

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-1-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) The question paper contains **three** Sections – Section A, B and C.
- (ii) Each Section is **Compulsory**.
- (iii) **Section – A** has **6** short answer type-I questions of **2** marks each.
- (iv) **Section – B** has **4** short answer type-II questions of **3** marks each.
- (v) **Section – C** has **4** long answer type questions of **4** marks each.
- (vi) There is an internal choice in some questions.
- (vii) Question No. **14** is a case based problem with **2** subparts of **2** marks each.

Section A (12 marks)

Question numbers 1 to 6 carry 2 marks each.

1(a). If $\frac{d}{dx}[F(x)] = \frac{\sec^4 x}{\csc^4 x}$ and $F\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$, then find $F(x)$.

[2 marks]

Solution:

Concept:

- The ratio $\sec^4 x / \csc^4 x = \tan^4 x$, wherever the given derivative is defined.
- Use $\tan^2 x = \sec^2 x - 1$ and $\int \sec^2 x \tan^2 x dx = \tan^3 x / 3$.
- An antiderivative contains a constant C ; the prescribed value of F determines it.

Step 1: Simplify and integrate the derivative First write the derivative entirely in terms of $\tan x$. Since $\tan^4 x = \tan^2 x(\sec^2 x - 1)$, we obtain

$$\begin{aligned} F(x) &= \int \tan^4 x dx \\ &= \int \tan^2 x \sec^2 x dx - \int (\sec^2 x - 1) dx \\ &= \frac{\tan^3 x}{3} - \tan x + x + C. \end{aligned}$$

In the first integral, the substitution $u = \tan x$ gives $du = \sec^2 x dx$, so its value is $u^3/3$.

Step 2: Apply the prescribed value At $x = \pi/4$, we have $\tan(\pi/4) = 1$. Thus

$$\frac{\pi}{4} = \frac{1}{3} - 1 + \frac{\pi}{4} + C, \quad C = \frac{2}{3}.$$

As a check, differentiating the antiderivative gives $\tan^2 x \sec^2 x - \sec^2 x + 1 = \tan^4 x$, which reproduces the given derivative.

Step 3: Conclusion $F(x) = \frac{\tan^3 x}{3} - \tan x + x + \frac{2}{3}$ on the interval of definition containing $\pi/4$.

Quick Tip to Solve: Rewrite an even power of tangent using $\tan^2 x = \sec^2 x - 1$ before integrating; substitute the boundary value only after integration.

OR

1(b). Find: $\int \frac{\log x - 3}{(\log x)^4} dx$.

[2 marks]

Solution:

Concept:

- Take $\log x$ as the natural logarithm, so $d(\log x)/dx = 1/x$.
- The product rule gives $d[xg(x)]/dx = g(x) + xg'(x)$.
- The integrand is defined for $x > 0$ with $x \neq 1$; an antiderivative is considered separately on each

Quick Tip to Solve: For a numerator $\log x - n$ over $(\log x)^{n+1}$, check the derivative of $x/(\log x)^n$.

such interval.

Step 1: Rearrange the integrand Let I denote the required integral. Separate its numerator over the common denominator:

$$I = \int \left[\frac{1}{(\log x)^3} - \frac{3}{(\log x)^4} \right] dx.$$

The adjacent powers of $\log x$ suggest differentiating the product $x(\log x)^{-3}$. A direct substitution $u = \log x$ would introduce $dx = e^u du$, so recognising this product derivative is more efficient.

Step 2: Identify and verify the product derivative Using the product rule and the chain rule,

$$\begin{aligned} \frac{d}{dx} [x(\log x)^{-3}] &= (\log x)^{-3} + x[-3(\log x)^{-4}] \frac{1}{x} \\ &= \frac{1}{(\log x)^3} - \frac{3}{(\log x)^4} \\ &= \frac{\log x - 3}{(\log x)^4}. \end{aligned}$$

This is exactly the given integrand, including the negative sign and the coefficient 3. Adding a constant gives the entire family of antiderivatives on any interval avoiding $x = 1$.

Step 3: Conclusion $\int \frac{\log x - 3}{(\log x)^4} dx = \frac{x}{(\log x)^3} + C.$

Quick Tip to Solve: For a numerator $\log x - n$ over $(\log x)^{n+1}$, check the derivative of $x/(\log x)^n$.

2. Let A and B be two events such that $P(A) = \frac{5}{8}$, $P(B) = \frac{1}{2}$ and $P(A/B) = \frac{3}{4}$. Find the value of $P(B/A)$. [2 marks]

Solution:

Concept:

- Conditional probability is $P(A | B) = P(A \cap B)/P(B)$, provided $P(B) > 0$.
- Similarly, $P(B | A) = P(A \cap B)/P(A)$, provided $P(A) > 0$.
- The intersection $A \cap B$ is the same event as $B \cap A$, although the two conditional probabilities need not be equal.

Step 1: Find the probability of the intersection The printed notation $P(A/B)$ means $P(A | B)$. We are given $P(A) = 5/8$, $P(B) = 1/2$ and $P(A | B) = 3/4$. Applying the definition with event B as the conditioning event gives

$$P(A \cap B) = P(A | B)P(B) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}.$$

This value is no larger than either marginal probability, as an intersection probability must be.

