

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-1-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) The question paper contains **three** Sections – Section A, B and C.
- (ii) **Each** Section is **Compulsory**.
- (iii) **Section – A** has **6** short answer type-I questions of **2** marks each.
- (iv) **Section – B** has **4** short answer type-II questions of **3** marks each.
- (v) **Section – C** has **4** long answer type questions of **4** marks each.
- (vi) There is an internal choice in some questions.
- (vii) Question No. **14** is a case based problem with **2** subparts of **2** marks each.

Section A (12 marks)

Question numbers 1 to 6 carry 2 marks each.

1. Let A and B be two events such that $P(A) = \frac{5}{8}$, $P(B) = \frac{1}{2}$ and $P(A/B) = \frac{3}{4}$. Find the value of $P(B/A)$. [2 marks]

Solution:

Concept:

- Conditional probability is $P(A/B) = P(A \cap B)/P(B)$, provided $P(B) > 0$. Here A/B means that event B has already occurred.
- Reversing the conditioning gives $P(B/A) = P(A \cap B)/P(A)$; the intersection is unchanged, but the denominator changes.

Step 1: Find the probability of the intersection Since $P(B) = \frac{1}{2} > 0$, use the given conditional probability to obtain

$$P(A \cap B) = P(A/B)P(B) = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}.$$

This is the probability that both events occur. It does not require the events to be independent; in fact, the given $P(A/B)$ differs from $P(A)$.

Step 2: Reverse the conditioning Now $P(A) = \frac{5}{8} > 0$, so division by $P(A)$ is valid. Thus

$$P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{3/8}{5/8} = \frac{3}{8} \times \frac{8}{5} = \frac{3}{5}.$$

The result lies between zero and one, as a probability must. Notice that changing the conditioning event changes the sample space and therefore changes the denominator.

Step 3: Conclusion Hence, $P(B/A) = \frac{3}{5}$.

Quick Tip to Solve: Find the common intersection first, then divide by the probability of the event after the slash.

2. Two balls are drawn at random from a bag containing 2 red balls and 3 blue balls, without replacement. Let the variable X denote the number of red balls. Find the probability distribution of X. [2 marks]

Solution:

Concept:

- A probability distribution lists every possible value of a random variable and its probability. Here X counts red balls in the two-ball selection, so $X = 0, 1, 2$.
- Without replacement, every unordered pair of distinct balls is equally likely. Thus probabilities equal favourable pairs divided by $\binom{5}{2} = 10$.

Quick Tip to Solve: For a two-ball draw without replacement, count unordered pairs and check that the three probabilities add to one.

Step 1: Count the possible selections There are five balls in all, with two red and three blue. The number of unordered pairs drawn is $\binom{5}{2} = 10$. For $X = 0$, choose both balls from the three blue balls; for $X = 1$, choose one red and one blue; for $X = 2$, choose both red balls. Consequently,

$$P(X = 0) = \frac{\binom{2}{0}\binom{3}{2}}{\binom{5}{2}} = \frac{3}{10},$$

$$P(X = 1) = \frac{\binom{2}{1}\binom{3}{1}}{\binom{5}{2}} = \frac{6}{10} = \frac{3}{5},$$

$$P(X = 2) = \frac{\binom{2}{2}\binom{3}{0}}{\binom{5}{2}} = \frac{1}{10}.$$

Step 2: Check that the distribution is complete These three cases are disjoint and cover every possible draw of two balls. Their probabilities are nonnegative and sum to

$$\frac{3}{10} + \frac{3}{5} + \frac{1}{10} = \frac{3+6+1}{10} = 1.$$

The count of red balls cannot be negative or exceed two, so all other values of X have probability zero.

Step 3: Conclusion The probability distribution is

x	0	1	2
$P(X = x)$	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{1}{10}$

Quick Tip to Solve: For a two-ball draw without replacement, count unordered pairs and check that the three probabilities add to one.

3. Find the sum of the order and the degree of the differential equation: $\left(x + \frac{dy}{dx}\right)^2 = \left(\frac{dy}{dx}\right)^2 + 1$. [2 marks]

Solution:

Concept:

- The order of a differential equation is the order of its highest derivative. A derivative dy/dx is of first order.
- The degree is the exponent of the highest-order derivative in the simplified polynomial equation in derivatives. Expand and cancel terms before determining it.

Step 1: Simplify the printed differential equation Write $p = dy/dx$ temporarily. Expanding the square on the left of the given equation gives

$$(x + p)^2 = p^2 + 1 \implies x^2 + 2xp + p^2 = p^2 + 1.$$

Quick Tip to Solve: Cancel equal derivative powers on both sides before reading the degree; the displayed square can be misleading.

The p^2 terms cancel exactly on the two sides. Hence the equation is equivalently

$$2x \frac{dy}{dx} + x^2 - 1 = 0.$$

The initially visible square therefore does not determine the degree.

Step 2: Identify the order and degree The only derivative in the simplified equation is dy/dx , so the highest derivative is first order. The equation is polynomial in this derivative, and its exponent is one. Thus the order is 1 and the degree is 1. The factor $2x$ is a coefficient depending on the independent variable; it is not a power of the derivative and does not increase the degree. There is no need to solve the differential equation to identify these quantities.

Step 3: Conclusion Therefore, the sum of the order and degree is $\boxed{1 + 1 = 2}$.

Quick Tip to Solve: Cancel equal derivative powers on both sides before reading the degree; the displayed square can be misleading.

4(a). If $\frac{d}{dx}[F(x)] = \frac{\sec^4 x}{\csc^4 x}$ and $F\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$, then find $F(x)$.

