

CBSE Class 12 Mathematics Question Paper 2020 with Solution PDF-Set 65-3-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper comprises **four** Sections A, B, C and D. This question paper carries **36** questions. **All** questions are compulsory.
- (ii) **Section A** – Questions no. **1 to 20** comprises of **20** questions of **1** mark each.
- (iii) **Section B** – Questions no. **21 to 26** comprises of **6** questions of **2** marks each.
- (iv) **Section C** – Questions no. **27 to 32** comprises of **6** questions of **4** marks each.
- (v) **Section D** – Questions no. **33 to 36** comprises of **4** questions of **6** marks each.
- (vi) There is no overall choice in the question paper. However, an internal choice has been provided in 3 questions of one mark, 2 questions of two marks, 2 questions of four marks and 2 questions of six marks. Only one of the choices in such questions have to be attempted.
- (vii) In addition to this, separate instructions are given with each section and question, wherever necessary.
- (viii) Use of calculators is **not** permitted.

Section A (20 marks)

Question numbers 1 to 20 carry 1 mark each. Question numbers 1 to 10 are multiple choice type questions. Select the correct option.

1. The value of p for which $p(\hat{i} + \hat{j} + \hat{k})$ is a unit vector is

- (A) 0
- (B) $\frac{1}{\sqrt{3}}$
- (C) 1
- (D) $\sqrt{3}$

[1 mark]

Solution:

Concept:

- The magnitude of $a\hat{i} + b\hat{j} + c\hat{k}$ is $\sqrt{a^2 + b^2 + c^2}$. A unit vector has magnitude 1.
- Magnitude involves $|p|$, so both signs must be considered before checking the printed choices.

Step 1: Calculate the magnitude For $\vec{v} = p(\hat{i} + \hat{j} + \hat{k})$, all three Cartesian components are p . Hence

$$|\vec{v}| = \sqrt{p^2 + p^2 + p^2} = \sqrt{3} |p|.$$

Step 2: Impose the unit condition and check choices Thus $\sqrt{3}|p| = 1$, giving $p = \pm 1/\sqrt{3}$. Among the printed choices, only (B) satisfies this condition. Choice (A) gives the zero vector, (C) gives magnitude $\sqrt{3}$, and (D) gives magnitude 3.

Step 3: Conclusion Hence the correct printed option is (B), $p = 1/\sqrt{3}$. The negative value $-1/\sqrt{3}$ also gives a unit vector but is not listed.

Quick Tip to Solve: Square the three equal components; remember that magnitude determines the absolute value of the scalar.

2. $\tan\left(\sin^{-1} \frac{3}{5} + \tan^{-1} \frac{3}{4}\right)$ is equal to

- (A) $\frac{7}{24}$
- (B) $\frac{24}{7}$
- (C) $\frac{3}{2}$
- (D) $\frac{3}{4}$

[1 mark]

Solution:

Concept:

- For a principal angle $\alpha = \sin^{-1}(3/5)$, the angle is acute and $\cos \alpha = 4/5$.
- The addition formula is $\tan(\alpha + \beta) = (\tan \alpha + \tan \beta)/(1 - \tan \alpha \tan \beta)$.

Quick Tip to Solve: Recognise the 3-4-5 triangle to convert the inverse sine into a tangent before adding angles.

Step 1: Express both tangents Let $\alpha = \sin^{-1}(3/5)$ and $\beta = \tan^{-1}(3/4)$. Since both arguments are positive, both angles are acute. Thus $\tan \alpha = (3/5)/(4/5) = 3/4$, and $\tan \beta = 3/4$.

Step 2: Apply the addition formula

$$\tan(\alpha + \beta) = \frac{3/4 + 3/4}{1 - (3/4)(3/4)} = \frac{3/2}{7/16} = \frac{24}{7}.$$

The denominator is positive and nonzero. Option (A) is the reciprocal, (C) keeps only the numerator, and (D) is the tangent of a single angle.

Step 3: Conclusion The required value is $24/7$, so the correct option is (B).

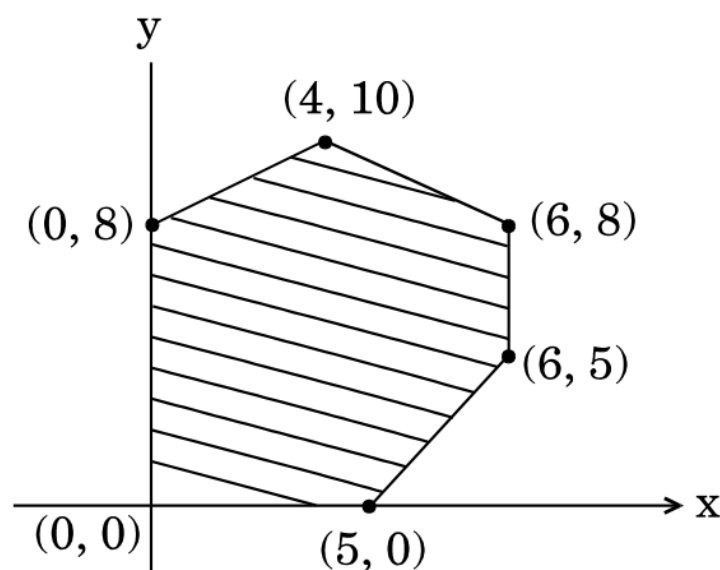
Quick Tip to Solve: Recognise the 3-4-5 triangle to convert the inverse sine into a tangent before adding angles.

3. The feasible region for an LPP is shown below:

Let $z = 3x - 4y$ be the objective function. Minimum of z occurs at

- (A) (0, 0)
- (B) (0, 8)
- (C) (5, 0)
- (D) (4, 10)

[1 mark]



Solution:

Concept:

- A linear objective function on a bounded polygonal feasible region attains its minimum at a corner point.
- Evaluate $z = 3x - 4y$ at every vertex of the shaded region, including vertices absent from the choices.

Quick Tip to Solve: For a negative y coefficient, a high vertex may help, but compare the x contribution at every corner.

Step 1: Read the vertices and evaluate the objective The figure gives the vertices $(0, 0)$, $(5, 0)$, $(6, 5)$, $(6, 8)$, $(4, 10)$ and $(0, 8)$. Their objective values are

$$\begin{aligned}z(0, 0) &= 0, & z(5, 0) &= 15, \\z(6, 5) &= -2, & z(6, 8) &= -14, \\z(4, 10) &= -28, & z(0, 8) &= -32.\end{aligned}$$

Step 2: Compare all corner values The feasible polygon is closed and bounded, so the corner-point test applies. Among $0, 15, -2, -14, -28, -32$, the least value is -32 . Although $(4, 10)$ has the largest ordinate, its positive $3x$ term raises the objective above the value at $(0, 8)$.

Step 3: Conclusion The minimum is $z_{\min} = -32$ at $(0, 8)$. Hence the correct option is (B).

Quick Tip to Solve: For a negative y coefficient, a high vertex may help, but compare the x contribution at every corner.

4. If f and g are two functions from $\mathbb{R}$ to $\mathbb{R}$ defined as $f(x) = |x| + x$ and $g(x) = |x| - x$, then $f \circ g(x)$ for $x < 0$ is

- (A) $4x$
- (B) $2x$
- (C) 0
- (D) $-4x$

[1 mark]

Solution:

Concept:

- The composition $(f \circ g)(x)$ means $f(g(x))$: apply g first and then f .
- For $x < 0$, $|x| = -x$, while the output $-2x$ is positive.

Step 1: Evaluate the inner function Using $x < 0$,

$$g(x) = |x| - x = -x - x = -2x.$$

Since x is negative, $g(x) > 0$; this sign determines the absolute value in the next function.

Step 2: Evaluate the outer function For a positive input u , $f(u) = |u| + u = 2u$. Putting $u = -2x$,

$$f(g(x)) = |-2x| + (-2x) = -2x - 2x = -4x.$$

Choices (A) and (B) are negative for $x < 0$, whereas this output is positive; choice (C) incorrectly treats the inner output as negative.

Step 3: Conclusion Therefore $(f \circ g)(x) = -4x$ for $x < 0$, which is option (D).

Quick Tip to Solve: After evaluating the inner function, check its sign afresh before removing the next absolute value.

5. $\int \frac{1}{x \log x} dx$ is equal to

- (A) $\frac{(\log x)^2}{2} + c$
(B) $\log |\log x| + c$
(C) $\log |x \log x| + c$
(D) $\frac{1}{\log x} + c$

[1 mark]

Solution:

Concept:

- The substitution $u = \log x$ gives $du = dx/x$. Here $\log$ denotes the natural logarithm.
- The standard antiderivative $\int du/u = \log |u| + c$ is valid on intervals where $u \neq 0$.

Step 1: Make the logarithmic substitution The integrand is real for $x > 0$ and defined only when $x \neq 1$. Set $u = \log x$. Then

$$\int \frac{dx}{x \log x} = \int \frac{du}{u} = \log |u| + c.$$

Step 2: Verify and check the alternatives Replacing u gives $\log |\log x| + c$, whose derivative is $1/(x \log x)$. Choice (A) differentiates to $(\log x)/x$, (C) to $1/x + 1/(x \log x)$, and (D) to $-1/[x(\log x)^2]$, so these fail the derivative test.

Step 3: Conclusion The integral is $\log |\log x| + c$, so the correct option is (B).

Quick Tip to Solve: When dx/x accompanies a function of $\log x$, substitute $\log x$ and retain the absolute value in $\log |u|$.

6. The order of the differential equation of the family of circles touching x-axis at the origin is

- (A) 1
(B) 2
(C) 3
(D) 4

[1 mark]

Solution:

Concept:

- A circle tangent to the x-axis at the origin has its centre on the y-axis, say $(0, a)$, and radius $|a|$.
- The order is the highest derivative present after eliminating the arbitrary parameter.

Step 1: Write the one-parameter family The equation is

$$x^2 + (y - a)^2 = a^2, \quad \text{or} \quad x^2 + y^2 = 2ay.$$

There is only one arbitrary parameter a , since tangency fixes the horizontal coordinate of the centre and the radius in terms of a .

Step 2: Differentiate once and eliminate the parameter Differentiating with respect to x gives

Quick Tip to Solve: Count independent parameters after imposing tangency, then eliminate them by differentiation.

$2x + 2yy' = 2ay'$. Multiply this by y and use $2ay = x^2 + y^2$:

$$2xy + 2y^2y' = (x^2 + y^2)y',$$

hence $2xy + (y^2 - x^2)y' = 0$. Only the first derivative occurs. A general circle has more parameters, but the specified tangency removes them; thus orders 2, 3 and 4 do not apply.

Step 3: Conclusion The differential equation is of order 1. Hence the correct option is (A).

Quick Tip to Solve: Count independent parameters after imposing tangency, then eliminate them by differentiation.