Step 2: Reverse the conditioning event Now condition on A , so the denominator must change to $P(A)$. Since $5/8 > 0$,

$$P(B | A) = \frac{P(A \cap B)}{P(A)} = \frac{3/8}{5/8} = \frac{3}{8} \times \frac{8}{5} = \frac{3}{5}.$$

Quick Tip to Solve: First compute the shared intersection from the given conditional probability, then divide by the probability of the new conditioning event.

The common denominator 8 cancels. Notice that reversing the conditional probability does not mean simply copying $3/4$; it changes the reference event from B to A . The result lies between zero and one.

Step 3: Conclusion $P(B/A) = \frac{3}{5}$.

Quick Tip to Solve: First compute the shared intersection from the given conditional probability, then divide by the probability of the new conditioning event.

3. Two balls are drawn at random from a bag containing 2 red balls and 3 blue balls, without replacement. Let the variable X denote the number of red balls. Find the probability distribution of X . [2 marks]

Solution:

Concept:

- The number X of red balls drawn can take the values 0, 1, 2.
- For sampling without replacement, count equally likely unordered pairs: $P(X = k) = \frac{\binom{2}{k} \binom{3}{2-k}}{\binom{5}{2}}$.

Step 1: Count all pairs The bag contains five distinct balls. There are $\binom{5}{2} = 10$ equally likely unordered pairs. Although balls share a colour, they are distinct objects. Counting unordered pairs counts each selection once, and the order of drawing does not affect the number of red balls.

Step 2: Find the three probabilities For no red balls choose two of the three blue balls. For one red ball choose one of two reds and one of three blues. For two reds select the only pair of red balls. Thus

$$\begin{aligned}P(X = 0) &= \frac{\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10}, \\P(X = 1) &= \frac{\binom{2}{1} \binom{3}{1}}{\binom{5}{2}} = \frac{6}{10} = \frac{3}{5}, \\P(X = 2) &= \frac{\binom{2}{2}}{\binom{5}{2}} = \frac{1}{10}.\end{aligned}$$

These events are mutually exclusive and exhaustive. The sum $3/10 + 6/10 + 1/10 = 1$ verifies that this is a probability distribution.

Step 3: Conclusion The required distribution is

x	0	1	2
$P(X = x)$	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{1}{10}$

Quick Tip to Solve: Count unordered pairs with combinations; without replacement, the second draw does not have the original colour probabilities.

4. Find the values of λ , for which the distance of the point $(2, 1, \lambda)$ from the plane $3x + 5y + 4z = 11$ is $2\sqrt{2}$ units. [2 marks]

Solution:

Concept:

- The distance of (x_0, y_0, z_0) from $Ax + By + Cz + D = 0$ is $d = |Ax_0 + By_0 + Cz_0 + D| / \sqrt{A^2 + B^2 + C^2}$.
- The absolute value accounts for points on either side of the plane. A positive distance therefore permits both signs of the numerator.

Step 1: Substitute the point and plane Write the plane as $3x + 5y + 4z - 11 = 0$. Here $A = 3, B = 5, C = 4, D = -11$, and the point is $(2, 1, \lambda)$. Therefore

$$d = \frac{|3(2) + 5(1) + 4\lambda - 11|}{\sqrt{3^2 + 5^2 + 4^2}} = \frac{|4\lambda|}{\sqrt{50}} = \frac{|4\lambda|}{5\sqrt{2}}.$$

The constant terms cancel because $(2, 1, 0)$ lies in the plane.

Step 2: Use the prescribed distance Setting the distance equal to $2\sqrt{2}$ gives

$$\frac{|4\lambda|}{5\sqrt{2}} = 2\sqrt{2} \implies |4\lambda| = 20.$$

Hence $4\lambda = 20$ or $4\lambda = -20$, giving $\lambda = 5$ or $\lambda = -5$. For either value the numerator has absolute value 20, so the distance is $20/(5\sqrt{2}) = 2\sqrt{2}$, verifying both possibilities.

Step 3: Conclusion The required values are $\lambda = 5$ or $\lambda = -5$.

Quick Tip to Solve: Retain the absolute value in the point-to-plane distance formula, then solve both signs to avoid losing one value of lambda.

5. Solve the differential equation: $\log\left(\frac{dy}{dx}\right) = x - y$. [2 marks]

Solution:

Concept:

- Exponentiating $\log(dy/dx) = x - y$ gives $dy/dx = e^{x-y}$, with $dy/dx > 0$ because the logarithm has a positive argument.
- A separable differential equation is solved by collecting the factors in y with dy , and the factors in x with dx , then integrating.

Step 1: Separate the variables Taking the exponential of both sides of the given equation yields

$$\frac{dy}{dx} = e^{x-y} = e^x e^{-y}.$$

Multiplication by e^y , which is always positive and never zero for real y , gives

$$e^y dy = e^x dx.$$

Quick Tip to Solve: Exponentiate first, then multiply by e to the power y to separate variables; keep the positivity condition when taking the logarithm.

Thus the variables are separated without discarding any real solution.

Step 2: Integrate and verify the domain Integrating both sides gives

$$\int e^y dy = \int e^x dx \implies e^y = e^x + C,$$

where C is an arbitrary real constant. Equivalently, $y = \log(e^x + C)$, on any interval where $e^x + C > 0$. Differentiating gives $dy/dx = e^x/(e^x + C) > 0$; therefore

$$\log\left(\frac{dy}{dx}\right) = \log e^x - \log(e^x + C) = x - y.$$

This verifies the original equation, including its logarithmic domain condition.

