

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-1-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) The question paper contains **three** Sections – Section A, B and C.
- (ii) Each Section is **Compulsory**.
- (iii) Section – A has 6 short answer type-I questions of 2 marks each.
- (iv) Section – B has 4 short answer type-II questions of 3 marks each.
- (v) Section – C has 4 long answer type questions of 4 marks each.
- (vi) There is an internal choice in some questions.
- (vii) Question No. 14 is a case based problem with 2 subparts of 2 marks each.

Section A (12 marks)

Question numbers 1 to 6 carry 2 marks each.

1. Find the sum of the order and the degree of the differential equation: $\left(x + \frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right)^2 + 1$. [2 marks]

Solution:

Concept:

- The order of a differential equation is the order of its highest derivative. Here the only derivative occurring is the first derivative $y' = \frac{dy}{dx}$.
- The degree is the power of the highest-order derivative after the equation is expressed as a polynomial in derivatives and simplified. Terms that cancel must be removed before reading the degree.

Step 1: Expand and simplify the equation Write the given equation using y' , and expand the square:

$$x^2 + 2xy' + (y')^2 = (y')^2 + 1.$$

The quadratic derivative terms on the two sides cancel exactly, giving

$$2x \frac{dy}{dx} + x^2 - 1 = 0.$$

Thus the apparent square in the original expression does not make the degree two.

Step 2: Identify the order and degree In the simplified polynomial equation the highest derivative is $\frac{dy}{dx}$, so the order is 1. Its power is also 1, so the degree is 1. There are no radicals or fractions involving derivatives requiring any further elimination. Hence their sum is $1 + 1$.

Step 3: Conclusion The order is 1, the degree is 1, and their sum is $\boxed{2}$.

Quick Tip to Solve: Expand both sides and cancel the identical derivative squares before deciding the degree.

2. In a parallelogram PQRS, $\vec{PQ} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{PS} = -\hat{i} - 2\hat{k}$. Find $|\vec{PR}|$ and $|\vec{QS}|$. [2 marks]

Solution:

Concept:

- In a parallelogram, the diagonal starting at the common vertex of two adjacent sides is their vector sum: $\vec{PR} = \vec{PQ} + \vec{PS}$.
- The other diagonal is the difference $\vec{QS} = \vec{PS} - \vec{PQ}$. The magnitude of $a\hat{i} + b\hat{j} + c\hat{k}$, in the orthonormal unit-vector basis, is $\sqrt{a^2 + b^2 + c^2}$.

Step 1: Find the diagonal from P to R Opposite sides of a parallelogram have equal directed vectors,

Quick Tip to Solve: Use the sum for the diagonal from the common vertex and the difference for the other diagonal.

so $\vec{QR} = \vec{PS}$. Following the path from P through Q to R ,

$$\begin{aligned}\vec{PR} &= \vec{PQ} + \vec{QR} = \vec{PQ} + \vec{PS} \\ &= (3\hat{i} - 2\hat{j} + 2\hat{k}) + (-\hat{i} - 2\hat{k}) \\ &= 2\hat{i} - 2\hat{j}.\end{aligned}$$

Consequently, $|\vec{PR}| = \sqrt{2^2 + (-2)^2 + 0^2} = \sqrt{8} = 2\sqrt{2}$.

Step 2: Find the diagonal from Q to S Following the path from Q through P to S , use $\vec{QP} = -\vec{PQ}$:

$$\begin{aligned}\vec{QS} &= \vec{QP} + \vec{PS} = \vec{PS} - \vec{PQ} \\ &= (-\hat{i} - 2\hat{k}) - (3\hat{i} - 2\hat{j} + 2\hat{k}) \\ &= -4\hat{i} + 2\hat{j} - 4\hat{k}.\end{aligned}$$

Hence $|\vec{QS}| = \sqrt{(-4)^2 + 2^2 + (-4)^2} = \sqrt{36} = 6$. Reversing the direction of a diagonal would change its vector sign but not this magnitude.

Step 3: Conclusion The diagonal lengths are $|\vec{PR}| = 2\sqrt{2}$ and $|\vec{QS}| = 6$.

Quick Tip to Solve: Use the sum for the diagonal from the common vertex and the difference for the other diagonal.

3(a). If $\frac{d}{dx}[F(x)] = \frac{\sec^4 x}{\csc^4 x}$ and $F\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$, then find $F(x)$.

[2 marks]

Solution:

Concept:

- An antiderivative $F(x)$ satisfies $F'(x) = f(x)$; integrating $f(x)$ gives $F(x)$ up to an additive constant C , which is fixed by the supplied value of F .
- Use $\sec^4 x / \csc^4 x = \tan^4 x$ and $\tan^2 x = \sec^2 x - 1$. Since $d(\tan x) = \sec^2 x dx$, the product $\tan^2 x \sec^2 x$ integrates by direct substitution.

Step 1: Integrate the derivative On the domain of the printed expression,

$$F'(x) = \frac{1/\cos^4 x}{1/\sin^4 x} = \tan^4 x.$$

Rewrite the fourth power as

$$\tan^4 x = \tan^2 x (\sec^2 x - 1) = \tan^2 x \sec^2 x - \sec^2 x + 1.$$

Integrating term by term, with $u = \tan x$ in the first integral, gives

$$F(x) = \frac{\tan^3 x}{3} - \tan x + x + C.$$

Quick Tip to Solve: Split tan to the fourth power into tan squared times sec squared, minus sec squared, plus one.