7. If $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$, then the value of $|\text{adj } A|$ is

- (A) 64
- (B) 16
- (C) 0
- (D) -8

[1 mark]

Solution:

Concept:

- For an invertible $n \times n$ matrix, $\det(\text{adj } A) = (\det A)^{n-1}$.
- The determinant of a diagonal matrix is the product of its diagonal entries.

Step 1: Calculate the determinant of A The given matrix is $A = -2I_3$, so

$$|A| = (-2)(-2)(-2) = -8.$$

It is nonzero, and therefore the adjugate determinant identity can be applied.

Step 2: Apply the identity and verify directly As the matrix has order 3,

$$|\text{adj } A| = (-8)^{3-1} = 64.$$

Also $\text{adj } A = |A|A^{-1} = (-8)(-I_3/2) = 4I_3$, whose determinant is $4^3 = 64$. Choice (B) is the square of a single adjugate diagonal entry; (D) is $|A|$, and (C) would require a singular adjugate.

Step 3: Conclusion The value of $|\text{adj } A|$ is 64, so the correct option is (A).

Quick Tip to Solve: For a 3 by 3 matrix, square its determinant to obtain the determinant of its adjugate.

8. The image of the point $(2, -1, 4)$ in the YZ-plane is

- (A) $(0, -1, 4)$
- (B) $(-2, -1, 4)$
- (C) $(2, 1, -4)$
- (D) $(2, 0, 4)$

[1 mark]

Solution:

Concept:

- The YZ-plane has equation $x = 0$. Reflection in this plane reverses the coordinate perpendicular to it.
- The image of (x, y, z) in the YZ-plane is $(-x, y, z)$; the coordinates parallel to the plane stay unchanged.

Step 1: Reflect the given point For $P = (2, -1, 4)$, change only the sign of its x -coordinate. Thus its reflected point is $P' = (-2, -1, 4)$. The midpoint of PP' is $(0, -1, 4)$, which lies in the YZ-plane.

Step 2: Check the alternatives Option (A) is the projection onto the plane rather than the reflected image. Option (C) changes the wrong coordinates, and option (D) sets the y -coordinate to zero. Only (B) preserves y, z and reverses x .

Step 3: Conclusion The image is $(-2, -1, 4)$. Hence, the correct option is (B).

Quick Tip to Solve: For reflection in a coordinate plane, reverse only the coordinate absent from its name.

9. The maximum value of slope of the curve $y = -x^3 + 3x^2 + 12x - 5$ is

- (A) 15
- (B) 12
- (C) 9
- (D) 0

[1 mark]

Solution:

Concept:

- The slope of a curve $y = f(x)$ at a point is $f'(x)$.
- A quadratic of the form $M - a(x - h)^2$, with $a > 0$, has maximum value M .

Step 1: Find the slope function Differentiate the given cubic term by term:

$$m(x) = \frac{dy}{dx} = -3x^2 + 6x + 12.$$

We must maximize this derivative, not the original ordinate y .

Step 2: Complete the square

$$m(x) = -3(x^2 - 2x) + 12 = 15 - 3(x - 1)^2.$$

Since $(x - 1)^2 \geq 0$ for every real x , the slope cannot exceed 15. Equality is attained at $x = 1$. Although

Quick Tip to Solve: Differentiate first, then complete the square in the derivative to maximize the slope.

12, 9 and 0 can be slope values, each is smaller than the attained upper bound.

Step 3: Conclusion The maximum slope is $\boxed{15}$, attained at $x = 1$. Hence, the correct option is (A).

Quick Tip to Solve: Differentiate first, then complete the square in the derivative to maximize the slope.

10. The vector equation of XY-plane is

(A) $\vec{r} \cdot \hat{k} = 0$

(B) $\vec{r} \cdot \hat{j} = 0$

(C) $\vec{r} \cdot \hat{i} = 0$

(D) $\vec{r} \cdot \vec{n} = 1$

[1 mark]

Solution:

Concept:

- The XY-plane consists of all points $(x, y, 0)$, so its Cartesian equation is $z = 0$.
- For $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, the dot product $\vec{r} \cdot \hat{k} = z$.

Step 1: Express the plane using its normal The unit vector $\hat{k}$ is perpendicular to the XY-plane. Since this plane passes through the origin, its vector equation has zero on the right-hand side:

$$\vec{r} \cdot \hat{k} = 0.$$

Step 2: Check all alternatives Option (B) imposes $y = 0$, describing the XZ-plane. Option (C) imposes $x = 0$, describing the YZ-plane. In option (D), $\vec{r} \cdot \vec{n} = 1$ does not pass through the origin, whereas the XY-plane does. Thus only option (A) describes exactly the required plane.

Step 3: Conclusion The vector equation is $\boxed{\vec{r} \cdot \hat{k} = 0}$. Hence, the correct option is (A).

Quick Tip to Solve: The missing coordinate names the normal: XY-plane has normal along the z-axis.

11(a). The area of the parallelogram whose diagonals are $2\hat{i}$ and $-3\hat{k}$ is _____ square units. [1 mark]

Solution:

Concept:

- If $\vec{d}_1, \vec{d}_2$ are diagonals of a parallelogram, its area is $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$.
- The factor $\frac{1}{2}$ occurs because, for adjacent sides $\vec{a}, \vec{b}$, the diagonals are $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, whose cross product has twice the side cross-product magnitude.

Step 1: Evaluate the cross product Here $\vec{d}_1 = 2\hat{i}$ and $\vec{d}_2 = -3\hat{k}$. Using $\hat{i} \times \hat{k} = -\hat{j}$,

$$\vec{d}_1 \times \vec{d}_2 = -6(\hat{i} \times \hat{k}) = 6\hat{j}.$$

Its magnitude is 6.

Quick Tip to Solve: When vectors are diagonals, take half the cross-product magnitude, not the full magnitude.

Step 2: Apply the diagonal area formula The required area is

$$A = \frac{1}{2} \times 6 = 3.$$

Equivalently, these perpendicular diagonals have lengths 2 and 3, giving half their product. Using their full product would wrongly treat the diagonals as adjacent sides.

Step 3: Conclusion The blank is $\boxed{3}$ square units.

Quick Tip to Solve: When vectors are diagonals, take half the cross-product magnitude, not the full magnitude.

OR

11(b). The value of λ for which the vectors $2\hat{i} - \lambda\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} - \hat{k}$ are orthogonal is _____. [1 mark]

Solution:

Concept:

- Two nonzero vectors are orthogonal if and only if their scalar product is zero.
- For Cartesian vectors, multiply corresponding components and add: $\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$.

Step 1: Form the orthogonality equation The given vectors have components $(2, -\lambda, 1)$ and $(1, 2, -1)$. Hence

$$\vec{a} \cdot \vec{b} = 2(1) + (-\lambda)(2) + 1(-1) = 1 - 2\lambda.$$

Both vectors are nonzero for every possible value of λ .

Step 2: Solve and check For perpendicular vectors, set $1 - 2\lambda = 0$, which gives $\lambda = \frac{1}{2}$. Substitution into the scalar product gives $2 - 1 - 1 = 0$, confirming orthogonality. Retain the negative sign of the first vector's $\hat{j}$ -component while multiplying.

Step 3: Conclusion The required value is $\boxed{\lambda = \frac{1}{2}}$.

Quick Tip to Solve: Write the component triples before taking the dot product to preserve the sign of the parameter.

12. A bag contains 3 black, 4 red and 2 green balls. If three balls are drawn simultaneously at random, then the probability that the balls are of different colours is _____. [1 mark]

Solution:

Concept:

- For equally likely selections without replacement, probability equals favourable selections divided by all selections.
- As the balls are drawn simultaneously, use unordered combinations. Three different colours means one black, one red and one green ball.

Quick Tip to Solve: For one ball of each colour, multiply the colour counts and divide by the total three-ball combinations.

Step 1: Count all possible selections There are $3 + 4 + 2 = 9$ balls in the bag. The number of ways to select any three balls is

$$\binom{9}{3} = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84.$$

Each subset of three individual balls is equally likely.

Step 2: Count favourable selections Choose one of the three black balls, one of the four red balls and one of the two green balls. These choices can be made in

$$\binom{3}{1} \binom{4}{1} \binom{2}{1} = 3 \cdot 4 \cdot 2 = 24$$

ways. Therefore the required probability is $24/84 = 2/7$. No ordering factor is needed because both counts refer to simultaneous selections.

Step 3: Conclusion The probability is $\boxed{\frac{2}{7}}$.

Quick Tip to Solve: For one ball of each colour, multiply the colour counts and divide by the total three-ball combinations.

13. The minimum value of the function $f(x) = |x + 3| - 1$ is _____. [1 mark]

Solution:

Concept:

- An absolute value is always nonnegative: $|u| \geq 0$, with equality precisely when $u = 0$.
- Subtracting 1 from $|x + 3|$ shifts every function value down by 1.

Step 1: Establish a lower bound For every real x , $|x + 3| \geq 0$. Consequently

$$f(x) = |x + 3| - 1 \geq -1.$$

This bound applies on both sides of $x = -3$, so no differentiability argument at the corner is required.

Step 2: Check that the bound is attained Set $x + 3 = 0$, giving $x = -3$. Substitution yields

$$f(-3) = |-3 + 3| - 1 = 0 - 1 = -1.$$

Since a function value equals the universal lower bound, that bound is its minimum. The number -3 is the location of the minimum, not the minimum value itself.

Step 3: Conclusion The minimum value is $\boxed{-1}$, attained at $x = -3$.

Quick Tip to Solve: Set the expression inside the absolute value to zero, then include the outside constant.

14(a). If $y = \tan^{-1} x + \cot^{-1} x$, $x \in \mathbb{R}$, then $\frac{dy}{dx}$ is equal to ____.

[1 mark]

Solution:

Concept:

- The standard derivatives are $\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$ and $\frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$.
- The derivative of a sum is the sum of the derivatives, with the usual principal branches of the inverse trigonometric functions.

Step 1: Differentiate both terms Applying the sum rule to the given function gives

$$\frac{dy}{dx} = \frac{1}{1+x^2} - \frac{1}{1+x^2}.$$

The two terms have equal magnitudes and opposite signs.

Step 2: Check the domain For every real x , the denominator $1+x^2$ is positive. Both derivative formulas therefore apply everywhere on the stated domain, and the cancellation is valid for all real x . This also agrees with the identity $\tan^{-1} x + \cot^{-1} x = \pi/2$, whose right side is constant.

Step 3: Conclusion $\frac{dy}{dx} = 0$ for every $x \in \mathbb{R}$.

Quick Tip to Solve: Remember that the derivatives of arctan and arccot have opposite signs; their sum is constant.

OR

14(b). If $\cos(xy) = k$, where k is a constant and $xy \neq n\pi$, $n \in \mathbb{Z}$, then $\frac{dy}{dx}$ is equal to ____.