Step 3: Conclusion The general solution is $e^y - e^x = C$, or $y = \log(e^x + C)$ where $e^x + C > 0$.

Quick Tip to Solve: Exponentiate first, then multiply by e to the power y to separate variables; keep the positivity condition when taking the logarithm.

6. In a parallelogram PQRS, $\vec{PQ} = 3\hat{i} - 2\hat{j} + 2\hat{k}$ and $\vec{PS} = -\hat{i} - 2\hat{k}$. Find $|\vec{PR}|$ and $|\vec{QS}|$. [2 marks]

Solution:

Concept:

- Opposite sides of a parallelogram have equal directed vectors, so $\vec{QR} = \vec{PS}$. The diagonal starting at P is the sum of its adjacent side vectors.
- For $\vec{a} = a\hat{i} + b\hat{j} + c\hat{k}$, the magnitude is $|\vec{a}| = \sqrt{a^2 + b^2 + c^2}$. A vector directed along the reverse path has every component negated.

Step 1: Find the diagonal PR Follow the path $P \rightarrow Q \rightarrow R$. Since the opposite sides are equal as vectors,

$$\begin{aligned}\vec{PR} &= \vec{PQ} + \vec{QR} = \vec{PQ} + \vec{PS} \\ &= (3\hat{i} - 2\hat{j} + 2\hat{k}) + (-\hat{i} - 2\hat{k}) \\ &= 2\hat{i} - 2\hat{j}.\end{aligned}$$

The $\hat{k}$ components cancel. Squaring and adding the components gives

$$|\vec{PR}| = \sqrt{2^2 + (-2)^2 + 0^2} = \sqrt{8} = 2\sqrt{2}.$$

Step 2: Find the diagonal QS The path $Q \rightarrow P \rightarrow S$ requires reversing the first given vector. Thus

$$\begin{aligned}\vec{QS} &= \vec{QP} + \vec{PS} = \vec{PS} - \vec{PQ} \\ &= (-\hat{i} - 2\hat{k}) - (3\hat{i} - 2\hat{j} + 2\hat{k}) \\ &= -4\hat{i} + 2\hat{j} - 4\hat{k}.\end{aligned}$$

Quick Tip to Solve: Add adjacent side vectors for PR and subtract them for QS; negate every component when reversing PQ.

Each negative component also contributes a positive square to the length:

$$|\vec{QS}| = \sqrt{(-4)^2 + 2^2 + (-4)^2} = \sqrt{36} = 6.$$

Step 3: Conclusion The required lengths are $|\vec{PR}| = 2\sqrt{2}$, $|\vec{QS}| = 6$.

Quick Tip to Solve: Add adjacent side vectors for PR and subtract them for QS; negate every component when reversing PQ.

Section B (12 marks)

Question numbers 7 to 10 carry 3 marks each.

7. Find the particular solution of the differential equation $x \frac{dy}{dx} + x \cos^2\left(\frac{y}{x}\right) = y$; given that when $x = 1$, $y = \frac{\pi}{4}$. [3 marks]

Solution:

Concept:

- For an equation involving the ratio y/x , substitute $y = vx$, where v is a function of x . The product rule gives $dy/dx = v + x dv/dx$.
- Separate the resulting variables and integrate, using $\int \sec^2 v dv = \tan v$ and $\int dx/x = \ln|x|$. The given initial value determines the integration constant.

Step 1: Reduce the differential equation Since the initial point has $x = 1$, work on the interval $x > 0$. Put $v = y/x$, so that $y = vx$ and $dy/dx = v + x dv/dx$. Substituting in the printed equation gives

$$x \left(v + x \frac{dv}{dx} \right) + x \cos^2 v = xv.$$

Cancel xv from both sides and divide by $x \neq 0$ to obtain

$$x \frac{dv}{dx} = -\cos^2 v.$$

Step 2: Separate variables and integrate At the initial point $v = \pi/4$, so $\cos v \neq 0$. Separating variables on the solution branch through this point,

$$\sec^2 v dv = -\frac{dx}{x}.$$

Integrating both sides yields

$$\tan v = -\ln x + C.$$

Replacing v by y/x , the family relevant to the initial condition is

$$\tan\left(\frac{y}{x}\right) + \ln x = C.$$

Quick Tip to Solve: Substitute $y=vx$ before separating variables: the xv terms cancel, leaving a secant-squared integral.

Step 3: Apply and check the initial condition For $x = 1$ and $y = \pi/4$, we have

$$C = \tan(\pi/4) + \ln 1 = 1.$$

The continuous branch through the given point is $y = x \arctan(1 - \ln x)$, with the inverse tangent taking values in $(-\pi/2, \pi/2)$. To check the differential equation, differentiating $\tan v + \ln x = 1$ gives $\sec^2 v v' + 1/x = 0$, hence $xv' = -\cos^2 v$. With $y' = v + xv'$, this implies $xy' + x \cos^2 v = xv = y$, as required. At $x = 1$, the branch gives $y = \arctan 1 = \pi/4$.

Step 4: Conclusion The particular solution is $\tan\left(\frac{y}{x}\right) + \ln x = 1$, on the branch through $(1, \pi/4)$. Equivalently, $y = x \arctan(1 - \ln x)$, $x > 0$.

Quick Tip to Solve: Substitute $y=vx$ before separating variables: the xv terms cancel, leaving a secant-squared integral.