Step 2: Apply the given value of F Since $\tan(\pi/4) = 1$, the condition implies

$$\frac{1}{3} - 1 + \frac{\pi}{4} + C = \frac{\pi}{4}, \quad C = \frac{2}{3}.$$

This determines F on the interval $0 < x < \pi/2$ containing the given point, where the original derivative is defined. Differentiating the resulting expression gives $\tan^2 x \sec^2 x - \sec^2 x + 1 = \tan^4 x$, verifying the required derivative and the integration signs.

Step 3: Conclusion $F(x) = \frac{\tan^3 x}{3} - \tan x + x + \frac{2}{3}$, on the interval containing $\pi/4$ where the given derivative is defined.

Quick Tip to Solve: Split $\tan$ to the fourth power into $\tan$ squared times $\sec$ squared, minus $\sec$ squared, plus one.

OR

3(b). Find: $\int \frac{\log x - 3}{(\log x)^4} dx$.

[2 marks]

Solution:

Concept:

- Reverse the product rule: the integral of a derivative of a product is that product plus an arbitrary constant.
- The chain rule gives $\frac{d}{dx} \log x = 1/x$. The real domain here is $x > 0, x \neq 1$.

Step 1: Identify a suitable product Separate the integrand into two terms:

$$\frac{\log x - 3}{(\log x)^4} = \frac{1}{(\log x)^3} - \frac{3}{(\log x)^4}.$$

The first term suggests the product $x(\log x)^{-3}$. When the logarithmic factor is differentiated, the chain rule produces $1/x$, which cancels the factor x . This explains why this product can generate both terms of the integrand.

Step 2: Verify the antiderivative by differentiation By the product rule followed by the chain rule,

$$\begin{aligned} \frac{d}{dx} [x(\log x)^{-3}] &= (\log x)^{-3} + x(-3)(\log x)^{-4} \frac{1}{x} \\ &= \frac{1}{(\log x)^3} - \frac{3}{(\log x)^4} \\ &= \frac{\log x - 3}{(\log x)^4}. \end{aligned}$$

The derivative agrees exactly with the given integrand. The antiderivative is valid separately on $(0, 1)$ and $(1, \infty)$, and the constant of integration is arbitrary on each interval.

Quick Tip to Solve: For a numerator $\log x - n$ over $(\log x)^{n+1}$, try differentiating $x/(\log x)^n$.

Step 3: Conclusion

$$\int \frac{\log x - 3}{(\log x)^4} dx = \frac{x}{(\log x)^3} + C, \quad x > 0, x \neq 1.$$

Quick Tip to Solve: For a numerator $\log x - n$ over $(\log x)^{n+1}$, try differentiating $x/(\log x)^n$.

4. Let A and B be two events such that $P(A) = \frac{5}{8}$, $P(B) = \frac{1}{2}$ and $P(A/B) = \frac{3}{4}$. Find the value of $P(B/A)$. [2 marks]

Solution:**Concept:**

- Conditional probability is $P(A | B) = P(A \cap B)/P(B)$, provided $P(B) > 0$.
- Reversing the conditioning event preserves the intersection: $P(B | A) = P(A \cap B)/P(A)$, provided $P(A) > 0$.

Step 1: Calculate the intersection probability The notation $P(A/B)$ in the question means $P(A | B)$. Using the given conditional probability and $P(B) = 1/2$,

$$P(A \cap B) = P(A | B)P(B) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}.$$

This is the probability that both events occur. It does not depend on the order in which the events are named, since $A \cap B = B \cap A$.

Step 2: Change the conditioning event to A Now the event forming the denominator is A, whose probability is $5/8$. Therefore,

$$P(B | A) = \frac{P(B \cap A)}{P(A)} = \frac{3/8}{5/8} = \frac{3}{8} \times \frac{8}{5} = \frac{3}{5}.$$

Both denominators are positive, so both conditional probabilities are defined. Notice that the reverse conditional probability need not equal $3/4$, because the conditioning events have different probabilities.

Step 3: Conclusion The required probability is $P(B/A) = \frac{3}{5}$.

Quick Tip to Solve: First find the common intersection probability, then divide it by the probability of the new conditioning event.

5. Two balls are drawn at random from a bag containing 2 red balls and 3 blue balls, without replacement. Let the variable X denote the number of red balls. Find the probability distribution of X . [2 marks]

Solution:

Concept:

- A probability distribution lists every possible value of a random variable and the probability of each value.
- For a random selection without replacement, count unordered selections using $\binom{n}{r}$, the number of ways to choose r objects from n objects.

Step 1: List the possible values and count the sample space There are 2 red balls and 3 blue balls, making 5 balls in total. Since two balls are selected, the number X of red balls can be 0, 1, or 2. Treating the individual balls as distinct, all unordered pairs are equally likely. The total number of such pairs is

$$\binom{5}{2} = \frac{5 \times 4}{2} = 10.$$

We count pairs rather than multiply unchanged colour probabilities, because the first ball is not replaced before the second draw.

Step 2: Count each possible colour combination For $X = 0$, both balls are blue; for $X = 1$, choose one red and one blue; and for $X = 2$, both balls are red. Thus,

$$P(X = 0) = \frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10},$$

$$P(X = 1) = \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \frac{6}{10} = \frac{3}{5},$$

$$P(X = 2) = \frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}} = \frac{1}{10}.$$

These events are mutually exclusive and exhaust all possible pairs. The check $3/10 + 6/10 + 1/10 = 1$ confirms that the probabilities form a complete distribution.

