid unnecessary division by x.

This expresses the constant in terms of the derivative.

Step 2: Eliminate the arbitrary constant Substitute $m = dy/dx$ into the original equation $y = mx$. This gives $y = x\,dy/dx$. Writing the result with all terms on one side eliminates m without dividing by x, so the equation also remains meaningful at the origin.

Step 3: Conclusion The differential equation is $\boxed{x\dfrac{dy}{dx} - y = 0}$.

Quick Tip to Solve: Replace m by dy/dx in $y = mx$; avoid unnecessary division by x.

20(a). Differentiate $\sec^2(x^2)$ with respect to x^2. [1 mark]

Solution:

Concept:

- To differentiate with respect to a composite expression, take that expression as the independent variable: here put $u = x^2$.
- The chain rule gives $\dfrac{d}{du}(\sec u)^2 = 2\sec u\dfrac{d}{du}(\sec u)$, where $\dfrac{d}{du}\sec u = \sec u\tan u$.

Step 1: Choose the required variable Let $u = x^2$. The given function becomes $y = \sec^2 u$. Since the derivative is requested with respect to x^2, we need dy/du, rather than dy/dx.

Step 2: Apply the power and trigonometric rules On every interval where $\cos u \neq 0$, the function is differentiable. Thus

$$\frac{dy}{du} = 2\sec u(\sec u\tan u) = 2\sec^2 u\tan u.$$

Substitute $u = x^2$ after completing the differentiation. No factor $2x$ is needed, because we are differentiating with respect to x^2 itself.

Step 3: Conclusion

$$\boxed{\frac{d\{\sec^2(x^2)\}}{d(x^2)} = 2\sec^2(x^2)\tan(x^2)}.$$

Quick Tip to Solve: Name the variable of differentiation first; a factor $2x$ appears only when differentiating with respect to x.

OR

20(b). If $y = f(x^2)$ and $f'(x) = e^{\sqrt{x}}$, then find $\dfrac{dy}{dx}$. [1 mark]

Solution:

Concept:

- For $y = f(u)$ with $u = x^2$, the chain rule is $\dfrac{dy}{dx} = f\prime(u)\dfrac{du}{dx}$.
- The real principal square root satisfies $\sqrt{x^2} = |x|$. It equals x only when $x \geq 0$.

Quick Tip to Solve: Keep $\sqrt{x^2}$ as $|x|$ unless the question restricts x to nonnegative values.

Step 1: Evaluate the derivative of the outer function The given rule is $f\prime(t) = e^{\sqrt{t}}$, with real argument $t \geq 0$. Replacing its argument by x^2, which is always nonnegative, gives

$$f\prime(x^2) = e^{\sqrt{x^2}} = e^{|x|}.$$

The symbol inside the rule is a dummy argument; it must be replaced throughout.

Step 2: Include the inner derivative Since $\frac{d}{dx}(x^2) = 2x$, the chain rule gives

$$\frac{dy}{dx} = 2xe^{\sqrt{x^2}} = 2xe^{|x|}.$$

For $x \geq 0$ this is $2xe^x$; for $x < 0$ it is $2xe^{-x}$. At $x = 0$ it gives zero, consistently with the derivative of the composition.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = 2xe^{\sqrt{x^2}} = 2xe^{|x|}}.$$

In particular, for $x \geq 0$, this is $2xe^x$.

Quick Tip to Solve: Keep $\sqrt{x^2}$ as $|x|$ unless the question restricts x to nonnegative values.

Section B (12 marks)

Question numbers 21 to 26 carry 2 marks each.

21(a). Find a vector $\vec{r}$ equally inclined to the three axes and whose magnitude is $3\sqrt{3}$ units. [2 marks]

Solution:

Concept:

- If a nonzero vector makes angles α, β, γ with the positive coordinate axes, its direction cosines are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$, and $\vec{r} = |\vec{r}|(l\hat{i} + m\hat{j} + n\hat{k})$.

Step 1: Use equal inclinations Equal angles with the three axes give $\alpha = \beta = \gamma$, hence $l = m = n$. Substituting in the direction-cosine identity,

$$3l^2 = 1 \quad \Longrightarrow \quad l = \pm\frac{1}{\sqrt{3}}.$$

The three direction cosines have the same sign because they are equal. For a requested vector, we may choose the positive sign.

Step 2: Construct and verify a vector Using $|\vec{r}| = 3\sqrt{3}$ and $l = m = n = 1/\sqrt{3}$,

$$\vec{r} = 3\sqrt{3}\left(\frac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}\right) = 3\hat{i} + 3\hat{j} + 3\hat{k}.$$

Quick Tip to Solve: For equal inclinations, set the three direction cosines equal before using their sum-of-squares identity.

Its magnitude is $\sqrt{3^2+3^2+3^2} = 3\sqrt{3}$. Its scalar product with each coordinate unit vector is 3, so the cosine of each inclination is $3/(3\sqrt{3}) = 1/\sqrt{3}$, verifying equal inclinations.

Step 3: Conclusion A suitable vector is $\boxed{\vec{r} = 3\hat{i} + 3\hat{j} + 3\hat{k}}$. The vector $-3\hat{i} - 3\hat{j} - 3\hat{k}$ also has equal inclinations and the stated magnitude.

Quick Tip to Solve: For equal inclinations, set the three direction cosines equal before using their sum-of-squares identity.

OR

21(b). Find the angle between unit vectors $\vec{a}$ and $\vec{b}$ so that $\sqrt{3}\,\vec{a} - \vec{b}$ is also a unit vector. [2 marks]

Solution:

Concept:

- The scalar product obeys $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, where $0 \le \theta \le \pi$ is the angle between the vectors.
- For any vector $\vec{v}$, $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$. Unit vectors have squared magnitude 1.

Step 1: Translate the unit-vector condition We are given $|\vec{a}| = |\vec{b}| = 1$ and $|\sqrt{3}\vec{a} - \vec{b}| = 1$. Squaring the last relation using the dot product,

$$\begin{aligned} 1 &= (\sqrt{3}\vec{a} - \vec{b}) \cdot (\sqrt{3}\vec{a} - \vec{b}) \\ &= 3|\vec{a}|^2 + |\vec{b}|^2 - 2\sqrt{3}\,\vec{a} \cdot \vec{b}. \end{aligned}$$

The cross term is negative because the vector expression contains subtraction.

Step 2: Determine the cosine and the angle Substitution of the two unit magnitudes gives

$$1 = 4 - 2\sqrt{3}\cos\theta, \qquad \cos\theta = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}.$$

In the standard angle range $[0, \pi]$, this cosine corresponds uniquely to $\theta = \pi/6$. Checking the resulting squared magnitude gives $4 - 2\sqrt{3}(\sqrt{3}/2) = 1$, so the constructed condition is satisfied.

Step 3: Conclusion

$$\boxed{\theta = \frac{\pi}{6} = 30^\circ}.$$

Quick Tip to Solve: Square the vector magnitude using a dot product; retain the minus sign in the cross term.

22. If $A = \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, find scalar k so that $A^2 + I = kA$. [2 marks]

Solution:

Concept:

- Matrix multiplication uses the scalar product of a row of the first matrix and a column of the second; squaring a matrix does not mean squaring its entries.
- Two matrices are equal exactly when all corresponding entries are equal. Multiplication by the scalar k multiplies each entry by k.