8(a). If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four non-zero vectors such that $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = 4\vec{b} \times \vec{d}$, then show that $(\vec{a} - 2\vec{d})$ is parallel to $(2\vec{b} - \vec{c})$, where $\vec{a} \neq 2\vec{d}$, $\vec{c} \neq 2\vec{b}$. [3 marks]

Solution:

Concept:

- Non-zero vectors are parallel if their cross product is zero: $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \theta$, so the angle is 0 or π .
- The cross product distributes over vector addition and subtraction. Reversing its factors changes its sign: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.

Step 1: Expand the cross product of the target vectors Write $\vec{u} = \vec{a} - 2\vec{d}$ and $\vec{v} = 2\vec{b} - \vec{c}$. The cross product that tests the required parallelism is

$$\begin{aligned}\vec{u} \times \vec{v} &= (\vec{a} - 2\vec{d}) \times (2\vec{b} - \vec{c}) \\ &= 2(\vec{a} \times \vec{b}) - \vec{a} \times \vec{c} - 4(\vec{d} \times \vec{b}) + 2(\vec{d} \times \vec{c}).\end{aligned}$$

Each term preserves the order of the original factors. In particular, the coefficient -4 comes from multiplying $-2\vec{d}$ by $2\vec{b}$.

Step 2: Rearrange the terms using the given equalities Reverse the factors in the last two products and collect the terms:

$$\begin{aligned}\vec{u} \times \vec{v} &= 2(\vec{a} \times \vec{b} - \vec{c} \times \vec{d}) - (\vec{a} \times \vec{c} - 4\vec{b} \times \vec{d}) \\ &= 2\vec{0} - \vec{0} = \vec{0}.\end{aligned}$$

The first bracket vanishes because $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$, and the second because $\vec{a} \times \vec{c} = 4\vec{b} \times \vec{d}$. The conditions $\vec{a} \neq 2\vec{d}$ and $\vec{c} \neq 2\vec{b}$ ensure that neither target vector is zero, so the criterion for parallel directions applies.

Step 3: Conclusion Both target vectors are non-zero and their cross product vanishes. Hence $\vec{a} - 2\vec{d} \parallel 2\vec{b} - \vec{c}$.

Quick Tip to Solve: Expand the cross product of the target vectors, then reverse the products involving d to expose both given equalities.

8(b). The two adjacent sides of a parallelogram are represented by $2\hat{i} - 4\hat{j} - 5\hat{k}$ and $2\hat{i} + 2\hat{j} + 3\hat{k}$. Find the unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram also. [3 marks]

Solution:

Concept:

- If adjacent sides from a common vertex are $\vec{u}$ and $\vec{v}$, the diagonals have vectors $\vec{p} = \vec{u} + \vec{v}$ and $\vec{q} = \vec{u} - \vec{v}$. Reversing a diagonal reverses its vector.
- The unit vectors parallel to a non-zero vector $\vec{w}$ are $\pm\vec{w}/|\vec{w}|$. The parallelogram area is $\frac{1}{2}|\vec{p} \times \vec{q}|$, in square units.

Step 1: Determine both diagonal vectors Take $\vec{u} = 2\hat{i} - 4\hat{j} - 5\hat{k}$ and $\vec{v} = 2\hat{i} + 2\hat{j} + 3\hat{k}$. Adding and subtracting their corresponding components gives

$$\vec{p} = 4\hat{i} - 2\hat{j} - 2\hat{k}, \quad \vec{q} = -6\hat{j} - 8\hat{k}.$$

The first vector joins the common vertex to the opposite vertex, while the second joins the endpoint of $\vec{v}$ to the endpoint of $\vec{u}$. Either orientation is acceptable for a diagonal.

Step 2: Normalize the diagonal vectors Their magnitudes are

$$|\vec{p}| = \sqrt{16 + 4 + 4} = 2\sqrt{6}, \quad |\vec{q}| = \sqrt{36 + 64} = 10.$$

Thus the unit vectors along the first diagonal are $\pm(2\hat{i} - \hat{j} - \hat{k})/\sqrt{6}$, and those along the second are $\pm(3\hat{j} + 4\hat{k})/5$. Both signs are included because parallel vectors can point in the same or the opposite direction. The squared component sums of these normalized vectors are respectively 6/6 and 25/25, verifying unit magnitude.

Step 3: Calculate the area from the diagonals Compute the diagonal cross product:

$$\vec{p} \times \vec{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & -6 & -8 \end{vmatrix} = 4\hat{i} + 32\hat{j} - 24\hat{k}.$$

Consequently,

$$A = \frac{1}{2}\sqrt{4^2 + 32^2 + (-24)^2} = \frac{1}{2}\sqrt{1616} = 2\sqrt{101}.$$

The factor 1/2 is essential: $(\vec{u} + \vec{v}) \times (\vec{u} - \vec{v}) = -2\vec{u} \times \vec{v}$, whose magnitude is twice the parallelogram area.

Step 4: Conclusion The required unit vectors are $\pm \frac{2\hat{i} - \hat{j} - \hat{k}}{\sqrt{6}}$ and $\pm \frac{3\hat{j} + 4\hat{k}}{5}$. The area is $2\sqrt{101}$ square units.

Quick Tip to Solve: Add and subtract the adjacent side vectors to find the diagonals; halve the magnitude of their cross product for the area.