Step 3: Conclusion The probability distribution is

x	0	1	2
$P(X = x)$	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{1}{10}$

Quick Tip to Solve: For sampling without replacement, use $P(X = r) = \frac{\binom{2}{r}\binom{3}{2-r}}{\binom{5}{2}}$ and check the sum is 1.

6. Find the values of λ , for which the distance of the point $(2, 1, \lambda)$ from the plane $3x + 5y + 4z = 11$ is $2\sqrt{2}$ units. [2 marks]

Solution:

Concept:

- The perpendicular distance of a point (x_1, y_1, z_1) from a plane $Ax + By + Cz + D = 0$ is $d = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}$.
- The absolute value allows the point to lie on either side of the plane. Thus $|u| = k$, for $k > 0$, gives both $u = k$ and $u = -k$.

Step 1: Substitute the point into the distance formula Write the plane as $3x + 5y + 4z - 11 = 0$. Its normal vector is $(3, 5, 4)$, whose length is $\sqrt{9 + 25 + 16} = \sqrt{50} = 5\sqrt{2}$. For the point $(2, 1, \lambda)$,

$$d = \frac{|3(2) + 5(1) + 4\lambda - 11|}{5\sqrt{2}} = \frac{|4\lambda|}{5\sqrt{2}}.$$

Step 2: Solve the absolute-value equation Equating this distance to the given $2\sqrt{2}$ gives

$$|4\lambda| = (2\sqrt{2})(5\sqrt{2}) = 20.$$

Hence $4\lambda = 20$ or $4\lambda = -20$. Both cases must be retained because distance is non-negative. Substituting either candidate into the formula gives $20/(5\sqrt{2}) = 2\sqrt{2}$, verifying the prescribed distance.

Step 3: Conclusion The required values are $\lambda = 5$ or $\lambda = -5$.

Quick Tip to Solve: Simplify the numerator before solving: here the constant terms cancel, but retain the absolute value to obtain both values of the parameter.

Section B (12 marks)

Question numbers 7 to 10 carry 3 marks each.

7(a). If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four non-zero vectors such that $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = 4\vec{b} \times \vec{d}$, then show that $(\vec{a} - 2\vec{d})$ is parallel to $(2\vec{b} - \vec{c})$, where $\vec{a} \neq 2\vec{d}, \vec{c} \neq 2\vec{b}$. [3 marks]

Solution:

Concept:

- For two non-zero vectors $\vec{u}$ and $\vec{v}$, $\vec{u} \times \vec{v} = \vec{0}$ implies that they are parallel, since $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$.
- The cross product distributes over addition and scalar multiplication, but its order matters: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

Step 1: Form the cross product of the required vectors To prove the requested parallelism, consider

$$\begin{aligned} (\vec{a} - 2\vec{d}) \times (2\vec{b} - \vec{c}) &= 2(\vec{a} \times \vec{b}) - \vec{a} \times \vec{c} \\ &\quad - 4(\vec{d} \times \vec{b}) + 2(\vec{d} \times \vec{c}). \end{aligned}$$

Quick Tip to Solve: Expand the cross product of the two target vectors, then reverse the order of products containing the last vector so the given identities can be substituted directly.

Each scalar coefficient multiplies its corresponding cross product. In particular, the product of the two negative terms is positive, giving the last term $2(\vec{d} \times \vec{c})$.

Step 2: Apply the two given identities Reverse the order of the last two cross products using anti-commutativity. Then

$$\begin{aligned}(\vec{a} - 2\vec{d}) \times (2\vec{b} - \vec{c}) &= 2(\vec{a} \times \vec{b}) - \vec{a} \times \vec{c} \\&\quad + 4(\vec{b} \times \vec{d}) - 2(\vec{c} \times \vec{d}) \\&= 2(\vec{c} \times \vec{d}) - 4(\vec{b} \times \vec{d}) \\&\quad + 4(\vec{b} \times \vec{d}) - 2(\vec{c} \times \vec{d}) \\&= \vec{0}.\end{aligned}$$

Here the first given identity replaces $\vec{a} \times \vec{b}$, and the second replaces $\vec{a} \times \vec{c}$. The resulting terms cancel in pairs.

Step 3: Check the non-zero condition The hypotheses $\vec{a} \neq 2\vec{d}$ and $\vec{c} \neq 2\vec{b}$ ensure that both $\vec{a} - 2\vec{d}$ and $2\vec{b} - \vec{c}$ are non-zero. Their zero cross product therefore gives $\sin \theta = 0$, so their directions are either the same or opposite; either case is parallelism.

Step 4: Conclusion Hence $\boxed{(\vec{a} - 2\vec{d}) \parallel (2\vec{b} - \vec{c})}$, as required.

Quick Tip to Solve: Expand the cross product of the two target vectors, then reverse the order of products containing the last vector so the given identities can be substituted directly.

OR

7(b). The two adjacent sides of a parallelogram are represented by $2\hat{i} - 4\hat{j} - 5\hat{k}$ and $2\hat{i} + 2\hat{j} + 3\hat{k}$. Find the unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram also. [3 marks]

Solution:

Concept:

- For adjacent side vectors $\vec{a}$ and $\vec{b}$, the diagonal vectors are $\vec{d}_1 = \vec{a} + \vec{b}$ and $\vec{d}_2 = \vec{a} - \vec{b}$, up to reversal of direction.
- A unit vector parallel to a nonzero vector $\vec{d}$ is $\pm \vec{d}/|\vec{d}|$. Both signs describe the two possible orientations along the same diagonal.
- The area of the parallelogram is $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$, since the cross product of the diagonal vectors has twice the magnitude of the cross product of the sides.