[2 marks]

Solution:

Concept:

- An antiderivative satisfies $F'(x) = f(x)$; the given value of F determines its arbitrary constant.
- Use $\sec x = 1/\cos x$, $\csc x = 1/\sin x$ and $\tan^2 x = \sec^2 x - 1$ before integrating.

Step 1: Simplify and integrate The derivative simplifies to

$$\frac{\sec^4 x}{\csc^4 x} = \frac{\sin^4 x}{\cos^4 x} = \tan^4 x.$$

Write $\tan^4 x = \tan^2 x(\sec^2 x - 1)$. Substituting $u = \tan x$ in the first integral gives

$$\begin{aligned} F(x) &= \int \tan^2 x \sec^2 x \, dx - \int \tan^2 x \, dx \\ &= \frac{\tan^3 x}{3} - \int (\sec^2 x - 1) \, dx \\ &= \frac{\tan^3 x}{3} - \tan x + x + C. \end{aligned}$$

Step 2: Determine the constant Since $\tan(\pi/4) = 1$, the supplied condition gives

$$\frac{\pi}{4} = \frac{1}{3} - 1 + \frac{\pi}{4} + C, \quad C = \frac{2}{3}.$$

For verification, differentiating the expression yields $\tan^2 x \sec^2 x - \sec^2 x + 1 = \tan^4 x$, exactly the required derivative. Substitution at $\pi/4$ also yields $\pi/4$. The formula holds on the interval of definition containing that point.

Quick Tip to Solve: For an even power of tangent, replace one factor $\tan^2 x$ by $\sec^2 x - 1$ before integrating.

Step 3: Conclusion Hence $F(x) = \frac{\tan^3 x}{3} - \tan x + x + \frac{2}{3}$.

Quick Tip to Solve: For an even power of tangent, replace one factor $\tan^2 x$ by $\sec^2 x - 1$ before integrating.

OR

4(b). Find: $\int \frac{\log x - 3}{(\log x)^4} dx$.

[2 marks]

Solution:

Concept:

- The product rule gives $\frac{d}{dx}[xg(x)] = g(x) + xg'(x)$. Recognise the integrand as the derivative of a product.
- For real logarithms require $x > 0$; the denominator further excludes $x = 1$. Here $\log x$ denotes the natural logarithm.

Step 1: Recognise a useful product Write the integrand as

$$\frac{\log x - 3}{(\log x)^4} = \frac{1}{(\log x)^3} - \frac{3}{(\log x)^4}.$$

The first term suggests differentiating $x(\log x)^{-3}$. The chain rule supplies the missing second term because $\frac{d}{dx} \log x = 1/x$. Thus the factor x cancels that denominator in the product rule.

Step 2: Differentiate to verify the antiderivative Using both the product rule and chain rule explicitly,

$$\begin{aligned} \frac{d}{dx} \left[\frac{x}{(\log x)^3} \right] &= (\log x)^{-3} + x \left[-3(\log x)^{-4} \frac{1}{x} \right] \\ &= \frac{1}{(\log x)^3} - \frac{3}{(\log x)^4} \\ &= \frac{\log x - 3}{(\log x)^4}. \end{aligned}$$

This equals the given integrand on either interval $(0, 1)$ or $(1, \infty)$. Adding an arbitrary constant accounts for all antiderivatives on each such interval.

Step 3: Conclusion Therefore $\int \frac{\log x - 3}{(\log x)^4} dx = \frac{x}{(\log x)^3} + C$.

Quick Tip to Solve: When the numerator is $\log x - n$, test the derivative of $x/(\log x)^n$.

5. Find the values of λ , for which the distance of the point $(2, 1, \lambda)$ from the plane $3x + 5y + 4z = 11$ is $2\sqrt{2}$ units. [2 marks]

Solution:

Concept:

- The perpendicular distance of (x_0, y_0, z_0) from $ax + by + cz + d = 0$ is $D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$.
- An absolute-value equation $|u| = k$, with $k > 0$, has the two cases $u = k$ and $u = -k$; these represent points on opposite sides of the plane.

Step 1: Apply the point-to-plane formula Write the plane as $3x + 5y + 4z - 11 = 0$, so its normal vector is $(3, 5, 4)$. Substituting the coordinates $(2, 1, \lambda)$,

$$D = \frac{|3(2) + 5(1) + 4\lambda - 11|}{\sqrt{3^2 + 5^2 + 4^2}} \\ = \frac{|4\lambda|}{\sqrt{50}} = \frac{|4\lambda|}{5\sqrt{2}}.$$

The constant terms cancel because $6 + 5 - 11 = 0$. This also means the point with $\lambda = 0$ lies on the plane.

Step 2: Solve both absolute-value cases Equate this distance to the specified $2\sqrt{2}$:

$$\frac{|4\lambda|}{5\sqrt{2}} = 2\sqrt{2} \implies |4\lambda| = 20.$$

Thus $4\lambda = 20$ or $4\lambda = -20$. For either resulting value, substitution back into the distance expression gives $20/(5\sqrt{2}) = 2\sqrt{2}$, confirming that both signs satisfy the condition. Keeping both cases is essential since distance is always non-negative.

Step 3: Conclusion The required values are $\lambda = 5$ or $\lambda = -5$.

Quick Tip to Solve: Simplify the plane expression at the point first, then retain both signs when solving the absolute-value equation.

6. If $\vec{a} = 2\hat{i} + y\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are two vectors for which the vector $(\vec{a} + \vec{b})$ is perpendicular to the vector $(\vec{a} - \vec{b})$, then find all the possible values of y . [2 marks]

Solution:

Concept:

- Perpendicular vectors have zero scalar product: $\vec{u} \cdot \vec{v} = 0$.
- By distributivity and commutativity, $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$. The squared magnitude of a vector is the sum of the squares of its three components.