9(a). Find: $\int \frac{x^3 + x}{x^4 - 9} dx.$

[3 marks]

Solution:

Concept:

- Substitution converts an integral into a simpler variable: for $t = x^2$, $dt = 2x dx$, so $x dx = dt/2$.
- Factor a quadratic denominator and resolve the rational function into partial fractions. Each simple fraction is integrated using $\int du/u = \ln |u| + C$.

Step 1: Use the substitution suggested by the even powers Write the numerator as $x(x^2 + 1)$. With $t = x^2$, both the numerator remaining after $x dx$ and the denominator become polynomials in t :

$$I = \int \frac{x(x^2 + 1)}{x^4 - 9} dx = \frac{1}{2} \int \frac{t + 1}{t^2 - 9} dt.$$

The original integrand is defined wherever $x^4 \neq 9$, that is, away from $x = \pm\sqrt{3}$. An antiderivative is understood separately on each interval of its domain.

Step 2: Resolve the transformed rational function Since $t^2 - 9 = (t - 3)(t + 3)$, set

$$\frac{t + 1}{(t - 3)(t + 3)} = \frac{A}{t - 3} + \frac{B}{t + 3}.$$

Comparing numerators gives $t + 1 = (A + B)t + 3(A - B)$. Thus $A + B = 1$ and $A - B = 1/3$, yielding $A = 2/3$, $B = 1/3$. Consequently,

$$I = \frac{1}{3} \int \frac{dt}{t - 3} + \frac{1}{6} \int \frac{dt}{t + 3}.$$

Step 3: Integrate and return to the original variable Applying the logarithmic integration rule gives

$$I = \frac{1}{3} \ln |t - 3| + \frac{1}{6} \ln |t + 3| + C.$$

Substitute $t = x^2$. The second logarithm may be written without absolute-value signs because $x^2 + 3 > 0$ for every real x . As a check, differentiating the two terms produces

$$\frac{2x}{3(x^2 - 3)} + \frac{x}{3(x^2 + 3)} = \frac{x(2x^2 + 6 + x^2 - 3)}{3(x^4 - 9)} = \frac{x^3 + x}{x^4 - 9},$$

exactly the printed integrand.

Step 4: Conclusion

$$\int \frac{x^3 + x}{x^4 - 9} dx = \frac{1}{3} \ln |x^2 - 3| + \frac{1}{6} \ln(x^2 + 3) + C.$$

Quick Tip to Solve: Factor out x from the numerator before using $t = x^2$; retain the factor $1/2$ from $x dx = dt/2$ throughout partial fractions.

9(b). Evaluate, using properties: $\int_{-\pi}^{\pi} (3 \sin x - 2)^2 dx$.

[3 marks]

Solution:

Concept:

- On a symmetric interval, an odd function integrates to zero, whereas an even function satisfies $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$.
- The identities $\sin(-x) = -\sin x$ and $\sin^2 x = (1 - \cos 2x)/2$ identify parity and evaluate the squared-sine term.

Step 1: Expand before applying symmetry Let $I = \int_{-\pi}^{\pi} (3 \sin x - 2)^2 dx$. Expanding the square carefully,

$$(3 \sin x - 2)^2 = 9 \sin^2 x - 12 \sin x + 4.$$

The entire squared expression is not even: replacing x by $-x$ changes the sign of the middle term. Therefore, apply the definite-integral properties to the separate terms, using linearity:

$$I = 9 \int_{-\pi}^{\pi} \sin^2 x dx - 12 \int_{-\pi}^{\pi} \sin x dx + 4 \int_{-\pi}^{\pi} 1 dx.$$

Step 2: Eliminate the odd term and evaluate the even term Since $\sin x$ is odd, $\int_{-\pi}^{\pi} \sin x dx = 0$. Since $\sin^2 x$ is even, the symmetric-interval property and the double-angle identity give

$$\begin{aligned} \int_{-\pi}^{\pi} \sin^2 x dx &= 2 \int_0^{\pi} \sin^2 x dx \\ &= \int_0^{\pi} (1 - \cos 2x) dx \\ &= \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} = \pi. \end{aligned}$$

The endpoint sine terms vanish because $\sin 0 = \sin 2\pi = 0$.

Step 3: Combine the three contributions The constant function integrates to the length of the interval:

$$\int_{-\pi}^{\pi} 1 dx = \pi - (-\pi) = 2\pi.$$

Thus the squared-sine term contributes 9π , the odd sine term contributes zero, and the constant term contributes 8π . Substituting these values into the expanded integral gives $I = 9\pi - 12(0) + 4(2\pi)$. No arbitrary constant is added, because this is a definite integral with fixed limits.

Step 4: Conclusion

$$\boxed{\int_{-\pi}^{\pi} (3 \sin x - 2)^2 dx = 17\pi.}$$

Quick Tip to Solve: Expand the square first: the sine cross term is odd and vanishes on symmetric limits, while the constant contributes its value times 2 pi.

10. Find the co-ordinates of the point on the line $\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2}$ which is at a distance of 5 units from the point (1, 3, 3). [3 marks]

Solution:

Concept:

- In the symmetric equation of a line, equate each ratio to a common real parameter t . This expresses all three coordinates of a general point on the line in terms of the same parameter.
- For points $P(x, y, z)$ and $A(x_1, y_1, z_1)$, the squared distance is given by

$$PA^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2.$$

A prescribed positive distance can be imposed using this formula.

