

CBSE Class 12 Mathematics Question Paper 2020 with Solution PDF-Set 65-3-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper comprises **four** Sections A, B, C and D. This question paper carries **36** questions. **All** questions are compulsory.
- (ii) **Section A** – Questions no. **1 to 20** comprises of **20** questions of **1** mark each.
- (iii) **Section B** – Questions no. **21 to 26** comprises of **6** questions of **2** marks each.
- (iv) **Section C** – Questions no. **27 to 32** comprises of **6** questions of **4** marks each.
- (v) **Section D** – Questions no. **33 to 36** comprises of **4** questions of **6** marks each.
- (vi) There is no overall choice in the question paper. However, an internal choice has been provided in 3 questions of one mark, 2 questions of two marks, 2 questions of four marks and 2 questions of six marks. Only one of the choices in such questions have to be attempted.
- (vii) In addition to this, separate instructions are given with each section and question, wherever necessary.
- (viii) Use of calculators is **not** permitted.

Section A (20 marks)

Question numbers 1 to 20 carry 1 mark each.

Question numbers 1 to 10 are multiple choice type questions. Select the correct option.

1. The value of p for which $p(\hat{i} + \hat{j} + \hat{k})$ is a unit vector is

- (A) 0
- (B) $\frac{1}{\sqrt{3}}$
- (C) 1
- (D) $\sqrt{3}$

[1 mark]

Solution:

Concept:

- A unit vector has magnitude 1. For mutually perpendicular unit vectors, $|a\hat{i} + b\hat{j} + c\hat{k}| = \sqrt{a^2 + b^2 + c^2}$.
- The magnitude contains $|p|$, so both signs must be considered before comparing the printed choices.

Step 1: Compute the magnitude For $\vec{v} = p(\hat{i} + \hat{j} + \hat{k})$, its three components are all p . Hence

$$|\vec{v}| = \sqrt{p^2 + p^2 + p^2} = \sqrt{3} |p|.$$

The unit-vector condition gives $3p^2 = 1$, or $p = \pm 1/\sqrt{3}$.

Step 2: Check the choices Option (A) gives magnitude 0; option (B) gives magnitude 1; option (C) gives $\sqrt{3}$; and option (D) gives 3. Only (B) is a listed value that makes the vector a unit vector. The negative value also works mathematically, but is absent from the choices.

Step 3: Conclusion Among the printed choices, $p = 1/\sqrt{3}$, option (B).

Quick Tip to Solve: Square the magnitude first: three equal components give $3p^2 = 1$; then check the available signs.

2. $\left(\tan^{-1} \frac{7}{9} + \tan^{-1} \frac{1}{8}\right)$ is equal to

(A) $\tan^{-1} \left(\frac{65}{72}\right)$

(B) $\tan^{-1} \left(\frac{63}{65}\right)$

(C) $\frac{\pi}{4}$

(D) $\frac{\pi}{2}$

[1 mark]

Solution:

Concept:

- For angles α, β , $\tan(\alpha + \beta) = (\tan \alpha + \tan \beta) / (1 - \tan \alpha \tan \beta)$.
- The principal inverse tangent lies in $(-\pi/2, \pi/2)$; a tangent value must be combined with a quadrant check.

Step 1: Use the addition formula Put $\alpha = \tan^{-1}(7/9)$ and $\beta = \tan^{-1}(1/8)$. Then

$$\tan(\alpha + \beta) = \frac{7/9 + 1/8}{1 - (7/9)(1/8)} = \frac{65/72}{65/72} = 1.$$

Step 2: Check the angle and options Both angles are positive and less than $\pi/4$, since their arguments lie between 0 and 1. Thus $0 < \alpha + \beta < \pi/2$, where tangent is one-to-one. The angle with tangent 1 is $\pi/4$. Options (A) and (B) have tangents $65/72$ and $63/65$, both different from 1; tangent is undefined at the angle in (D).

Step 3: Conclusion $\tan^{-1}(7/9) + \tan^{-1}(1/8) = \pi/4$, option (C).

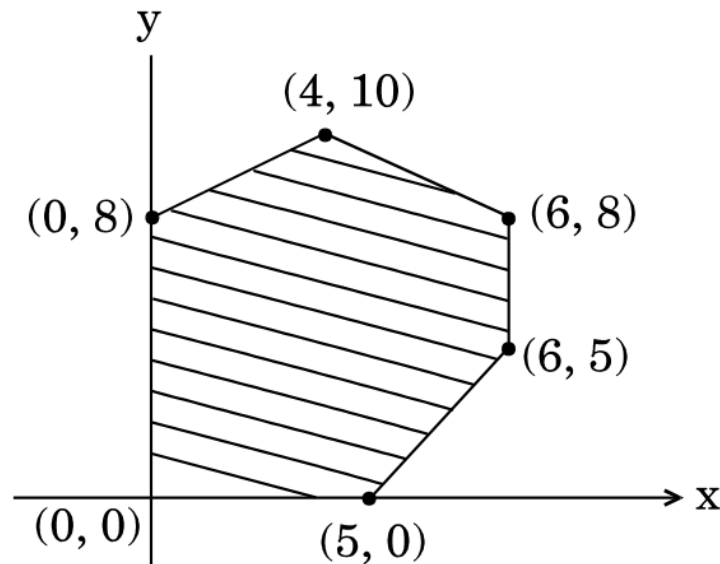
Quick Tip to Solve: If the numerator and denominator of the tangent addition formula agree, check the quadrant to obtain $\pi/4$.

3. The feasible region for an LPP is shown below:

Let $z = 3x - 4y$ be the objective function. Minimum of z occurs at

- (A) $(0, 0)$
- (B) $(0, 8)$
- (C) $(5, 0)$
- (D) $(4, 10)$

[1 mark]



Solution:

Concept:

- A linear objective on a bounded polygonal feasible region attains its minimum at a vertex.
- Use the coordinates shown in the figure and evaluate $z = 3x - 4y$ at every vertex, including vertices absent from the options.

Step 1: Evaluate the objective at the vertices The figure gives the vertices $(0, 0)$, $(5, 0)$, $(6, 5)$, $(6, 8)$, $(4, 10)$, and $(0, 8)$. Their objective values are

$$\begin{aligned} z(0, 0) &= 0, & z(5, 0) &= 15, \\ z(6, 5) &= -2, & z(6, 8) &= -14, \\ z(4, 10) &= -28, & z(0, 8) &= -32. \end{aligned}$$

Step 2: Compare all candidates The least of these six values is -32 . Among the printed options, (A) gives 0, (B) gives -32 , (C) gives 15, and (D) gives -28 . The higher point $(4, 10)$ does not minimize this objective: its positive $3x$ contribution offsets part of the reduction from $-4y$.

Step 3: Conclusion The minimum is $z_{\min} = -32$ at $(0, 8)$, option (B).

Quick Tip to Solve: Compare $3x - 4y$ at every vertex; the highest vertex need not give the smallest objective.

4. If f and g are two functions from $\mathbb{R}$ to $\mathbb{R}$ defined as $f(x) = |x| + x$ and $g(x) = |x| - x$, then $f \circ g(x)$ for $x < 0$ is

- (A) $4x$
- (B) $2x$
- (C) 0
- (D) $-4x$

[1 mark]

Solution:

Concept:

- Composition means $(f \circ g)(x) = f(g(x))$: the output of g is the input to f .
- For $x < 0$, $|x| = -x$; the sign of the intermediate output must be checked separately.

Step 1: Find the inner function Since $x < 0$,

$$g(x) = |x| - x = -x - x = -2x.$$

Because x is negative, this output $-2x$ is positive.

Step 2: Apply the outer function Let $u = g(x) = -2x > 0$. For positive u , $f(u) = |u| + u = 2u$. Therefore

$$f(g(x)) = |-2x| + (-2x) = -2x - 2x = -4x.$$

The values $4x$ and $2x$ in (A) and (B) are negative for this domain, whereas $f(u)$ here is positive. Option (C) would result from applying f directly to a negative input, rather than to $g(x)$.

Step 3: Conclusion For $x < 0$, $(f \circ g)(x) = -4x$, option (D).

Quick Tip to Solve: Check the sign after the inner function: $x < 0$ makes $g(x) = -2x > 0$.

5. $\int e^x \left(\frac{x \log x + 1}{x} \right) dx$ is equal to

- (A) $\log(e^x \log x) + c$
- (B) $\frac{e^x}{x} + c$
- (C) $x \log x + e^x + c$
- (D) $e^x \log x + c$

[1 mark]

Solution:

Concept:

- The product rule gives $\frac{d}{dx}[e^x h(x)] = e^x [h(x) + h'(x)]$.
- Here $h(x) = \log x$, with $h'(x) = 1/x$ on $x > 0$.

Step 1: Recognize the product derivative Simplify the factor in the integrand:

$$\frac{x \log x + 1}{x} = \log x + \frac{1}{x}.$$

Quick Tip to Solve: Split the fraction before integrating and look for the pattern $e^x(h + h')$.

Consequently

$$\frac{d}{dx}(e^x \log x) = e^x \log x + \frac{e^x}{x},$$

exactly the given integrand.

Step 2: Check the alternatives by differentiation Option (A), where defined, differentiates to $1 + 1/(x \log x)$, which is different. Option (B) differentiates to $e^x/x - e^x/x^2$. Option (C) differentiates to $\log x + 1 + e^x$. Only (D) differentiates to $e^x(\log x + 1/x)$ throughout the domain $x > 0$.

Step 3: Conclusion $\int e^x \frac{x \log x + 1}{x} dx = e^x \log x + c$, option (D).

Quick Tip to Solve: Split the fraction before integrating and look for the pattern $e^x(h + h')$.

6. The integrating factor of the differential equation $(x + 3y^2) \frac{dy}{dx} = y$ is

- (A) y
- (B) $-y$
- (C) $\frac{1}{y}$
- (D) $-\frac{1}{y}$

[1 mark]

Solution:

Concept:

- For a linear equation $dx/dy + P(y)x = Q(y)$, the integrating factor is $e^{\int P(y) dy}$.
- The equation is linear in x as a function of y , rather than in y as a function of x .

Step 1: Rewrite with y as the independent variable From $(x + 3y^2) dy/dx = y$, the differential form is $(x + 3y^2) dy = y dx$. On an interval with $y \neq 0$, this becomes

$$\frac{dx}{dy} - \frac{x}{y} = 3y.$$

Hence $P(y) = -1/y$.

Step 2: Compute and identify the standard factor The conventional factor is

$$\text{I.F.} = e^{\int -1/y dy} = e^{-\log |y|} = \frac{1}{|y|}.$$

On either interval of fixed sign, a nonzero constant multiple gives $1/y$, the standard printed answer. Multiplication gives $d(x/y)/dy = 3$. The factors y and $-y$ do not produce this product derivative. The factor $-1/y$ is also a valid nonzero constant multiple; the MCQ convention selects the standard positive-form expression $1/y$.