Step 1: Compute the matrix square For the printed matrix,

$$\begin{aligned} A^2 &= \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} (-3)(-3) + 2(1) & (-3)(2) + 2(-1) \\ 1(-3) + (-1)(1) & 1(2) + (-1)(-1) \end{bmatrix} \\ &= \begin{bmatrix} 11 & -8 \\ -4 & 3 \end{bmatrix}. \end{aligned}$$

Each entry includes both products in its row-column calculation.

Step 2: Add the identity and compare every entry Adding I changes only the diagonal entries:

$$A^2 + I = \begin{bmatrix} 12 & -8 \\ -4 & 4 \end{bmatrix}, \qquad kA = \begin{bmatrix} -3k & 2k \\ k & -k \end{bmatrix}.$$

The lower-left entry requires $k = -4$. For this value, the remaining entries are $-3k = 12$, $2k = -8$, and $-k = 4$. Thus all four entries satisfy the required matrix equation, and the nonzero lower-left entry establishes uniqueness.

Step 3: Conclusion The required scalar is $\boxed{k = -4}$, since $A^2 + I = -4A$.

Quick Tip to Solve: After computing $A^2 + I$, read k from the entry where A equals 1 and verify the other entries.

23(a). If $f(x) = \sqrt{\dfrac{\sec x - 1}{\sec x + 1}}$, find $f'\left(\dfrac{\pi}{3}\right)$. [2 marks]

Solution:

Concept:

- The half-angle identity is $\dfrac{1 - \cos x}{1 + \cos x} = \tan^2(x/2)$, wherever the expressions are defined.
- The principal square root satisfies $\sqrt{u^2} = |u|$. On an interval where $\tan(x/2) > 0$, this absolute value can be removed before differentiating.

Quick Tip to Solve: First convert the secant ratio to a half-angle square, then check the sign near the evaluation point.

Step 1: Simplify and determine the local sign Dividing numerator and denominator by $\sec x$, near $x = \pi/3$ we obtain

$$f(x) = \sqrt{\frac{1-\cos x}{1+\cos x}} = \sqrt{\tan^2(x/2)} = |\tan(x/2)|.$$

At the required point, $x/2 = \pi/6$, so $\tan(x/2) = 1/\sqrt{3} > 0$. By continuity it remains positive in a neighbourhood of this point. Hence on that neighbourhood $f(x) = \tan(x/2)$. The sign check matters because replacing a square root of a square by the unsquared expression globally would be incorrect.

Step 2: Differentiate and substitute Apply the chain rule on this neighbourhood:

$$f'(x) = \frac{1}{2}\sec^2(x/2), \qquad f'(\pi/3) = \frac{1}{2}\sec^2(\pi/6) = \frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2 = \frac{2}{3}.$$

The function is differentiable at the required point since the half-angle expression is positive and finite there.

Step 3: Conclusion $\boxed{f'(\pi/3) = \frac{2}{3}}$.

Quick Tip to Solve: First convert the secant ratio to a half-angle square, then check the sign near the evaluation point.

OR

23(b). Find $f'(x)$ if $f(x) = (\tan x)^{\tan x}$. [2 marks]

Solution:

Concept:

- For a variable base and a variable exponent, logarithmic differentiation converts the power into a product: $\ln(u^v) = v \ln u$, for $u > 0$.
- Use the product rule and $\frac{d}{dx}\tan x = \sec^2 x$, with angles measured in radians.

Step 1: Take the logarithm Put $y = (\tan x)^{\tan x}$. On its standard real differentiable domain, $\tan x > 0$, so both the logarithm and the variable power are well defined. Taking natural logarithms gives

$$\ln y = (\tan x)\ln(\tan x).$$

Both factors on the right depend on x. Therefore treating the exponent as a constant, or differentiating only the exponent, would omit part of the derivative.

Step 2: Differentiate the product and recover the derivative Differentiate implicitly and apply the product and chain rules:

$$\begin{aligned}\frac{y'}{y} &= \sec^2 x \ln(\tan x) + \tan x \frac{\sec^2 x}{\tan x} \\ &= \sec^2 x\,[\ln(\tan x) + 1].\end{aligned}$$

Multiplying by y and substituting the original function yields the required derivative. Division by $\tan x$ is valid on the stated domain, where it is strictly positive.

Quick Tip to Solve: When both the base and exponent vary, take logarithms before applying the product rule.

Step 3: Conclusion $\boxed{f'(x) = (\tan x)^{\tan x} \sec^2 x\,[1 + \ln(\tan x)]}$, for $\tan x > 0$.

Quick Tip to Solve: When both the base and exponent vary, take logarithms before applying the product rule.

24. Find: $\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\,dx$. [2 marks]

Solution:

Concept:

- Use $\tan^2 x = \sec^2 x - 1$ to rewrite powers of tangent in terms of secant.
- The substitution $u = \sec x$ gives $du = \sec x \tan x\,dx$. The power rule is $\int u^n\,du = u^{n+1}/(n+1) + C$ for $n \neq -1$.

Step 1: Arrange the integrand for substitution Since $1/\cos^3 x = \sec^3 x$, write

$$\begin{aligned} I &= \int \tan^3 x \sec^3 x\,dx \\ &= \int (\tan^2 x)(\sec^2 x)(\sec x \tan x\,dx) \\ &= \int (\sec^2 x - 1)\sec^2 x(\sec x \tan x\,dx). \end{aligned}$$

This separates precisely the factor whose differential is $d(\sec x)$. The remaining factors become a polynomial in $\sec x$.

Step 2: Integrate the polynomial and verify Set $u = \sec x$. Then

$$I = \int (u^2 - 1)u^2\,du = \int (u^4 - u^2)\,du = \frac{u^5}{5} - \frac{u^3}{3} + C.$$

After substituting back, differentiation of the obtained expression gives

$$(\sec^4 x - \sec^2 x)\sec x \tan x = \sec^3 x \tan x(\sec^2 x - 1) = \sec^3 x \tan^3 x,$$

which is the original integrand. The result applies on any interval where $\cos x \neq 0$.

Step 3: Conclusion $\boxed{\displaystyle\int \frac{\tan^3 x}{\cos^3 x}\,dx = \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C}$.

Quick Tip to Solve: Keep one factor of secant times tangent for the differential, and replace the remaining tangent square.

25. Find the angle between the line $\vec{r} = 2\hat{i} - 3\hat{j} + \lambda\hat{k}$ and the plane $\vec{r} \cdot (\hat{j} - \hat{k}) = 7$. [2 marks]

Solution:

Concept:

- For $\vec{r} = \vec{a} + \lambda\vec{b}$, the vector $\vec{b}$ gives the direction of the line; the fixed vector $\vec{a}$ locates a point on it.
- If $\vec{n}$ is normal to a plane, the acute angle θ between the line and plane satisfies $\sin\theta = |\vec{b}\cdot\vec{n}|/(|\vec{b}||\vec{n}|)$.

Step 1: Identify the direction and normal vectors The line is $\vec{r} = 2\hat{i} - 3\hat{j} + \lambda\hat{k}$, so its direction vector is $\vec{b} = \hat{k} = (0, 0, 1)$. The plane has equation $\vec{r}\cdot(\hat{j} - \hat{k}) = 7$, hence a normal vector is $\vec{n} = \hat{j} - \hat{k} = (0, 1, -1)$. The coordinates 2 and -3 in the fixed point do not change the direction of the line. Likewise, the plane constant 7 changes its position but does not change its normal direction.