Step 1: Parametrise the point on the given line Let the common value of the three ratios be t :

$$\frac{x+2}{3} = \frac{y+1}{2} = \frac{z-3}{2} = t.$$

Thus a general point on the line is

$$P = (3t - 2, 2t - 1, 2t + 3), \quad t \in \mathbb{R}.$$

We must determine every real value of t for which this point is 5 units from $A = (1, 3, 3)$. The coordinate differences $P - A$ are $3t - 3$, $2t - 4$, and $2t$.

Step 2: Apply the distance condition and solve for the parameter Since $PA = 5$, the squared distance gives

$$\begin{aligned} (3t - 3)^2 + (2t - 4)^2 + (2t)^2 &= 25, \\ 9t^2 - 18t + 9 + 4t^2 - 16t + 16 + 4t^2 &= 25, \\ 17t^2 - 34t + 25 &= 25, \\ 17t(t - 2) &= 0. \end{aligned}$$

Therefore $t = 0$ or $t = 2$. Both are real and must be retained: the distance condition can select two different points on a line. Squaring introduces no extraneous solution here, because distance is nonnegative and its required value is positive.

Step 3: Substitute both values and verify the distances For $t = 0$, the coordinates are $(-2, -1, 3)$; for $t = 2$, they are $(4, 3, 7)$. Each lies on the given line because its three ratios have the common value 0 or 2, respectively. Their distances from A are

$$\begin{aligned} P_0A &= \sqrt{(-3)^2 + (-4)^2 + 0^2} = 5, \\ P_2A &= \sqrt{3^2 + 0^2 + 4^2} = 5. \end{aligned}$$

These checks confirm that both roots satisfy the original conditions, and the quadratic has no further roots.

Step 4: Conclusion The required points are $(-2, -1, 3)$ and $(4, 3, 7)$.

Quick Tip to Solve: Set all three ratios equal to one parameter, then use the squared distance. Retain both real roots and substitute each into all three coordinates.

Section C (16 marks)

Question numbers 11 to 14 carry 4 marks each.

11. Evaluate: $\int_0^{\pi} \frac{x}{1 + \sin x} dx.$

[4 marks]

Solution:

Concept:

- Reflection property: for a continuous integrand on $[0, a]$, $\int_0^a f(x) dx = \int_0^a f(a - x) dx$. Here $\sin(\pi - x) = \sin x$, so reflection leaves the denominator unchanged.
- The half-angle substitution $t = \tan(x/2)$ gives $\sin x = 2t/(1 + t^2)$ and $dx = 2 dt/(1 + t^2)$. This turns the remaining trigonometric integral into a rational integral.

Step 1: Reflect the given integral Put $I = \int_0^{\pi} \frac{x}{1 + \sin x} dx$. The denominator is at least 1 throughout this interval, so the integral is well defined. Using the substitution $u = \pi - x$, the limits reverse and the negative differential restores their order. Thus

$$I = \int_0^{\pi} \frac{\pi - u}{1 + \sin(\pi - u)} du = \int_0^{\pi} \frac{\pi - x}{1 + \sin x} dx.$$

The variable of integration is a dummy variable; hence we may rename u as x .

Step 2: Add the two expressions Adding the original and reflected expressions eliminates the factor x :

$$2I = \int_0^{\pi} \frac{x + (\pi - x)}{1 + \sin x} dx = \pi \int_0^{\pi} \frac{dx}{1 + \sin x}.$$

Write $J = \int_0^{\pi} \frac{dx}{1 + \sin x}$. We therefore need only evaluate J , after which $I = (\pi/2)J$.

Step 3: Evaluate the remaining integral Take $t = \tan(x/2)$. As x goes from 0 to π , t goes from 0 to $+\infty$. The upper endpoint is understood as a limit. Since

$$1 + \sin x = 1 + \frac{2t}{1 + t^2} = \frac{(t + 1)^2}{1 + t^2},$$

we obtain

$$\begin{aligned} J &= \lim_{R \rightarrow \infty} \int_0^R \frac{2}{(t + 1)^2} dt \\ &= \lim_{R \rightarrow \infty} \left[-\frac{2}{t + 1} \right]_0^R \\ &= \lim_{R \rightarrow \infty} \left(2 - \frac{2}{R + 1} \right) = 2. \end{aligned}$$

This limiting computation comes from the substitution; the original integral itself has a continuous integrand at both endpoints.

Step 4: Conclusion Substituting $J = 2$ into $I = (\pi/2)J$, the required value is $\boxed{\pi}$.

Quick Tip to Solve: Reflect about $\pi/2$ first: because $\sin(\pi - x) = \sin x$, adding the two integrals replaces the numerator x by the constant π .

12. A man is known to speak truth 7 out of 10 times. He threw a pair of dice and reports that a doublet appeared. Find the probability that it was actually a doublet. [4 marks]

Solution:

Concept:

- A doublet means that both dice show the same number. For two fair dice there are 36 equally likely ordered outcomes, of which 6 are doublets.
- Bayes' theorem gives the probability of an event after a report about it. If D denotes an actual doublet and R denotes the report of a doublet, then $P(D | R) = \frac{P(D)P(R | D)}{P(D)P(R | D) + P(D^c)P(R | D^c)}$.
- The probability that the man tells the truth is $7/10$; the probability that he tells a lie is $3/10$. A report that a doublet appeared is true when a doublet occurred and false when it did not.