Step 3: Conclusion The standard integrating factor is $\boxed{1/y}$, option (C). Option (D) is its constant multiple and is mathematically valid as well.

Quick Tip to Solve: When the equation is not linear in y , interchange the dependent variable and identify the coefficient of x .

7. If $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, then the value of $|\text{adj } A|$ is

- (A) 64
(B) 16
(C) 0
(D) -8

[1 mark]

Solution:

Concept:

- For an invertible $n \times n$ matrix, $\text{adj } A = |A|A^{-1}$ and $|\text{adj } A| = |A|^{n-1}$.
- The determinant of a diagonal matrix is the product of its diagonal entries.

Step 1: Find the determinant of A The given matrix is $A = -2I_3$, so

$$|A| = (-2)(-2)(-2) = -8.$$

Since this is nonzero, the adjoint identity applies.

Step 2: Find and check the adjoint determinant For this matrix of order 3,

$$|\text{adj } A| = |A|^2 = (-8)^2 = 64.$$

Directly, $A^{-1} = -\frac{1}{2}I_3$, so $\text{adj } A = (-8)(-\frac{1}{2}I_3) = 4I_3$, whose determinant is $4^3 = 64$. Option (B) omits one diagonal factor, (C) would require singularity, and (D) is the determinant of A itself.

Step 3: Conclusion $|\text{adj } A| = 64$, option (A).

Quick Tip to Solve: For a matrix of order three, square its determinant to obtain the determinant of its adjoint.

8. The distance of the point $(2, 3, 4)$ from the plane $\vec{r} \cdot (3\hat{i} - 6\hat{j} + 2\hat{k}) = -11$ is

- (A) 0 unit
(B) 1 unit
(C) 2 units
(D) $\frac{15}{7}$ units

[1 mark]

Solution:

Concept:

- The perpendicular distance from (x_0, y_0, z_0) to $ax + by + cz + d = 0$ is $D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$.
- The normal vector gives the plane coefficients; a point lies on the plane precisely when the numerator is zero.

Step 1: Write the Cartesian plane For $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, the given equation becomes $3x - 6y + 2z = -11$, or $3x - 6y + 2z + 11 = 0$. Thus the normal vector is $(3, -6, 2)$, whose magnitude is $\sqrt{9 + 36 + 4} = 7$.

Quick Tip to Solve: First move the constant to the left; the numerator here is 7 and the normal length is also 7.

Step 2: Substitute the point and compare the options At $(2, 3, 4)$, the expression on the left is $3(2) - 6(3) + 2(4) + 11 = 6 - 18 + 8 + 11 = 7$. Therefore $D = |7|/7 = 1$. Option (A) would require the point to lie on the plane, which it does not. Options (C) and (D) do not equal the computed perpendicular distance.

Step 3: Conclusion The distance is $\boxed{1 \text{ unit}}$, option (B).

Quick Tip to Solve: First move the constant to the left; the numerator here is 7 and the normal length is also 7.

9. The maximum value of slope of the curve $y = -x^3 + 3x^2 + 12x - 5$ is

- (A) 15
- (B) 12
- (C) 9
- (D) 0

[1 mark]

Solution:

Concept:

- The slope of the tangent to $y = f(x)$ is the derivative $m(x) = f'(x)$.
- A quadratic with a negative coefficient of x^2 attains its greatest value at its vertex; completing the square proves the bound.

Step 1: Differentiate the curve Differentiating each term gives

$$m(x) = \frac{dy}{dx} = -3x^2 + 6x + 12.$$

The constant -5 contributes nothing to the slope. We must maximize this derivative, rather than maximize the original cubic.

Step 2: Find the global maximum of the slope Completing the square,

$$m(x) = -3(x^2 - 2x) + 12 = 15 - 3(x - 1)^2.$$

Since $(x - 1)^2 \geq 0$ for every real x , the slope is at most 15, and this value is attained at $x = 1$. Although 12, 9 and 0 may occur as slopes at other points, none is the greatest possible slope, so options (B), (C) and (D) fail.

Step 3: Conclusion The maximum slope is $\boxed{15}$, option (A).

Quick Tip to Solve: Complete the square in the derivative: 15 minus a nonnegative term immediately reveals the maximum.

10. The vector equation of XY-plane is

- (A) $\vec{r} \cdot \hat{k} = 0$
(B) $\vec{r} \cdot \hat{j} = 0$
(C) $\vec{r} \cdot \hat{i} = 0$
(D) $\vec{r} \cdot \hat{n} = 1$

[1 mark]

Solution:

Concept:

- The position vector of a point (x, y, z) is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
- A plane through the origin with normal vector $\vec{n}$ has equation $\vec{r} \cdot \vec{n} = 0$.

Step 1: Identify the coordinate condition Every point of the XY-plane has zero third coordinate, so its Cartesian equation is $z = 0$. Both x and y are unrestricted. The unit vector $\hat{k}$ points along the Z-axis and is perpendicular to this plane.

Step 2: Express the condition as a dot product Using the orthogonality of the unit coordinate vectors gives

$$\vec{r} \cdot \hat{k} = x(\hat{i} \cdot \hat{k}) + y(\hat{j} \cdot \hat{k}) + z(\hat{k} \cdot \hat{k}) = z.$$

Thus option (A) imposes exactly $z = 0$. Option (B) imposes $y = 0$, the XZ-plane, while option (C) imposes $x = 0$, the YZ-plane. Option (D) has a nonzero right-hand side and excludes the origin, so it cannot represent the XY-plane.

Step 3: Conclusion The equation is $\boxed{\vec{r} \cdot \hat{k} = 0}$, option (A).

Quick Tip to Solve: For a coordinate plane, dot the position vector with the unit vector along the missing axis.

11(a). The area of the parallelogram whose diagonals are $2\hat{i}$ and $-3\hat{k}$ is _____ square units. [1 mark]

Solution:

Concept:

- If $\vec{d}_1$ and $\vec{d}_2$ are the diagonals of a parallelogram, its area is $A = \frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$.
- The cross-product magnitude is $|\vec{d}_1||\vec{d}_2|\sin\theta$, where θ is the angle between the diagonals.

Step 1: Determine the diagonal lengths and angle The given diagonals are $\vec{d}_1 = 2\hat{i}$ and $\vec{d}_2 = -3\hat{k}$. Their lengths are 2 and 3 respectively. As $\hat{i} \cdot \hat{k} = 0$, these diagonals are perpendicular; the negative sign changes the direction but not the length or the area.

Step 2: Apply the diagonal area formula We have

$$(2\hat{i}) \times (-3\hat{k}) = -6(\hat{i} \times \hat{k}) = 6\hat{j},$$

whose magnitude is 6. Hence the area is $\frac{1}{2}(6) = 3$. The factor $\frac{1}{2}$ is essential because the vectors supplied are diagonals, not adjacent sides of the parallelogram.

Step 3: Conclusion The required area is $\boxed{3 \text{ square units}}$.

Quick Tip to Solve: When the supplied vectors are diagonals, halve their cross-product magnitude to obtain the area.

11(b). The value of λ for which the vectors $2\hat{i} - \lambda\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} - \hat{k}$ are orthogonal is _____. [1 mark]

Solution:

Concept:

- Two nonzero vectors are orthogonal if and only if their scalar product is zero.
- For Cartesian vectors, the scalar product is the sum of products of corresponding components:
 $(a, b, c) \cdot (p, q, r) = ap + bq + cr$.

Step 1: Form the orthogonality equation The vectors have components $(2, -\lambda, 1)$ and $(1, 2, -1)$. Both are nonzero because their first components are nonzero. Their dot product is

$$2(1) + (-\lambda)(2) + 1(-1) = 1 - 2\lambda.$$

Orthogonality therefore requires $1 - 2\lambda = 0$.

Step 2: Solve and check the parameter The equation gives $2\lambda = 1$, hence $\lambda = \frac{1}{2}$. Substitution into the first vector gives $(2, -\frac{1}{2}, 1)$, whose dot product with $(1, 2, -1)$ is $2 - 1 - 1 = 0$. This confirms the required perpendicularity and checks the signs of the second and third components.

Step 3: Conclusion The required value is $\lambda = \frac{1}{2}$.

Quick Tip to Solve: Keep the component signs when forming the dot product: the variable product is minus 2 lambda.

12. A bag contains 3 black, 4 red and 2 green balls. If three balls are drawn simultaneously at random, then the probability that the balls are of different colours is _____. [1 mark]

Solution:

Concept:

- For an equally likely random selection, probability equals the number of favourable selections divided by the total number of selections.
- When balls are drawn simultaneously, selections are unordered and counted by combinations $\binom{n}{r}$.

Step 1: Count all selections of three balls The bag contains $3 + 4 + 2 = 9$ balls. Each unordered selection of three distinct balls is equally likely. Thus the total number of outcomes is

$$\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84.$$

Individual balls are counted as distinct even when they share a colour.

Step 2: Count the selections with different colours With three balls and exactly three available

Quick Tip to Solve: For a simultaneous draw of three different colours, count one choice from each colour and divide by $C(9,3)$.

colours, different colours means one black, one red and one green. We can choose these in

$$\binom{3}{1} \binom{4}{1} \binom{2}{1} = 3 \cdot 4 \cdot 2 = 24$$

ways. No additional factor of $3!$ is used, since the draw is simultaneous and no order is recorded. The desired probability is therefore $24/84 = 2/7$.

Step 3: Conclusion The required probability is $\boxed{\frac{2}{7}}$.

Quick Tip to Solve: For a simultaneous draw of three different colours, count one choice from each colour and divide by $C(9,3)$.

13. The absolute minimum value of $f(x) = 2 \sin x$ in $\left[0, \frac{3\pi}{2}\right]$ is _____. [1 mark]

Solution:

Concept:

- For all real x , $-1 \leq \sin x \leq 1$, so $-2 \leq 2 \sin x \leq 2$.
- An absolute minimum on a specified interval must be attained within that interval; endpoints must be included when the interval is closed.

Step 1: Establish a lower bound on the interval The given interval is $\left[0, \frac{3\pi}{2}\right]$. The universal sine bound gives $f(x) = 2 \sin x \geq -2$ at every point of this interval. A lower bound is the absolute minimum only if some permitted value of x achieves it.

Step 2: Check attainment and the relevant values The endpoint $x = \frac{3\pi}{2}$ is included, and

$$f\left(\frac{3\pi}{2}\right) = 2 \sin \frac{3\pi}{2} = 2(-1) = -2.$$

For comparison, $f(0) = 0$ and the interior stationary point $x = \frac{\pi}{2}$ gives $f\left(\frac{\pi}{2}\right) = 2$, a maximum. Since no sine value is smaller than -1 , no interior point can produce a value below -2 .

Step 3: Conclusion The absolute minimum value is $\boxed{-2}$, attained at $x = \frac{3\pi}{2}$.