Step 1: Form the diagonal vectors Let $\vec{a} = 2\hat{i} - 4\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + 2\hat{j} + 3\hat{k}$. Adding and subtracting the adjacent side vectors gives

$$\vec{d}_1 = 4\hat{i} - 2\hat{j} - 2\hat{k}, \quad \vec{d}_2 = -6\hat{j} - 8\hat{k}.$$

The second diagonal can equally be written as $\vec{b} - \vec{a}$; this only reverses its direction and does not affect

Quick Tip to Solve: Add and subtract the adjacent side vectors first; take half the magnitude of their diagonal cross product for the area.

its length or the area.

Step 2: Normalize each diagonal The magnitudes are

$$|\vec{d}_1| = \sqrt{16 + 4 + 4} = 2\sqrt{6}, \quad |\vec{d}_2| = \sqrt{36 + 64} = 10.$$

Hence the unit vectors parallel to the first diagonal are $\pm(2\hat{i} - \hat{j} - \hat{k})/\sqrt{6}$, and those parallel to the second are $\pm(3\hat{j} + 4\hat{k})/5$. The latter expression absorbs the minus sign in $\vec{d}_2$ into the choice of orientation.

Step 3: Calculate area using the diagonals Using the diagonal vectors found above,

$$\begin{aligned} \vec{d}_1 \times \vec{d}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & -6 & -8 \end{vmatrix} \\ &= 4\hat{i} + 32\hat{j} - 24\hat{k}. \end{aligned}$$

Therefore

$$\frac{1}{2}|\vec{d}_1 \times \vec{d}_2| = \frac{1}{2}\sqrt{16 + 1024 + 576} = 2\sqrt{101}.$$

The factor $1/2$ is necessary because $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = -2(\vec{a} \times \vec{b})$.

Step 4: Conclusion The required unit vectors are $\pm \frac{2\hat{i} - \hat{j} - \hat{k}}{\sqrt{6}}$ and $\pm \frac{3\hat{j} + 4\hat{k}}{5}$. The area is $2\sqrt{101}$ square units.

Quick Tip to Solve: Add and subtract the adjacent side vectors first; take half the magnitude of their diagonal cross product for the area.

8. Find the vector equation of the plane passing through the intersection of the planes $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$ and $\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$ and through the point $(2, 1, 3)$. [3 marks]

Solution:

Concept:

- A plane containing the line of intersection of $P_1 = 0$ and $P_2 = 0$ belongs to the family $P_1 + \lambda P_2 = 0$, where λ is a real parameter (with $P_2 = 0$ as the omitted family member).
- The position vector of the point (x, y, z) is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. A plane with normal vector $\vec{n}$ has vector equation $\vec{r} \cdot \vec{n} = d$.
- Substituting the specified point in the plane family determines the parameter. Multiplying an entire plane equation by a nonzero constant leaves the plane unchanged.

Step 1: Write the family containing the common line The given planes in Cartesian form are $P_1 = 2x + 2y - 3z - 7 = 0$ and $P_2 = 2x + 5y + 3z - 9 = 0$. Thus consider

$$2x + 2y - 3z - 7 + \lambda(2x + 5y + 3z - 9) = 0.$$

Quick Tip to Solve: Keep the constants inside the plane-family expressions, substitute the point, then clear the parameter denominator before collecting terms.

Every point on their intersection satisfies both $P_1 = 0$ and $P_2 = 0$, and consequently satisfies this equation for every λ . The specified point is not on the omitted member $P_2 = 0$, so a finite value of λ suffices.

Step 2: Use the point to determine the parameter At $(2, 1, 3)$, the two left-hand expressions evaluate to

$$P_1 = 4 + 2 - 9 - 7 = -10, \quad P_2 = 4 + 5 + 9 - 9 = 9.$$

The required plane must therefore satisfy $-10 + 9\lambda = 0$, giving $\lambda = 10/9$. Multiplying the resulting family equation by 9 gives

$$9(2x + 2y - 3z - 7) + 10(2x + 5y + 3z - 9) = 0.$$

Collecting coefficients yields $38x + 68y + 3z = 153$.

Step 3: Express and check the vector equation The normal vector is $38\hat{i} + 68\hat{j} + 3\hat{k}$, so the equation can be written with a dot product. As a check on the point condition, $38(2) + 68(1) + 3(3) = 76 + 68 + 9 = 153$. As a check on the intersection condition, its left-hand side minus 153 equals $9P_1 + 10P_2$, which vanishes at every point common to the original planes. The normals of the original planes are not parallel, so their intersection is a line and these conditions determine the required plane.

Step 4: Conclusion The required vector equation is $\boxed{\vec{r} \cdot (38\hat{i} + 68\hat{j} + 3\hat{k}) = 153}$.

Quick Tip to Solve: Keep the constants inside the plane-family expressions, substitute the point, then clear the parameter denominator before collecting terms.

9(a). Find: $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}.$

[3 marks]

Solution:

Concept:

- A substitution using the least common multiple of the root indices removes both fractional powers. Here take $t = x^{1/6}$, so $x = t^6$, $\sqrt{x} = t^3$ and $\sqrt[3]{x} = t^2$.
- Polynomial division turns an improper rational expression into polynomial terms and a logarithmic term. Use $\int t^n dt = t^{n+1}/(n+1)$ and $\int dt/(t+1) = \ln|t+1|$.