[1 mark]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x .
- The chain rule gives $\frac{d}{dx} \cos u = -\sin u \cdot u'$, and the product rule gives $(xy)' = y + xy'$.

Step 1: Differentiate the equation Since k is constant, its derivative is zero. Applying the chain rule and then the product rule gives

$$-\sin(xy) \left(y + x \frac{dy}{dx} \right) = 0.$$

Both x and y must be included when differentiating the product xy .

Step 2: Use the stated restriction The condition $xy \neq n\pi$, for every integer n , implies $\sin(xy) \neq 0$. Division by this factor is therefore valid, yielding $y + x \frac{dy}{dx} = 0$. Also $x \neq 0$: otherwise $xy = 0$, contrary to the restriction. Thus we can divide by x to isolate the derivative.

Step 3: Conclusion $\frac{dy}{dx} = -\frac{y}{x}$.

Quick Tip to Solve: First differentiate the product inside cosine; use the restriction before cancelling its sine factor.

15. The value of λ so that the function f defined by

$$f(x) = \begin{cases} \lambda x, & \text{if } x \leq \pi \\ \cos x, & \text{if } x > \pi \end{cases}$$

is continuous at $x = \pi$ is _____.

[1 mark]

Solution:

Concept:

- Continuity at $x = a$ requires the left-hand limit, right-hand limit and function value to be equal.
- Use the branch λx at $x = \pi$, since that branch includes equality.

Step 1: Evaluate the two sides From the branch on $x \leq \pi$,

$$\lim_{x \rightarrow \pi^-} f(x) = \lambda\pi, \quad f(\pi) = \lambda\pi.$$

From the branch on $x > \pi$, continuity of cosine gives

$$\lim_{x \rightarrow \pi^+} f(x) = \cos \pi = -1.$$

Step 2: Impose continuity Equating the two one-sided limits gives $\lambda\pi = -1$. Since $\pi \neq 0$, division by π determines a unique parameter. For this choice, the function value from the first branch also equals the right-hand limit, so all parts of the continuity condition hold.

Step 3: Conclusion $\lambda = -\frac{1}{\pi}$.

Quick Tip to Solve: At a piecewise junction, equate the two branch values and then check the branch containing equality.

16(a). Evaluate: $\int_{-2}^2 |x| dx$.

[1 mark]

Solution:

Concept:

- The modulus satisfies $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$.
- A definite integral can be split at a point where the formula for its integrand changes.

Step 1: Split at the sign change The integrand changes its expression at zero, so

$$\int_{-2}^2 |x| dx = \int_{-2}^0 (-x) dx + \int_0^2 x dx.$$

The negative part must use $-x$; using x throughout would incorrectly cancel the two halves.

Quick Tip to Solve: On symmetric limits, use the evenness of modulus to write twice the integral from zero to the upper limit.

Step 2: Evaluate both nonnegative contributions Applying the power rule for integration gives

$$\int_{-2}^0 (-x) dx = \left[-\frac{x^2}{2} \right]_{-2}^0 = 2,$$
$$\int_0^2 x dx = \left[\frac{x^2}{2} \right]_0^2 = 2.$$

The equal contributions also follow from the evenness of $|x|$. Adding them gives the required value.

Step 3: Conclusion $\int_{-2}^2 |x| dx = 4$.

Quick Tip to Solve: On symmetric limits, use the evenness of modulus to write twice the integral from zero to the upper limit.

OR

16(b). Find: $\int \frac{dx}{9 + 4x^2}$.

[1 mark]

Solution:

Concept:

- The standard formula is $\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$, where $a > 0$.
- A linear substitution must include its differential scale factor.

Step 1: Reduce to the standard form Let $u = 2x$, so $du = 2 dx$ and $dx = du/2$. Then

$$\int \frac{dx}{9 + 4x^2} = \frac{1}{2} \int \frac{du}{3^2 + u^2}.$$

Here the constant in the standard formula is $a = 3$, not 9.

Step 2: Integrate and check the scale Using the inverse tangent formula gives

$$\frac{1}{2} \cdot \frac{1}{3} \tan^{-1} \frac{u}{3} + C.$$

Substitute $u = 2x$. As a check, differentiating the resulting expression gives

$$\frac{1}{6} \frac{2/3}{1 + 4x^2/9} = \frac{1}{9 + 4x^2},$$

which is precisely the given integrand.

Step 3: Conclusion $\int \frac{dx}{9 + 4x^2} = \frac{1}{6} \tan^{-1} \frac{2x}{3} + C$.

Quick Tip to Solve: Set u equal to the linear term inside the square, and retain the differential factor before applying arctan.

17. Find the interval in which the function f given by $f(x) = 7 - 4x - x^2$ is strictly increasing. [1 mark]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval where its derivative is positive.
- For a polynomial, the derivative exists for every real x , so the sign condition determines the interval.

Step 1: Find the derivative Differentiate the three terms of $f(x) = 7 - 4x - x^2$ separately:

$$f'(x) = 0 - 4 - 2x = -2(x + 2).$$

The constant differentiates to zero; the quadratic term gives $-2x$.

Step 2: Determine where the derivative is positive The increasing condition is

$$-2(x + 2) > 0.$$

Dividing by the negative number -2 reverses the inequality, giving $x + 2 < 0$, or $x < -2$. At $x = -2$ the derivative vanishes, and for $x > -2$ it is negative, so the usual open interval found from $f'(x) > 0$ lies to the left of -2 .

Step 3: Conclusion The function is strictly increasing on $\boxed{(-\infty, -2)}$.

Quick Tip to Solve: Factor the derivative and remember to reverse the inequality when dividing by a negative coefficient.

18. Differentiate $\sin^2(\sqrt{x})$ with respect to x .

[1 mark]

Solution:

Concept:

- The chain rule differentiates each layer in a composite function, multiplying the resulting factors.
- For $x > 0$, $(\sqrt{x})' = 1/(2\sqrt{x})$; also $\sin(2t) = 2 \sin t \cos t$.

Step 1: Differentiate through all three layers Write $y = [\sin(\sqrt{x})]^2$. Differentiate the outer square, then sine, then the square root:

$$\frac{dy}{dx} = 2 \sin(\sqrt{x}) \cos(\sqrt{x}) \frac{1}{2\sqrt{x}}, \quad x > 0.$$

The factor from the innermost square root is essential.

Step 2: Simplify the expression Cancel the factors of two to obtain $\sin(\sqrt{x}) \cos(\sqrt{x})/\sqrt{x}$. Using the double-angle identity provides the equivalent compact expression $\sin(2\sqrt{x})/(2\sqrt{x})$. Both forms are valid for positive x ; the displayed quotient is interpreted only where its denominator is nonzero.

Step 3: Conclusion $\frac{d}{dx} \sin^2(\sqrt{x}) = \frac{\sin(2\sqrt{x})}{2\sqrt{x}}$, for $x > 0$.

Quick Tip to Solve: For a squared sine of a square root, multiply the square, sine and square-root derivative factors in order.

19. Construct a 2×2 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = |(i)^2 - j|$.

[1 mark]

Solution:

Concept:

- The entry a_{ij} lies in row i , column j . For a 2×2 matrix, both indices take the values 1 and 2.
- An absolute value is non-negative: evaluate $i^2 - j$ first, then take its absolute value.

Step 1: Calculate the first row For row 1, keep $i = 1$ and substitute each column index:

$$a_{11} = |1^2 - 1| = 0, \quad a_{12} = |1^2 - 2| = |-1| = 1.$$

Thus the first row is $(0, 1)$.

Step 2: Calculate the second row For row 2, keep $i = 2$, so $i^2 = 4$. Therefore

$$a_{21} = |4 - 1| = 3, \quad a_{22} = |4 - 2| = 2.$$

Place these entries in the second row, preserving the row and column order.

Step 3: Conclusion

$$A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}.$$

Quick Tip to Solve: Fix the row index before varying the column index; square only i , then apply the absolute value.

20. A black die and a red die are rolled together. Find the conditional probability of obtaining a sum greater than 9 given that the black die resulted in a 5.

[1 mark]

Solution:

Concept:

- For events E and B , $P(E | B) = P(E \cap B)/P(B)$, provided $P(B) > 0$.
- The dice are distinguished by their colours. Given the black result, the six possible red results remain equally likely.

Step 1: Restrict the sample space Let B be the event that the black die shows 5 and E the event that the sum exceeds 9. Under B , the possible ordered results (black, red) are

$$(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6).$$

Each has conditional probability $1/6$.

Step 2: Count the favourable outcomes If the red result is r , the required inequality is $5 + r > 9$, giving $r > 4$. Only $r = 5, 6$ qualify; $r = 4$ gives exactly 9 and is excluded. Thus two of the six conditional outcomes are favourable. Equivalently, $P(E \cap B) = 2/36$ and $P(B) = 6/36$.

Quick Tip to Solve: Once the black die is fixed at 5, count red outcomes only and remember that a sum of 9 is excluded.

Step 3: Conclusion

$$P(E | B) = \frac{2/36}{6/36} = \frac{1}{3}.$$

Quick Tip to Solve: Once the black die is fixed at 5, count red outcomes only and remember that a sum of 9 is excluded.

Section B (12 marks)

Question numbers 21 to 26 carry 2 marks each.

21(a). Show that for any two non-zero vectors $\vec{a}$ and $\vec{b}$, $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ if and only if $\vec{a}$ and $\vec{b}$ are perpendicular vectors. [2 marks]

Solution:

Concept:

- For any vector $\vec{v}$, $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$. Expand dot products using distributivity.
- For non-zero vectors, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$; hence a zero dot product is equivalent to perpendicularity.

Step 1: Prove that equal magnitudes imply perpendicularity Assume $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$. Squaring and expanding gives

$$\begin{aligned} |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} + |\vec{b}|^2 &= |\vec{a}|^2 - 2\vec{a} \cdot \vec{b} + |\vec{b}|^2, \\ 4\vec{a} \cdot \vec{b} &= 0. \end{aligned}$$

Since both vectors are non-zero, their magnitudes have positive product. Thus $\cos \theta = 0$, so $\theta = 90^\circ$.

Step 2: Prove the converse Conversely, suppose $\vec{a} \perp \vec{b}$. Then $\vec{a} \cdot \vec{b} = 0$, and

$$|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 = |\vec{a} - \vec{b}|^2.$$

Both magnitudes are non-negative, so taking their non-negative square roots yields the required equality. This establishes the reverse implication as well.

Step 3: Conclusion Hence $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}| \iff \vec{a} \perp \vec{b}$ for non-zero $\vec{a}, \vec{b}$.

Quick Tip to Solve: Subtract the squared magnitudes: their difference is $4\vec{a} \cdot \vec{b}$, so both directions follow.