Quick Tip to Solve: Check the closed endpoint $3\pi/2$: sine equals minus one there and reaches its universal lower bound.

14(a). If $y = \tan^{-1} x + \cot^{-1} x$, $x \in \mathbb{R}$, then $\frac{dy}{dx}$ is equal to _____. [1 mark]

Solution:

Concept:

- The principal inverse functions satisfy $\tan^{-1} x + \cot^{-1} x = \pi/2$ for every real x .

Quick Tip to Solve: Remember the complementary-angle identity; the derivative of its constant sum is immediately zero.

- Their derivatives are $1/(1+x^2)$ and $-1/(1+x^2)$, respectively.

Step 1: Differentiate both terms Using the sum rule, differentiate each inverse trigonometric function with respect to the same variable:

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{1}{1+x^2}.$$

The denominator is positive for every real x , so both derivatives exist throughout the stated domain.

Step 2: Check by the constant identity The two derivative contributions cancel exactly. This also agrees with the identity $y = \pi/2$: changing x changes the two angles by equal and opposite amounts, leaving their sum constant.

Step 3: Conclusion Thus, $\boxed{dy/dx = 0}$.

Quick Tip to Solve: Remember the complementary-angle identity; the derivative of its constant sum is immediately zero.

OR

14(b). If $\cos(xy) = k$, where k is a constant and $xy \neq n\pi$, $n \in \mathbb{Z}$, then $\frac{dy}{dx}$ is equal to _____. [1 mark]

Solution:

Concept:

- For a differentiable implicit function $y = y(x)$, the chain rule gives $d[\cos u]/dx = -\sin u \, du/dx$.
- The product rule gives $d(xy)/dx = y + x \, dy/dx$.

Step 1: Differentiate the implicit equation Because k is constant, its derivative is zero. Applying the chain and product rules gives

$$-\sin(xy) \left(y + x \frac{dy}{dx} \right) = 0.$$

The condition $xy \neq n\pi$, for every integer n , ensures that $\sin(xy) \neq 0$.

Step 2: Isolate the derivative Dividing by the nonzero sine factor gives $y + x \, dy/dx = 0$. Also $x \neq 0$, since $x = 0$ would give $xy = 0$, an excluded integral multiple of π . We may therefore divide by x without losing an admissible point.

Step 3: Conclusion Hence, $\boxed{\frac{dy}{dx} = -\frac{y}{x}}$.

Quick Tip to Solve: Use the stated exclusion to cancel the sine factor before solving the product-rule equation.

15. The value of λ so that the function f defined by

$$f(x) = \begin{cases} \lambda x, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$$

is continuous at $x = \pi$ is _____.

[1 mark]

Solution:

Concept:

- Continuity at $x = a$ requires the left limit, right limit and function value to be equal.
- Each branch here is continuous on its own domain, so substitute the joining point into each branch.

Step 1: Find the value and left limit The branch λx applies at $x = \pi$, because its condition includes equality. Thus

$$f(\pi) = \lambda\pi, \quad \lim_{x \rightarrow \pi^-} f(x) = \lambda\pi.$$

The left limit already equals the function value for every fixed λ .

Step 2: Match the right limit For $x > \pi$, the branch is $\cos x$. Therefore $\lim_{x \rightarrow \pi^+} f(x) = \cos \pi = -1$. Continuity requires $\lambda\pi = -1$. Since $\pi \neq 0$, dividing by π determines the unique value of the parameter.

Step 3: Conclusion The required value is $\lambda = -1/\pi$.

Quick Tip to Solve: At a piecewise joining point, first identify which branch contains equality, then match both limits.

16(a). Evaluate: $\int_{-2}^2 |x| dx$.

[1 mark]

Solution:

Concept:

- The absolute-value function is even: $|-x| = |x|$.
- For an even integrand, $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.

Step 1: Use symmetry and the correct branch On $[0, 2]$, the absolute value satisfies $|x| = x$. The negative half contributes the same positive area as the positive half. Hence

$$\int_{-2}^2 |x| dx = 2 \int_0^2 x dx.$$

Cancellation does not occur: $|x|$ is nonnegative on both halves.

Step 2: Evaluate the elementary integral An antiderivative of x is $x^2/2$. Consequently

$$2 \left[\frac{x^2}{2} \right]_0^2 = 2 \left(\frac{4}{2} - 0 \right).$$

This agrees with adding the areas of the two triangles, each with base and height equal to 2.

Quick Tip to Solve: Distinguish x from its absolute value: symmetric limits cancel an odd integrand, but double an even one.

Step 3: Conclusion Therefore, $\int_{-2}^2 |x| dx = 4$.

Quick Tip to Solve: Distinguish x from its absolute value: symmetric limits cancel an odd integrand, but double an even one.

OR

16(b). Find: $\int \frac{dx}{9 + 4x^2}$.

[1 mark]

Solution:

Concept:

- The standard form is $\int du/(a^2 + u^2) = (1/a) \tan^{-1}(u/a) + C$, for $a > 0$.
- A linear substitution must include its differential factor.

Step 1: Reduce to the standard form Let $u = 2x$, so $du = 2 dx$ and $dx = du/2$. With $9 = 3^2$, the integral becomes

$$\int \frac{dx}{9 + 4x^2} = \frac{1}{2} \int \frac{du}{3^2 + u^2}.$$

The factor $1/2$ comes from the differential, not from the denominator.

Step 2: Integrate and substitute back Apply the standard formula with $a = 3$: the expression is $\frac{1}{2} \cdot \frac{1}{3} \tan^{-1}(u/3) + C$. Replace u by $2x$. Differentiating the resulting expression gives $\frac{1}{6} \frac{2/3}{1+4x^2/9} = 1/(9 + 4x^2)$, confirming the coefficient.

Step 3: Conclusion The integral is $\frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right) + C$.

Quick Tip to Solve: For a squared linear term, include both the standard-form coefficient and the substitution differential.

17. Find the interval in which the function f given by $f(x) = 7 - 4x - x^2$ is strictly increasing. [1 mark]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval where its derivative is positive.
- For this polynomial, the derivative exists for every real x .

Step 1: Differentiate the polynomial Apply the power rule term by term to $f(x) = 7 - 4x - x^2$. The constant contributes zero, the linear term contributes -4 , and the quadratic term contributes $-2x$. Thus

$$f'(x) = -4 - 2x = -2(x + 2).$$

Step 2: Solve the positive-derivative condition For increasing behaviour, require $-2(x + 2) > 0$.

Quick Tip to Solve: Factor the derivative before its sign test, and reverse the inequality when dividing by a negative number.

Dividing by the negative number -2 reverses the inequality and gives $x + 2 < 0$, hence $x < -2$. At $x = -2$, the derivative is zero; for $x > -2$ it is negative, so the function decreases there.

Step 3: Conclusion The required interval is $(-\infty, -2]$.

Quick Tip to Solve: Factor the derivative before its sign test, and reverse the inequality when dividing by a negative number.

18. Differentiate $\sin^2(\sqrt{x})$ with respect to x .

[1 mark]

Solution:

Concept:

- For $y = (\sin u)^2$, the chain rule gives $dy/dx = 2 \sin u \cos u \, du/dx$.
- For $u = \sqrt{x}$, $du/dx = 1/(2\sqrt{x})$ when $x > 0$.

Step 1: Identify the nested functions Write the function as $y = (\sin u)^2$, where $u = \sqrt{x}$. Differentiating the outer square first and then the sine gives

$$\frac{dy}{dx} = 2 \sin u \cos u \frac{du}{dx}.$$

Both inner factors are required: differentiating only the outer square would omit the chain-rule contribution.

Step 2: Insert the square-root derivative For $x > 0$, substitute $u = \sqrt{x}$ and $du/dx = 1/(2\sqrt{x})$. The factors of 2 cancel. Using $2 \sin u \cos u = \sin(2u)$ gives an equivalent double-angle expression. These formulas describe the derivative on the interior of the real domain.

Step 3: Conclusion For $x > 0$, $\frac{dy}{dx} = \frac{\sin \sqrt{x} \cos \sqrt{x}}{\sqrt{x}} = \frac{\sin(2\sqrt{x})}{2\sqrt{x}}$.

Quick Tip to Solve: Differentiate from the outside inward: square, sine, then square root; simplify only after all factors appear.

19. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = |(i)^2 - j|$.

[1 mark]

Solution:

Concept:

- In a matrix $A = [a_{ij}]$, the first index i identifies the row and the second index j identifies the column.
- For a 2×2 matrix both indices take values 1, 2. Evaluate the absolute value after calculating $i^2 - j$.

Step 1: Find the entries in the first row Set $i = 1$, so that $i^2 = 1$. For columns $j = 1$ and $j = 2$, respectively,

$$a_{11} = |1^2 - 1| = 0, \quad a_{12} = |1^2 - 2| = |-1| = 1.$$

Quick Tip to Solve: Fix the row index first, square it, and then vary the column index before taking absolute values.

The absolute value makes the second entry positive.

Step 2: Find the entries in the second row Set $i = 2$, so that $i^2 = 4$. Hence

$$a_{21} = |2^2 - 1| = 3, \quad a_{22} = |2^2 - 2| = 2.$$

Place these entries in row order: the entries with first index 1 belong above those with first index 2.

Step 3: Conclusion The required matrix is $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$.

Quick Tip to Solve: Fix the row index first, square it, and then vary the column index before taking absolute values.

20. A problem is given to three students whose probabilities of solving it are $\frac{1}{3}$, $\frac{1}{4}$ and $\frac{1}{6}$ respectively. If the events of solving the problem are independent, find the probability that at least one of them solves it. [1 mark]

Solution:

Concept:

- For independent events, the probability that all fail is the product of their individual failure probabilities.
- The event that at least one student succeeds is the complement of the event that none succeeds.

Step 1: Calculate the failure probabilities Let E_1, E_2, E_3 denote the events that the respective students solve the problem. The printed success probabilities are $1/3, 1/4, 1/6$. Thus

$$P(E_1^c) = \frac{2}{3}, \quad P(E_2^c) = \frac{3}{4}, \quad P(E_3^c) = \frac{5}{6}.$$

Complements of independent events remain independent.

Step 2: Apply the complement rule The probability that no student solves the problem is

$$P(E_1^c \cap E_2^c \cap E_3^c) = \frac{2}{3} \times \frac{3}{4} \times \frac{5}{6} = \frac{5}{12}.$$

Therefore

$$P(E_1 \cup E_2 \cup E_3) = 1 - P(E_1^c \cap E_2^c \cap E_3^c) = 1 - \frac{5}{12}.$$

Adding the three success probabilities would count outcomes with several successes more than once.

Step 3: Conclusion The required probability is $\frac{7}{12}$.

Quick Tip to Solve: For at least one success, calculate the probability of all failures and subtract it from one.

Section B (12 marks)

Question numbers 21 to 26 carry 2 marks each.