Step 1: Remove the fractional powers The real integrand is defined for $x > 0$: the square root requires $x \geq 0$, and the denominator vanishes at zero. Put $x = t^6$, with $t > 0$. Then $dx = 6t^5 dt$, and the integral becomes

$$I = \int \frac{6t^5}{t^3 + t^2} dt = 6 \int \frac{t^3}{t+1} dt.$$

The factor t^2 cancels because $t > 0$. This substitution changes the differential as well as the two roots; replacing only the roots would give an incorrect power in the numerator.

Quick Tip to Solve: For square and cube roots together, put x equal to the sixth power of a new variable, then divide the resulting rational expression before integrating.

Step 2: Divide and integrate term by term Since $(t+1)(t^2-t+1) = t^3+1$, polynomial division gives

$$\frac{t^3}{t+1} = t^2 - t + 1 - \frac{1}{t+1}.$$

Hence

$$\begin{aligned} I &= 6 \int \left(t^2 - t + 1 - \frac{1}{t+1} \right) dt \\ &= 2t^3 - 3t^2 + 6t - 6 \ln |t+1| + C. \end{aligned}$$

On the present domain $t+1 > 0$, so the absolute value may be omitted. Substitute $t = x^{1/6}$ in every term. As a check, differentiating the expression in t gives $6t^3/(t+1)$, precisely the transformed integrand.

Step 3: Conclusion

$$I = 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6 \ln(1 + \sqrt[6]{x}) + C, \quad x > 0.$$

Quick Tip to Solve: For square and cube roots together, put x equal to the sixth power of a new variable, then divide the resulting rational expression before integrating.

OR

9(b). Evaluate: $\int_0^{\pi/2} \frac{\cos x}{(1+\sin x)(4+\sin x)} dx.$

[3 marks]

Solution:

Concept:

- When the numerator is $\cos x$ and the denominator is a function of $\sin x$, use $t = \sin x$, giving $dt = \cos x dx$. For a definite integral, transform the limits at the same time.
- Resolve the rational function into partial fractions: $1/((t+1)(t+4)) = A/(t+1) + B/(t+4)$. Integrating each reciprocal linear factor produces a natural logarithm.

Step 1: Substitute and transform the limits Let $t = \sin x$. Then $dt = \cos x dx$. At the lower limit $x = 0$, $t = 0$; at the upper limit $x = \pi/2$, $t = 1$. Thus

$$I = \int_0^1 \frac{dt}{(1+t)(4+t)}.$$

Both denominator factors are positive throughout $[0, 1]$, so this is a proper definite integral with no singularities. The new bounds must be used with the new variable; retaining the old angle bounds after this substitution would change the value of the integral.

Quick Tip to Solve: Substitute $\sin x$ immediately when its derivative $\cos x$ is in the numerator; the difference of the two denominator constants gives the partial-fraction factor $1/3$.

Step 2: Find and integrate the partial fractions Write

$$\frac{1}{(t+1)(t+4)} = \frac{A}{t+1} + \frac{B}{t+4}.$$

Equating numerators gives $1 = A(t+4) + B(t+1)$. Taking $t = -1$ gives $A = 1/3$, and taking $t = -4$ gives $B = -1/3$. These substitutions determine the algebraic identity; neither value lies in the integration interval. Therefore

$$\begin{aligned} I &= \frac{1}{3} \int_0^1 \left(\frac{1}{t+1} - \frac{1}{t+4} \right) dt \\ &= \frac{1}{3} [\ln(t+1) - \ln(t+4)]_0^1 \\ &= \frac{1}{3} (\ln 2 - \ln 5 - \ln 1 + \ln 4) \\ &= \frac{1}{3} \ln \frac{8}{5}. \end{aligned}$$

All logarithm arguments are positive on the interval. Also $8/5 > 1$, so the value is positive, consistent with the positive original integrand.

Step 3: Conclusion

$$\int_0^{\pi/2} \frac{\cos x}{(1 + \sin x)(4 + \sin x)} dx = \frac{1}{3} \ln \frac{8}{5}.$$

Quick Tip to Solve: Substitute $\sin x$ immediately when its derivative $\cos x$ is in the numerator; the difference of the two denominator constants gives the partial-fraction factor $1/3$.

10. Find the particular solution of the differential equation $x \frac{dy}{dx} + x \cos^2\left(\frac{y}{x}\right) = y$; given that when $x = 1, y = \frac{\pi}{4}$. [3 marks]

Solution:

Concept:

- A first-order equation whose derivative is a function of y/x is homogeneous. Put $v = y/x$, so that $y = xv$ and the product rule gives $dy/dx = v + x dv/dx$.
- After substitution, separate the variables. The identity $1/\cos^2 v = \sec^2 v$ and the integral $\int \sec^2 v dv = \tan v + C$ are used. Since the initial value is at $x = 1$, work on the positive x -axis, where $\int dx/x = \ln x + C$.

Step 1: Reduce the equation using the homogeneous substitution The printed equation, divided by $x \neq 0$, is

$$\frac{dy}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right).$$

Quick Tip to Solve: Put $y=xv$ first: the v terms cancel, and dividing by $\cos^2 v$ turns the separated integral directly into $\tan v$.

Substitute $y = xv$ and $dy/dx = v + x dv/dx$:

$$v + x \frac{dv}{dx} + \cos^2 v = v,$$
$$x \frac{dv}{dx} = -\cos^2 v.$$

Thus the terms containing v cancel, leaving a separable equation. At the initial point $v = \pi/4$, $\cos v \neq 0$, so division by $\cos^2 v$ is valid on the branch through that point.