21(b). Show that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $3\hat{i} + 7\hat{j} + \hat{k}$ and $5\hat{i} + 6\hat{j} + 2\hat{k}$ form the sides of a right-angled triangle. [2 marks]

Solution:

Concept:

- Vectors representing successive sides of a triangle must close: if $\vec{u} + \vec{v} = \vec{w}$, then $\vec{u}, \vec{v}, -\vec{w}$ give a closed triangle.
- Two non-zero vectors are perpendicular when their dot product is zero. Their magnitudes then obey the Pythagorean relation.

Step 1: Establish that the vectors form triangle sides Write $\vec{u} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{v} = 3\hat{i} + 7\hat{j} + \hat{k}$ and $\vec{w} = 5\hat{i} + 6\hat{j} + 2\hat{k}$. Componentwise addition gives

$$\vec{u} + \vec{v} = (2 + 3)\hat{i} + (-1 + 7)\hat{j} + (1 + 1)\hat{k} = \vec{w}.$$

Choose vertices with $\overrightarrow{AB} = \vec{u}$, $\overrightarrow{BC} = \vec{v}$; then $\overrightarrow{AC} = \vec{w}$. The third directed boundary side is $-\vec{w}$.

Step 2: Show that the included angle is a right angle The dot product is

$$\vec{u} \cdot \vec{v} = 2(3) + (-1)(7) + 1(1) = 6 - 7 + 1 = 0.$$

At vertex B , the outgoing sides are $\overrightarrow{BA} = -\vec{u}$ and $\overrightarrow{BC} = \vec{v}$. Their dot product is also zero, proving $\angle ABC = 90^\circ$. As a check, the squared lengths are $|\vec{u}|^2 = 6$, $|\vec{v}|^2 = 59$, $|\vec{w}|^2 = 65 = 6 + 59$.

Step 3: Conclusion The vectors form the sides of a right-angled triangle, with the side represented by $\vec{w}$ as the hypotenuse.

Quick Tip to Solve: First check the side closure relation, then a zero dot product; perpendicularity alone does not establish a triangle.

22. Find the matrix A such that $A \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & 6 \end{bmatrix}$. [2 marks]

Solution:

Concept:

- Two matrices are equal only when their corresponding entries are equal. Introduce four unknown entries for the required 2×2 matrix.
- In a matrix product, each entry is the dot product of a row of the first matrix with a column of the second.

Quick Tip to Solve: Use the second column first: its zero lower entry immediately gives $2a$ and $2c$ in the product.

Step 1: Multiply using unknown entries Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The given equation becomes

$$\begin{bmatrix} a-b & 2a \\ c-d & 2c \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & 6 \end{bmatrix}.$$

Comparing the second column first gives $2a = 4$ and $2c = 6$, so $a = 2$ and $c = 3$.

Step 2: Find the remaining entries and verify From the first column, $a - b = 3$ gives $2 - b = 3$, hence $b = -1$. Also $c - d = -1$ gives $3 - d = -1$, hence $d = 4$. Verify directly:

$$\begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2+1 & 4 \\ 3-4 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -1 & 6 \end{bmatrix}.$$

This reproduces every entry of the specified right-hand side.

Step 3: Conclusion

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}.$$

Quick Tip to Solve: Use the second column first: its zero lower entry immediately gives $2a$ and $2c$ in the product.

23. If $y = \tan^{-1}\left(\frac{x}{\sqrt{a^2 - x^2}}\right)$, $|x| < a$, then find $\frac{dy}{dx}$. [2 marks]

Solution:

Concept:

- The chain rule gives $\frac{d}{dx} \tan^{-1} u = \frac{u'}{1+u^2}$, where u is a differentiable function of x .
- The condition $|x| < a$ implies $a > 0$ and $a^2 - x^2 > 0$, so the square root and all denominators below are well defined.

Step 1: Differentiate the inner function Put $u = x(a^2 - x^2)^{-1/2}$. By the product and chain rules,

$$\begin{aligned} u' &= (a^2 - x^2)^{-1/2} + x \left(-\frac{1}{2}\right) (a^2 - x^2)^{-3/2} (-2x) \\ &= \frac{a^2 - x^2 + x^2}{(a^2 - x^2)^{3/2}} = \frac{a^2}{(a^2 - x^2)^{3/2}}. \end{aligned}$$

Step 2: Apply the inverse tangent derivative Also,

$$1 + u^2 = 1 + \frac{x^2}{a^2 - x^2} = \frac{a^2}{a^2 - x^2}.$$

Quick Tip to Solve: Compute $1 + u^2$ before multiplying: its factor a^2 cancels the numerator of u' .

Hence the chain rule yields

$$\frac{dy}{dx} = \frac{a^2}{(a^2 - x^2)^{3/2}} \cdot \frac{a^2 - x^2}{a^2}.$$

Cancelling the positive factors leaves the reciprocal of the positive square root.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = \frac{1}{\sqrt{a^2 - x^2}}}, \quad |x| < a.$$

Quick Tip to Solve: Compute $1 + u^2$ before multiplying; its factor a^2 cancels the numerator of u' .

24. If A and B are two events such that $P(A) = 0.4$, $P(B) = 0.3$ and $P(A \cup B) = 0.6$, then find $P(B' \cap A)$. [2 marks]

Solution:

Concept:

- For any two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. The intersection is subtracted because it has been counted twice.
- The events $A \cap B$ and $A \cap B'$ are disjoint and together form A ; thus their probabilities add to $P(A)$.

Step 1: Find the overlap of the events Substitute the given probabilities into the addition rule:

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= 0.4 + 0.3 - 0.6 = 0.1. \end{aligned}$$

This is the probability that both events occur, rather than the probability of either event alone.

Step 2: Remove the overlap from event A The required event $B' \cap A$ means that A occurs while B does not occur. Since

$$A = (A \cap B) \cup (A \cap B'),$$

with the two parts disjoint,

$$P(B' \cap A) = P(A) - P(A \cap B) = 0.4 - 0.1.$$

The subtraction removes precisely those outcomes of A that also belong to B .

Step 3: Conclusion

$$\boxed{P(B' \cap A) = 0.3}.$$

Quick Tip to Solve: For A but not B , subtract the overlap from $P(A)$; do not subtract $P(B)$ itself.

25(a). Solve for x : $\sin^{-1} 4x + \sin^{-1} 3x = -\frac{\pi}{2}$.

[2 marks]

Solution:

Concept:

- The principal values of $\sin^{-1} t$ lie in $[-\pi/2, \pi/2]$, and real values require $-1 \leq t \leq 1$.
- Use $\sin(-\pi/2 - \theta) = -\cos \theta$ and $\cos(\sin^{-1} t) = \sqrt{1 - t^2}$, where the nonnegative square root follows from the principal-value range.

Step 1: Use the principal-value equation The common domain is $|x| \leq 1/4$. Since the given sum is negative and both inverse sine terms have the sign of x , we must have $x < 0$. Rearrange the equation as

$$\sin^{-1}(4x) = -\frac{\pi}{2} - \sin^{-1}(3x).$$

Taking sine on both sides gives

$$4x = -\cos(\sin^{-1}(3x)) = -\sqrt{1 - 9x^2}.$$

Step 2: Solve and verify the admissible value Squaring yields

$$16x^2 = 1 - 9x^2, \quad 25x^2 = 1, \quad x = \pm \frac{1}{5}.$$

The earlier sign requirement rejects $x = 1/5$. For $x = -1/5$, put $\theta = \sin^{-1}(3/5)$, so $0 < \theta < \pi/2$ and $\cos \theta = 4/5$. Therefore $\sin^{-1}(4/5) = \pi/2 - \theta$, and

$$\sin^{-1}(-4/5) + \sin^{-1}(-3/5) = -(\pi/2 - \theta) - \theta = -\pi/2.$$

The candidate satisfies both the domain and the original equation.

Step 3: Conclusion

$$x = -\frac{1}{5}.$$

Quick Tip to Solve: Before squaring, record that the inverse sine sum forces $x < 0$; this eliminates the extraneous positive root.

OR

25(b). Express $\tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right)$, $-\frac{3\pi}{2} < x < \frac{\pi}{2}$ in the simplest form.

[2 marks]

Solution:

Concept:

- The half-angle identity gives $\tan(\pi/4 + x/2) = \frac{\cos x}{1 - \sin x}$ wherever the displayed expressions are defined.

Quick Tip to Solve: Always translate the given interval into the interval for $\pi/4 + x/2$ before cancelling tangent with inverse tangent.

- The principal-value range of $\tan^{-1}$ is $(-\pi/2, \pi/2)$. Thus $\tan^{-1}(\tan \theta) = \theta$ only after checking that θ lies in this interval.

Step 1: Rewrite the ratio using a half-angle identity Using the sine and cosine of $\pi/4 + x/2$,

$$\tan\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{\cos(x/2) + \sin(x/2)}{\cos(x/2) - \sin(x/2)} = \frac{\cos x}{1 - \sin x}.$$

The excluded endpoints are the adjacent values where $\sin x = 1$, so the denominator is nonzero throughout the given open interval.

Step 2: Check the inverse tangent branch From $-3\pi/2 < x < \pi/2$, dividing by two and then adding $\pi/4$ gives

$$-\frac{\pi}{2} < \frac{\pi}{4} + \frac{x}{2} < \frac{\pi}{2}.$$

Hence the angle lies strictly inside the principal-value range. Applying $\tan^{-1}$ to its tangent therefore returns that angle directly, without adding or subtracting any multiple of π .

Step 3: Conclusion

$$\boxed{\tan^{-1}\left(\frac{\cos x}{1 - \sin x}\right) = \frac{\pi}{4} + \frac{x}{2}}, \quad -\frac{3\pi}{2} < x < \frac{\pi}{2}.$$

Quick Tip to Solve: Always translate the given interval into the interval for $\pi/4 + x/2$ before cancelling tangent with inverse tangent.

26. Find the coordinates of the point where the line through $(-1, 1, -8)$ and $(5, -2, 10)$ crosses the ZX-plane. [2 marks]

Solution:

Concept:

- A line through points A and B has vector form $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, where t is a real parameter.
- The ZX-plane consists of all points whose y -coordinate is zero. Its equation is $y = 0$, while x and z are unrestricted.

Step 1: Write the parametric equation of the line Let $A = (-1, 1, -8)$ and $B = (5, -2, 10)$. A direction vector is

$$\overrightarrow{AB} = (5 + 1, -2 - 1, 10 + 8) = (6, -3, 18).$$

Thus a general point of the line has coordinates

$$(x, y, z) = (-1 + 6t, 1 - 3t, -8 + 18t).$$

At $t = 0$ this gives A , and at $t = 1$ it gives B , confirming the line passes through both specified points.

Step 2: Impose the equation of the ZX-plane At the intersection with the ZX-plane,

$$y = 1 - 3t = 0 \implies t = \frac{1}{3}.$$

Quick Tip to Solve: For the ZX-plane set $y = 0$, then substitute its parameter into both x and z .