Step 1: Use the perpendicularity condition Taking the scalar product of the given perpendicular vectors gives

$$0 = (\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = |\vec{a}|^2 - \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} - |\vec{b}|^2.$$

Quick Tip to Solve: When the sum and difference of two vectors are perpendicular, equate their squared magnitudes and retain both roots of the parameter equation.

Since $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$, the mixed terms cancel. Thus the two original vectors have equal squared magnitudes.

Step 2: Substitute the components and check both roots From the given components,

$$|\vec{a}|^2 = 2^2 + y^2 + 1^2 = y^2 + 5, \quad |\vec{b}|^2 = 1^2 + 2^2 + 3^2 = 14.$$

Therefore $y^2 + 5 = 14$, so $y^2 = 9$. Taking square roots requires both the positive and negative roots. To verify directly, the sum and difference have components $(3, y + 2, 4)$ and $(1, y - 2, -2)$, respectively. Their scalar product is $3 + (y + 2)(y - 2) - 8 = y^2 - 9$, which is zero for each of the two roots. Hence neither sign is excluded by the original condition.

Step 3: Conclusion The possible values are $y = 3$ or $y = -3$.

Quick Tip to Solve: When the sum and difference of two vectors are perpendicular, equate their squared magnitudes and retain both roots of the parameter equation.

Section B (12 marks)

Question numbers 7 to 10 carry 3 marks each.

7(a). Find: $\int \frac{dx}{\sqrt{x} + \sqrt[3]{x}}.$

[3 marks]

Solution:

Concept:

- For fractional powers with denominators 2 and 3, substitute $t = x^{1/6}$, using their least common multiple. Then $x = t^6$, $\sqrt{x} = t^3$ and $\sqrt[3]{x} = t^2$.
- After substitution, polynomial division reduces a rational integrand to polynomial terms and a logarithmic term. Use $\int t^n dt = t^{n+1}/(n+1)$ for $n \neq -1$, and $\int dt/(t+1) = \ln|t+1| + C$.

Step 1: Remove the fractional powers by substitution Let I denote the required integral. Its real domain is $x > 0$: the square root requires $x \geq 0$, but the denominator vanishes at zero. Put $t = x^{1/6} > 0$. Consequently $dx = 6t^5 dt$, and

$$I = \int \frac{6t^5}{t^3 + t^2} dt = 6 \int \frac{t^3}{t+1} dt.$$

Cancelling t^2 is valid since $t > 0$.

Step 2: Divide and integrate term by term Polynomial division gives the exact identity

$$\frac{t^3}{t+1} = t^2 - t + 1 - \frac{1}{t+1},$$

Quick Tip to Solve: Choose the sixth root substitution to clear both fractional powers, then divide the resulting cubic by the linear denominator before integrating.

because $(t+1)(t^2-t+1)-1=t^3$. Thus

$$\begin{aligned} I &= 6 \int \left(t^2 - t + 1 - \frac{1}{t+1} \right) dt \\ &= 2t^3 - 3t^2 + 6t - 6 \ln|t+1| + C. \end{aligned}$$

Here C is an arbitrary constant of integration. Since $t > 0$, the logarithm's argument is positive.

Step 3: Return to the original variable and verify Substitute $t^3 = \sqrt{x}$, $t^2 = \sqrt[3]{x}$ and $t = x^{1/6}$. For an exact derivative check, the derivative of the expression in t is $6t^3/(t+1)$. Multiplying by $dt/dx = 1/(6t^5)$ gives $1/[t^2(t+1)]$, which equals $1/(\sqrt{x} + \sqrt[3]{x})$. Therefore the antiderivative matches the original integrand throughout its real domain.

Step 4: Conclusion

$$I = 2\sqrt{x} - 3\sqrt[3]{x} + 6x^{1/6} - 6 \ln(1 + x^{1/6}) + C, \quad x > 0.$$

Quick Tip to Solve: Choose the sixth root substitution to clear both fractional powers, then divide the resulting cubic by the linear denominator before integrating.

OR

7(b). Evaluate: $\int_0^{\pi/2} \frac{\cos x}{(1 + \sin x)(4 + \sin x)} dx.$

[3 marks]

Solution:

Concept:

- Substitution changes both the integrand and the limits: if $t = \sin x$, then $dt = \cos x dx$. Thus the numerator is precisely the differential needed to remove the trigonometric functions.
- A rational function with two distinct linear factors can be decomposed into partial fractions. Integrate each term using $\int dt/(t+a) = \ln|t+a| + C$.

Step 1: Substitute and transform the limits Denote the required integral by I . Put $t = \sin x$, so that $dt = \cos x dx$. At the lower limit $x = 0$, we have $t = 0$; at the upper limit $x = \pi/2$, we have $t = 1$. Consequently,

$$I = \int_0^1 \frac{dt}{(1+t)(4+t)}.$$

The substitution is valid on this interval because $\sin x$ increases from zero to one. Neither denominator factor vanishes for $0 \leq t \leq 1$, so the transformed definite integral is proper.

Step 2: Decompose the rational expression Write

$$\frac{1}{(t+1)(t+4)} = \frac{A}{t+1} + \frac{B}{t+4}.$$

Quick Tip to Solve: When a rational expression in $\sin x$ has $\cos x dx$ in its numerator, use $t = \sin x$ first and immediately replace both limits.