Step 2: Integrate and use the initial value Separating and integrating gives

$$\sec^2 v dv = -\frac{dx}{x},$$
$$\tan v = -\ln x + C.$$

At $x = 1$, $v = y/x = \pi/4$. Therefore

$$1 = -\ln 1 + C, \quad C = 1.$$

Hence the branch satisfying the given condition has $\tan(y/x) = 1 - \ln x$. The initial ratio $\pi/4$ lies in $(-\pi/2, \pi/2)$, so the principal inverse tangent selects the required continuous branch.

Step 3: Conclusion The particular solution is $\tan(y/x) + \ln x = 1$. Equivalently, on the branch through the prescribed point,

$$y = x \tan^{-1}(1 - \ln x), \quad x > 0.$$

Quick Tip to Solve: Put $y=xv$ first: the v terms cancel, and dividing by $\cos$ squared v turns the separated integral directly into $\tan v$.

Section C (16 marks)

Question numbers 11 to 14 carry 4 marks each.

11(a). Using integration, find the area of the region $\{(x, y) : 4x^2 + 9y^2 \leq 36, 2x + 3y \geq 6\}$. [4 marks]

Solution:

Concept:

- The ellipse $4x^2 + 9y^2 = 36$ has intercepts $(\pm 3, 0)$ and $(0, \pm 2)$. Its upper boundary is $y = \frac{2}{3}\sqrt{9 - x^2}$.
- For vertical strips, the area between an upper curve $y = u(x)$ and a lower curve $y = l(x)$ is $A = \int_a^b [u(x) - l(x)] dx$, where a, b are the limiting abscissae.
- The inequality $2x + 3y \geq 6$ selects the side above the line $y = 2 - \frac{2x}{3}$. The standard antiderivative is $\int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) + C$.

Quick Tip to Solve: First find the chord's intercepts: here they are the ellipse's positive axis intercepts, so integrate only from 0 to 3 and subtract the area under the straight line.

Step 1: Determine the boundary intersections Substitute $y = 2 - \frac{2x}{3}$ into the ellipse equation:

$$\begin{aligned}4x^2 + 9\left(2 - \frac{2x}{3}\right)^2 &= 36, \\4x^2 + (6 - 2x)^2 &= 36, \\8x^2 - 24x &= 0, \\x(x - 3) &= 0.\end{aligned}$$

Thus the line meets the ellipse at $(0, 2)$ and $(3, 0)$. These give the integration limits $x = 0$ and $x = 3$.

Step 2: Identify the selected region and form the integral For $0 < x < 3$, the upper ellipse lies above the line, while the line is nonnegative. The lower half of the ellipse cannot satisfy the inequality on this interval. For $x < 0$, the line has ordinate greater than 2, which exceeds the largest ordinate on the ellipse. Hence the required region is exactly the cap between the upper ellipse and its chord in the first quadrant. Using vertical strips,

$$\begin{aligned}A &= \int_0^3 \left[\frac{2}{3} \sqrt{9 - x^2} - \left(2 - \frac{2x}{3}\right) \right] dx \\&= \frac{2}{3} \int_0^3 \sqrt{9 - x^2} dx - \int_0^3 \left(2 - \frac{2x}{3}\right) dx.\end{aligned}$$

Step 3: Evaluate the two integrals Apply the standard formula with $a = 3$:

$$\begin{aligned}\int_0^3 \sqrt{9 - x^2} dx &= \left[\frac{x}{2} \sqrt{9 - x^2} + \frac{9}{2} \sin^{-1}\left(\frac{x}{3}\right) \right]_0^3 \\&= \frac{9}{2} \cdot \frac{\pi}{2} = \frac{9\pi}{4}.\end{aligned}$$

The area below the chord is

$$\int_0^3 \left(2 - \frac{2x}{3}\right) dx = \left[2x - \frac{x^2}{3} \right]_0^3 = 6 - 3 = 3.$$

Therefore the difference of the two evaluated integrals is $\frac{2}{3} \cdot \frac{9\pi}{4} - 3$.

Step 4: Conclusion The required area is $\frac{3\pi}{2} - 3$ square units.

Quick Tip to Solve: First find the chord's intercepts: here they are the ellipse's positive axis intercepts, so integrate only from 0 to 3 and subtract the area under the straight line.

11(b). Using integration, find the area of the region bounded by lines $x - y + 1 = 0$, $x = -2$, $x = 3$ and x -axis. [4 marks]

Solution:

Concept:

- The area between a curve $y = f(x)$ and the x -axis is obtained by integrating the non-negative vertical height $|f(x)|$ over the given interval.
- If the curve crosses the x -axis inside the interval, split the integral at that crossing. Below the axis the height is $0 - f(x)$; above it the height is $f(x) - 0$.

Step 1: Express the line as a function and locate its crossing From $x - y + 1 = 0$, the slant boundary is $y = x + 1$. The other boundaries are the vertical lines $x = -2$, $x = 3$, and the horizontal axis $y = 0$. Setting $y = 0$ gives $x + 1 = 0$, so the line meets the axis at $(-1, 0)$, inside the interval $[-2, 3]$. For $-2 \leq x \leq -1$, the line lies below the axis, while for $-1 \leq x \leq 3$, it lies above the axis. Hence the bounded region consists of two portions meeting at this crossing.