Step 1: Calculate the prior probabilities Let D be the event that the actual throw is a doublet, and let D^c be the event that it is not. The doublets are $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)$. Thus

$$P(D) = \frac{6}{36} = \frac{1}{6}, \quad P(D^c) = 1 - \frac{1}{6} = \frac{5}{6}.$$

These two events are mutually exclusive and exhaustive, so they form the partition needed for Bayes' theorem.

Step 2: Relate the report to truth and falsehood Let R be the event that the man reports a doublet. If a doublet actually occurred, this report is truthful. If no doublet occurred, this report is a lie. Therefore

$$P(R | D) = \frac{7}{10}, \quad P(R | D^c) = \frac{3}{10}.$$

A report can consequently arise in two ways: an actual doublet accompanied by truth, or a non-doublet accompanied by a false report.

Step 3: Apply total probability and Bayes' theorem The probability of receiving the report is

$$\begin{aligned} P(R) &= P(D)P(R | D) + P(D^c)P(R | D^c) \\ &= \frac{1}{6} \cdot \frac{7}{10} + \frac{5}{6} \cdot \frac{3}{10} \\ &= \frac{7}{60} + \frac{15}{60} = \frac{22}{60}. \end{aligned}$$

The joint probability that the throw is a doublet and the report is made is $P(D \cap R) = 7/60$. Hence

$$P(D | R) = \frac{P(D \cap R)}{P(R)} = \frac{7/60}{22/60} = \frac{7}{22}.$$

The man's truthfulness alone is not the requested probability: the relative frequencies of doublets and non-doublets must also be included in the conditioning.

Step 4: Conclusion The probability that the throw was actually a doublet, given his report, is $\boxed{\frac{7}{22}}$.

Quick Tip to Solve: Weight the truthful-doublet case by $1/6$ and the false-report case by $5/6$; divide the first weight by their sum.

13(a). Using integration, find the area of the region $\{(x, y) : 4x^2 + 9y^2 \leq 36, 2x + 3y \geq 6\}$. [4 marks]

Solution:

Concept:

- The ellipse $4x^2 + 9y^2 = 36$ has intercepts $(\pm 3, 0)$ and $(0, \pm 2)$. Its upper boundary is $y = \frac{2}{3}\sqrt{9 - x^2}$.
- The area between two curves over an interval is $A = \int_a^b (y_{\text{upper}} - y_{\text{lower}}) dx$. The inequality $2x + 3y \geq 6$ selects the side above the line $y = 2 - \frac{2x}{3}$.

Step 1: Locate the intersections and the selected region Substitute $y = (6 - 2x)/3$ into the ellipse equation:

$$\begin{aligned}4x^2 + (6 - 2x)^2 &= 36, \\8x^2 - 24x &= 0, \\8x(x - 3) &= 0.\end{aligned}$$

Thus the boundaries meet at $(0, 2)$ and $(3, 0)$. For $-3 \leq x < 0$, the line lies above $y = 2$, whereas every point inside the ellipse has $y \leq 2$; hence no selected region occurs there. For $0 \leq x \leq 3$, the line is nonnegative and lies below the upper ellipse. The required region therefore has the upper ellipse as its upper boundary and the line segment joining these intersections as its lower boundary.

Step 2: Set up and evaluate the area integral Using vertical strips, the area is

$$\begin{aligned}A &= \int_0^3 \left[\frac{2}{3}\sqrt{9 - x^2} - \left(2 - \frac{2x}{3}\right) \right] dx \\&= \frac{2}{3} \int_0^3 \sqrt{9 - x^2} dx - \left[2x - \frac{x^2}{3} \right]_0^3.\end{aligned}$$

To evaluate the first integral by integration, put $x = 3 \sin \theta$, so $dx = 3 \cos \theta d\theta$. The limits become 0 and $\pi/2$, and $\sqrt{9 - x^2} = 3 \cos \theta$ on this interval. Therefore,

$$\begin{aligned}\int_0^3 \sqrt{9 - x^2} dx &= 9 \int_0^{\pi/2} \cos^2 \theta d\theta \\&= \frac{9}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} = \frac{9\pi}{4}.\end{aligned}$$

The line contributes $\left[2x - \frac{x^2}{3} \right]_0^3 = 6 - 3 = 3$, which must be subtracted from the ellipse contribution.

Step 3: Conclusion The required area is $\frac{2}{3} \cdot \frac{9\pi}{4} - 3 = \boxed{\frac{3\pi}{2} - 3}$ square units.

Quick Tip to Solve: First find the chord endpoints: they are the positive-axis intercepts. Integrate upper ellipse minus chord over $0 \leq x \leq 3$, and use $x = 3 \sin \theta$ for the radical.

13(b). Using integration, find the area of the region bounded by lines $x - y + 1 = 0$, $x = -2$, $x = 3$ and x-axis. [4 marks]

Solution:

Concept:

- The area between a curve $y = f(x)$ and the x-axis is $\int |f(x)| dx$. Geometrical area is non-negative even when the curve is below the x-axis.
- For a line crossing the x-axis within the given interval, split the integral at its zero. On each interval, integrate the upper ordinate minus the lower ordinate.

Step 1: Locate the line and its intercept The equation $x - y + 1 = 0$ gives $y = x + 1$. Its intersection with the x-axis is obtained by putting $y = 0$, which gives $x = -1$. This point lies between the two vertical boundaries $x = -2$ and $x = 3$, so the bounded region has two portions. For $-2 \leq x \leq -1$, the line is below the x-axis; for $-1 \leq x \leq 3$, it is above the x-axis. Thus the vertical height of the region changes from $-(x + 1)$ to $x + 1$ at $x = -1$.