Step 2: Calculate the acute line-plane angle Compute the scalar product and magnitudes:

$$\vec{b}\cdot\vec{n} = -1, \qquad |\vec{b}| = 1, \qquad |\vec{n}| = \sqrt{2}.$$

Therefore

$$\sin\theta = \frac{|-1|}{1\cdot\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

The angle between a line and a plane is taken in $[0, \pi/2]$. Thus this sine determines $\theta = \pi/4$. The sine is used because the line-plane angle is complementary to the acute angle between the line direction and the plane normal.

Step 3: Conclusion The angle between the line and the plane is $\boxed{\pi/4 = 45^\circ}$.

Quick Tip to Solve: Read the coefficient of the parameter as the line direction; use sine for the angle with a plane.

26. A and B throw a pair of dice alternately till one of them gets the sum of the numbers as multiples of 6 and wins the game. If A starts first, find the probability of B winning the game. [2 marks]

Solution:

Concept:

- A pair of fair dice has 36 equally likely ordered outcomes. A winning sum must be a positive multiple of 6 among sums 2 to 12.
- Independent throws allow the probabilities of successive failures and a success to be multiplied. Repeated unsuccessful rounds lead to a geometric series.

Step 1: Find the success probability for one throw The possible winning sums are 6 and 12. Sum 6 occurs for

$$(1, 5), (2, 4), (3, 3), (4, 2), (5, 1),$$

and sum 12 occurs only for $(6, 6)$. Thus there are 6 winning outcomes out of 36, giving success probability $p = 1/6$ and failure probability $q = 5/6$. Both players have the same probabilities on every independent throw.

Step 2: Sum the events in which B wins Since A starts, B wins on the first round when A fails and B

Quick Tip to Solve: Include the initial failure by A, then repeat whole two-player rounds with geometric ratio q squared.

succeeds, with probability qp. If both fail, the next round begins in the same situation, with probability q^2. Consequently

$$P(B \text{ wins}) = qp + q^3p + q^5p + \cdots$$
$$= \frac{qp}{1-q^2} = \frac{(5/6)(1/6)}{1-25/36} = \frac{5/36}{11/36} = \frac{5}{11}.$$

The series converges because $q^2 < 1$; the probability of infinitely many unsuccessful rounds is zero. The value is below $1/2$, as expected because A has the first opportunity to win.

Step 3: Conclusion The probability of B winning the game is $\boxed{5/11}$.

Quick Tip to Solve: Include the initial failure by A, then repeat whole two-player rounds with geometric ratio q squared.

Section C (24 marks)

Question numbers 27 to 32 carry 4 marks each.

27. Solve the following differential equation: $\left(1+e^{y/x}\right)dy + e^{y/x}\left(1-\frac{y}{x}\right)dx = 0\ (x \neq 0)$**.** [4 marks]

Solution:

Concept:

- A homogeneous first-order equation depends on the ratio y/x. Put $y = vx$, where $v = y/x$, so that $dy = v\,dx + x\,dv$.
- Separation uses $\int f'(v)/f(v)\,dv = \ln|f(v)| + C$ wherever $f(v) \neq 0$. Solutions excluded by division must be checked separately.

Step 1: Use the homogeneous substitution Work on an interval on which $x \neq 0$. Set $v = y/x$, so $y = vx$ and $dy = v\,dx + x\,dv$. Substitution into the printed equation gives

$$(1+e^v)(v\,dx + x\,dv) + e^v(1-v)\,dx = 0.$$

The coefficient of dx simplifies as $v + ve^v + e^v - ve^v = v + e^v$. Therefore,

$$x(1+e^v)\,dv + (v+e^v)\,dx = 0.$$

Step 2: Separate and integrate For $v + e^v \neq 0$, divide by $x(v+e^v)$:

$$\frac{1+e^v}{v+e^v}\,dv = -\frac{dx}{x}.$$

The derivative of the denominator $v + e^v$ is precisely the numerator $1 + e^v$. Hence integration yields

$$\ln|v+e^v| = -\ln|x| + C_1, \qquad \ln|x(v+e^v)| = C_1.$$

Absorbing the constant sign into a nonzero real constant C, this is $x(v+e^v) = C$. Substitute $v = y/x$ to express the relation in the original variables.

Quick Tip to Solve: After setting $y = vx$, look for cancellation of the two ve^v terms; the remaining numerator is the derivative of $v + e^v$.

Step 3: Check the divided-out case and verify the result If $v + e^v = 0$, a constant value $v = r$ gives $y = rx$, and substitution into the original equation leaves $(r + e^r)\,dx = 0$. This straight-line solution is included by allowing $C = 0$ in the same family. There is exactly one such real r, since $v + e^v$ is strictly increasing and changes sign between -1 and 0. To verify the entire family directly, differentiate its left side:

$$\begin{aligned} d\big(y + xe^{y/x}\big) &= dy + e^{y/x}dx + xe^{y/x}d(y/x) \\ &= (1 + e^{y/x})dy + e^{y/x}(1 - y/x)dx. \end{aligned}$$

Thus its differential equals the given expression exactly, including its signs and coefficients.

Step 4: Conclusion The required general solution is $\boxed{y + xe^{y/x} = C}$, where C is an arbitrary real constant and $x \neq 0$.

Quick Tip to Solve: After setting $y = vx$, look for cancellation of the two ve^v terms; the remaining numerator is the derivative of $v + e^v$.

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28. A cottage industry manufactures pedestal lamps and wooden shades. Both the products require machine time as well as craftsman time in the making. The number of hour(s) required for producing 1 unit of each and the corresponding profit is given in the following table:

Pedestal lamp: Machine Time 1.5 hours, Craftsman time 3 hours, Profit ₹ 30.

Wooden shades: Machine Time 3 hours, Craftsman time 1 hour, Profit ₹ 20.

In a day, the factory has availability of not more than 42 hours of machine time and 24 hours of craftsman time.

Assuming that all items manufactured are sold, how should the manufacturer schedule his daily production in order to maximise the profit? Formulate it as an LPP and solve it graphically. [4 marks]

Solution:

Concept:

- Let x and y be the daily numbers of pedestal lamps and wooden shades, respectively. Production quantities satisfy $x \geq 0$ and $y \geq 0$.
- Available machine time and craftsman time give linear inequalities. Since all products are sold, total daily profit is $Z = 30x + 20y$, in rupees.
- For a bounded feasible polygon, a linear objective attains its maximum at a corner point. Compare the profit at every vertex of the common feasible region.

Step 1: Formulate the production constraints Each pedestal lamp requires 1.5 hours of machine time and 3 hours of craftsman time. Each wooden shade requires 3 hours of machine time and 1 hour of craftsman time. Therefore the resources used are $1.5x + 3y$ machine hours and $3x + y$ craftsman hours. Neither may exceed the daily availability:

$$1.5x + 3y \leq 42, \qquad 3x + y \leq 24.$$

Quick Tip to Solve: Simplify the machine constraint to $x + 2y \leq 28$ before plotting; test the intersection as well as both feasible axis vertices.