Multiplication by $(t+1)(t+4)$ gives $1 = A(t+4) + B(t+1)$. Comparing coefficients yields $A + B = 0$ and $4A + B = 1$, hence $A = 1/3$ and $B = -1/3$. Therefore,

$$I = \frac{1}{3} \int_0^1 \left(\frac{1}{t+1} - \frac{1}{t+4} \right) dt.$$

Step 3: Integrate and apply the endpoints Both logarithm arguments are positive throughout the interval. Using natural logarithms,

$$\begin{aligned} I &= \frac{1}{3} [\ln(t+1) - \ln(t+4)]_0^1 \\ &= \frac{1}{3} [(\ln 2 - \ln 5) - (\ln 1 - \ln 4)] \\ &= \frac{1}{3} \ln \left(\frac{2 \cdot 4}{5 \cdot 1} \right). \end{aligned}$$

Subtracting the entire lower-limit expression is essential: the term $-\ln 4$ at the lower endpoint contributes $+\ln 4$.

Step 4: Conclusion The value of the integral is $\boxed{\frac{1}{3} \ln \left(\frac{8}{5} \right)}$.

Quick Tip to Solve: When a rational expression in $\sin x$ has $\cos x dx$ in its numerator, use $t = \sin x$ first and immediately replace both limits.

8. Find the co-ordinates of the point where the line $\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6}$ crosses the plane passing through the points $\left(\frac{7}{2}, 0, 0\right)$, $(0, 7, 0)$ and $(0, 0, 7)$. [3 marks]

Solution:

Concept:

- The plane with nonzero intercepts a, b, c on the coordinate axes has equation $x/a + y/b + z/c = 1$. Its normal vector is proportional to the coefficients of x, y, z .
- Set the three ratios in a line's symmetric equation equal to a parameter λ . Substitution of the resulting coordinates into the plane equation gives the parameter at the intersection.

Step 1: Obtain the equation of the plane The given points $(7/2, 0, 0)$, $(0, 7, 0)$ and $(0, 0, 7)$ are the intercepts on the three coordinate axes. The intercept form therefore gives

$$\frac{x}{7/2} + \frac{y}{7} + \frac{z}{7} = 1.$$

Multiplying by 7, the required plane equation is

$$2x + y + z = 7.$$

Quick Tip to Solve: Use the three axis intercepts to write the plane directly, then set all line ratios equal to one parameter; verify your point in both original equations.

Each of the three supplied points satisfies this equation. They are noncollinear and consequently determine a unique plane, whose normal vector can be taken as $(2, 1, 1)$.

Step 2: Express the line using one parameter Let

$$\frac{x-3}{-1} = \frac{y+4}{1} = \frac{z+5}{6} = \lambda.$$

The coordinates of a general point of the line are then

$$x = 3 - \lambda, \quad y = \lambda - 4, \quad z = 6\lambda - 5.$$

In particular, the constants -4 and -5 arise by solving the numerators $y + 4$ and $z + 5$; their signs must be retained. The direction vector of the line is $(-1, 1, 6)$.

Step 3: Impose the plane equation and verify the point At the intersection the line coordinates must satisfy $2x + y + z = 7$. Thus

$$2(3 - \lambda) + (\lambda - 4) + (6\lambda - 5) = 7,$$

$$5\lambda - 3 = 7,$$

$$\lambda = 2.$$

Substitution in the parametric equations gives $x = 1$, $y = -2$ and $z = 7$. To check, the three line ratios are all 2, and the plane expression is $2(1) + (-2) + 7 = 7$. Moreover, $(2, 1, 1) \cdot (-1, 1, 6) = 5 \neq 0$, so the line crosses the plane at exactly one point.

Step 4: Conclusion The line crosses the plane at $(1, -2, 7)$.

Quick Tip to Solve: Use the three axis intercepts to write the plane directly, then set all line ratios equal to one parameter; verify your point in both original equations.

9. Solve the differential equation: $\frac{dy}{dx} = \frac{x^2 + y^2}{xy}$.

[3 marks]

Solution:

Concept:

- A first-order equation is homogeneous when its right-hand side depends only on y/x . Substitute $y = vx$, where v is a function of x , so that $dy/dx = v + x dv/dx$.
- Separation of variables reduces the transformed equation to integrals in v and x . Use $\int dx/x = \ln|x| + C$, valid on an interval that does not contain zero.

Step 1: Recognise the homogeneous equation Since the denominator in the given equation is xy , we work where $x \neq 0$ and $y \neq 0$. Rewrite its right-hand side as

$$\frac{dy}{dx} = \frac{x}{y} + \frac{y}{x} = \frac{1}{y/x} + \frac{y}{x}.$$

Thus it depends only on the ratio y/x . Put $y = vx$; then $v \neq 0$ and the product rule gives $dy/dx = v + x dv/dx$.

Quick Tip to Solve: Divide the numerator by xy first: the terms $x/y + y/x$ immediately suggest $y = vx$, after which the v terms cancel.

Step 2: Separate and integrate Substitute into the differential equation and cancel the common term v :

$$v + x \frac{dv}{dx} = \frac{1}{v} + v \implies v dv = \frac{dx}{x}.$$

Integrating both sides gives

$$\frac{v^2}{2} = \ln |x| + C_1, \quad v^2 = 2 \ln |x| + C,$$

where $C = 2C_1$ is an arbitrary real constant. Substituting $v = y/x$ yields $y^2/x^2 = 2 \ln |x| + C$.

Step 3: Verify the implicit family Differentiate $y^2 = x^2(2 \ln |x| + C)$ on an interval where $x \neq 0$. Using the relation itself,

$$2y \frac{dy}{dx} = 2x(2 \ln |x| + C) + 2x = \frac{2y^2}{x} + 2x.$$

Division by $2y$ gives $dy/dx = y/x + x/y$, exactly the original equation. The solution is used on intervals where its real branch has $y \neq 0$; zero is not a solution because the original right-hand side is undefined there.