21(a). Show that for any two non-zero vectors $\vec{a}$ and $\vec{b}$, $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular vectors. [2 marks]

Solution:

Concept:

- The square of a vector magnitude is its dot product with itself: $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$.
- Two non-zero vectors are perpendicular exactly when their dot product is zero, since $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta$.

Step 1: Expand both squared magnitudes By distributivity and symmetry of the dot product,

$$\begin{aligned} |\vec{a} + \vec{b}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}, \\ |\vec{a} - \vec{b}|^2 &= |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}. \end{aligned}$$

The difference of these expressions is $4\vec{a} \cdot \vec{b}$.

Step 2: Prove the forward implication If $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, their squares are equal. Subtracting the expansions gives $4\vec{a} \cdot \vec{b} = 0$. As both vectors are non-zero, $|\vec{a}||\vec{b}| > 0$, so $\cos\theta = 0$. For the angle $0 \leq \theta \leq \pi$, this means $\theta = \pi/2$.

Step 3: Prove the converse If $\vec{a} \perp \vec{b}$, then $\vec{a} \cdot \vec{b} = 0$. Both expansions reduce to $|\vec{a}|^2 + |\vec{b}|^2$. Taking non-negative square roots yields $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$, proving the reverse implication as well.

Step 4: Conclusion Hence $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}| \iff \vec{a} \perp \vec{b}$.

Quick Tip to Solve: Subtract the squared magnitudes; the common norm terms cancel and leave four times the dot product.

OR

21(b). Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $3\hat{i} + 7\hat{j} + \hat{k}$ and $5\hat{i} + 6\hat{j} + 2\hat{k}$ form the sides of a right-angled triangle. [2 marks]

Solution:

Concept:

- Two vectors representing adjacent sides form a right angle when their dot product is zero.
- If $\vec{w} = \vec{u} + \vec{v}$, then $\vec{u}, \vec{v}, -\vec{w}$ close a triangle when placed consecutively. Reversing a side changes its direction but not its length.

Step 1: Establish that the vectors form a triangle Write $\vec{u} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{v} = 3\hat{i} + 7\hat{j} + \hat{k}$, and

Quick Tip to Solve: First verify that two vectors add to the third; then take their dot product to identify the right angle.

$\vec{w} = 5\hat{i} + 6\hat{j} + 2\hat{k}$. Componentwise addition gives

$$\vec{u} + \vec{v} = (2 + 3)\hat{i} + (-1 + 7)\hat{j} + (1 + 1)\hat{k} = \vec{w}.$$

Choose points O, P, Q with $\overrightarrow{OP} = \vec{u}$ and $\overrightarrow{PQ} = \vec{v}$. Then $\overrightarrow{OQ} = \vec{w}$, so the three vectors represent its sides.

Step 2: Check the angle between adjacent sides The dot product is

$$\vec{u} \cdot \vec{v} = 2(3) + (-1)(7) + 1(1) = 6 - 7 + 1 = 0.$$

At vertex P , the sides directed outward are $\overrightarrow{PO} = -\vec{u}$ and $\overrightarrow{PQ} = \vec{v}$. Their dot product is also zero. Both vectors are non-zero, so the angle at P is 90° . As a check, the squared side lengths are 6, 59, 65, and $6 + 59 = 65$.

Step 3: Conclusion The vectors form a right-angled triangle, with the side $5\hat{i} + 6\hat{j} + 2\hat{k}$ as its hypotenuse.

Quick Tip to Solve: First verify that two vectors add to the third; then take their dot product to identify the right angle.

22. Find $(AB)^{-1}$ if $A = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$.

[2 marks]

Solution:

Concept:

- For invertible square matrices, $(AB)^{-1} = B^{-1}A^{-1}$; the order of the factors is reversed.
- For $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $ad - bc \neq 0$, $M^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Step 1: Find the inverse of A The printed matrix is $A = \begin{bmatrix} 1 & 0 \\ -4 & 2 \end{bmatrix}$. Its determinant is

$$|A| = 1(2) - 0(-4) = 2 \neq 0.$$

Therefore

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & \frac{1}{2} \end{bmatrix}.$$

The supplied B^{-1} also has non-zero determinant $3(2) - 1(5) = 1$, consistently defining an invertible matrix B .

Quick Tip to Solve: For an inverse of a product, reverse the factor order before multiplying; use the supplied inverse directly.

Step 2: Multiply in the reversed order Using the supplied $B^{-1} = \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$,

$$\begin{aligned}(AB)^{-1} &= B^{-1}A^{-1} \\ &= \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 3(1) + 1(2) & 3(0) + 1(\frac{1}{2}) \\ 5(1) + 2(2) & 5(0) + 2(\frac{1}{2}) \end{bmatrix}.\end{aligned}$$

The factor order follows because $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I$.

Step 3: Conclusion Thus $(AB)^{-1} = \begin{bmatrix} 5 & \frac{1}{2} \\ 9 & 1 \end{bmatrix}$.

Quick Tip to Solve: For an inverse of a product, reverse the factor order before multiplying; use the supplied inverse directly.

23. If $x = a \sec \theta$, $y = b \tan \theta$, then find $\frac{dy}{dx}$ at $\theta = \frac{\pi}{3}$.

[2 marks]

Solution:

Concept:

- For a parametrically defined curve, $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$, provided $dx/d\theta \neq 0$.
- The derivatives are $\frac{d}{d\theta}(\sec \theta) = \sec \theta \tan \theta$ and $\frac{d}{d\theta}(\tan \theta) = \sec^2 \theta$.

Step 1: Differentiate with respect to the parameter Treat a and b as constants. Differentiating the two equations separately gives

$$\frac{dx}{d\theta} = a \sec \theta \tan \theta, \quad \frac{dy}{d\theta} = b \sec^2 \theta.$$

Both derivatives must be taken with respect to the same parameter before forming their ratio.

Step 2: Form the ratio and substitute the angle Using the chain rule,

$$\frac{dy}{dx} = \frac{b \sec^2 \theta}{a \sec \theta \tan \theta} = \frac{b \sec \theta}{a \tan \theta} = \frac{b}{a \sin \theta}.$$

At the printed angle $\theta = \pi/3$, use $\sin(\pi/3) = \sqrt{3}/2$. Thus the slope is obtained by dividing b by $a\sqrt{3}/2$, assuming $a \neq 0$ so that the derivative is defined.

Step 3: Conclusion $\left. \frac{dy}{dx} \right|_{\theta=\pi/3} = \frac{2b}{a\sqrt{3}}.$

Quick Tip to Solve: Simplify $\sec \theta / \tan \theta$ to $1 / \sin \theta$ before substituting the angle.

24. If A and B are two events such that $P(A) = 0.4$, $P(B) = 0.3$ and $P(A \cup B) = 0.6$, then find $P(B' \cap A)$. [2 marks]

Solution:

Concept:

- The addition rule is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- The event A splits into disjoint events $A \cap B$ and $A \cap B'$.

Step 1: Find the probability of the intersection Rearrange the addition rule to isolate the overlap of the two events:

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) = 0.4 + 0.3 - 0.6 = 0.1.$$

The overlap is subtracted in the union formula because it was counted once in each of $P(A)$ and $P(B)$.

Step 2: Remove the overlap from event A The requested event $B' \cap A$ means that A occurs while B does not occur. Since

$$A = (A \cap B) \cup (A \cap B'),$$

and these events are disjoint, their probabilities add to $P(A)$. Hence

$$P(B' \cap A) = P(A) - P(A \cap B) = 0.4 - 0.1.$$

This is the portion of A outside B , so subtract from $P(A)$, not from $P(B)$.

Step 3: Conclusion $P(B' \cap A) = 0.3$.

Quick Tip to Solve: For A occurring without B , subtract the overlap from $P(A)$; no independence assumption is needed.

25(a). Solve for x : $\sin^{-1} 4x + \sin^{-1} 3x = -\frac{\pi}{2}$.

[2 marks]

Solution:

Concept:

- The principal range of $\sin^{-1} t$ is $[-\pi/2, \pi/2]$, and its real domain is $[-1, 1]$.
- Use $\sin(-\pi/2 - u) = -\cos u$ and $\cos(\sin^{-1} t) = \sqrt{1 - t^2}$.

Step 1: Use the principal range and isolate one inverse sine Both inverse sines exist only when $|x| \leq 1/4$. Their sum is negative, so $x < 0$, because both terms have the same sign as x . Put $u = \sin^{-1}(3x)$. The equation gives

$$\sin^{-1}(4x) = -\frac{\pi}{2} - u.$$

Taking sine and using the nonnegative cosine on the principal range yields

$$4x = -\cos u = -\sqrt{1 - 9x^2}.$$

Quick Tip to Solve: Before squaring an inverse sine equation, record the sign of x and check the retained root in the principal ranges.

Step 2: Solve and check the candidate Squaring gives

$$16x^2 = 1 - 9x^2, \quad 25x^2 = 1.$$

The earlier sign condition selects $x = -1/5$. To check it in the original equation, let $\alpha = \sin^{-1}(3/5)$. Since $\alpha \in (0, \pi/2)$ and $\cos \alpha = 4/5$, we have $\sin^{-1}(4/5) = \pi/2 - \alpha$. Using the oddness of inverse sine, the sum at $x = -1/5$ is $-\alpha - (\pi/2 - \alpha) = -\pi/2$, as required.

Step 3: Conclusion $x = -\frac{1}{5}$.

Quick Tip to Solve: Before squaring an inverse sine equation, record the sign of x and check the retained root in the principal ranges.

OR

25(b). Express $\tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right)$, $-\frac{3\pi}{2} < x < \frac{\pi}{2}$ **in the simplest form.**

[2 marks]

Solution:

Concept:

- The principal range of $\tan^{-1} t$ is $(-\pi/2, \pi/2)$.
- The half-angle identity gives $\frac{\cos x}{1 - \sin x} = \tan(\pi/4 + x/2)$ wherever the original denominator is nonzero.

Step 1: Rewrite the ratio using half angles Write $c = \cos(x/2)$ and $s = \sin(x/2)$. Then

$$\cos x = (c - s)(c + s), \quad 1 - \sin x = (c - s)^2.$$

Consequently,

$$\frac{\cos x}{1 - \sin x} = \frac{c + s}{c - s} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

The denominator is nonzero throughout the given open interval: the nearest values with $\sin x = 1$ are precisely its endpoints.

Step 2: Check the angle against the principal range The supplied interval is

$$-\frac{3\pi}{2} < x < \frac{\pi}{2}.$$

Divide each term by two and add $\pi/4$ to obtain

$$-\frac{\pi}{2} < \frac{\pi}{4} + \frac{x}{2} < \frac{\pi}{2}.$$

Thus the transformed angle lies entirely in the principal range of inverse tangent. Therefore applying $\tan^{-1}$ to its tangent returns the angle itself, with no addition or subtraction of π .

Quick Tip to Solve: Always transform the given x interval along with the half-angle expression before cancelling inverse tangent and tangent.