Step 2: Integrate the portion below the x-axis In the left portion the upper boundary is $y = 0$ and the lower boundary is $y = x + 1$. Therefore,

$$\begin{aligned} A_1 &= \int_{-2}^{-1} \{0 - (x + 1)\} dx \\ &= - \left[\frac{x^2}{2} + x \right]_{-2}^{-1} \\ &= - \left\{ \left(\frac{1}{2} - 1 \right) - (2 - 2) \right\} = \frac{1}{2}. \end{aligned}$$

The negative sign in the integrand is necessary: integrating the line's negative ordinate directly would produce a signed integral rather than the positive area of this portion.

Step 3: Integrate the portion above the x-axis and add In the right portion the upper boundary is $y = x + 1$ and the lower boundary is $y = 0$. Hence,

$$\begin{aligned} A_2 &= \int_{-1}^3 (x + 1) dx \\ &= \left[\frac{x^2}{2} + x \right]_{-1}^3 \\ &= \left(\frac{9}{2} + 3 \right) - \left(\frac{1}{2} - 1 \right) = 8. \end{aligned}$$

The two portions have disjoint interiors, so their geometrical areas must be added: $A = A_1 + A_2 = \frac{1}{2} + 8$. A single integral of $x + 1$ from -2 to 3 would subtract the left portion and therefore would not give the required bounded area.

Step 4: Conclusion The required area is $\boxed{\frac{17}{2}}$ square units.

Quick Tip to Solve: First solve $x + 1 = 0$; splitting at the intercept $x = -1$ prevents the area below the x-axis from cancelling part of the area above it.

14. Electrical transmission wires which are laid down in winters are stretched tightly to accommodate expansion in summers.

Two such wires lie along the following lines:

$$l_1 : \frac{x+1}{3} = \frac{y-3}{-2} = \frac{z+2}{-1}$$

$$l_2 : \frac{x}{-1} = \frac{y-7}{3} = \frac{z+7}{-2}$$

Based on the given information, answer the following questions:

[4 marks]

14(i). Are the lines l_1 and l_2 coplanar? Justify your answer.

[2 marks]

Solution:

Concept:

- Two lines with points A, B and direction vectors $\mathbf{b}_1, \mathbf{b}_2$ are coplanar if $\vec{AB} \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = 0$. This scalar triple product measures the volume determined by the three vectors.
- For a line in symmetric form, its numerator shifts give a point on the line and its denominators give a direction vector.

Step 1: Identify points and direction vectors From l_1 , take $A = (-1, 3, -2)$ and $\mathbf{b}_1 = (3, -2, -1)$. From l_2 , take $B = (0, 7, -7)$ and $\mathbf{b}_2 = (-1, 3, -2)$. Therefore $\vec{AB} = B - A = (1, 4, -5)$. The order of subtraction is kept consistent with the scalar triple product.

Step 2: Evaluate the coplanarity condition First calculate the cross product:

$$\mathbf{b}_1 \times \mathbf{b}_2 = (4 + 3, 1 + 6, 9 - 2) = (7, 7, 7).$$

Hence

$$\vec{AB} \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = 7 + 28 - 35 = 0.$$

The cross product is nonzero, so the direction vectors are not parallel. The zero scalar triple product shows that the displacement between the lines lies in the plane of their direction vectors.

Step 3: Conclusion The lines l_1 and l_2 are coplanar because their scalar triple product is zero.

Quick Tip to Solve: Use the denominator triples as direction vectors and subtract the two base points; here the cross product has equal components, so check $1 + 4 - 5 = 0$.

14(ii). Find the point of intersection of the lines l_1 and l_2 .

[2 marks]

Solution:

Concept:

- At an intersection, the coordinates obtained from the two parametric equations must agree for all three coordinates.
- Use independent parameters λ and μ for the two lines; solve two coordinate equations and verify the third.

Step 1: Write the lines in parametric form Set the common ratios of l_1 and l_2 equal to λ and μ ,

Quick Tip to Solve: Solve using the x- and y-coordinate equations, then check z in both lines; agreement of only two coordinates does not establish an intersection.

respectively. This gives

$$l_1 : (x, y, z) = (-1 + 3\lambda, 3 - 2\lambda, -2 - \lambda),$$

$$l_2 : (x, y, z) = (-\mu, 7 + 3\mu, -7 - 2\mu).$$

The parameters need not have the same value, since the two lines have different base points and direction vectors.

Step 2: Solve the coordinate equations and verify Equating the x -coordinates gives $\mu = 1 - 3\lambda$. Equating the y -coordinates then gives

$$3 - 2\lambda = 7 + 3(1 - 3\lambda) = 10 - 9\lambda,$$

so $7\lambda = 7$, yielding $\lambda = 1$ and $\mu = -2$. Substituting in l_1 gives $(2, 1, -3)$. In l_2 , the same parameters give $x = 2$, $y = 7 - 6 = 1$ and $z = -7 + 4 = -3$. Thus the third coordinate agrees as well, confirming an actual intersection rather than merely a solution of two projected equations.

Step 3: Conclusion The point of intersection is $\boxed{(2, 1, -3)}$.

Quick Tip to Solve: Solve using the x - and y -coordinate equations, then check z in both lines; agreement of only two coordinates does not establish an intersection.