Step 4: Conclusion The general solution is

$$\boxed{y^2 = x^2(2 \ln |x| + C)},$$

on any interval where $x \neq 0$ and $y \neq 0$.

Quick Tip to Solve: Divide the numerator by xy first: the terms $x/y + y/x$ immediately suggest $y = vx$, after which the v terms cancel.

10(a). If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ are four non-zero vectors such that $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = 4\vec{b} \times \vec{d}$, then show that $(\vec{a} - 2\vec{d})$ is parallel to $(2\vec{b} - \vec{c})$, where $\vec{a} \neq 2\vec{d}, \vec{c} \neq 2\vec{b}$. [3 marks]

Solution:

Concept:

- For two non-zero vectors $\vec{u}$ and $\vec{v}$, $\vec{u} \times \vec{v} = \vec{0}$ implies that their directions are parallel, since $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \theta$.
- The cross product distributes over addition and subtraction, and is anti-commutative: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$. Scalars may be taken outside the cross product.

Step 1: Form the cross product of the required vectors Let $\vec{u} = \vec{a} - 2\vec{d}$ and $\vec{v} = 2\vec{b} - \vec{c}$. The given inequalities ensure that both $\vec{u}$ and $\vec{v}$ are non-zero, so we can establish parallelism by showing that their cross product vanishes. Expanding in the stated order gives

$$\begin{aligned} \vec{u} \times \vec{v} &= (\vec{a} - 2\vec{d}) \times (2\vec{b} - \vec{c}) \\ &= 2\vec{a} \times \vec{b} - \vec{a} \times \vec{c} - 4\vec{d} \times \vec{b} + 2\vec{d} \times \vec{c}. \end{aligned}$$

Each coefficient follows by multiplying the scalar factors in the corresponding pair of terms.

Quick Tip to Solve: Expand the required cross product and reverse $\vec{d} \times \vec{b}$ and $\vec{d} \times \vec{c}$ before grouping terms into the two given identities.

Step 2: Reverse the products involving the second vector By anti-commutativity, $\vec{d} \times \vec{b} = -\vec{b} \times \vec{d}$ and $\vec{d} \times \vec{c} = -\vec{c} \times \vec{d}$. Therefore

$$\begin{aligned}\vec{u} \times \vec{v} &= 2\vec{a} \times \vec{b} - \vec{a} \times \vec{c} + 4\vec{b} \times \vec{d} - 2\vec{c} \times \vec{d} \\ &= 2(\vec{a} \times \vec{b} - \vec{c} \times \vec{d}) - (\vec{a} \times \vec{c} - 4\vec{b} \times \vec{d}).\end{aligned}$$

This grouping matches the two given equalities exactly; in particular, reversing the order of a cross product changes its sign.

Step 3: Apply the hypotheses The first given equality makes $\vec{a} \times \vec{b} - \vec{c} \times \vec{d} = \vec{0}$, and the second makes $\vec{a} \times \vec{c} - 4\vec{b} \times \vec{d} = \vec{0}$. Hence $\vec{u} \times \vec{v} = 2\vec{0} - \vec{0} = \vec{0}$. Since neither vector is zero, the magnitude formula forces $\sin \theta = 0$, so their directions are parallel, whether in the same or opposite sense.

Step 4: Conclusion Thus

$$(\vec{a} - 2\vec{d}) \parallel (2\vec{b} - \vec{c}).$$

Quick Tip to Solve: Expand the required cross product and reverse $\vec{d} \times \vec{b}$ and $\vec{d} \times \vec{c}$ before grouping terms into the two given identities.

OR

10(b). The two adjacent sides of a parallelogram are represented by $2\hat{i} - 4\hat{j} - 5\hat{k}$ and $2\hat{i} + 2\hat{j} + 3\hat{k}$. Find the unit vectors parallel to its diagonals. Using the diagonal vectors, find the area of the parallelogram also. [3 marks]

Solution:

Concept:

- If adjacent sides from one vertex are $\vec{a}$ and $\vec{b}$, the two diagonals have direction vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$.
- The area of the parallelogram is $\frac{1}{2}|\vec{d}_1 \times \vec{d}_2|$, where $\vec{d}_1, \vec{d}_2$ are its diagonal vectors. This follows from $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = -2(\vec{a} \times \vec{b})$.

Step 1: Obtain the diagonal vectors Write $\vec{a} = 2\hat{i} - 4\hat{j} - 5\hat{k}$ and $\vec{b} = 2\hat{i} + 2\hat{j} + 3\hat{k}$. The diagonals are:

$$\begin{aligned}\vec{d}_1 &= \vec{a} + \vec{b} = 4\hat{i} - 2\hat{j} - 2\hat{k}, \\ \vec{d}_2 &= \vec{a} - \vec{b} = -6\hat{j} - 8\hat{k}.\end{aligned}$$

Step 2: Normalize each diagonal Their lengths are

$$|\vec{d}_1| = \sqrt{4^2 + (-2)^2 + (-2)^2} = 2\sqrt{6}, \quad |\vec{d}_2| = \sqrt{(-6)^2 + (-8)^2} = 10.$$

Quick Tip to Solve: Form the sum and difference of the adjacent sides first; normalize them for direction, but use their full vectors and halve the cross-product magnitude for area.