Step 3: Conclusion

$$\tan^{-1} \left(\frac{\cos x}{1 - \sin x} \right) = \frac{\pi}{4} + \frac{x}{2}.$$

Quick Tip to Solve: Always transform the given x interval along with the half-angle expression before cancelling inverse tangent and tangent.

26. Find the coordinates of the point where the line through $(-1, 1, -8)$ and $(5, -2, 10)$ crosses the ZX-plane. [2 marks]

Solution:

Concept:

- A line through points A and B can be written as $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$.
- The ZX-plane contains the x and z axes; every point on it has $y = 0$.

Step 1: Write the parametric equation of the line Let $A = (-1, 1, -8)$ and $B = (5, -2, 10)$. The direction vector from A to B is

$$B - A = (5 + 1, -2 - 1, 10 + 8) = (6, -3, 18).$$

Therefore every point on the line has coordinates

$$(x, y, z) = (-1 + 6t, 1 - 3t, -8 + 18t), \quad t \in \mathbb{R}.$$

At $t = 0$ and $t = 1$, this gives the two supplied points, confirming the parametrisation.

Step 2: Impose the equation of the ZX-plane At the crossing with the ZX-plane, the y -coordinate must vanish. Thus

$$1 - 3t = 0 \implies t = \frac{1}{3}.$$

Substitute this same parameter into the other coordinates:

$$x = -1 + 6 \left(\frac{1}{3} \right) = 1, \quad z = -8 + 18 \left(\frac{1}{3} \right) = -2.$$

The nonzero y -component of the direction vector ensures that this crossing is unique. The resulting point has y -coordinate zero and lies on the required line.

Step 3: Conclusion The line crosses the ZX-plane at $(1, 0, -2)$.

Quick Tip to Solve: Identify the missing coordinate in the plane name: for the ZX-plane set y equal to zero in the line equation.

Section C (24 marks)

Question numbers 27 to 32 carry 4 marks each.

27. Solve the following LPP graphically:

Minimise $z = 5x + 7y$

subject to the constraints

$$2x + y \geq 8$$

$$x + 2y \geq 10$$

$$x, y \geq 0$$

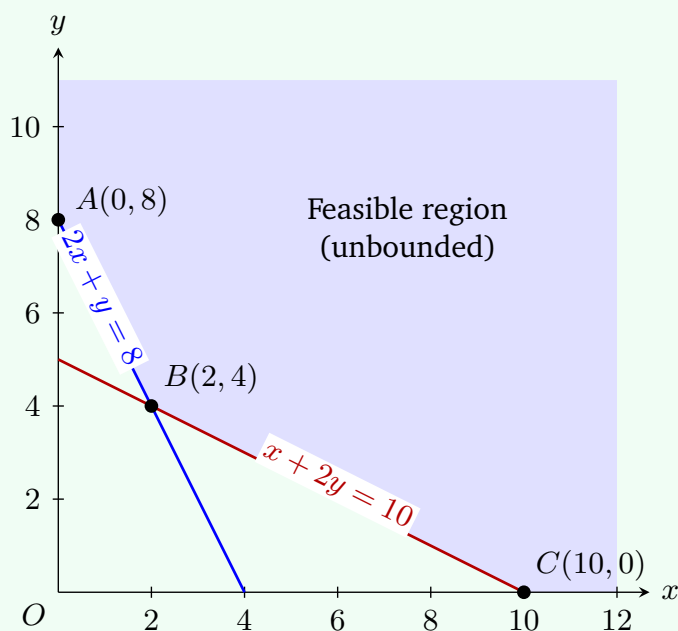
[4 marks]

Solution:

Concept:

- The feasible region is the intersection of the half-planes representing all constraints. The conditions $x, y \geq 0$ restrict the graph to the first quadrant.
- An attained optimum of a linear objective function occurs at a corner point. For an unbounded feasible region, also check whether the open half-plane giving smaller objective values meets the region.

Step 1: Plot the constraints and shade the common region The boundary $2x + y = 8$ passes through $(4, 0)$ and $(0, 8)$. The boundary $x + 2y = 10$ passes through $(10, 0)$ and $(0, 5)$. Join each pair of intercepts. The origin satisfies neither inequality, so select the side of each boundary away from the origin. The common region in the first quadrant is shaded in the graph. Both inequalities include equality, so the boundaries belong to the feasible region. The region extends indefinitely upwards and to the right.



Shaded feasible region with its three corners A, B and C.

Step 2: Determine all corner points On the y -axis the two inequalities together require $y \geq 8$, giving

Quick Tip to Solve: Check the unbounded region using $5x + 7y = (2x + y) + 3(x + 2y)$: positive coefficients immediately give the lower bound 38.

$A(0, 8)$. On the x -axis they require $x \geq 10$, giving $C(10, 0)$. Solve the boundary equations for their intersection:

$$\begin{aligned} 2x + y &= 8, \\ x + 2y &= 10. \end{aligned}$$

Twice the first equation minus the second yields $3x = 6$, hence $x = 2$. Substitution gives $y = 4$, so the intersection is $B(2, 4)$. The feasible boundary follows the vertical ray above A , the segments AB and BC , and the horizontal ray to the right of C .

Step 3: Compare objective values and verify the minimum Evaluate $z = 5x + 7y$ at the three corners:

Corner point	z
$A(0, 8)$	$5(0) + 7(8) = 56$
$B(2, 4)$	$5(2) + 7(4) = 38$
$C(10, 0)$	$5(10) + 7(0) = 50$

Since the feasible region is unbounded, confirm that its unbounded part gives no smaller objective value. For every feasible point,

$$5x + 7y = (2x + y) + 3(x + 2y) \geq 8 + 3(10) = 38.$$

Consequently the half-plane $5x + 7y < 38$ has no point in common with the shaded feasible region. Equality requires both original inequalities to be equalities, which happens only at their intersection $B(2, 4)$.

Step 4: Conclusion The minimum is $z_{\min} = 38$, attained uniquely at $x = 2, y = 4$.

Quick Tip to Solve: Check the unbounded region using $5x + 7y = (2x + y) + 3(x + 2y)$: positive coefficients immediately give the lower bound 38.

28. Evaluate: $\int_{-1}^2 |x^3 - x| dx$.

[4 marks]

Solution:

Concept:

- The absolute value satisfies $|f(x)| = f(x)$ where $f(x) \geq 0$, and $|f(x)| = -f(x)$ where $f(x) < 0$. Locate all zeros inside the integration interval before removing the modulus.
- Definite integrals are additive over adjacent intervals. The power rule gives the antiderivative $F(x) = \frac{x^4}{4} - \frac{x^2}{2}$ of $x^3 - x$, and $\int_a^b (x^3 - x) dx = F(b) - F(a)$.

Step 1: Factor the cubic and determine its sign The printed expression is $x^3 - x = x(x - 1)(x + 1)$, whose zeros on $[-1, 2]$ are $-1, 0, 1$. For $-1 < x < 0$, both x and $x - 1$ are negative, while $x + 1$ is positive, giving a positive product. For $0 < x < 1$, only $x - 1$ is negative, giving a negative product. For $1 < x \leq 2$, all factors are positive. At each zero, the modulus is zero, so either neighbouring formula gives the same endpoint value.

Step 2: Split the integral and remove the modulus Denote the required integral by I . Using the signs

Quick Tip to Solve: Factor the cubic first, split at both interior zeros 0 and 1, and reverse the sign only between 0 and 1.

on the three intervals gives

$$I = \int_{-1}^0 (x^3 - x) dx + \int_0^1 (x - x^3) dx + \int_1^2 (x^3 - x) dx.$$

The sign must be reversed on $(0, 1)$. Integrating the cubic directly over the whole interval would incorrectly cancel positive and negative contributions. Here each contribution is nonnegative, as required by an absolute value integrand.

Step 3: Substitute the endpoints in the antiderivative For $F(x) = \frac{x^4}{4} - \frac{x^2}{2}$, calculate

$$F(-1) = -\frac{1}{4}, \quad F(0) = 0, \quad F(1) = -\frac{1}{4}, \quad F(2) = 2.$$

Thus,

$$\begin{aligned} \int_{-1}^0 (x^3 - x) dx &= 0 - \left(-\frac{1}{4}\right) = \frac{1}{4}, \\ \int_0^1 (x - x^3) dx &= -\left(-\frac{1}{4} - 0\right) = \frac{1}{4}, \\ \int_1^2 (x^3 - x) dx &= 2 - \left(-\frac{1}{4}\right) = \frac{9}{4}. \end{aligned}$$

The first two contributions agree because $|x^3 - x|$ is even on $[-1, 1]$; this independently checks the signs and endpoint arithmetic.

Step 4: Conclusion Therefore, $\int_{-1}^2 |x^3 - x| dx = \frac{1}{4} + \frac{1}{4} + \frac{9}{4} = \boxed{\frac{11}{4}}.$

Quick Tip to Solve: Factor the cubic first, split at both interior zeros 0 and 1, and reverse the sign only between 0 and 1.

29(a). A bag contains two coins, one biased and the other unbiased. When tossed, the biased coin has a 60% chance of showing heads. One of the coins is selected at random and on tossing it shows tails. What is the probability it was an unbiased coin? [4 marks]

Solution:

Concept:

- Conditional probability is $P(U | T) = P(U \cap T)/P(T)$, where U denotes selection of the unbiased coin and T denotes observing tails. The condition restricts the sample space to trials that produced tails.
- If B denotes selection of the biased coin, then B and U are mutually exclusive and exhaustive. The total probability formula is $P(T) = P(B)P(T | B) + P(U)P(T | U)$.

Step 1: Define the events and record the given probabilities Let B be the event that the biased coin is selected, U the event that the unbiased coin is selected, and T the event that the selected coin shows tails. Random selection of one of two coins gives $P(B) = P(U) = \frac{1}{2}$. The biased coin has probability $0.60 = \frac{3}{5}$ of heads. Since a toss gives either heads or tails, $P(T | B) = 1 - \frac{3}{5} = \frac{2}{5}$. For the unbiased

Quick Tip to Solve: With equal selection probabilities, compare tails likelihoods directly: the required ratio is $(1/2)/[(2/5) + (1/2)]$.

coin, $P(T | U) = \frac{1}{2}$. The required conditional probability reverses the order: infer the coin from observed tails.

Step 2: Calculate the total probability of observing tails Tails occurs via two disjoint routes, one for each selected coin. Multiply the selection probability by its tails probability, then add:

$$\begin{aligned} P(T) &= P(B)P(T | B) + P(U)P(T | U) \\ &= \frac{1}{2} \cdot \frac{2}{5} + \frac{1}{2} \cdot \frac{1}{2} = \frac{9}{20}. \end{aligned}$$

In particular, the contribution from the unbiased coin is $P(U \cap T) = \frac{1}{4}$, and the denominator includes tails from both coins.