Step 2: Integrate the portion below the axis On $[-2, -1]$, the upper boundary is $y = 0$ and the lower boundary is $y = x + 1$. Therefore the vertical height is $-(x + 1)$, which is non-negative throughout this interval. Its area is

$$\begin{aligned} A_1 &= \int_{-2}^{-1} [0 - (x + 1)] dx \\ &= \left[-\frac{x^2}{2} - x \right]_{-2}^{-1} \\ &= \left(-\frac{1}{2} + 1 \right) - (-2 + 2) = \frac{1}{2}. \end{aligned}$$

Step 3: Integrate the portion above the axis and combine On $[-1, 3]$, the upper boundary is $y = x + 1$ and the lower boundary is the axis. Thus

$$\begin{aligned} A_2 &= \int_{-1}^3 (x + 1) dx \\ &= \left[\frac{x^2}{2} + x \right]_{-1}^3 \\ &= \left(\frac{9}{2} + 3 \right) - \left(\frac{1}{2} - 1 \right) = 8. \end{aligned}$$

Add the two positive areas: $A = A_1 + A_2 = \frac{1}{2} + 8$. A single integral of $x + 1$ from -2 to 3 would subtract the area below the axis, so it would give signed area instead of the required total area.

Step 4: Conclusion The required area is $\boxed{\frac{17}{2}}$ square units.

Quick Tip to Solve: Solve $x + 1 = 0$ first and split at $x = -1$; reverse the sign of the integrand on $[-2, -1]$ to count both areas positively.

12. A card from a pack of 52 playing cards is lost. From the remaining cards, 2 cards are drawn at random without replacement, and are found to be both aces. Find the probability that the lost card being an ace. [4 marks]

Solution:

Concept:

- Conditional probability: $P(A | B)$ is the probability of event A when event B has already occurred.
- Bayes' theorem for complementary events A and A^c is $P(A | B) = \frac{P(A)P(B | A)}{P(A)P(B | A) + P(A^c)P(B | A^c)}$.
- When cards are drawn without replacement, the second draw is made from one fewer card, and the number of available aces also decreases if the first card is an ace.

Step 1: Define the events and their prior probabilities Let A be the event that the lost card is an ace, and let B be the event that the two cards drawn from the remaining pack are both aces. We need $P(A | B)$, since the information about the drawn cards is known. A standard pack contains 4 aces and 48 other cards, and each of the 52 cards is equally likely to be lost. Hence,

$$P(A) = \frac{4}{52} = \frac{1}{13}, \quad P(A^c) = \frac{48}{52} = \frac{12}{13}.$$

These events exhaust the possibilities for the lost card.

Step 2: Find the probability of drawing two aces in each case If the lost card is an ace, the remaining 51 cards contain only 3 aces. Therefore, using successive draws without replacement,

$$P(B | A) = \frac{3}{51} \cdot \frac{2}{50} = \frac{6}{2550}.$$

If the lost card is not an ace, all 4 aces remain among the 51 cards. In this case,

$$P(B | A^c) = \frac{4}{51} \cdot \frac{3}{50} = \frac{12}{2550}.$$

Both cases use denominators 51 and 50 because precisely one card is lost before the two draws take place. The numerators differ because losing an ace reduces the available aces by one.

Step 3: Apply Bayes' theorem to the observed draw The event B can occur with either kind of lost card. Thus the law of total probability gives

$$P(B) = \frac{1}{13} \cdot \frac{6}{2550} + \frac{12}{13} \cdot \frac{12}{2550}.$$

Substituting into Bayes' theorem and cancelling the common factor $1/(13 \cdot 2550)$,

$$\begin{aligned} P(A | B) &= \frac{(1/13)(6/2550)}{(1/13)(6/2550) + (12/13)(12/2550)} \\ &= \frac{6}{6 + 144} = \frac{6}{150} = \frac{1}{25}. \end{aligned}$$

The observation of two aces makes a lost ace less likely: two aces are twice as likely to be drawn when the lost card is a non-ace as when it is an ace.

Quick Tip to Solve: Use 51 and 50 as the draw denominators, and compare 3 remaining aces with 4 before applying Bayes' theorem.

Step 4: Conclusion The probability that the lost card is an ace, given that both drawn cards are aces, is $\boxed{\frac{1}{25}}$.

Quick Tip to Solve: Use 51 and 50 as the draw denominators, and compare 3 remaining aces with 4 before applying Bayes' theorem.

13. Evaluate: $\int_0^\pi \frac{x}{1 + \sin x} dx$.

[4 marks]

Solution:

Concept:

- For a definite integral, the substitution $x = a + b - t$ reverses the limits. In particular, $\int_0^\pi f(x) dx = \int_0^\pi f(\pi - x) dx$.
- The symmetry $\sin(\pi - x) = \sin x$ lets us combine the numerators x and $\pi - x$, replacing a variable numerator by a constant.
- The half-angle identity $1 + \cos u = 2 \cos^2(u/2)$ reduces the remaining integral to the standard formula $\int \sec^2 v dv = \tan v + C$.

Step 1: Use the symmetry of the interval Let the required integral be I . Substitute $x = \pi - t$, so $dx = -dt$. The limits $x = 0, \pi$ become $t = \pi, 0$, respectively. Reversing these limits cancels the minus sign, giving

$$I = \int_0^\pi \frac{\pi - t}{1 + \sin(\pi - t)} dt = \int_0^\pi \frac{\pi - t}{1 + \sin t} dt.$$

Since the variable of integration is a dummy variable, we can write this expression using x in place of t .

Step 2: Add the two expressions for the integral Add the original expression for I to its transformed expression. Both have the same limits and the same denominator:

$$\begin{aligned} 2I &= \int_0^\pi \frac{x + (\pi - x)}{1 + \sin x} dx \\ &= \pi \int_0^\pi \frac{dx}{1 + \sin x}. \end{aligned}$$

Put $J = \int_0^\pi (1 + \sin x)^{-1} dx$. Then $I = \frac{\pi}{2} J$. The original factor x has now been removed by symmetry, so it remains to evaluate a standard trigonometric integral.