Dividing the first inequality by 1.5, the required LPP is

$$\text{Maximise } Z = 30x + 20y \quad \text{subject to} \quad \begin{aligned} x + 2y &\le 28, \\ 3x + y &\le 24, \\ x, y &\ge 0. \end{aligned}$$

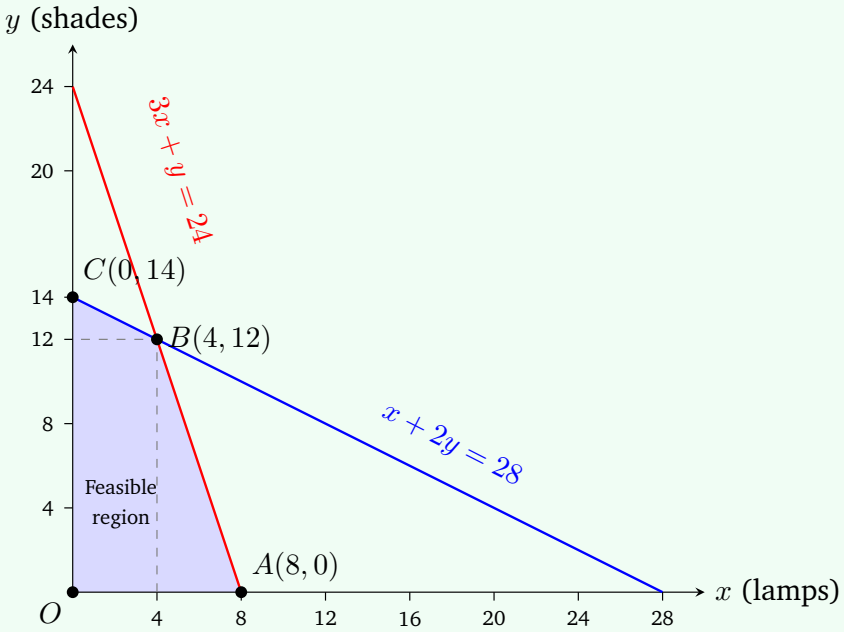
Step 2: Draw the boundary lines and locate the feasible region The line $x + 2y = 28$ has intercepts $(28, 0)$ and $(0, 14)$; the line $3x + y = 24$ has intercepts $(8, 0)$ and $(0, 24)$. Both inequalities include the origin, so their common region lies below both lines in the first quadrant. The graph shows this bounded feasible region $OABC$. To find their intersection B, solve

$$x + 2y = 28, \qquad 3x + y = 24.$$

From the second equation, $y = 24 - 3x$. Substituting into the first gives

$$x + 2(24 - 3x) = 28 \implies -5x = -20 \implies x = 4, \quad y = 12.$$

Thus the vertices are $O(0, 0)$, $A(8, 0)$, $B(4, 12)$ and $C(0, 14)$.



Shaded feasible region for daily production; B is the optimal corner point.

Step 3: Compare the objective at all corner points Evaluate the same profit expression at each vertex:

Corner point (x, y)	$Z = 30x + 20y$ (rupees)
$(0, 0)$	0
$(8, 0)$	240
$(4, 12)$	$120 + 240 = 360$
$(0, 14)$	280

The largest value occurs at $B(4, 12)$. This schedule uses $1.5(4) + 3(12) = 42$ machine hours and

Quick Tip to Solve: Simplify the machine constraint to $x + 2y \le 28$ before plotting; test the intersection as well as both feasible axis vertices.

$3(4) + 12 = 24$ craftsman hours, exactly the available daily times. Both production quantities are whole numbers, so the schedule is directly practicable.

Step 4: Conclusion The optimal daily schedule is $\boxed{x = 4,\ y = 12}$: produce 4 pedestal lamps and 12 wooden shades. The maximum daily profit is $\boxed{₹\,360}$.

Quick Tip to Solve: Simplify the machine constraint to $x + 2y \le 28$ before plotting; test the intersection as well as both feasible axis vertices.

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29. Find: $\displaystyle\int \left(\sqrt{\cot x} + \frac{1}{\sqrt{\cot x}}\right) dx$. [4 marks]

Solution:

Concept:

- Combine the two terms over their common denominator, then use a substitution whose differential contains the numerator.
- The identity $(\sin x - \cos x)^2 = 1 - 2\sin x \cos x$ converts the resulting integral into $\int du/\sqrt{1-u^2} = \sin^{-1} u + C$.

Step 1: Simplify the integrand Let I denote the required integral. First work on $0 < x < \pi/2$, where both sine and cosine are positive, so that the square roots can be combined without a sign change. We have

$$\begin{aligned}\sqrt{\cot x} + \frac{1}{\sqrt{\cot x}} &= \sqrt{\frac{\cos x}{\sin x}} + \sqrt{\frac{\sin x}{\cos x}} \\ &= \frac{\cos x + \sin x}{\sqrt{\sin x \cos x}}.\end{aligned}$$

The numerator is the derivative of the difference of sine and cosine.

Step 2: Make the substitution and integrate Put $u = \sin x - \cos x$. Then $du = (\cos x + \sin x)\,dx$, and

$$u^2 = 1 - 2\sin x \cos x, \qquad \sin x \cos x = \frac{1 - u^2}{2}.$$

Therefore

$$\begin{aligned}I &= \int \frac{du}{\sqrt{(1-u^2)/2}} \\ &= \sqrt{2}\int \frac{du}{\sqrt{1-u^2}} \\ &= \sqrt{2}\sin^{-1} u + C.\end{aligned}$$

Here $-1 < u < 1$, so the real inverse-sine formula applies.

Step 3: Check the result and its domain Differentiating gives

$$\frac{d}{dx}\left[\sqrt{2}\sin^{-1}(\sin x - \cos x)\right] = \frac{\sqrt{2}(\cos x + \sin x)}{\sqrt{2\sin x \cos x}},$$

which equals the simplified integrand. For the other real intervals of definition, $k\pi < x < k\pi + \pi/2$,

Quick Tip to Solve: After combining the radicals, choose the difference $\sin x - \cos x$: its derivative is the numerator, and its square removes the product in the denominator.

use $t = x - k\pi$. Since the integrand has period π, the same computation applies to t; each interval has its own constant.

Step 4: Conclusion

$$\boxed{I = \sqrt{2}\sin^{-1}(\sin x - \cos x) + C}, \qquad 0 < x < \frac{\pi}{2}.$$

On any other real interval of definition, use $\sqrt{2}\sin^{-1}[\sin(x - k\pi) - \cos(x - k\pi)] + C$, where $k\pi < x < k\pi + \pi/2$.

Quick Tip to Solve: After combining the radicals, choose the difference $\sin x - \cos x$: its derivative is the numerator, and its square removes the product in the denominator.

30(a). Check whether the relation R in the set N of natural numbers given by $R = \{(a, b) : a$ is divisor of $b\}$ is reflexive, symmetric or transitive. Also determine whether R is an equivalence relation. [4 marks]

Solution:

Concept:

- Divisibility means $a \mid b$ if $b = ak$ for an integer k. Thus $(a, b) \in R$ precisely when $a \mid b$.
- Reflexivity requires $(a, a) \in R$ for every element; symmetry requires the reverse of every related pair; transitivity requires $(a, c) \in R$ whenever (a, b) and (b, c) belong to R.
- An equivalence relation must satisfy all three properties: reflexivity, symmetry and transitivity.

Step 1: Test reflexivity Take an arbitrary natural number a. Since $a = a \cdot 1$, the number a divides itself. Hence $(a, a) \in R$ for every $a \in N$. This establishes reflexivity universally, rather than merely checking a few examples. Therefore R is reflexive.

Step 2: Test symmetry To be symmetric, R would have to satisfy $b \mid a$ whenever $a \mid b$. Choose $a = 2$ and $b = 4$. Since $4 = 2 \cdot 2$, we have $(2, 4) \in R$. However, 2 is not an integer multiple of 4, so 4 does not divide 2, and $(4, 2) \notin R$. This single counterexample disproves the universal condition for symmetry. Therefore R is not symmetric.