Substituting this same parameter in the other two coordinates,

$$x = -1 + 6\left(\frac{1}{3}\right) = 1, \quad z = -8 + 18\left(\frac{1}{3}\right) = -2.$$

Since the direction vector has a nonzero y -component, the plane is crossed at a unique point.

Step 3: Conclusion

$$(1, 0, -2)$$

is the point where the line crosses the ZX-plane.

Quick Tip to Solve: For the ZX-plane set $y = 0$, then substitute its parameter into both x and z .

Section C (24 marks)

Question numbers 27 to 32 carry 4 marks each.

27. Solve the following LPP graphically:

Minimise $z = 5x + 7y$

subject to the constraints

$$2x + y \geq 8$$

$$x + 2y \geq 10$$

$$x, y \geq 0$$

[4 marks]

Solution:

Concept:

- Each linear inequality represents a half-plane. The feasible region is the intersection of these half-planes with the first quadrant, where $x \geq 0$ and $y \geq 0$.
- For a linear objective $z = 5x + 7y$, a minimum, if attained, occurs at a corner point. For an unbounded feasible region, also check that the half-plane giving a smaller objective value has no feasible point.

Step 1: Plot the boundary lines and identify the feasible side The line $2x + y = 8$ meets the coordinate axes at $(4, 0)$ and $(0, 8)$. The line $x + 2y = 10$ meets them at $(10, 0)$ and $(0, 5)$. Join each pair of intercepts to draw the two lines. The origin does not satisfy either inequality: $0 < 8$ and $0 < 10$. Therefore, for each line, choose the half-plane away from the origin. Their common part in the first quadrant is the shaded feasible region. It extends indefinitely upwards and to the right.

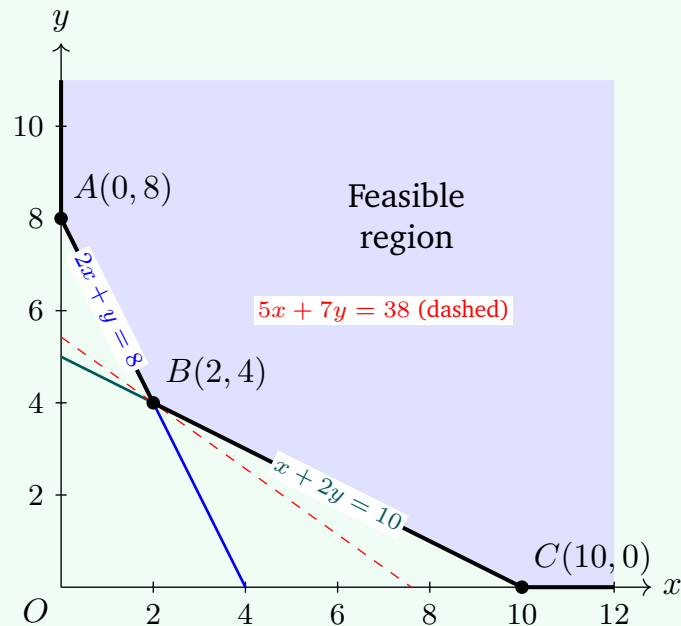
Step 2: Determine the corner points The two boundary lines intersect where

$$\begin{aligned} 2x + y &= 8, \\ x + 2y &= 10. \end{aligned}$$

Multiplying the first equation by 2 and subtracting the second gives $3x = 6$, so $x = 2$ and $y = 4$. Thus the intersection is $B(2, 4)$. On the y -axis, both constraints require $y \geq 8$, giving corner $A(0, 8)$. On the x -axis they require $x \geq 10$, giving corner $C(10, 0)$. The lower boundary of the feasible region consists

Quick Tip to Solve: For an unbounded feasible region, verify the smallest corner value by checking its lower objective half-plane; here adding the first constraint to three times the second proves the lower bound directly.

of the segments AB , BC and the outward axis rays starting at A and C .



Shaded feasible region with corner points A , B and C ; the dashed objective line touches it at B .

Step 3: Evaluate the objective and verify the minimum At the corner points,

Corner point	$z = 5x + 7y$
$A(0, 8)$	56
$B(2, 4)$	38
$C(10, 0)$	50

The smallest corner value is 38. Since the region is unbounded, check the half-plane $5x + 7y < 38$: it lies towards the origin from the dashed line $5x + 7y = 38$ and has no intersection with the feasible region. Indeed, adding the first constraint to three times the second gives

$$(2x + y) + 3(x + 2y) \geq 8 + 30, \quad 5x + 7y \geq 38.$$

Equality holds only when both constraints are equalities, at $(2, 4)$. Hence the graphical candidate is the attained global minimum.

Step 4: Conclusion The minimum value is $z_{\min} = 38$, attained at $x = 2, y = 4$.

Quick Tip to Solve: For an unbounded feasible region, verify the smallest corner value by checking its lower objective half-plane; here adding the first constraint to three times the second proves the lower bound directly.

28. Evaluate: $\int_{-1}^{3/2} |x \sin \pi x| dx$.

[4 marks]

Solution:

Concept:

- For a real expression $f(x)$, $|f(x)| = f(x)$ where $f(x) \geq 0$, and $|f(x)| = -f(x)$ where $f(x) < 0$. Split the interval where the sign changes.
- Integration by parts gives $\int u dv = uv - \int v du$. Here take $u = x$ and $dv = \sin(\pi x) dx$.
- The function $x \sin(\pi x)$ is even because both factors are odd, so its integral from -1 to 1 is twice its integral from 0 to 1 .

Step 1: Determine the signs and remove the modulus On $(-1, 0)$, both x and $\sin(\pi x)$ are negative, so their product is positive. On $(0, 1)$, both are positive. On $(1, 3/2)$, x is positive but $\sin(\pi x)$ is negative. The zeros at the endpoints do not affect the integral. Thus, writing the required integral as I ,

$$\begin{aligned} I &= \int_{-1}^1 x \sin(\pi x) dx - \int_1^{3/2} x \sin(\pi x) dx \\ &= 2 \int_0^1 x \sin(\pi x) dx - \int_1^{3/2} x \sin(\pi x) dx. \end{aligned}$$

Step 2: Integrate by parts With $du = dx$ and $v = -\cos(\pi x)/\pi$, integration by parts gives the primitive

$$F(x) = -\frac{x \cos(\pi x)}{\pi} + \frac{\sin(\pi x)}{\pi^2}.$$

The second factor of $1/\pi$ comes from integrating the cosine. Differentiating this expression cancels the cosine terms and leaves $x \sin(\pi x)$, verifying the calculation.

Step 3: Substitute the limits Since $\cos \pi = -1$, $\sin \pi = 0$, $\cos(3\pi/2) = 0$ and $\sin(3\pi/2) = -1$,

$$F(0) = 0, \quad F(1) = \frac{1}{\pi}, \quad F(3/2) = -\frac{1}{\pi^2}.$$

The first interval contributes $2[F(1) - F(0)] = 2/\pi$. The second contributes

$$-[F(3/2) - F(1)] = \frac{1}{\pi^2} + \frac{1}{\pi}.$$

Both contributions are positive, as an integral of an absolute value requires.

Step 4: Conclusion Adding the contributions,

$$\boxed{\int_{-1}^{3/2} |x \sin(\pi x)| dx = \frac{3}{\pi} + \frac{1}{\pi^2} = \frac{3\pi + 1}{\pi^2}}.$$

Quick Tip to Solve: Check the sign of the product, not just the sine: on the negative interval both factors are negative, so only the interval beyond 1 needs a minus sign.

29(a). A bag contains two coins, one biased and the other unbiased. When tossed, the biased coin has a 60% chance of showing heads. One of the coins is selected at random and on tossing it shows tails. What is the probability it was an unbiased coin? [4 marks]

Solution:

Concept:

- If U and B denote selection of the unbiased and biased coins, random selection gives $P(U) = P(B) = 1/2$. These events are mutually exclusive and exhaustive.
- The total probability of tails T is $P(T) = P(U)P(T | U) + P(B)P(T | B)$.
- Bayes' theorem gives $P(U | T) = P(U)P(T | U)/P(T)$, updating the selection probability after observing tails.

Step 1: Identify the prior and conditional probabilities Let U be the event that the unbiased coin is selected, B the event that the biased coin is selected, and T the event that the toss shows tails. Each coin has the same probability of selection:

$$P(U) = \frac{1}{2}, \quad P(B) = \frac{1}{2}.$$

The unbiased coin has equal chances of heads and tails, so $P(T | U) = 1/2$. The biased coin has heads probability $60/100 = 3/5$, hence

$$P(T | B) = 1 - \frac{3}{5} = \frac{2}{5}.$$

The requested probability is $P(U | T)$; it conditions on the observed outcome, so it is different from the original selection probability.

Step 2: Compute the total probability of tails Tails can arise either from selecting the unbiased coin and then obtaining tails, or from selecting the biased coin and obtaining tails. The two routes are disjoint, so

$$\begin{aligned} P(T) &= P(U)P(T | U) + P(B)P(T | B) \\ &= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{5} \\ &= \frac{1}{4} + \frac{1}{5} = \frac{9}{20}. \end{aligned}$$

The joint probability of selecting the unbiased coin and observing tails is $P(U \cap T) = 1/4$. This must be divided by the probability of all possible tails outcomes to obtain the conditional probability.

Step 3: Apply Bayes' theorem Conditioning on tails gives

$$P(U | T) = \frac{P(U \cap T)}{P(T)} = \frac{1/4}{9/20} = \frac{1}{4} \times \frac{20}{9} = \frac{5}{9}.$$

This is greater than $1/2$, which is consistent with the observation: tails is more likely with the unbiased coin than with the biased coin, so observing tails increases the probability of the unbiased coin having been selected.

Step 4: Conclusion The probability that the selected coin was unbiased, given that it showed tails, is

$$\boxed{\frac{5}{9}}.$$

Quick Tip to Solve: With equal selection probabilities, the posterior odds come directly from the tails probabilities: $1/2 : 2/5 = 5 : 4$, so the unbiased coin has probability $5/(5 + 4)$.

29(b). The probability distribution of a random variable X , where k is a constant is given below:

$$P(X = x) = \begin{cases} 0.1, & \text{if } x = 0 \\ kx^2, & \text{if } x = 1 \\ kx, & \text{if } x = 2 \text{ or } 3 \\ 0, & \text{otherwise} \end{cases}$$

Determine: (a) the value of k ; (b) $P(X \leq 2)$; (c) Mean of the variable X .

[4 marks]

Solution:

Concept:

- Probabilities sum to one: $\sum_x P(X = x) = 1$. For mutually exclusive values, add probabilities to find the probability of an event.
- The mean is $E(X) = \sum_x xP(X = x)$: multiply each possible value by its own probability, then sum.