Step 3: Apply Bayes theorem to the observed outcome Restrict to trials producing tails and apply Bayes theorem:

$$\begin{aligned} P(U | T) &= \frac{P(U)P(T | U)}{P(T)} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{2}}{\frac{9}{20}} = \frac{5}{9}. \end{aligned}$$

Check: the biased-coin posterior is $(\frac{1}{5})/(\frac{9}{20}) = \frac{4}{9}$; both posteriors sum to one. Tails favours the unbiased coin because its tails probability is higher.

Step 4: Conclusion The probability that the selected coin was unbiased, given that it showed tails, is

$$\boxed{\frac{5}{9}}.$$

Quick Tip to Solve: With equal selection probabilities, compare tails likelihoods directly: the required ratio is $(1/2)/[(2/5) + (1/2)]$.

OR

29(b). The probability distribution of a random variable X , where k is a constant is given below:

$$P(X = x) = \begin{cases} 0.1, & \text{if } x = 0 \\ kx^2, & \text{if } x = 1 \\ kx, & \text{if } x = 2 \text{ or } 3 \\ 0, & \text{otherwise} \end{cases}$$

Determine: (a) the value of k ; (b) $P(X \leq 2)$; (c) Mean of the variable X .

[4 marks]

Solution:

Concept:

- For a discrete random variable, all probabilities are non-negative and their sum over its possible values is 1.
- An event probability is the sum of the probabilities of its included values. The mean is the weighted sum $E(X) = \sum_x xP(X = x)$.

Quick Tip to Solve: List the four probabilities first; include the endpoint 2 in the event and multiply each value by its probability for the mean.

Step 1: (a) Determine the constant using normalisation The possible values are 0, 1, 2, 3; all other values have probability zero. Substituting each value into the appropriate branch gives

$$P(X = 0) = 0.1, \quad P(X = 1) = k, \quad P(X = 2) = 2k, \quad P(X = 3) = 3k.$$

These outcomes are mutually exclusive and exhaustive, so

$$0.1 + k + 2k + 3k = 1 \implies 6k = 0.9 \implies k = 0.15 = \frac{3}{20}.$$

The probabilities are thus 0.10, 0.15, 0.30, 0.45. Each is non-negative, and their sum is 1, confirming that the constant gives a valid distribution.

Step 2: (b) Find the probability of the specified event The event $X \leq 2$ contains the values 0, 1, 2, including the endpoint of the inequality. Add the probabilities of these three disjoint outcomes:

$$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 0.10 + 0.15 + 0.30 = 0.55.$$

Step 3: (c) Calculate the weighted mean Multiply each possible value by its corresponding probability and sum the products:

$$\begin{aligned} E(X) &= 0(0.10) + 1(0.15) + 2(0.30) + 3(0.45) \\ &= 0 + 0.15 + 0.60 + 1.35 = 2.10. \end{aligned}$$

Equivalently, the weighted sum before substituting the constant is $k + 4k + 9k = 14k$; substituting $k = 0.15$ again gives 2.10.

Step 4: Conclusion $k = 0.15, \quad P(X \leq 2) = 0.55, \quad E(X) = 2.10$.

Quick Tip to Solve: List the four probabilities first; include the endpoint 2 in the event and multiply each value by its probability for the mean.

30. Find the particular solution of the differential equation

$$\cos y \, dx + (1 + e^{-x}) \sin y \, dy = 0$$

given that $y = \frac{\pi}{4}$ when $x = 0$.

[4 marks]

Solution:

Concept:

- A separable differential equation can be integrated after placing all terms in x with dx , and all terms in y with dy .
- The identities $\int \tan y \, dy = -\ln |\cos y|$ and $\int f'(x)/f(x) \, dx = \ln |f(x)|$ give the required antiderivatives. An initial condition determines the integration constant.

Step 1: Separate the variables The given equation is

$$\cos y \, dx + (1 + e^{-x}) \sin y \, dy = 0.$$

Near the initial point $(0, \pi/4)$, $\cos y \neq 0$. Also $1 + e^{-x} > 0$ for every real x . Dividing by $(1 + e^{-x}) \cos y$

Quick Tip to Solve: Rewrite $1/(1 + e^{-x})$ as $e^x/(1 + e^x)$; its numerator is the derivative of the denominator.

and rearranging gives

$$\frac{dx}{1+e^{-x}} = -\tan y \, dy, \quad \frac{e^x}{1+e^x} dx = -\tan y \, dy.$$

Multiplying the first fraction by e^x/e^x reveals the logarithmic derivative needed for integration.

Step 2: Integrate the separated equation For the left side, use $u = 1 + e^x$, so $du = e^x dx$. For the right side, $d(\cos y) = -\sin y \, dy$. Hence

$$\int \frac{e^x}{1+e^x} dx = \int -\tan y \, dy \implies \ln(1+e^x) = \ln |\cos y| + C.$$

The initial value has positive cosine; on the continuous branch through that point we may write $\ln |\cos y| = \ln(\cos y)$.

Step 3: Use the initial condition and check the relation At $x = 0$, $y = \pi/4$, and $\cos(\pi/4) = 1/\sqrt{2}$. Therefore

$$C = \ln 2 - \ln(1/\sqrt{2}) = \ln(2\sqrt{2}).$$

Exponentiating the integrated equation gives

$$1 + e^x = 2\sqrt{2} \cos y.$$

Differentiating this relation yields $e^x dx = -2\sqrt{2} \sin y \, dy$. Since $2\sqrt{2} = (1 + e^x)/\cos y$, division by e^x recovers the original equation. Substitution of $x = 0$, $y = \pi/4$ also gives equality on both sides.

Step 4: Conclusion The particular solution through the prescribed initial point is $1 + e^x = 2\sqrt{2} \cos y$.

Quick Tip to Solve: Rewrite $1/(1 + e^{-x})$ as $e^x/(1 + e^x)$; its numerator is the derivative of the denominator.

31(a). Show that the function $f : (-\infty, 0) \rightarrow (-1, 0)$ defined by $f(x) = \frac{x}{1+|x|}$, $x \in (-\infty, 0)$ is **one-one** and **onto**. [4 marks]

Solution:

Concept:

- A function is one-one if equal outputs imply equal inputs: $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.
- A function is onto if every element of its codomain has a preimage in its domain.
- For $x < 0$, $|x| = -x$, so the given function becomes $f(x) = x/(1-x)$.

Step 1: Check that the function maps into the stated codomain Let $x \in (-\infty, 0)$. Since $|x| = -x$, we have

$$f(x) = \frac{x}{1-x}.$$

Quick Tip to Solve: Replace $|x|$ by $-x$ before working; for onto, solve for the preimage and explicitly check its domain.

The denominator $1 - x$ is positive and the numerator is negative, giving $f(x) < 0$. Further,

$$f(x) + 1 = \frac{x + 1 - x}{1 - x} = \frac{1}{1 - x} > 0.$$

Thus $-1 < f(x) < 0$, so the output lies in the given codomain. Neither endpoint is attained, consistently with the open interval.

Step 2: Prove that equal images imply equal inputs Take $x_1, x_2 \in (-\infty, 0)$ and suppose $f(x_1) = f(x_2)$. Then

$$\begin{aligned}\frac{x_1}{1 - x_1} &= \frac{x_2}{1 - x_2}, \\ x_1(1 - x_2) &= x_2(1 - x_1), \\ x_1 - x_1x_2 &= x_2 - x_1x_2, \\ x_1 &= x_2.\end{aligned}$$

Both denominators are strictly positive, so cross multiplication is valid. The cancellation shows that two distinct inputs cannot have the same image. Therefore f is one-one.

Step 3: Construct a preimage for every codomain element Let $y \in (-1, 0)$ be arbitrary. Solving the equation $f(x) = y$ suggests

$$\begin{aligned}\frac{x}{1 - x} &= y, \\ x &= y - xy, \\ x(1 + y) &= y, \\ x &= \frac{y}{1 + y}.\end{aligned}$$

Here $1 + y > 0$ and $y < 0$, so the chosen preimage is negative and belongs to $(-\infty, 0)$. The denominator cannot vanish because $y > -1$. To verify the preimage rather than merely obtain a candidate, substitute back:

$$f\left(\frac{y}{1 + y}\right) = \frac{\frac{y}{1 + y}}{1 - \frac{y}{1 + y}} = \frac{\frac{y}{1 + y}}{\frac{1}{1 + y}} = y.$$

Thus each element of the entire stipulated codomain, and not only a subset of it, is attained by an input in the domain. This proves the onto property.

Step 4: Conclusion Hence $f : (-\infty, 0) \rightarrow (-1, 0)$, defined by $f(x) = x/(1 + |x|)$, is one-one and onto, and therefore bijective.

Quick Tip to Solve: Replace $|x|$ by $-x$ before working; for onto, solve for the preimage and explicitly check its domain.

31(b). Show that the relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ given by $R = \{(a, b) : |a - b| \text{ is divisible by } 2\}$ is an equivalence relation. [4 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation when it is reflexive, symmetric and transitive.
- An integer is divisible by 2 precisely when it is even. Two integers have an even difference precisely when they have the same parity.
- Divisibility is unaffected by a change of sign: $2 \mid |a - b|$ is equivalent to $2 \mid (a - b)$.

Step 1: Prove reflexivity Let a be any element of $A = \{1, 2, 3, 4, 5, 6\}$. Then

$$|a - a| = 0 = 2 \cdot 0.$$

Since zero is divisible by 2, the defining condition of R holds for (a, a) . This is true for every element $a \in A$, so all diagonal pairs belong to R . Hence R is reflexive.

Step 2: Prove symmetry Suppose $a, b \in A$ and $(a, b) \in R$. By the definition of the relation, $|a - b|$ is divisible by 2. But absolute value gives

$$|b - a| = |-(a - b)| = |a - b|.$$

Therefore $|b - a|$ is also divisible by 2, and consequently $(b, a) \in R$. Thus reversing any related pair preserves the condition, proving that R is symmetric.

Step 3: Prove transitivity using signed differences Suppose $a, b, c \in A$, with $(a, b) \in R$ and $(b, c) \in R$. Then both $|a - b|$ and $|b - c|$ are divisible by 2. Their corresponding signed differences are also even, so there are integers m, n such that

$$a - b = 2m, \quad b - c = 2n.$$

Adding these equations eliminates the intermediate element b :

$$a - c = (a - b) + (b - c) = 2m + 2n = 2(m + n).$$

Taking absolute values now gives $|a - c| = 2|m + n|$. Since $|m + n|$ is an integer, $|a - c|$ is divisible by 2. Hence $(a, c) \in R$, establishing transitivity. This argument uses signed differences; adding absolute differences directly would not generally give $|a - c|$.

Step 4: Identify the equivalence classes The condition means that related elements have the same parity. Thus the class containing 1 is $\{1, 3, 5\}$, and the class containing 2 is $\{2, 4, 6\}$. Every odd member belongs to the first class and every even member to the second. These disjoint classes together cover A , consistently with the properties just proved.