Dividing each diagonal by its own length gives the unit direction vectors

$$\hat{u}_1 = \frac{2\hat{i} - \hat{j} - \hat{k}}{\sqrt{6}}, \quad \hat{u}_2 = \frac{-3\hat{j} - 4\hat{k}}{5}.$$

Step 3: Calculate the area using the diagonal vectors Take the cross product of the diagonal vectors:

$$\begin{aligned} \vec{d}_1 \times \vec{d}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & -2 \\ 0 & -6 & -8 \end{vmatrix} \\ &= (16 - 12)\hat{i} - (-32)\hat{j} - 24\hat{k} \\ &= 4\hat{i} + 32\hat{j} - 24\hat{k}. \end{aligned}$$

Therefore

$$|\vec{d}_1 \times \vec{d}_2| = \sqrt{4^2 + 32^2 + (-24)^2} = \sqrt{1616} = 4\sqrt{101}.$$

Hence the area is half this magnitude.

Step 4: Conclusion The parallel unit vectors are $\pm(2\hat{i} - \hat{j} - \hat{k})/\sqrt{6}$ and $\pm(3\hat{j} + 4\hat{k})/5$. The area is $2\sqrt{101}$ square units.

Quick Tip to Solve: Form the sum and difference of the adjacent sides first; normalize them for direction, but use their full vectors and halve the cross-product magnitude for area.

Section C (16 marks)

Question numbers 11 to 14 carry 4 marks each.

11. A card from a pack of 52 playing cards is lost. From the remaining cards, 2 cards are drawn at random without replacement, and are found to be both aces. Find the probability that the lost card being an ace. [4 marks]

Solution:

Concept:

- Bayes' theorem updates the probability of an event after an observation: $P(A | B) = \frac{P(A)P(B | A)}{P(B)}$, where $P(B) > 0$.
- If two cards are drawn without replacement from n cards containing r aces, the probability of two aces is $\frac{r}{n} \cdot \frac{r-1}{n-1}$. The second draw has one fewer card and one fewer ace.
- For complementary possibilities A and A^c , the total probability is $P(B) = P(A)P(B | A) + P(A^c)P(B | A^c)$.

Step 1: Define the lost-card possibilities and their prior probabilities Let A be the event that the lost card is an ace, and let B be the event that the two cards drawn from the remaining pack are both aces. Each card of the original pack is equally likely to be lost. Since the pack contains four aces and

Quick Tip to Solve: Cancel the common factor $1/(51 \cdot 50)$ in Bayes' theorem: the two posterior weights are $4 \cdot 3 \cdot 2$ and $48 \cdot 4 \cdot 3$, so the required fraction is $24/(24 + 576)$.

forty-eight other cards,

$$P(A) = \frac{4}{52} = \frac{1}{13}, \quad P(A^c) = \frac{48}{52} = \frac{12}{13}.$$

The information that both drawn cards are aces is the observation B ; therefore the required probability is conditional on B , rather than merely the original probability of losing an ace.

Step 2: Find the likelihood of drawing two aces in each case If the lost card is an ace, the remaining fifty-one cards contain three aces. Drawing without replacement gives

$$P(B | A) = \frac{3}{51} \cdot \frac{2}{50} = \frac{6}{2550}.$$

If the lost card is not an ace, all four aces remain among the fifty-one cards. Thus

$$P(B | A^c) = \frac{4}{51} \cdot \frac{3}{50} = \frac{12}{2550}.$$

The second denominator is fifty because the first selected card is not returned. Drawing two aces is twice as likely when the lost card is not an ace.

Step 3: Apply the total-probability formula and Bayes' theorem The two possibilities for the lost card are mutually exclusive and exhaustive, so

$$\begin{aligned} P(B) &= \frac{1}{13} \cdot \frac{6}{2550} + \frac{12}{13} \cdot \frac{12}{2550} \\ &= \frac{150}{13 \cdot 2550}. \end{aligned}$$

Hence, using Bayes' theorem,

$$\begin{aligned} P(A | B) &= \frac{\frac{1}{13} \cdot \frac{6}{2550}}{\frac{150}{13 \cdot 2550}} \\ &= \frac{6}{150}. \end{aligned}$$

This calculation weights both possible kinds of lost card by how likely each is to produce the observed pair of aces.

Step 4: Conclusion The probability that the lost card is an ace, given that both drawn cards are aces, is $\boxed{\frac{1}{25}}$.

Quick Tip to Solve: Cancel the common factor $1/(51 \cdot 50)$ in Bayes' theorem: the two posterior weights are $4 \cdot 3 \cdot 2$ and $48 \cdot 4 \cdot 3$, so the required fraction is $24/(24 + 576)$.

12. Evaluate: $\int_0^\pi \frac{x}{1 + \sin x} dx.$

[4 marks]

Solution:

Concept:

- Reflection property of a definite integral: $\int_0^a f(x) dx = \int_0^a f(a-x) dx$. It follows by substituting $t = a - x$, reversing the limits and accounting for the negative differential.
- The denominator is unchanged by reflection because $\sin(\pi - x) = \sin x$. Adding the original and reflected integrals therefore replaces the numerator x by the constant π .
- The half-angle identities give $1 + \sin x = (\sin(x/2) + \cos(x/2))^2 = 2 \cos^2(x/2 - \pi/4)$. This converts the remaining integral into an elementary secant-squared integral.

Step 1: Reflect the integral about the midpoint Denote the required integral by I . Its integrand is continuous throughout $[0, \pi]$, since $\sin x \geq 0$ there and the denominator never vanishes. Using the reflection property and then the identity for the sine of supplementary angles,

$$I = \int_0^\pi \frac{x}{1 + \sin x} dx = \int_0^\pi \frac{\pi - x}{1 + \sin(\pi - x)} dx = \int_0^\pi \frac{\pi - x}{1 + \sin x} dx.$$

The limits remain the same after the substitution because the reversal of limits cancels the negative sign in the differential.