Step 1: Determine the constant by normalisation The squared branch applies only at one. Substituting into the appropriate branches gives

$$P(0) = 0.1, \quad P(1) = k, \quad P(2) = 2k, \quad P(3) = 3k,$$

where $P(x)$ denotes $P(X = x)$. Normalisation gives

$$0.1 + k + 2k + 3k = 1 \implies 6k = 0.9 \implies k = 0.15.$$

The probabilities are therefore 0.10, 0.15, 0.30, 0.45. They are nonnegative and sum to one, verifying a valid distribution.

Step 2: Find the probability of the stated event The inequality includes its endpoint two. The event consists of the three mutually exclusive values zero, one and two, so

$$P(X \leq 2) = P(0) + P(1) + P(2) = 0.10 + 0.15 + 0.30 = 0.55.$$

The complementary event is $X = 3$, whose probability is 0.45. Thus $1 - 0.45 = 0.55$ checks the result independently.

Step 3: Calculate the mean The probabilities are unequal, so an unweighted average of the four values would be incorrect. Using the expectation formula,

$$\begin{aligned} E(X) &= 0(0.1) + 1(k) + 2(2k) + 3(3k) \\ &= k + 4k + 9k = 14k \\ &= 14(0.15) = 2.10. \end{aligned}$$

Zero contributes nothing to this weighted sum, but its probability is essential in the normalisation equation. The mean lies within the possible range zero to three.

Step 4: Conclusion Thus $\boxed{\text{(a) } k = 0.15, \quad \text{(b) } P(X \leq 2) = 0.55, \quad \text{(c) } E(X) = 2.1}$.

Quick Tip to Solve: List the probabilities as 0.1, k , $2k$ and $3k$ before calculating; the squared branch applies only at the value one.

30. Find the particular solution of the differential equation

$$\frac{dy}{dx} + y \sec x = \tan x, \text{ where } x \in \left[0, \frac{\pi}{2}\right)$$

given that $y = 1$, when $x = \frac{\pi}{4}$.

[4 marks]

Solution:

Concept:

- A first-order linear equation $y' + P(x)y = Q(x)$ has integrating factor $I(x) = e^{\int P(x) dx}$. Multiplication by this factor makes the left side $\frac{d}{dx}(Iy)$.
- Use $\int \sec x dx = \log |\sec x + \tan x|$, $\frac{d}{dx} \sec x = \sec x \tan x$, and $\tan^2 x = \sec^2 x - 1$.

Step 1: Find the integrating factor The equation is already linear, with $P(x) = \sec x$ and $Q(x) = \tan x$. On the given interval $0 \leq x < \pi/2$, both functions are defined and $\sec x + \tan x > 0$. Therefore

$$I(x) = e^{\int \sec x dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x.$$

Its derivative is $\sec x \tan x + \sec^2 x = I(x) \sec x$, so the product-rule identity required for this integrating factor holds throughout the stated domain.

Step 2: Integrate the equation after multiplication Multiplying the original equation by $I(x)$ gives

$$\frac{d}{dx} \{y(\sec x + \tan x)\} = (\sec x + \tan x) \tan x.$$

Integrate both sides and simplify the right side using the trigonometric identity:

$$\begin{aligned} y(\sec x + \tan x) &= \int (\sec x \tan x + \tan^2 x) dx \\ &= \int (\sec x \tan x + \sec^2 x - 1) dx \\ &= \sec x + \tan x - x + C. \end{aligned}$$

Here C is a single arbitrary constant. The term $-x$ is essential because the integral of $\tan^2 x$ is $\tan x - x$.

Step 3: Use the initial value and verify At $x = \pi/4$, we have $y = 1$, $\sec x = \sqrt{2}$ and $\tan x = 1$. Substitution gives

$$\sqrt{2} + 1 = \sqrt{2} + 1 - \frac{\pi}{4} + C, \quad C = \frac{\pi}{4}.$$

Dividing by the positive integrating factor yields

$$y = 1 + \frac{\pi/4 - x}{\sec x + \tan x}.$$

For a check, put $A = \pi/4 - x$ and $I = \sec x + \tan x$. Since $I' = I \sec x$, differentiation gives $y' = -1/I - A \sec x/I$. Hence $y' + y \sec x = \sec x - 1/I = \tan x$, using $1/I = \sec x - \tan x$. Also $A = 0$ at the initial point, so the required value is satisfied.

Quick Tip to Solve: After multiplying by $\sec x + \tan x$, replace $\tan^2 x$ by $\sec^2 x - 1$ before integrating.

Step 4: Conclusion The particular solution is $y = 1 + \frac{\pi/4 - x}{\sec x + \tan x}$, for $0 \leq x < \pi/2$.

Quick Tip to Solve: After multiplying by $\sec x + \tan x$, replace $\tan^2 x$ by $\sec^2 x - 1$ before integrating.

31(a). Show that the function $f : (-\infty, 0) \rightarrow (-1, 0)$ defined by $f(x) = \frac{x}{1 + |x|}$, $x \in (-\infty, 0)$ is one-one and onto. [4 marks]

Solution:

Concept:

- One-one means that equal images imply equal inputs. Onto means that every element of the codomain has a preimage in the domain.
- For negative inputs, $|x| = -x$. Thus the function can be written as $f(x) = x/(1 - x)$ throughout its domain.

Step 1: Check that the values lie in the codomain For $x < 0$, the denominator $1 - x$ is positive. Therefore $f(x) < 0$. Moreover,

$$f(x) + 1 = \frac{x}{1 - x} + 1 = \frac{1}{1 - x} > 0.$$

Hence $-1 < f(x) < 0$ for every $x \in (-\infty, 0)$. The formula therefore gives a well-defined function into $(-1, 0)$. The endpoints are not included and neither is attained.

Step 2: Establish injectivity Take any $a, b \in (-\infty, 0)$ and suppose $f(a) = f(b)$. Since both inputs are negative, the simplified expression applies to both. Thus

$$\begin{aligned}\frac{a}{1 - a} &= \frac{b}{1 - b}, \\ a(1 - b) &= b(1 - a), \\ a - ab &= b - ab, \\ a &= b.\end{aligned}$$

Cross multiplication is valid because both denominators are positive and nonzero. Equal images force equal inputs, proving that f is one-one.

Step 3: Establish surjectivity by constructing a preimage Let y be any element of $(-1, 0)$. Solving the equation $y = x/(1 - x)$ gives

$$y(1 - x) = x \implies y = x(1 + y) \implies x = \frac{y}{1 + y}.$$

Here $y < 0$ and $1 + y > 0$, so this candidate x is negative and belongs to the required domain.

Quick Tip to Solve: Replace $|x|$ by $-x$ first; for onto, solve for the input and verify that $y/(1 + y) < 0$ whenever $-1 < y < 0$.

Substituting it into the function verifies the image exactly:

$$f\left(\frac{y}{1+y}\right) = \frac{\frac{y}{1+y}}{1 - \frac{y}{1+y}} = \frac{\frac{y}{1+y}}{\frac{1}{1+y}} = y.$$

Thus every element of the codomain has a preimage in the domain. This proves that f is onto, without relying merely on limiting values at the endpoints.

Step 4: Conclusion The given function $f : (-\infty, 0) \rightarrow (-1, 0)$ is one-one and onto, and therefore is bijective.

Quick Tip to Solve: Replace $|x|$ by $-x$ first; for onto, solve for the input and verify that $y/(1+y) < 0$ whenever $-1 < y < 0$.

OR

31(b). Show that the relation R in the set $A = \{1, 2, 3, 4, 5, 6\}$ given by $R = \{(a, b) : |a - b| \text{ is divisible by } 2\}$ is an equivalence relation. [4 marks]

Solution:

Concept:

- A relation is an equivalence relation if it is reflexive, symmetric and transitive on its underlying set.
- The condition that $|a - b|$ is divisible by 2 is equivalent to $a - b$ being an even integer. In particular, $0 = 2 \times 0$ is divisible by 2.

Step 1: Verify reflexivity Let a be any element of $A = \{1, 2, 3, 4, 5, 6\}$. Then

$$|a - a| = 0 = 2 \times 0.$$

Hence $|a - a|$ is divisible by 2, so $(a, a) \in R$. This holds for every element of A , proving that R is reflexive. Zero is included among the integer multiples of two, so the diagonal pairs satisfy the given condition.

Step 2: Verify symmetry Suppose $a, b \in A$ and $(a, b) \in R$. By the definition of R , $|a - b|$ is divisible by 2. Since

$$|b - a| = |-(a - b)| = |a - b|,$$

the number $|b - a|$ is also divisible by 2. Therefore $(b, a) \in R$. This implication holds for any related pair, proving that R is symmetric.

Step 3: Verify transitivity Suppose $a, b, c \in A$ with $(a, b) \in R$ and $(b, c) \in R$. The absolute differences are even, so their signed differences are also even. Thus there exist integers m, n such that

$$a - b = 2m, \quad b - c = 2n.$$

Quick Tip to Solve: Interpret the relation as having the same parity; for transitivity add the signed even differences and only then take the absolute value.

Adding these equalities gives

$$a - c = (a - b) + (b - c) = 2(m + n).$$

Consequently $|a - c| = 2|m + n|$ is divisible by 2, and hence $(a, c) \in R$. Thus R is transitive. The argument uses signed differences before taking the absolute value; it does not assume that absolute differences themselves always add.

Step 4: Conclusion Since R is reflexive, symmetric and transitive, it is an equivalence relation on A . Its equivalence classes are $\{1, 3, 5\}$ and $\{2, 4, 6\}$, the odd and even elements respectively.

Quick Tip to Solve: Interpret the relation as having the same parity; for transitivity add the signed even differences and only then take the absolute value.

32. If $y = x^3(\cos x)^x + \sin^{-1} \sqrt{x}$, then find $\frac{dy}{dx}$.

[4 marks]

Solution:

Concept:

- For a variable base raised to a variable power, logarithmic differentiation turns the power into a product: if $u = (\cos x)^x$, then $\ln u = x \ln(\cos x)$, on intervals where $\cos x > 0$.
- The product rule is $(fg)' = f'g + fg'$. The inverse-sine chain rule is $\frac{d}{dx} \sin^{-1} v = \frac{v'}{\sqrt{1-v^2}}$, whenever $|v| < 1$.

Step 1: Differentiate the variable power Set $u = (\cos x)^x$. Taking logarithms and differentiating both sides gives

$$\begin{aligned}\ln u &= x \ln(\cos x), \\ \frac{u'}{u} &= \ln(\cos x) + x \frac{-\sin x}{\cos x} \\ &= \ln(\cos x) - x \tan x.\end{aligned}$$

Therefore $u' = (\cos x)^x [\ln(\cos x) - x \tan x]$. Both the exponent and the base depend on x ; treating the exponent as a constant would omit the logarithm term.