Step 3: Test transitivity Let $a, b, c \in N$ and suppose $(a, b) \in R$ and $(b, c) \in R$. By the definition of divisibility, there are integers m, n such that

$$b = am, \qquad c = bn.$$

Substituting the first equality into the second gives

$$c = (am)n = a(mn).$$

Since the product mn is an integer, $a \mid c$. Thus $(a, c) \in R$ whenever the two given pairs belong to R. As the elements were arbitrary, this proves that R is transitive.

Step 4: Apply the equivalence-relation criterion All three defining properties must hold together for a relation to be an equivalence relation. The reflexivity and transitivity proofs hold for arbitrary elements of N, but the pair $(2, 4)$ violates symmetry. Consequently, the equivalence-relation criterion

Quick Tip to Solve: For the divisor relation, use $a = a \cdot 1$ for reflexivity, $2 \mid 4$ as the symmetry counterexample, and multiply the two integer factors to prove transitivity.

is not met; the two properties that do hold cannot compensate for the missing symmetry.

Step 5: Conclusion

$$\boxed{R \text{ is reflexive and transitive, but not symmetric.}}$$

Therefore $\boxed{R \text{ is not an equivalence relation.}}$

Quick Tip to Solve: For the divisor relation, use $a = a \cdot 1$ for reflexivity, $2 \mid 4$ as the symmetry counterexample, and multiply the two integer factors to prove transitivity.

OR

30(b). Prove that $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$. [4 marks]

Solution:

Concept:

- The principal value of $\tan^{-1} t$ lies in $(-\pi/2, \pi/2)$. For positive t, it is an acute angle.
- The addition formula is $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$, provided the denominator is nonzero.
- The identity $\sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta}$ determines the sine of twice an angle; its interval must also be checked before applying a principal inverse sine.

Step 1: Define the principal angles and determine their sum's interval Put $A = \tan^{-1}(1/4)$, $B = \tan^{-1}(2/9)$, and $\theta = A + B$. Both fractions lie strictly between 0 and 1, so $0 < A < \pi/4$ and $0 < B < \pi/4$. Therefore $0 < \theta < \pi/2$. This interval check prevents an incorrect multiple of π when using the tangent addition formula.

Step 2: Evaluate the tangent of the sum Using the addition formula gives

$$\tan\theta = \frac{\frac{1}{4} + \frac{2}{9}}{1 - \frac{1}{4}\cdot\frac{2}{9}} = \frac{\frac{17}{36}}{\frac{34}{36}} = \frac{1}{2}.$$

Since tangent increases on $(0, \pi/2)$ and $\tan(\pi/4) = 1$, the value $\tan\theta = 1/2$ further implies $0 < \theta < \pi/4$. Consequently $0 < 2\theta < \pi/2$.

Step 3: Apply the double-angle identity and check the inverse-sine branch Substituting the calculated tangent,

$$\sin 2\theta = \frac{2(1/2)}{1 + (1/2)^2} = \frac{1}{5/4} = \frac{4}{5}.$$

The principal inverse sine has values in $[-\pi/2, \pi/2]$. As 2θ lies in $(0, \pi/2)$, it is precisely the principal angle whose sine is $4/5$. Thus $2\theta = \sin^{-1}(4/5)$; the supplementary angle does not belong to the established interval.

Quick Tip to Solve: First reduce the tangent of the sum to $1/2$, then use the double-angle sine formula; check that twice the sum is acute before applying inverse sine.

Step 4: Conclusion Dividing by 2 and substituting the definition of θ, we obtain

$$\boxed{\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)}.$$

Hence the required identity is proved.

Quick Tip to Solve: First reduce the tangent of the sum to $1/2$, then use the double-angle sine formula; check that twice the sum is acute before applying inverse sine.

31. Find the cartesian equation of the plane passing through the intersection of the planes $x+2y-3=0$ and $2x-y+z=1$ and the origin i.e. $(0,0,0)$. Also write the equation of the plane so obtained in vector form. [4 marks]

Solution:

Concept:

- A plane through the intersection of two nonparallel planes $P_1=0$ and $P_2=0$ can be written as $P_1+\lambda P_2=0$, with the required member selected by an additional point.
- In the Cartesian equation $ax+by+cz=d$, the vector $a\hat{i}+b\hat{j}+c\hat{k}$ is normal to the plane. Its vector equation is $\vec{r}\cdot(a\hat{i}+b\hat{j}+c\hat{k})=d$, where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

Step 1: Form the family of planes through the intersection Write the given equations with zero on the right:

$$P_1=x+2y-3=0, \qquad P_2=2x-y+z-1=0.$$

Their normal vectors $(1,2,0)$ and $(2,-1,1)$ are not proportional, so the planes intersect in a line. Every point on that line satisfies both equations and hence satisfies the family

$$x+2y-3+\lambda(2x-y+z-1)=0.$$

We choose the parameter so that this member also contains the origin.

Step 2: Impose the condition that the origin lies in the plane Substitute $x=y=z=0$ into the family equation:

$$-3+\lambda(-1)=0 \quad\Longrightarrow\quad \lambda=-3.$$

Substituting this value and collecting like terms gives

$$\begin{aligned} x+2y-3-3(2x-y+z-1)&=0,\\ x+2y-3-6x+3y-3z+3&=0,\\ -5x+5y-3z&=0. \end{aligned}$$

Multiplying by -1 gives the equivalent Cartesian form $5x-5y+3z=0$. The constant term vanishes, as expected for a plane through the origin.

Step 3: Obtain the vector form and verify the required incidence A normal vector to the resulting plane is $\vec{n}=5\hat{i}-5\hat{j}+3\hat{k}$. For a variable point with position vector $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, its Cartesian

Quick Tip to Solve: For a plane through the origin, set the constant term of the plane family equal to zero first; the remaining coefficients immediately give its normal vector.

equation is equivalent to $\vec{r}\cdot\vec{n}=0$. Moreover the derived equation before changing its sign is exactly $P_1-3P_2=0$; every point of the given intersection therefore lies in it. The origin satisfies it as well, verifying both stated conditions.

Step 4: Conclusion The required plane has Cartesian equation

$$\boxed{5x-5y+3z=0},$$

and vector equation

$$\boxed{\vec{r}\cdot(5\hat{i}-5\hat{j}+3\hat{k})=0}.$$

Quick Tip to Solve: For a plane through the origin, set the constant term of the plane family equal to zero first; the remaining coefficients immediately give its normal vector.

32(a). **If** $\tan^{-1}\left(\frac{y}{x}\right)=\log\sqrt{x^2+y^2}$, **prove that** $\frac{dy}{dx}=\frac{x+y}{x-y}$. [4 marks]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x. Write $p=\frac{dy}{dx}$; the chain rule must therefore be used whenever a term contains y.
- The formulae needed are $\frac{d}{dx}\tan^{-1}u=\frac{u'}{1+u^2}$ and $\frac{d}{dx}\log v=\frac{v'}{v}$, where $v>0$. Also, $\log\sqrt{x^2+y^2}=\frac{1}{2}\log(x^2+y^2)$.

Step 1: Differentiate the inverse tangent The given equation is defined only when $x\neq 0$. Applying the quotient rule to $u=y/x$, we get

$$u'=\frac{xp-y}{x^2}.$$

Hence, by the chain rule,

$$\frac{d}{dx}\tan^{-1}\left(\frac{y}{x}\right)=\frac{1}{1+y^2/x^2}\,\frac{xp-y}{x^2}=\frac{xp-y}{x^2+y^2}.$$

Keeping the factor p here is essential because y varies with x.