Step 5: Conclusion Since R is reflexive, symmetric and transitive, it is an equivalence relation on A . Its two distinct equivalence classes are $\{1, 3, 5\}$ and $\{2, 4, 6\}$.

Quick Tip to Solve: For transitivity, remove the absolute values by using signed even differences, add them, and then take the absolute value.

32. If $y = x^3(\cos x)^x + \sin^{-1} \sqrt{x}$, then find $\frac{dy}{dx}$.

[4 marks]

Solution:

Concept:

- Logarithmic differentiation handles a variable base and exponent: if $v = u(x)^{w(x)}$, then $v'/v = w' \log u + w u'/u$, where $u > 0$.
- The product rule is $(fg)' = f'g + fg'$. The chain rule gives $\frac{d}{dx}(\sin^{-1} u) = \frac{u'}{\sqrt{1-u^2}}$, for $|u| < 1$.

Step 1: Differentiate the variable power Put $v = (\cos x)^x$. Taking natural logarithms, $\log v = x \log(\cos x)$. Differentiating both sides gives

$$\frac{v'}{v} = \log(\cos x) + x \frac{-\sin x}{\cos x} = \log(\cos x) - x \tan x.$$

Thus $v' = (\cos x)^x [\log(\cos x) - x \tan x]$. The logarithm term comes from differentiating the exponent, while the tangent term comes from differentiating the base. Both contributions are necessary.

Step 2: Apply the product and chain rules The derivative of the first summand is

$$\frac{d}{dx}[x^3 v] = 3x^2(\cos x)^x + x^3(\cos x)^x [\log(\cos x) - x \tan x].$$

For the second summand, set $u = \sqrt{x}$. Then $u' = 1/(2\sqrt{x})$ and $u^2 = x$, so

$$\frac{d}{dx}(\sin^{-1} \sqrt{x}) = \frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

Combine these derivatives and factor $x^2(\cos x)^x$ from the first two terms. The real inverse sine expression requires $0 \leq x \leq 1$. On the interior $0 < x < 1$, the derivative exists and $\cos x > 0$, justifying the logarithms above; the endpoints are excluded from the finite derivative formula.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = x^2(\cos x)^x [3 + x \log(\cos x) - x^2 \tan x] + \frac{1}{2\sqrt{x(1-x)}}}, \quad 0 < x < 1.$$

Quick Tip to Solve: Differentiate $(\cos x)^x$ separately first: treating its exponent as constant misses the $\log(\cos x)$ contribution.

Section D (24 marks)

Question numbers 33 to 36 carry 6 marks each.

33. Find the points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with both the axes. Also find the equation of the normals. [6 marks]

Solution:

Concept:

- Implicit differentiation of a curve gives its tangent slope $m_t = dy/dx$. At a regular point with finite nonzero tangent slope, its normal slope is $m_n = -1/m_t$.
- A line cutting equal distances on the coordinate axes has intercepts of equal magnitude. Its slope is therefore $+1$ or -1 , including both possible signs of the intercepts.

Step 1: Express the normal slope at a general point Let $P = (a, b)$ lie on $9y^2 = x^3$, so $9b^2 = a^3$. Differentiating implicitly,

$$18y \frac{dy}{dx} = 3x^2, \quad \frac{dy}{dx} = \frac{x^2}{6y}.$$

At a nonzero point of the curve, $a > 0$, $b \neq 0$, and the normal slope is $m_n = -6b/a^2$. The origin is a cusp: its limiting tangent is the x -axis and the perpendicular is the y -axis. That perpendicular does not cut two finite nonzero equal intercepts, so the origin is not a required point.

Step 2: Impose equal intercept lengths Write a normal as $y = m_n x + c$. Its intercepts are $-c/m_n$ on the x -axis and c on the y -axis. Equal intercept lengths require $|-c/m_n| = |c|$, hence $|m_n| = 1$ for nonzero intercepts. Thus

$$\left| \frac{-6b}{a^2} \right| = 1, \quad 36b^2 = a^4.$$

Using $9b^2 = a^3$, we obtain $4a^3 = a^4$, or $a^3(a - 4) = 0$. The zero value has already been excluded, so $a = 4$. Substitution into the curve gives $b^2 = 64/9$, hence $b = \pm 8/3$.

Step 3: Obtain and verify the two normals At $(4, 8/3)$, the normal slope is -1 . Its equation is

$$y - \frac{8}{3} = -(x - 4), \quad x + y = \frac{20}{3}.$$

Its intercepts on both axes are $20/3$. At $(4, -8/3)$, the normal slope is $+1$, giving

$$y + \frac{8}{3} = x - 4, \quad x - y = \frac{20}{3}.$$

This normal has intercepts $20/3$ and $-20/3$, whose lengths are equal. In both cases, the tangent slope is respectively $+1$ and -1 , so the tangent and normal slopes multiply to -1 , verifying perpendicularity.

Step 4: Conclusion The required points and corresponding normals are

$$\left(4, \frac{8}{3}\right) : x + y = \frac{20}{3}, \quad \left(4, -\frac{8}{3}\right) : x - y = \frac{20}{3}.$$

Quick Tip to Solve: Equal intercept lengths allow slopes $+1$ and -1 ; retain both signs of y after substituting $x = 4$ into the curve.

34. Show that the lines

$\frac{x-2}{1} = \frac{y-2}{3} = \frac{z-3}{1}$ and $\frac{x-2}{1} = \frac{y-3}{4} = \frac{z-4}{2}$ intersect.

Also, find the coordinates of the point of intersection. Find the equation of the plane containing the two lines. [6 marks]

Solution:

Concept:

- A symmetric equation of a line is converted to parametric form by equating its three ratios to a parameter. An intersection must satisfy all three coordinate equations simultaneously.
- The plane containing two intersecting lines has a normal perpendicular to both direction vectors. Thus its normal is their cross product, and its equation through (x_0, y_0, z_0) is $\vec{n} \cdot (x - x_0, y - y_0, z - z_0) = 0$.

Step 1: Express both lines using independent parameters Let the common ratio for the first line be t , and that for the second be s . Then their general points are

$$L_1 : (x, y, z) = (2 + t, 2 + 3t, 3 + t), \quad L_2 : (x, y, z) = (2 + s, 3 + 4s, 4 + 2s).$$

The direction vectors are $\vec{a} = (1, 3, 1)$ and $\vec{b} = (1, 4, 2)$. These are not proportional: their first components would require a proportionality factor of 1, whereas their second components differ. Hence the lines are distinct and cannot coincide.

Step 2: Solve and verify all intersection conditions Equality of the x -coordinates requires $t = s$. Equality of the y -coordinates then gives

$$2 + 3t = 3 + 4t \implies t = -1, \quad s = -1.$$

The remaining coordinate must also agree; otherwise the lines would be skew. At these parameter values,

$$z_{L_1} = 3 + (-1) = 2, \quad z_{L_2} = 4 + 2(-1) = 2.$$

Thus all three coordinate conditions are satisfied. In fact, substitution into either parametric form gives the same point $P = (1, -1, 2)$. This establishes intersection, rather than merely matching two coordinates.

Step 3: Compute a normal to the required plane Since the plane contains both lines, a normal vector is

$$\begin{aligned} \vec{n} &= \vec{a} \times \vec{b} = (3 \cdot 2 - 1 \cdot 4, 1 \cdot 1 - 1 \cdot 2, 1 \cdot 4 - 3 \cdot 1) \\ &= (2, -1, 1). \end{aligned}$$

This is nonzero, so the two intersecting directions determine a unique plane. Check perpendicularity explicitly: $\vec{n} \cdot \vec{a} = 2 - 3 + 1 = 0$ and $\vec{n} \cdot \vec{b} = 2 - 4 + 2 = 0$.

Step 4: Form and check the plane equation Using the common point P , the point-normal equation is

$$2(x - 1) - (y + 1) + (z - 2) = 0.$$

On simplifying its left side, it becomes $2x - y + z - 5$. For the first line, direct substitution gives

Quick Tip to Solve: After matching two coordinates, always verify the third; then cross the two direction vectors to get the containing plane's normal.

$2(2+t) - (2+3t) + (3+t) - 5 = 0$ for every t . For the second line it gives $2(2+s) - (3+4s) + (4+2s) - 5 = 0$ for every s . Hence the equation contains each entire line.

Step 5: Conclusion The lines intersect at $(1, -1, 2)$, and the plane containing them is $2x - y + z - 5 = 0$.

Quick Tip to Solve: After matching two coordinates, always verify the third; then cross the two direction vectors to get the containing plane's normal.

35(a). Using integration, find the area of the region bounded by the lines $x - y = 0$, $3x - y = 0$ and $x + y = 12$. [6 marks]

Solution:

Concept:

- The area between an upper curve $y = u(x)$ and a lower curve $y = l(x)$ is $\int_a^b [u(x) - l(x)] dx$, provided $u(x) \geq l(x)$ throughout that interval.
- For a region bounded by three lines, first find the pairwise intersection points. Split the integral wherever the upper or lower boundary changes, so that every vertical strip belongs to the bounded region.

Step 1: Find the vertices of the bounded region Write the three lines in the forms $y = x$, $y = 3x$, and $y = 12 - x$. The first two meet where $x = 3x$, hence $x = 0$ and $y = 0$; call this point $O = (0, 0)$. The intersection of $y = 3x$ with $y = 12 - x$ satisfies

$$3x = 12 - x \implies x = 3, \quad y = 9.$$

Call it $A = (3, 9)$. Similarly, the intersection of $y = x$ with $y = 12 - x$ satisfies

$$x = 12 - x \implies x = 6, \quad y = 6,$$

giving $B = (6, 6)$. These three non-collinear points enclose the required triangular region, whose x -values run from 0 to 6.

Step 2: Identify the upper and lower boundaries The lower boundary throughout the region is $y = x$. From $x = 0$ to $x = 3$, the upper boundary is $y = 3x$, because it lies below $y = 12 - x$ there. Indeed, $3x \leq 12 - x$ is equivalent to $x \leq 3$. From $x = 3$ to $x = 6$, the upper boundary becomes $y = 12 - x$. Thus the vertical-strip heights are $3x - x = 2x$ on the first interval and $(12 - x) - x = 12 - 2x$ on the second. Both heights are nonnegative on their respective intervals. A single upper-line formula over the whole interval would include points outside the triangle.

Quick Tip to Solve: Find where the two possible upper lines meet first: here $x=3$ is the exact point at which the area integral must be split.

Step 3: Integrate the two portions of the region The area is the sum of the two integrals:

$$\begin{aligned}\mathcal{A} &= \int_0^3 (3x - x) dx + \int_3^6 (12 - x - x) dx \\&= \int_0^3 2x dx + \int_3^6 (12 - 2x) dx \\&= [x^2]_0^3 + [12x - x^2]_3^6 \\&= (9 - 0) + [(72 - 36) - (36 - 9)] \\&= 9 + (36 - 27) = 18.\end{aligned}$$

The separate areas are positive, as expected from the positive strip heights, and their sum accounts for the whole bounded region without overlap.