Step 2: Apply the product rule to the first term Since the first term of y is $x^3 u$, its derivative is

$$\begin{aligned}\frac{d}{dx}(x^3 u) &= 3x^2 u + x^3 u' \\ &= 3x^2 (\cos x)^x + x^3 (\cos x)^x [\ln(\cos x) - x \tan x].\end{aligned}$$

Differentiating u alone is insufficient: the factor x^3 also changes with x , producing the first summand above.

Step 3: Differentiate the inverse-sine term and combine With $v = \sqrt{x}$, we have $v' = 1/(2\sqrt{x})$ and $1 - v^2 = 1 - x$. Thus

$$\frac{d}{dx} \sin^{-1} \sqrt{x} = \frac{1}{\sqrt{1-x}} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}\sqrt{1-x}}.$$

Quick Tip to Solve: Differentiate the variable power separately by logarithms, then use the product rule for its factor x^3 .

Add this derivative to the product-rule expression. For the real-valued function, the inverse-sine term requires $0 \leq x \leq 1$; the displayed finite derivative applies on $0 < x < 1$, where $\cos x > 0$ as needed for the logarithm.

Step 4: Conclusion

$$\frac{dy}{dx} = x^2(\cos x)^x[3 + x \ln(\cos x) - x^2 \tan x] + \frac{1}{2\sqrt{x}\sqrt{1-x}}, \quad 0 < x < 1.$$

Quick Tip to Solve: Differentiate the variable power separately by logarithms, then use the product rule for its factor x^3 .

Section D (24 marks)

Question numbers 33 to 36 carry 6 marks each.

33. Find the points on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with both the axes. Also find the equation of the normals. [6 marks]

Solution:

Concept:

- A line with equal intercept lengths on the coordinate axes has slope 1 or -1 : in $x/a + y/b = 1$, equal lengths mean $|a| = |b|$, so $-b/a = \pm 1$.
- At a regular point of a curve, the normal is perpendicular to the tangent. For a finite nonzero tangent slope m_t , its slope is $m_n = -1/m_t$.
- A line with slope m through (x_0, y_0) has equation $y - y_0 = m(x - x_0)$. Candidate points must satisfy both the original curve and the slope condition.

Step 1: Obtain the tangent and normal slopes Differentiate the given curve implicitly with respect to x :

$$9y^2 = x^3 \implies 18y \frac{dy}{dx} = 3x^2.$$

Thus, at a nonzero point of the curve,

$$m_t = \frac{x^2}{6y}, \quad m_n = -\frac{6y}{x^2}.$$

Since $x^3 = 9y^2 \geq 0$, every real point has $x \geq 0$. At a nonzero point $x > 0$, so dividing by x^2 is valid. The origin will be checked separately because this slope formula is undefined there.

Step 2: Enforce equal intercept lengths Equal lengths of the two axis intercepts require $m_n = \pm 1$. Consequently

$$-\frac{6y}{x^2} = \pm 1 \implies y = \pm \frac{x^2}{6}.$$

Quick Tip to Solve: Equal axis-intercept lengths allow both normal slopes 1 and -1 ; retain both signs of y and check the cusp separately.

Squaring this condition and substituting in the original curve gives

$$9\frac{x^4}{36} = x^3 \implies x^3(x-4) = 0.$$

For the nonzero points under consideration, $x = 4$. Hence the ordinates are $y = \pm 16/6 = \pm 8/3$. Substitution confirms $9(8/3)^2 = 64 = 4^3$, so both signs give actual points on the curve.

Step 3: Check the singular point separately The algebra also produced $x = 0$, which gives the cusp $(0, 0)$. Near the origin the two branches are $y = \pm x^{3/2}/3$. Their slopes are $\pm \sqrt{x}/2$, tending to zero as x tends to zero from the right. The tangent at the cusp is therefore the x -axis, and the normal is the y -axis. This normal does not cut both axes in finite, nonzero equal intercepts, so the origin is excluded.

Step 4: Form the two normal equations and verify their intercepts At $(4, 8/3)$, the normal slope is -1 . Its equation is

$$y - \frac{8}{3} = -(x - 4) \implies 3x + 3y = 20.$$

Its axis intercepts are $(20/3, 0)$ and $(0, 20/3)$. At $(4, -8/3)$, the normal slope is 1 , giving

$$y + \frac{8}{3} = x - 4 \implies 3x - 3y = 20.$$

Its axis intercepts are $(20/3, 0)$ and $(0, -20/3)$. The intercept lengths are $20/3$ in each case. The tangent slopes at the upper and lower points are respectively 1 and -1 , confirming perpendicularity to the corresponding normals.

Step 5: Conclusion The required points and corresponding normals are

$$\left(4, \frac{8}{3}\right) : 3x + 3y = 20, \quad \left(4, -\frac{8}{3}\right) : 3x - 3y = 20.$$

Quick Tip to Solve: Equal axis-intercept lengths allow both normal slopes 1 and -1 ; retain both signs of y and check the cusp separately.

34. Show that the lines

$$\frac{x-2}{1} = \frac{y-2}{3} = \frac{z-3}{1} \text{ and } \frac{x-2}{1} = \frac{y-3}{4} = \frac{z-4}{2} \text{ intersect.}$$

Also, find the coordinates of the point of intersection. Find the equation of the plane containing the two lines. [6 marks]

Solution:

Concept:

- A line through (x_0, y_0, z_0) with direction ratios (l, m, n) has parametric equations $x = x_0 + lt$, $y = y_0 + mt$, $z = z_0 + nt$.
- Two lines intersect when some parameter on each line gives the same three coordinates. A plane containing two intersecting lines has a normal perpendicular to both direction vectors.

Quick Tip to Solve: Use the identical x -coordinate equations to set the two parameters equal, then check z after solving y ; cross the two direction vectors for the plane normal.

- If a plane has normal (A, B, C) and passes through (x_0, y_0, z_0) , its equation is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$.

Step 1: Express both lines parametrically Let the common ratio on the first line be t , and that on the second be s . The two lines are therefore

$$L_1 : (x, y, z) = (2 + t, 2 + 3t, 3 + t),$$

$$L_2 : (x, y, z) = (2 + s, 3 + 4s, 4 + 2s).$$

Their direction vectors are $\mathbf{d}_1 = (1, 3, 1)$ and $\mathbf{d}_2 = (1, 4, 2)$. These are not proportional, so the lines are distinct and cannot coincide.

Step 2: Solve the common-coordinate conditions Equality of the x -coordinates gives $2 + t = 2 + s$, hence $t = s$. Equality of the y -coordinates then gives

$$2 + 3t = 3 + 4t \implies t = -1,$$

and consequently $s = -1$. It remains essential to check the third coordinate: two coordinate equations alone do not prove intersection in space. At these parameters, the first line gives $z = 3 - 1 = 2$, while the second gives $z = 4 + 2(-1) = 2$. Thus all three coordinates agree. Substitution into either line gives $P = (1, -1, 2)$.

Step 3: Obtain a normal to the containing plane A normal vector is the cross product of the two direction vectors:

$$\begin{aligned} \mathbf{n} &= \mathbf{d}_1 \times \mathbf{d}_2 \\ &= (3 \cdot 2 - 1 \cdot 4, 1 \cdot 1 - 1 \cdot 2, 1 \cdot 4 - 3 \cdot 1) \\ &= (2, -1, 1). \end{aligned}$$

This is nonzero, so the intersecting lines determine a unique plane. Using P in the point-normal form gives

$$2(x - 1) - (y + 1) + (z - 2) = 0.$$

Step 4: Verify that both entire lines lie in the plane On simplifying, the plane expression is $2x - y + z - 5$. For the first line it becomes

$$2(2 + t) - (2 + 3t) + (3 + t) - 5 = 0,$$

identically for every t . For the second line it becomes

$$2(2 + s) - (3 + 4s) + (4 + 2s) - 5 = 0,$$

identically for every s . This confirms containment of both lines, rather than merely passage through their common point.

Step 5: Conclusion The lines intersect at $(1, -1, 2)$, and the plane containing them is $2x - y + z - 5 = 0$.

Quick Tip to Solve: Use the identical x -coordinate equations to set the two parameters equal, then check z after solving y ; cross the two direction vectors for the plane normal.

35(a). Using integration, find the area of the parabola $y^2 = 4ax$ bounded by its latus rectum. [6 marks]

Solution:

Concept:

- For the standard parabola $y^2 = 4ax$, where $a > 0$, the vertex is the origin, the axis is the x-axis and the focus is $(a, 0)$.
- The latus rectum is the chord through the focus perpendicular to the axis. Its line is $x = a$; its endpoints follow by substituting this value into the parabola.
- For horizontal strips, area is $\int (x_{\text{right}} - x_{\text{left}}) dy$. Reflection symmetry about the x-axis allows the upper-half area to be doubled.

Step 1: Locate the latus rectum and its endpoints The axis of $y^2 = 4ax$ is the x-axis. Hence the latus rectum, passing through the focus $(a, 0)$ perpendicular to that axis, lies on $x = a$. At its intersections with the parabola,

$$y^2 = 4a \cdot a = 4a^2,$$

so $y = 2a$ or $y = -2a$. The endpoints are therefore $(a, 2a)$ and $(a, -2a)$. The required bounded region lies between the parabolic arc and this vertical chord.

Step 2: Determine the width of a horizontal strip Solve the parabola equation for x :

$$x = \frac{y^2}{4a}.$$

At a height y with $-2a \leq y \leq 2a$, the left boundary is the parabola and the right boundary is $x = a$. The strip width is consequently

$$a - \frac{y^2}{4a}.$$

This width is nonnegative throughout the interval because $y^2 \leq 4a^2$. It vanishes at the two endpoints and equals a at $y = 0$, consistent with the geometry of the region.

Step 3: Integrate over the bounded region Using strips of thickness dy , the area is

$$A = \int_{-2a}^{2a} \left(a - \frac{y^2}{4a} \right) dy.$$

The integrand is even, so the areas above and below the x-axis are equal. Thus

$$\begin{aligned} A &= 2 \int_0^{2a} \left(a - \frac{y^2}{4a} \right) dy \\ &= 2 \left[ay - \frac{y^3}{12a} \right]_0^{2a} \\ &= 2 \left(2a^2 - \frac{8a^3}{12a} \right) \\ &= 2 \left(2a^2 - \frac{2a^2}{3} \right). \end{aligned}$$

The lower limit contributes zero to both terms of the antiderivative. The factor of two accounts for

Quick Tip to Solve: With a vertical latus rectum, horizontal strips give the simple polynomial width $a - y^2/4a$; use symmetry to halve the integration interval.

the lower half as well as the upper half.

Step 4: Check the integration using vertical strips Equivalently, $0 \leq x \leq a$, and the upper and lower branches are $y = 2\sqrt{ax}$ and $y = -2\sqrt{ax}$. Their vertical separation is $4\sqrt{ax}$, giving

$$\begin{aligned} A &= \int_0^a 4\sqrt{ax} \, dx \\ &= 4\sqrt{a} \left[\frac{2x^{3/2}}{3} \right]_0^a = \frac{8a^2}{3}. \end{aligned}$$

Both choices of strips agree, and the result has the expected dimensions of area.