Step 2: Differentiate the logarithm and equate derivatives Since $x\neq 0$, we have $x^2+y^2>0$, so the logarithm is defined. Its derivative is

$$\begin{aligned}\frac{d}{dx}\log\sqrt{x^2+y^2}&=\frac{1}{2}\,\frac{2x+2yp}{x^2+y^2}\\&=\frac{x+yp}{x^2+y^2}.\end{aligned}$$

Quick Tip to Solve: Rewrite the logarithm as one-half of a logarithm before differentiating; both sides then have the same denominator, which cancels immediately.

Differentiating both sides of the given equation therefore yields

$$\frac{xp - y}{x^2 + y^2} = \frac{x + yp}{x^2 + y^2}.$$

Multiplication by the nonzero common denominator gives $xp - y = x + yp$.

Step 3: Collect the derivative terms Move every term containing p to one side:

$$xp - yp = x + y, \qquad (x - y)p = x + y.$$

At any point where y is differentiable as a function of x, we can divide by $x - y$. Indeed, if $x = y$, the preceding differentiated identity would give $0 = 2x$, contradicting $x \neq 0$; such a point cannot have a finite derivative in this representation.

Step 4: Conclusion Thus, wherever the given relation defines a differentiable function $y(x)$,

$$\boxed{\frac{dy}{dx} = \frac{x + y}{x - y}}.$$

Quick Tip to Solve: Rewrite the logarithm as one-half of a logarithm before differentiating; both sides then have the same denominator, which cancels immediately.

OR

32(b). If $y = e^{a\cos^{-1}x}$, $-1 < x < 1$, then show that $(1 - x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - a^2y = 0$. [4 marks]

Solution:

Concept:

- The chain rule gives $\dfrac{d}{dx}e^u = e^u u'$. For $-1 < x < 1$, $\dfrac{d}{dx}\cos^{-1}x = -\dfrac{1}{\sqrt{1 - x^2}}$, and $1 - x^2$ is strictly positive.
- To obtain a second derivative, differentiate the first-derivative identity again using the product rule. Multiplying the first identity by $\sqrt{1 - x^2}$ first avoids unnecessary quotient-rule algebra.

Step 1: Find the first derivative Starting with $y = e^{a\cos^{-1}x}$, where a is a constant, differentiate by the chain rule:

$$\frac{dy}{dx} = e^{a\cos^{-1}x}\, a\left(-\frac{1}{\sqrt{1 - x^2}}\right)$$
$$= -\frac{ay}{\sqrt{1 - x^2}}.$$

Write $y' = dy/dx$. Multiplying by the positive square root gives the useful identity

$$\sqrt{1 - x^2}\, y' = -ay.$$

These denominators are nonzero on the stated interval.

Quick Tip to Solve: Differentiate the identity $\sqrt{1 - x^2}\, y' = -ay$ instead of expanding the quotient for y'; the product rule gives the target coefficients directly.

Step 2: Differentiate the first-derivative identity Applying the product rule to the left side and differentiating the right side, we obtain

$$\frac{-x}{\sqrt{1-x^2}}\,y' + \sqrt{1-x^2}\,y'' = -ay',$$

because $\dfrac{d}{dx}\sqrt{1-x^2} = -x/\sqrt{1-x^2}$. Multiply the entire equation by $\sqrt{1-x^2}$:

$$-xy' + (1-x^2)y'' = -a\sqrt{1-x^2}\,y'.$$

This clears the denominator and gives the required coefficient of y''.

Step 3: Substitute the first derivative back Use $\sqrt{1-x^2}\,y' = -ay$ from the first step on the right side:

$$\begin{aligned}(1-x^2)y'' - xy' &= -a\big(\sqrt{1-x^2}\,y'\big)\\ &= -a(-ay) = a^2y.\end{aligned}$$

Move a^2y to the left. The calculation also covers $a = 0$, because no division by a was used.

Step 4: Conclusion Therefore, for $-1 < x < 1$,

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - a^2y = 0}.$$

Quick Tip to Solve: Differentiate the identity $\sqrt{1-x^2}\,y' = -ay$ instead of expanding the quotient for y'; the product rule gives the target coefficients directly.

Section D (24 marks)

Question numbers 33 to 36 carry 6 marks each.

33. Amongst all open (from the top) right circular cylindrical boxes of volume 125π cm^3, find the dimensions of the box which has the least surface area. [6 marks]

Solution:

Concept:

- For an open cylindrical box of radius $r > 0$ and height $h > 0$, the volume is $V = \pi r^2 h$. The surface area consists of one circular base and the curved surface, so $S = \pi r^2 + 2\pi rh$.
- Use the fixed volume to express the height in terms of the radius. A stationary value of the resulting area function is a minimum when its second derivative is positive.

Step 1: Express the height using the fixed volume Let the radius and height of the box be r cm and h cm respectively. The prescribed volume gives

$$\pi r^2 h = 125\pi, \qquad h = \frac{125}{r^2}.$$

Quick Tip to Solve: For an open cylinder include only one base; eliminating height before differentiating immediately reduces the optimization to one variable.

Both dimensions must be positive, so the admissible domain for the radius is $r > 0$. There is no upper circular face because the box is open at the top.

Step 2: Form the area function and find its stationary point Substituting the height into the sum of the base area and curved surface area gives

$$S(r) = \pi r^2 + 2\pi r \frac{125}{r^2} = \pi r^2 + \frac{250\pi}{r}.$$

Differentiating with respect to r,

$$S'(r) = 2\pi r - \frac{250\pi}{r^2} = \frac{2\pi}{r^2}(r^3 - 125).$$

The stationary-point equation is therefore $r^3 = 125$, giving the unique positive candidate $r = 5$ cm.

Step 3: Verify the least area and determine the height The second derivative is

$$S''(r) = 2\pi + \frac{500\pi}{r^3} > 0 \qquad (r > 0).$$

Thus the area function is strictly convex throughout its admissible domain. Equivalently, $S'(r) < 0$ for $0 < r < 5$ and $S'(r) > 0$ for $r > 5$: the area decreases up to the candidate and increases afterwards. Also, $S(r)$ tends to infinity both as $r \to 0^+$ and as $r \to \infty$, confirming that this is the global minimum. At this radius,

$$h = \frac{125}{5^2} = 5\,\text{cm}, \qquad S = \pi(5)^2 + 2\pi(5)(5) = 75\pi\,\text{cm}^2.$$

The volume check is $\pi(5)^2(5) = 125\pi\,\text{cm}^3$, as required.

Step 4: Conclusion The box of least surface area has radius $\boxed{5\,\text{cm}}$ and height $\boxed{5\,\text{cm}}$ (diameter 10 cm). Its least surface area is $75\pi\,\text{cm}^2$.

Quick Tip to Solve: For an open cylinder include only one base; eliminating height before differentiating immediately reduces the optimization to one variable.

34(a). Using integration, find the area lying above x-axis and included between the circle $x^2 + y^2 = 8x$ and inside the parabola $y^2 = 4x$. [6 marks]

Solution:

Concept:

- For horizontal strips, the area between a right boundary $x_R(y)$ and a left boundary $x_L(y)$ is $\int [x_R(y) - x_L(y)]\,dy$. The limits come from the intersections and the restriction to the upper half-plane.
- The circle is $(x-4)^2 + y^2 = 16$, with centre $(4, 0)$ and radius 4. The region inside the right-opening parabola $y^2 = 4x$ satisfies $x \geq y^2/4$.

Step 1: Find the intersections in the upper half-plane Substitute $y^2 = 4x$ into the circle equation:

$$x^2 + 4x = 8x, \qquad x(x-4) = 0.$$

Quick Tip to Solve: Use horizontal strips here: the parabola is already expressed as x in terms of y, and the upper common region requires just one integral.