Step 4: Check the result using the vertices As a check on the integration, the triangle has vertices $(0, 0)$, $(3, 9)$ and $(6, 6)$. Its coordinate-area expression is

$$\frac{1}{2} |3 \cdot 6 - 9 \cdot 6| = \frac{1}{2} |18 - 54| = 18.$$

This agrees with the integral. The required method remains integration; the vertex calculation simply confirms the boundary choices and arithmetic.

Step 5: Conclusion By integration, the area bounded by the three lines is 18 square units.

Quick Tip to Solve: Find where the two possible upper lines meet first: here $x=3$ is the exact point at which the area integral must be split.

OR

35(b). Using integration, find the smaller area enclosed by the circle $x^2 + y^2 = 4$ and the line $x + y = 2$.

[6 marks]

Solution:

Concept:

- The area between two curves on $[a, b]$ is $\int_a^b (y_{\text{upper}} - y_{\text{lower}}) dx$. Intersection points determine the limits of integration.
- The circle $x^2 + y^2 = 4$ has radius 2. Its upper semicircle is $y = \sqrt{4 - x^2}$, while the given line is $y = 2 - x$.
- For radius a , an antiderivative is $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}(x/a) + C$, with the principal inverse sine used on $[-1, 1]$.

Step 1: Locate the endpoints of the chord Substitute $y = 2 - x$ into the circle equation:

$$\begin{aligned}x^2 + (2 - x)^2 &= 4, \\2x^2 - 4x &= 0, \\2x(x - 2) &= 0.\end{aligned}$$

Quick Tip to Solve: Find the chord endpoints first; here the required integral is the first-quadrant circle area minus the area beneath the chord.

Thus the intersection points are $(0, 2)$ and $(2, 0)$. The chord lies in the first quadrant. The smaller candidate region is the circular segment between this chord and the first-quadrant arc.

Step 2: Choose the correct upper and lower curves For $0 \leq x \leq 2$, both $\sqrt{4-x^2}$ and $2-x$ are nonnegative. Their squared difference is $(4-x^2) - (2-x)^2 = 2x(2-x) \geq 0$, so the arc is above the line throughout the interval. Therefore the segment area is

$$S = \int_0^2 \left(\sqrt{4-x^2} - (2-x) \right) dx.$$

Using the difference of heights avoids including the triangle beneath the chord in the required segment.

Step 3: Evaluate the definite integrals The circle contribution is

$$\begin{aligned} \int_0^2 \sqrt{4-x^2} dx &= \left[\frac{x}{2} \sqrt{4-x^2} + 2 \sin^{-1} \left(\frac{x}{2} \right) \right]_0^2 \\ &= 2 \left(\frac{\pi}{2} \right) = \pi. \end{aligned}$$

The line contribution is

$$\int_0^2 (2-x) dx = \left[2x - \frac{x^2}{2} \right]_0^2 = 4 - 2 = 2.$$

Subtracting gives $S = \pi - 2$.

Step 4: Verify that this segment is the smaller region The entire circle has area $\pi(2)^2 = 4\pi$. The complementary region therefore has area $4\pi - (\pi - 2) = 3\pi + 2$. Since $\pi - 2$ is positive and is less than $3\pi + 2$, the integral computed above indeed gives the smaller of the two regions enclosed by the circle and the line.

Step 5: Conclusion The smaller enclosed area is $\boxed{\pi - 2 \text{ square units}}$.

Quick Tip to Solve: Find the chord endpoints first; here the required integral is the first-quadrant circle area minus the area beneath the chord.

36(a). Solve the following system of equations by matrix method:

$$x - y + 2z = 7$$

$$2x - y + 3z = 12$$

$$3x + 2y - z = 5$$

[6 marks]

Solution:

Concept:

- Write a linear system as $AX = B$, where A is the coefficient matrix, X is the column of unknowns and B is the column of constants, all in matching equation order.
- If $|A| \neq 0$, the inverse exists and the unique solution is $X = A^{-1}B$, with $A^{-1} = \text{adj } A / |A|$.
- The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$ uses the minor formed by deleting row i and column j . The adjugate is the transpose of the cofactor matrix.

Quick Tip to Solve: Keep the constants column in equation order and transpose the cofactor matrix before multiplying it by that column.

Step 1: Express the printed system in matrix form Keeping the order x, y, z in each equation, the system becomes

$$\underbrace{\begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_X = \underbrace{\begin{pmatrix} 7 \\ 12 \\ 5 \end{pmatrix}}_B.$$

In particular, the last row comes from $3x + 2y - z = 5$.

Step 2: Check that the coefficient matrix is invertible Expanding the determinant along its first row,

$$\begin{aligned} |A| &= 1\{(-1)(-1) - 3(2)\} \\ &\quad - (-1)\{2(-1) - 3(3)\} \\ &\quad + 2\{2(2) - (-1)(3)\} \\ &= -5 - 11 + 14 = -2. \end{aligned}$$

The determinant is nonzero, so multiplication by A^{-1} is valid and the system has exactly one solution.

Step 3: Calculate the adjugate and inverse Computing the signed minors gives

$$(C_{ij}) = \begin{pmatrix} -5 & 11 & 7 \\ 3 & -7 & -5 \\ -1 & 1 & 1 \end{pmatrix}.$$

For example, $C_{12} = -\{2(-1) - 3(3)\} = 11$, while $C_{23} = -\{1(2) - (-1)(3)\} = -5$. Transpose this matrix to obtain

$$\text{adj } A = \begin{pmatrix} -5 & 3 & -1 \\ 11 & -7 & 1 \\ 7 & -5 & 1 \end{pmatrix}, \quad A^{-1} = -\frac{1}{2} \begin{pmatrix} -5 & 3 & -1 \\ 11 & -7 & 1 \\ 7 & -5 & 1 \end{pmatrix}.$$

Step 4: Multiply the inverse by the constants column Applying the matrix method,

$$\begin{aligned} X &= A^{-1}B = -\frac{1}{2} \begin{pmatrix} -5 & 3 & -1 \\ 11 & -7 & 1 \\ 7 & -5 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 12 \\ 5 \end{pmatrix} \\ &= -\frac{1}{2} \begin{pmatrix} -35 + 36 - 5 \\ 77 - 84 + 5 \\ 49 - 60 + 5 \end{pmatrix} \\ &= -\frac{1}{2} \begin{pmatrix} -4 \\ -2 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}. \end{aligned}$$

Step 5: Check all three equations Substitution gives $2 - 1 + 2(3) = 7$, $2(2) - 1 + 3(3) = 12$, and $3(2) + 2(1) - 3 = 5$. Thus every equation is satisfied, including the third equation used to form the constants column.

Quick Tip to Solve: Keep the constants column in equation order and transpose the cofactor matrix before multiplying it by that column.

Step 6: Conclusion The unique solution is $x = 2, y = 1, z = 3$.

Quick Tip to Solve: Keep the constants column in equation order and transpose the cofactor matrix before multiplying it by that column.

OR

36(b). Obtain the inverse of the following matrix using elementary operations:

$$A = \begin{bmatrix} 2 & 1 & -3 \\ -1 & -1 & 4 \\ 3 & 0 & 2 \end{bmatrix}.$$

[6 marks]

Solution:

Concept:

- To obtain the inverse by elementary row operations, write the augmented matrix $[A \mid I_3]$, where I_3 is the identity matrix of order three. Perform every row operation on both halves of this matrix.
- If the left half reduces to I_3 , the right half is A^{-1} . Indeed, the same sequence of operations multiplies both halves on the left by a matrix E ; thus $EA = I_3$ and the right half becomes $EI_3 = E = A^{-1}$.
- The permitted operations are interchanging two rows, multiplying a row by a nonzero constant, and adding a multiple of one row to another. Here R_i denotes the entire i th augmented row.

Step 1: Form the augmented matrix and obtain a unit first pivot We start with

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 2 & 1 & -3 & 1 & 0 & 0 \\ -1 & -1 & 4 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array} \right].$$

Interchange R_1 and R_2 , and then apply $R_1 \rightarrow -R_1$. This makes the first pivot equal to one without introducing fractions:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & -1 & 0 \\ 2 & 1 & -3 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array} \right].$$

Step 2: Eliminate the entries below the first pivot Apply $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 & 2 & 0 \\ 0 & -3 & 14 & 0 & 3 & 1 \end{array} \right].$$

Quick Tip to Solve: Start with the original second row, whose first entry is -1: interchange it with the first row and change its sign to keep all subsequent pivot calculations integral.

Now use $R_2 \rightarrow -R_2$, giving

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & -1 & 0 \\ 0 & 1 & -5 & -1 & -2 & 0 \\ 0 & -3 & 14 & 0 & 3 & 1 \end{array} \right].$$

The first two pivot entries are now one, and the first column has already been reduced to the corresponding column of the identity matrix.

Step 3: Obtain the third pivot and clear the third column Using the current second row, apply $R_3 \rightarrow R_3 + 3R_2$. The third row becomes $[0, 0, -1 \mid -3, -3, 1]$. Then apply $R_3 \rightarrow -R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & -1 & 0 \\ 0 & 1 & -5 & -1 & -2 & 0 \\ 0 & 0 & 1 & 3 & 3 & -1 \end{array} \right].$$

Clear the third-column entries above this pivot with $R_1 \rightarrow R_1 + 4R_3$ and $R_2 \rightarrow R_2 + 5R_3$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 12 & 11 & -4 \\ 0 & 1 & 0 & 14 & 13 & -5 \\ 0 & 0 & 1 & 3 & 3 & -1 \end{array} \right].$$

Step 4: Complete the reduction and verify the right half Finally, apply $R_1 \rightarrow R_1 - R_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -2 & 1 \\ 0 & 1 & 0 & 14 & 13 & -5 \\ 0 & 0 & 1 & 3 & 3 & -1 \end{array} \right] = [I_3 \mid B].$$

All three pivots are nonzero, so the reduction establishes that the inverse exists. As an arithmetic check, multiply the original matrix by the right half:

$$AB = \begin{bmatrix} -4 + 14 - 9 & -4 + 13 - 9 & 2 - 5 + 3 \\ 2 - 14 + 12 & 2 - 13 + 12 & -1 + 5 - 4 \\ -6 + 6 & -6 + 6 & 3 - 2 \end{bmatrix} = I_3.$$

This confirms that the signs and entries obtained by the elementary operations are consistent with the original matrix.

Step 5: Conclusion Hence, the required inverse is

$$A^{-1} = \begin{bmatrix} -2 & -2 & 1 \\ 14 & 13 & -5 \\ 3 & 3 & -1 \end{bmatrix}.$$

Quick Tip to Solve: Start with the original second row, whose first entry is -1: interchange it with the first row and change its sign to keep all subsequent pivot calculations integral.