Step 5: Conclusion The area bounded by the parabola and its latus rectum is $\boxed{\frac{8a^2}{3}}$ square units.

Quick Tip to Solve: With a vertical latus rectum, horizontal strips give the simple polynomial width a minus y squared divided by $4a$; use symmetry to halve the integration interval.

OR

35(b). Using integration, find the area of the region bounded by the curves $(x-1)^2 + y^2 = 1$ and $x^2 + y^2 = 1$. [6 marks]

Solution:

Concept:

- The area common to two circles is found by integrating the vertical distance between the upper and lower boundaries of their intersection.
- Both circles have radius 1, with centres $(0, 0)$ and $(1, 0)$. Their common region is symmetric about the x -axis and about $x = \frac{1}{2}$.
- For $|u| \leq 1$, $\int \sqrt{1-u^2} \, du = \frac{1}{2} \left(u\sqrt{1-u^2} + \sin^{-1} u \right) + C$, where the inverse sine takes its principal value in radians.

Step 1: Locate the points where the circles meet Subtract the two equations to identify the horizontal coordinate of every intersection:

$$(x-1)^2 - x^2 = 0 \Rightarrow 1 - 2x = 0 \Rightarrow x = \frac{1}{2}.$$

Substituting into $x^2 + y^2 = 1$ gives $y^2 = \frac{3}{4}$. Thus the intersection points are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$. The common region extends horizontally from $x = 0$ to $x = 1$.

Step 2: Select the boundary of the common region In the upper half-plane, the two circles give $y = \sqrt{1-x^2}$ and $y = \sqrt{1-(x-1)^2}$. A point inside both circles must lie below both upper arcs, so the smaller of these two ordinates forms the upper boundary. For $0 \leq x \leq \frac{1}{2}$, this is the arc $\sqrt{1-(x-1)^2}$; for $\frac{1}{2} \leq x \leq 1$, it is $\sqrt{1-x^2}$. Reflection in the x -axis doubles the upper area, and reflection in $x = \frac{1}{2}$

Quick Tip to Solve: Find the common chord first, then integrate the smaller upper ordinate; symmetry reduces this overlap to four equal pieces.

makes the two upper pieces equal. Consequently, if S denotes the required common area,

$$S = 4 \int_{1/2}^1 \sqrt{1-x^2} dx.$$

This choice integrates only the overlap, rather than adding the areas of the entire circles.

Step 3: Evaluate the integral with its limits Apply the standard antiderivative and use $\sin^{-1}(1) = \frac{\pi}{2}$, $\sin^{-1}(\frac{1}{2}) = \frac{\pi}{6}$:

$$\begin{aligned} S &= 2 \left[x\sqrt{1-x^2} + \sin^{-1} x \right]_{1/2}^1 \\ &= 2 \left[\frac{\pi}{2} - \left(\frac{\sqrt{3}}{4} + \frac{\pi}{6} \right) \right] \\ &= 2 \left(\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right). \end{aligned}$$

At the upper limit the product $x\sqrt{1-x^2}$ vanishes; at the lower limit it equals $\sqrt{3}/4$. Both limits therefore contribute to the result.

Step 4: Conclusion The required area common to the two circles is $\boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}$ square units.

Quick Tip to Solve: Find the common chord first, then integrate the smaller upper ordinate; symmetry reduces this overlap to four equal pieces.

36(a). Solve the following system of equations by matrix method:

$$x - y + 2z = 7$$

$$2x - y + 3z = 12$$

$$3x + 2y - z = 5$$

[6 marks]

Solution:

Concept:

- A system of linear equations can be written as $AX = B$, where A contains the coefficients, X is the column of unknowns and B contains the constant terms.
- If $|A| \neq 0$, the coefficient matrix is invertible, and the unique solution is $X = A^{-1}B$.
- The inverse is $A^{-1} = \frac{1}{|A|} \text{adj } A$. The adjugate is the transpose of the cofactor matrix, with $C_{ij} = (-1)^{i+j} M_{ij}$.

Step 1: Express the equations as a matrix equation Keeping the order of the unknowns as x, y, z in every row, write

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 7 \\ 12 \\ 5 \end{pmatrix}.$$

The given equations are precisely $AX = B$. In particular, the negative coefficients of y in the first two

Quick Tip to Solve: Keep the unknowns in the same column order, transpose the cofactor matrix, and check the resulting values in all three equations.

equations and of z in the third equation are retained in A .

Step 2: Calculate the determinant and establish invertibility Expanding along the first row gives

$$\begin{aligned}|A| &= 1[(-1)(-1) - 3(2)] - (-1)[2(-1) - 3(3)] + 2[2(2) - (-1)(3)] \\ &= -5 - 11 + 14 = -2.\end{aligned}$$

Since $|A| \neq 0$, multiplication by A^{-1} is valid and yields a unique solution.

Step 3: Find the cofactors and the inverse matrix The signed minors give the following nine cofactors:

$$\begin{aligned}C_{11} &= 1 - 6 = -5, & C_{12} &= -(-2 - 9) = 11, & C_{13} &= 4 + 3 = 7, \\ C_{21} &= -(1 - 4) = 3, & C_{22} &= -1 - 6 = -7, & C_{23} &= -(2 + 3) = -5, \\ C_{31} &= -3 + 2 = -1, & C_{32} &= -(3 - 4) = 1, & C_{33} &= -1 + 2 = 1.\end{aligned}$$

Transpose this cofactor matrix to obtain

$$\text{adj } A = \begin{pmatrix} -5 & 3 & -1 \\ 11 & -7 & 1 \\ 7 & -5 & 1 \end{pmatrix}, \quad A^{-1} = -\frac{1}{2} \begin{pmatrix} -5 & 3 & -1 \\ 11 & -7 & 1 \\ 7 & -5 & 1 \end{pmatrix}.$$

Transposition is essential: the cofactor matrix itself is not the adjugate.

Step 4: Multiply by the constant column and verify Thus

$$\begin{aligned}X &= A^{-1}B = -\frac{1}{2} \begin{pmatrix} -5 & 3 & -1 \\ 11 & -7 & 1 \\ 7 & -5 & 1 \end{pmatrix} \begin{pmatrix} 7 \\ 12 \\ 5 \end{pmatrix} \\ &= -\frac{1}{2} \begin{pmatrix} -35 + 36 - 5 \\ 77 - 84 + 5 \\ 49 - 60 + 5 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -4 \\ -2 \\ -6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.\end{aligned}$$

Substitution checks all three equations: $2 - 1 + 2(3) = 7$, $2(2) - 1 + 3(3) = 12$, and $3(2) + 2(1) - 3 = 5$. Hence the computed column satisfies the original system exactly.

Step 5: Conclusion The unique solution, obtained by the matrix method, is $\boxed{x = 2, y = 1, z = 3}$.

Quick Tip to Solve: Keep the unknowns in the same column order, transpose the cofactor matrix, and check the resulting values in all three equations.

36(b). Obtain the inverse of the following matrix using elementary operations, where

$$A = \begin{pmatrix} 2 & 1 & -3 \\ -1 & -1 & 4 \\ 3 & 0 & 2 \end{pmatrix}.$$

[6 marks]

Solution:

Concept:

- To find the inverse by elementary row operations, augment the given matrix A with the identity matrix I_3 and reduce $[A \mid I_3]$ to $[I_3 \mid A^{-1}]$.
- Every row operation must act on the entire augmented row, including the identity block. Row interchange, multiplication by a nonzero scalar and adding a multiple of another row are reversible operations.
- Obtaining three unit pivots on the left proves that A is invertible. The transformed right block is its inverse because the same sequence of operations has transformed A into I_3 .

Step 1: Form the augmented matrix and obtain the first pivot Start with

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 2 & 1 & -3 & 1 & 0 & 0 \\ -1 & -1 & 4 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array} \right].$$

First interchange R_1 and R_2 , then multiply the new R_1 by -1 . Thus

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & -1 & 0 \\ 2 & 1 & -3 & 1 & 0 & 0 \\ 3 & 0 & 2 & 0 & 0 & 1 \end{array} \right].$$

Choosing the existing entry -1 as the first pivot avoids fractions in the elimination.

Step 2: Eliminate the entries below the first pivot Apply $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$. The second row has third entry $-3 - 2(-4) = 5$, and the third row has third entry $2 - 3(-4) = 14$. We obtain

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & -1 & 0 \\ 0 & -1 & 5 & 1 & 2 & 0 \\ 0 & -3 & 14 & 0 & 3 & 1 \end{array} \right].$$

Now use $R_2 \rightarrow -R_2$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & -1 & 0 \\ 0 & 1 & -5 & -1 & -2 & 0 \\ 0 & -3 & 14 & 0 & 3 & 1 \end{array} \right].$$

Step 3: Obtain the third unit pivot Apply $R_3 \rightarrow R_3 + 3R_2$. Its third entry becomes $14 + 3(-5) = -1$,

Quick Tip to Solve: Move the second row to the top and reverse its sign to get a unit pivot immediately; carry each operation through both halves of the augmented matrix.

while its right block becomes $(-3, -3, 1)$. Hence

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & -1 & 0 \\ 0 & 1 & -5 & -1 & -2 & 0 \\ 0 & 0 & -1 & -3 & -3 & 1 \end{array} \right].$$

Multiplying R_3 by -1 gives

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -4 & 0 & -1 & 0 \\ 0 & 1 & -5 & -1 & -2 & 0 \\ 0 & 0 & 1 & 3 & 3 & -1 \end{array} \right].$$

Step 4: Clear the entries above the pivots Apply $R_1 \rightarrow R_1 + 4R_3$ and $R_2 \rightarrow R_2 + 5R_3$. This produces

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 12 & 11 & -4 \\ 0 & 1 & 0 & 14 & 13 & -5 \\ 0 & 0 & 1 & 3 & 3 & -1 \end{array} \right].$$

Finally apply $R_1 \rightarrow R_1 - R_2$:

$$[I_3 | A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -2 & 1 \\ 0 & 1 & 0 & 14 & 13 & -5 \\ 0 & 0 & 1 & 3 & 3 & -1 \end{array} \right].$$

As a check, multiplying the original matrix by the right block yields

$$\begin{pmatrix} 2 & 1 & -3 \\ -1 & -1 & 4 \\ 3 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & -2 & 1 \\ 14 & 13 & -5 \\ 3 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Step 5: Conclusion Therefore, using elementary row operations,

$$A^{-1} = \begin{pmatrix} -2 & -2 & 1 \\ 14 & 13 & -5 \\ 3 & 3 & -1 \end{pmatrix}.$$

Quick Tip to Solve: Move the second row to the top and reverse its sign to get a unit pivot immediately; carry each operation through both halves of the augmented matrix.