Step 2: Add the two representations Adding the original integral to its reflected representation yields

$$\begin{aligned} 2I &= \int_0^\pi \frac{x + (\pi - x)}{1 + \sin x} dx \\ &= \pi \int_0^\pi \frac{1}{1 + \sin x} dx. \end{aligned}$$

Thus it is enough to evaluate the simpler integral $J = \int_0^\pi (1 + \sin x)^{-1} dx$, after which $I = \pi J/2$.

Step 3: Evaluate the remaining integral using a half-angle identity Write $1 + \sin x = 2 \cos^2(x/2 - \pi/4)$. Set $u = x/2 - \pi/4$, so that $dx = 2 du$. The lower and upper limits become $-\pi/4$ and $\pi/4$, respectively. Hence

$$\begin{aligned} J &= \int_{-\pi/4}^{\pi/4} \sec^2 u du \\ &= [\tan u]_{-\pi/4}^{\pi/4} \\ &= 1 - (-1) = 2. \end{aligned}$$

This substitution has no singularity on the integration interval, since $\cos u$ is positive for $-\pi/4 \leq u \leq \pi/4$.

Step 4: Conclusion Substituting $J = 2$ into $I = \pi J/2$, the required value is $\boxed{\pi}$.

Quick Tip to Solve: When a numerator contains x and the denominator is unchanged by replacing x with $\pi - x$, add the reflected integral first to remove x .

13(a). Using integration, find the area of the region enclosed by the curve $y = x^2$, the x -axis and the ordinates $x = -2$ and $x = 1$. [4 marks]

Solution:

Concept:

- The area between a curve $y = f(x)$ and the x -axis over $[a, b]$ is $A = \int_a^b |f(x)| dx$. The absolute value ensures that geometric area is non-negative.
- When the curve lies on or above the x -axis throughout the interval, the area is simply $\int_a^b f(x) dx$. Here $x^2 \geq 0$ for every real x .
- By the power rule, an antiderivative of x^2 is $x^3/3$. A definite integral is evaluated by subtracting the value of an antiderivative at the lower limit from its value at the upper limit.

Step 1: Identify the enclosed region and its vertical height The ordinates $x = -2$ and $x = 1$ are vertical lines that give the left and right boundaries. Between these lines, the lower boundary is the x -axis, whose equation is $y = 0$, and the upper boundary is the parabola $y = x^2$. A thin vertical strip at coordinate x therefore has height $x^2 - 0 = x^2$ and area $dA = x^2 dx$. Although part of the interval lies to the left of the y -axis, its strip heights are still positive; negative x -coordinates do not make the area negative.

Step 2: Set up and evaluate the definite integral Adding the areas of the vertical strips across the full interval gives

$$\begin{aligned} A &= \int_{-2}^1 (x^2 - 0) dx \\ &= \left[\frac{x^3}{3} \right]_{-2}^1 \\ &= \frac{1^3}{3} - \frac{(-2)^3}{3} \\ &= \frac{1}{3} - \left(-\frac{8}{3} \right). \end{aligned}$$

The lower-limit value must be subtracted with its negative sign intact. No sign change of the integrand is needed at the origin because x^2 is non-negative on both sides.

Step 3: Check the contributions on either side of the origin The parabola touches the x -axis at $x = 0$. Splitting the region there provides a direct check:

$$\begin{aligned} A_{\text{left}} &= \int_{-2}^0 x^2 dx = \frac{8}{3}, \\ A_{\text{right}} &= \int_0^1 x^2 dx = \frac{1}{3}. \end{aligned}$$

Both contributions are positive and are added, since both are actual areas enclosed by the curve and the axis. Their sum agrees with the single definite integral above.

Step 4: Conclusion Hence, the required area is 3 square units.

Quick Tip to Solve: Check the sign of the curve's ordinate, not the sign of x : since x squared is non-negative, integrate directly from -2 to 1 and add both sides of the origin.

13(b). Using integration, find the area of the region enclosed by the line $y = \sqrt{3}x$, the semi-circle $y = \sqrt{4 - x^2}$ and the x-axis in the first quadrant. [4 marks]

Solution:

Concept:

- The area between a non-negative curve $y = f(x)$ and the x -axis from $x = a$ to $x = b$ is $\int_a^b f(x) dx$.
- When the upper boundary changes at an intersection, split the area integral there. The equation $y = \sqrt{4 - x^2}$ represents the upper semicircle $x^2 + y^2 = 4$, of radius 2.
- For the semicircle integral, the substitution $x = 2 \sin \theta$ gives $\sqrt{4 - x^2} = 2 \cos \theta$ in the first quadrant, with $0 \leq \theta \leq \pi/2$.

Step 1: Find the boundary intersections The line meets the x -axis at $O = (0, 0)$, and the semicircle meets the positive x -axis at $B = (2, 0)$. At the intersection of the line and the semicircle,

$$\sqrt{3}x = \sqrt{4 - x^2} \implies 3x^2 = 4 - x^2 \implies x^2 = 1.$$

Since the required region lies in the first quadrant, take $x = 1$, giving $P = (1, \sqrt{3})$. Both sides of the original intersection equation are non-negative, so this solution satisfies it.