Thus the intersection points are $(0,0)$, $(4,4)$ and $(4,-4)$. Since only the area above the x-axis is requested, use $0 \le y \le 4$. The negative intersection belongs to the excluded lower half-plane.

Step 2: Identify the horizontal boundaries of the required region Solving the circle equation for x gives

$$x = 4 \pm \sqrt{16 - y^2}.$$

On $0 \le y \le 4$, the left circle branch lies to the left of the parabola, whereas the right circle branch lies to its right. Indeed, putting $x = y^2/4$ into $x^2 + y^2 - 8x$ gives $y^2(y^2 - 16)/16 \le 0$, so the parabola boundary is inside the circle on this interval. The horizontal strip in the common region therefore extends from $x_L = y^2/4$ to $x_R = 4 + \sqrt{16 - y^2}$. Consequently,

$$A = \int_0^4 \left(4 + \sqrt{16 - y^2} - \frac{y^2}{4}\right) dy.$$

Step 3: Evaluate the definite integral Using the standard antiderivative for $\sqrt{a^2 - y^2}$,

$$A = \left[4y + \frac{y}{2}\sqrt{16 - y^2} + 8\sin^{-1}\left(\frac{y}{4}\right) - \frac{y^3}{12}\right]_0^4.$$

At the upper limit, $\sqrt{16 - y^2} = 0$ and $\sin^{-1}(1) = \pi/2$; all terms at the lower limit vanish. Hence the definite integral evaluates to

$$16 + 0 + 4\pi - \frac{16}{3} = \frac{32}{3} + 4\pi.$$

This calculation includes only the upper common region, so there is no factor of two for reflection across the x-axis.

Step 4: Conclusion The required area is $\boxed{\frac{32}{3} + 4\pi}$ square units.

Quick Tip to Solve: Use horizontal strips here: the parabola is already expressed as x in terms of y, and the upper common region requires just one integral.

OR

34(b). Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are $A(2,0)$, $B(4,5)$ and $C(6,3)$. [6 marks]

Solution:

Concept:

- The area between an upper boundary $y = u(x)$ and a lower boundary $y = l(x)$, for $a \le x \le b$, is $\int_a^b [u(x) - l(x)]\,dx$.
- A line through (x_1, y_1) and (x_2, y_2) has slope $m = (y_2 - y_1)/(x_2 - x_1)$ and equation $y - y_1 = m(x - x_1)$.
- Split an area integral at a vertex where the upper or lower boundary changes. Each integrand is a nonnegative vertical height inside the triangle.

Quick Tip to Solve: Split at the middle vertex's x-coordinate, here $x = 4$; side AC stays the lower boundary throughout.

Step 1: Find the equations of the three sides The vertices are $A(2,0)$, $B(4,5)$ and $C(6,3)$. Thus the side AB has slope $(5-0)/(4-2) = 5/2$, giving

$$AB: \quad y = \frac{5}{2}(x-2).$$

Similarly, the slope of AC is $(3-0)/(6-2) = 3/4$, so

$$AC: \quad y = \frac{3}{4}(x-2).$$

Finally, the slope of BC is $(3-5)/(6-4) = -1$, giving $y - 5 = -(x-4)$, or

$$BC: \quad y = 9 - x.$$

Substituting the relevant endpoint coordinates in each equation confirms the three boundaries.

Step 2: Identify the vertical strips and split the integral The triangle extends from $x = 2$ to $x = 6$. On $[2,4]$, a vertical strip starts on AC and ends on AB. On $[4,6]$, it starts on AC and ends on BC. Hence, if S denotes the area,

$$S = \int_2^4 \left[\frac{5}{2}(x-2) - \frac{3}{4}(x-2)\right] dx + \int_4^6 \left[9 - x - \frac{3}{4}(x-2)\right] dx.$$

The heights simplify to $\frac{7}{4}(x-2)$ and $\frac{7}{4}(6-x)$, respectively. Both are nonnegative on their respective intervals, so the integrals give areas directly.

Step 3: Evaluate the two area contributions For the first interval,

$$S_1 = \frac{7}{4}\int_2^4 (x-2)\,dx = \frac{7}{8}\left[(x-2)^2\right]_2^4 = \frac{7}{8}(4) = \frac{7}{2}.$$

For the second interval,

$$\begin{aligned} S_2 &= \frac{7}{4}\int_4^6 (6-x)\,dx \\ &= \frac{7}{4}\left[6x - \frac{x^2}{2}\right]_4^6 \\ &= \frac{7}{4}\left[(36-18) - (24-8)\right] = \frac{7}{2}. \end{aligned}$$

Each contribution is positive, and their equal values also follow from their equal base intervals and equal maximum strip height.

Step 4: Conclusion Therefore, by integration, the area of triangle ABC is $S = S_1 + S_2 = \frac{7}{2} + \frac{7}{2} =$ $\boxed{7 \text{ square units}}$.

Quick Tip to Solve: Split at the middle vertex's x-coordinate, here $x = 4$; side AC stays the lower boundary throughout.

35(a). **If** $A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}$, **find** A^{-1} **and use it to solve the following system of equations:**

$5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$ [6 marks]

Solution:

Concept:

- A square matrix A is invertible when its determinant $|A|$ is nonzero, and $A^{-1} = \operatorname{adj} A/|A|$.
- The cofactor C_{ij} equals $(-1)^{i+j}$ times the determinant obtained by deleting row i and column j. The adjugate is the transpose of the cofactor matrix.
- Write a linear system as $AX = B$, where X contains the unknowns and B the constants. For invertible A, its unique solution is $X = A^{-1}B$.

Step 1: Calculate the determinant For the given matrix, expand along its first row:

$$\begin{aligned} |A| &= 5\begin{vmatrix} 3 & 5 \\ -2 & 6 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 5 \\ 5 & 6 \end{vmatrix} + 4\begin{vmatrix} 2 & 3 \\ 5 & -2 \end{vmatrix} \\ &= 5(18+10) + (12-25) + 4(-4-15) \\ &= 140 - 13 - 76 = 51. \end{aligned}$$

Since $|A| \neq 0$, the inverse exists and the given system has exactly one solution.

Step 2: Obtain the cofactors and adjugate Calculate each signed minor:

$$\begin{aligned} &C_{11} = 18 + 10 = 28, && C_{12} = -(12-25) = 13, && C_{13} = -4 - 15 = -19, \\ &C_{21} = -(-6+8) = -2, && C_{22} = 30 - 20 = 10, && C_{23} = -(-10+5) = 5, \\ &C_{31} = -5 - 12 = -17, && C_{32} = -(25-8) = -17, && C_{33} = 15 + 2 = 17. \end{aligned}$$

Thus the cofactor matrix and its transpose are

$$C = \begin{bmatrix} 28 & 13 & -19 \\ -2 & 10 & 5 \\ -17 & -17 & 17 \end{bmatrix}, \qquad \operatorname{adj} A = \begin{bmatrix} 28 & -2 & -17 \\ 13 & 10 & -17 \\ -19 & 5 & 17 \end{bmatrix}.$$

In particular, transposing exchanges the off-diagonal cofactors; the inverse uses the adjugate, not the untransposed cofactor matrix.

Step 3: Form the inverse and solve the matrix equation The required inverse is

$$A^{-1} = \frac{1}{51}\begin{bmatrix} 28 & -2 & -17 \\ 13 & 10 & -17 \\ -19 & 5 & 17 \end{bmatrix}.$$

Quick Tip to Solve: Apply the alternating cofactor signs before transposing, then check the obtained values in all three original equations.