Step 3: Evaluate the remaining integral by a half-angle identity Set $u = x - \frac{\pi}{2}$. Then $dx = du$,

Quick Tip to Solve: When the denominator is unchanged by replacing x with $\pi - x$, add the original and reflected integrals to replace the numerator x by $\pi/2$.

$\sin x = \cos u$, and the new limits are $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Hence

$$\begin{aligned} J &= \int_{-\pi/2}^{\pi/2} \frac{du}{1 + \cos u} \\ &= \frac{1}{2} \int_{-\pi/2}^{\pi/2} \sec^2\left(\frac{u}{2}\right) du \\ &= \left[\tan\left(\frac{u}{2}\right) \right]_{-\pi/2}^{\pi/2} \\ &= \tan \frac{\pi}{4} - \tan\left(-\frac{\pi}{4}\right) = 1 - (-1) = 2. \end{aligned}$$

The chain-rule factor is included because $\frac{d}{du} \tan(u/2) = \frac{1}{2} \sec^2(u/2)$. Also, $u/2$ lies between $-\pi/4$ and $\pi/4$, so this antiderivative is defined throughout the integration interval.

Step 4: Conclusion Therefore, $I = \frac{\pi}{2} \times 2$, and the required value is

$$\boxed{\int_0^\pi \frac{x}{1 + \sin x} dx = \pi}.$$

Quick Tip to Solve: When the denominator is unchanged by replacing x with $\pi - x$, add the original and reflected integrals to replace the numerator x by $\pi/2$.

14. Electrical transmission wires which are laid down in winters are stretched tightly to accommodate expansion in summers.

Two such wires lie along the following lines:

$$l_1 : \frac{x+1}{3} = \frac{y-3}{-2} = \frac{z+2}{-1}$$

$$l_2 : \frac{x}{-1} = \frac{y-7}{3} = \frac{z+7}{-2}$$

Based on the given information, answer the following questions:

[4 marks]

14(i). Are the lines l_1 and l_2 coplanar? Justify your answer.

[2 marks]

Solution:

Concept:

- Two lines through position vectors $\mathbf{a}_1, \mathbf{a}_2$, with direction vectors $\mathbf{b}_1, \mathbf{b}_2$, are coplanar if $(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = 0$.
- The scalar triple product measures the volume formed by the joining vector and the two direction vectors. Zero volume means these vectors lie in one plane.

Step 1: Identify points and direction vectors From the symmetric equations, take $A = (-1, 3, -2)$ on l_1 and $B = (0, 7, -7)$ on l_2 . Their direction vectors are $\mathbf{b}_1 = (3, -2, -1)$ and $\mathbf{b}_2 = (-1, 3, -2)$. Thus the joining vector is $\overrightarrow{AB} = (1, 4, -5)$. The signs in these vectors follow directly from the denominators and the coordinates that make the numerators zero.

Quick Tip to Solve: Read a point from the zero numerators and a direction vector from the denominators before forming the scalar triple product.

Step 2: Evaluate the coplanarity test The cross product is

$$\mathbf{b}_1 \times \mathbf{b}_2 = (4 + 3, 1 + 6, 9 - 2) = (7, 7, 7).$$

Hence

$$\overrightarrow{AB} \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = 1(7) + 4(7) - 5(7) = 0.$$

The cross product is nonzero, so the direction vectors are not parallel. The zero scalar triple product shows that the joining vector belongs to the plane spanned by the two directions.

Step 3: Conclusion Yes, l_1 and l_2 are coplanar, since their scalar triple product is zero.

Quick Tip to Solve: Read a point from the zero numerators and a direction vector from the denominators before forming the scalar triple product.

14(ii). Find the point of intersection of the lines l_1 and l_2 .

[2 marks]

Solution:

Concept:

- At the intersection, the coordinates of a point on each line must agree. Use independent parameters for the two lines because their direction vectors differ.
- Two coordinate equations determine candidate parameter values; the third coordinate equation must also hold to confirm an intersection in space.

Step 1: Express each line using a parameter Let the common ratio on l_1 be t , and that on l_2 be s . Then their points have coordinates

$$\begin{aligned} l_1 : \quad x &= -1 + 3t, \quad y = 3 - 2t, \quad z = -2 - t, \\ l_2 : \quad x &= -s, \quad y = 7 + 3s, \quad z = -7 - 2s. \end{aligned}$$

Equating the first two coordinates gives $3t + s = 1$ and $2t + 3s = -4$.

Step 2: Solve the parameters and check the third coordinate From the first equation, $s = 1 - 3t$. Substitution in the second gives

$$2t + 3(1 - 3t) = -4 \quad \Rightarrow \quad -7t = -7,$$

so $t = 1$ and $s = -2$. The third coordinates agree, since $-2 - t = -3$ and $-7 - 2s = -7 + 4 = -3$. Substituting $t = 1$ into l_1 gives $x = 2, y = 1, z = -3$. As a direct check in l_2 , the three ratios are $2/(-1) = -2$, $(1 - 7)/3 = -2$, and $(-3 + 7)/(-2) = -2$.

Step 3: Conclusion The point of intersection of l_1 and l_2 is $\boxed{(2, 1, -3)}$.

Quick Tip to Solve: Use different parameters for the lines, solve two coordinate equations, and always verify the remaining coordinate.