Step 2: Split the area at the intersection The enclosed boundary follows the line from O to P , the semicircular arc from P to B , and the x -axis from B back to O . Thus the upper boundary is the line for $0 \leq x \leq 1$ and the semicircle for $1 \leq x \leq 2$. The required area is

$$A = \int_0^1 \sqrt{3}x \, dx + \int_1^2 \sqrt{4 - x^2} \, dx.$$

The first integral is

$$\int_0^1 \sqrt{3}x \, dx = \sqrt{3} \left[\frac{x^2}{2} \right]_0^1 = \frac{\sqrt{3}}{2}.$$

Step 3: Evaluate the semicircular part using integration Put $x = 2 \sin \theta$, so $dx = 2 \cos \theta \, d\theta$. When $x = 1$, $\theta = \pi/6$, and when $x = 2$, $\theta = \pi/2$. Therefore

$$\begin{aligned} \int_1^2 \sqrt{4 - x^2} \, dx &= 4 \int_{\pi/6}^{\pi/2} \cos^2 \theta \, d\theta \\ &= 2 \int_{\pi/6}^{\pi/2} (1 + \cos 2\theta) \, d\theta \\ &= [2\theta + \sin 2\theta]_{\pi/6}^{\pi/2} \\ &= \pi - \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2} \right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}. \end{aligned}$$

Quick Tip to Solve: Locate the intersection first: split at $x = 1$, using the line on $[0, 1]$ and the semicircle on $[1, 2]$.

On adding the two contributions, the terms involving $\sqrt{3}$ cancel:

$$A = \frac{\sqrt{3}}{2} + \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

Step 4: Conclusion The area enclosed by the line, the semicircle and the x -axis in the first quadrant is $\frac{2\pi}{3}$ square units.

Quick Tip to Solve: Locate the intersection first: split at $x = 1$, using the line on $[0, 1]$ and the semicircle on $[1, 2]$.

14. Electrical transmission wires which are laid down in winters are stretched tightly to accommodate expansion in summers.

Two such wires lie along the following lines:

$$l_1 : \frac{x+1}{3} = \frac{y-3}{-2} = \frac{z+2}{-1}$$

$$l_2 : \frac{x}{-1} = \frac{y-7}{3} = \frac{z+7}{-2}$$

Based on the given information, answer the following questions:

[4 marks]

14(i). Are the lines l_1 and l_2 coplanar? Justify your answer.

[2 marks]

Solution:

Concept:

- For two lines through points A and B , with direction vectors $\mathbf{a}$ and $\mathbf{b}$, coplanarity is equivalent to $\overrightarrow{AB} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.
- This scalar triple product measures the signed volume formed by the three vectors. Its vanishing means that the displacement between the lines and both directions lie in one plane.

Step 1: Identify the points and direction vectors Comparing the given equations with the symmetric form of a line, choose

$$A = (-1, 3, -2), \quad B = (0, 7, -7), \quad \mathbf{a} = (3, -2, -1), \quad \mathbf{b} = (-1, 3, -2).$$

Hence $\overrightarrow{AB} = B - A = (1, 4, -5)$. In particular, the numerators $x+1$ and $z+2$ give negative coordinates for A .

Step 2: Evaluate the coplanarity test Taking the cross product of the two direction vectors gives

$$\mathbf{a} \times \mathbf{b} = ((-2)(-2) - (-1)(3), (-1)(-1) - 3(-2), 3(3) - (-2)(-1)) = (7, 7, 7).$$

Therefore the scalar triple product is

$$\overrightarrow{AB} \cdot (\mathbf{a} \times \mathbf{b}) = 1(7) + 4(7) - 5(7) = 0.$$

The cross product is nonzero, so the directions are not parallel; the zero triple product places the two

Quick Tip to Solve: Extract base-point coordinates with the opposite signs to the constants in the numerators, then test the displacement using a scalar triple product.

lines in a common plane. As a check, both base points have coordinate sum zero, and both direction vectors have component sum zero, so both lines lie in the plane $x + y + z = 0$.

Step 3: Conclusion Yes, l_1 and l_2 are coplanar, since their scalar triple product is zero. A common plane is $x + y + z = 0$.

Quick Tip to Solve: Extract base-point coordinates with the opposite signs to the constants in the numerators, then test the displacement using a scalar triple product.

14(ii). Find the point of intersection of the lines l_1 and l_2 .

[2 marks]

Solution:

Concept:

- Introduce independent parameters for the two lines. A common point must have the same x , y and z coordinates in both parameterisations.
- Solve two coordinate equations for the parameters and verify the third equation; this checks that the apparent intersection satisfies both original lines.

Step 1: Write both lines in parametric form Set the three ratios for l_1 equal to λ , and those for l_2 equal to μ . Their coordinates are then

$$\begin{aligned}l_1 : x &= -1 + 3\lambda, & y &= 3 - 2\lambda, & z &= -2 - \lambda, \\l_2 : x &= -\mu, & y &= 7 + 3\mu, & z &= -7 - 2\mu.\end{aligned}$$

The parameters are independent: equal points do not generally require equal parameter values.

Step 2: Equate coordinates and verify the common point Equating the x -coordinates gives $\mu = 1 - 3\lambda$. Equating the y -coordinates and substituting this expression gives

$$\begin{aligned}3 - 2\lambda &= 7 + 3(1 - 3\lambda), \\3 - 2\lambda &= 10 - 9\lambda, \\7\lambda &= 7, & \lambda &= 1, & \mu &= -2.\end{aligned}$$

For these parameter values, the z -coordinates are $-2 - 1 = -3$ and $-7 - 2(-2) = -3$, so the third equation also holds. Substitution into either line now gives the coordinates $x = 2$, $y = 1$, $z = -3$. Directly in the original symmetric equations, the ratios for this point are 1, 1, 1 on l_1 and -2 , -2 , -2 on l_2 , verifying membership in both lines.

Step 3: Conclusion The point of intersection of l_1 and l_2 is $(2, 1, -3)$.

Quick Tip to Solve: Use different parameters for different lines, solve with two coordinates, and always substitute into the remaining coordinate before declaring an intersection.