The equations have precisely the coefficient matrix A, so set

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \qquad B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}, \qquad AX = B.$$

Multiplying on the left by A^{-1} gives

$$\begin{aligned} X &= \frac{1}{51}\begin{bmatrix} 28 & -2 & -17 \\ 13 & 10 & -17 \\ -19 & 5 & 17 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix} \\ &= \frac{1}{51}\begin{bmatrix} 140-4+17 \\ 65+20+17 \\ -95+10-17 \end{bmatrix} = \frac{1}{51}\begin{bmatrix} 153 \\ 102 \\ -102 \end{bmatrix}. \end{aligned}$$

Step 4: Verify the solution in every original equation Reading the components gives $x = 3$, $y = 2$, $z = -2$. Direct substitution yields

$$\begin{aligned} 5(3) - 2 + 4(-2) &= 15 - 2 - 8 = 5, \\ 2(3) + 3(2) + 5(-2) &= 6 + 6 - 10 = 2, \\ 5(3) - 2(2) + 6(-2) &= 15 - 4 - 12 = -1. \end{aligned}$$

All three constants agree with the printed system, confirming both the signs and the resulting solution.

Step 5: Conclusion $\boxed{A^{-1} = \frac{1}{51}\begin{bmatrix} 28 & -2 & -17 \\ 13 & 10 & -17 \\ -19 & 5 & 17 \end{bmatrix}}$, and the unique solution is $\boxed{x = 3,\ y = 2,\ z = -2}$.

Quick Tip to Solve: Apply the alternating cofactor signs before transposing, then check the obtained values in all three original equations.

OR

35(b). If x, y, z are different and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$, then using properties of determinants show that $1 + xyz = 0$. [6 marks]

Solution:

Concept:

- Adding a multiple of one column to another leaves a determinant unchanged. A common factor in a column or row can be taken outside the determinant.
- Since x, y, z are pairwise different, the product $(y - x)(z - x)(z - y)$ is nonzero. This allows cancellation after factorising the given determinant.

Quick Tip to Solve: For columns involving powers up to three, use the polynomial with roots x, y and z to turn the cubic column into a constant column.

Step 1: Construct a column operation from the three numbers Let the given determinant be D, and put $s_1 = x + y + z$ and $s_2 = xy + yz + zx$. The identity

$$(t - x)(t - y)(t - z) = t^3 - s_1 t^2 + s_2 t - xyz$$

gives $t^3 - s_1 t^2 + s_2 t = xyz$ whenever t is any one of x, y, z. Therefore the column operation $C_3 \to C_3 - s_1 C_2 + s_2 C_1$ makes all three entries of the third column equal to $1 + xyz$. Its coefficients are fixed scalars, so this operation does not change D.

Step 2: Factor the third column and simplify by row operations Taking the common factor from the third column yields

$$D = (1 + xyz) \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}.$$

Now perform $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$. Use $y^2 - x^2 = (y - x)(y + x)$ and $z^2 - x^2 = (z - x)(z + x)$, then factor $y - x$ and $z - x$ from the second and third rows:

$$D = (1 + xyz)(y - x)(z - x) \begin{vmatrix} x & x^2 & 1 \\ 1 & x + y & 0 \\ 1 & x + z & 0 \end{vmatrix}.$$

This is valid even if one of x, y, z is zero: no division by any of these numbers has been used.

Step 3: Expand the remaining determinant and use distinctness Expanding along the third column, the only nonzero entry has positive cofactor sign. Hence

$$\begin{vmatrix} x & x^2 & 1 \\ 1 & x + y & 0 \\ 1 & x + z & 0 \end{vmatrix} = \begin{vmatrix} 1 & x + y \\ 1 & x + z \end{vmatrix} = (x + z) - (x + y) = z - y.$$

The given equality $D = 0$ therefore becomes

$$(1 + xyz)(y - x)(z - x)(z - y) = 0.$$

Each difference is nonzero because the three numbers are different. Thus the zero product forces the remaining factor $1 + xyz$ to vanish.

Step 4: Conclusion Therefore, using properties of determinants, $\boxed{1 + xyz = 0}$, as required.

Quick Tip to Solve: For columns involving powers up to three, use the polynomial with roots x, y and z to turn the cubic column into a constant column.

36. A discrete random variable X has the following probability distribution:

X	0	1	2	3	4	5
$P(X)$	$4C^2$	$3C^2$	$2C^2$	C^2	C	$2C$

(a) Find the value of C.

(b) Find the mean of the distribution.

(c) Given $\sum x_i^2 p_i = 14$, find the variance of the distribution. [6 marks]

Solution:

Concept:

- In a probability distribution, every probability is nonnegative and their sum is one. Both conditions must be checked when solving for an unknown parameter.
- The mean is $\mu = E(X) = \sum x_i p_i$, where x_i is a possible value of the random variable and p_i is its probability.
- The variance is $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$. It measures the expected squared deviation from the mean.

Step 1: (a) Apply the total-probability condition Adding the six probabilities in the table gives

$$4C^2 + 3C^2 + 2C^2 + C^2 + C + 2C = 1,$$

so $10C^2 + 3C - 1 = 0$. Factorising,

$$(5C - 1)(2C + 1) = 0,$$

and therefore the algebraic candidates are $C = \frac{1}{5}$ and $C = -\frac{1}{2}$. The negative candidate would make $P(X = 4) = C$ and $P(X = 5) = 2C$ negative, which is impossible for probabilities. For the admissible candidate, the probabilities in order are

$$\frac{4}{25}, \quad \frac{3}{25}, \quad \frac{2}{25}, \quad \frac{1}{25}, \quad \frac{5}{25}, \quad \frac{10}{25}.$$

Each is nonnegative and their numerators sum to 25, verifying the distribution.

Step 2: (b) Calculate the weighted mean Multiply each possible value by its corresponding probability, including the zero-valued outcome:

$$\begin{aligned} E(X) &= 0(4C^2) + 1(3C^2) + 2(2C^2) + 3(C^2) + 4C + 5(2C) \\ &= 10C^2 + 14C. \end{aligned}$$

Substituting the admissible value of C,

$$E(X) = \frac{10}{25} + \frac{14}{5} = \frac{80}{25} = \frac{16}{5}.$$

This weighted mean lies between the smallest and largest possible values, 0 and 5, as expected. The outcomes 4 and 5 carry a substantial share of the probability, explaining why the mean exceeds the midpoint of the range.

Quick Tip to Solve: Reject any parameter root that makes a probability negative, and subtract the square of the mean from the second moment to find variance.

Step 3: (c) Use the given second moment to obtain the variance The supplied quantity $\sum x_i^2 p_i = 14$ is $E(X^2)$, not the variance itself. Subtract the square of the mean:

$$\text{Var}(X) = 14 - \left(\frac{16}{5}\right)^2 = \frac{350 - 256}{25} = \frac{94}{25}.$$

As a check, the table also gives

$$E(X^2) = (3 + 8 + 9)C^2 + (16 + 50)C = 20\left(\frac{1}{25}\right) + 66\left(\frac{1}{5}\right) = 14,$$

consistent with the value provided. The resulting variance is positive, as required.

Step 4: Conclusion $\boxed{\text{(a) } C = \frac{1}{5}, \qquad \text{(b) mean} = \frac{16}{5}, \qquad \text{(c) variance} = \frac{94}{25}}$.

Quick Tip to Solve: Reject any parameter root that makes a probability negative, and subtract the square of the mean from the second moment to find variance.