

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-2-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) This question paper contains **three** sections – **Section A, B** and **C**.
- (ii) Each section is **compulsory**.
- (iii) **Section A** has **6** short answer type I questions of **2** marks each.
- (iv) **Section B** has **4** short answer type II questions of **3** marks each.
- (v) **Section C** has **4** long answer type questions of **4** marks each.
- (vi) There is an internal choice in some questions.
- (vii) Question no. **14** is a case-study based question with **2** sub-parts of **2** marks each.

Section A (12 marks)

Questions number 1 to 6 carry 2 marks each.

1. A pair of dice is thrown. It is given that the sum of numbers appearing on both dice is an even number. Find the probability that the number appearing on at least one die is 3. [2 marks]

Solution:

Concept:

- For events A and B , $P(A | B) = P(A \cap B) / P(B)$, provided $P(B) > 0$.
- Two fair dice have 36 equally likely ordered outcomes. An even sum requires two odd faces or two even faces.

Step 1: Restrict the sample space Let B be the event that the sum is even. Each die has three odd faces, 1, 3, 5, and three even faces, 2, 4, 6. Both odd and both even outcomes must be included. Thus

$$n(B) = 3 \times 3 + 3 \times 3 = 18, \quad P(B) = \frac{18}{36}.$$

These 18 outcomes form the conditional sample space; using all 36 outcomes as the denominator would ignore the given information.

Step 2: Count the favourable outcomes Let A mean at least one die shows 3. Since 3 is odd, the other die must be odd for the sum to be even. The favourable ordered outcomes are

$$A \cap B = \{(3, 1), (3, 3), (3, 5), (1, 3), (5, 3)\}.$$

There are five outcomes because $(3, 3)$ is counted once, even though both dice show 3. Therefore

$$P(A | B) = \frac{P(A \cap B)}{P(B)} = \frac{5/36}{18/36} = \frac{5}{18}.$$

Step 3: Conclusion The required conditional probability is $\boxed{\frac{5}{18}}$.

Quick Tip to Solve: Restrict the sample space to the 18 even-sum outcomes, and count $(3,3)$ only once when using at least one.

2. $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|2\vec{a} + 3\vec{b}| = |3\vec{a} - 2\vec{b}|$. Find the angle between $\vec{a}$ and $\vec{b}$. [2 marks]

Solution:

Concept:

- For vectors $\vec{u}, \vec{v}$, $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 + 2\vec{u} \cdot \vec{v}$.
- If θ is the angle between two unit vectors, then $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta = \cos \theta$, with $0 \leq \theta \leq \pi$.

Step 1: Square and expand the equality Both magnitudes are nonnegative, so squaring the given

Quick Tip to Solve: Square both magnitudes: their constant terms both equal 13, while the dot-product terms have opposite signs.

equality preserves it. Using $|\vec{a}| = |\vec{b}| = 1$, we obtain

$$\begin{aligned} |2\vec{a} + 3\vec{b}|^2 &= 4|\vec{a}|^2 + 9|\vec{b}|^2 + 12\vec{a} \cdot \vec{b} = 13 + 12 \cos \theta, \\ |3\vec{a} - 2\vec{b}|^2 &= 9|\vec{a}|^2 + 4|\vec{b}|^2 - 12\vec{a} \cdot \vec{b} = 13 - 12 \cos \theta. \end{aligned}$$

The cross term on the right is negative because the second vector is subtracted.

Step 2: Determine the angle Equating the two expansions gives

$$13 + 12 \cos \theta = 13 - 12 \cos \theta \implies 24 \cos \theta = 0.$$

Hence $\cos \theta = 0$. On the allowed interval $[0, \pi]$, this gives $\theta = \pi/2$. As a check, perpendicular unit vectors make both squared magnitudes equal to $4 + 9 = 13$, so the original equality is satisfied.

Step 3: Conclusion The angle between the vectors is $\theta = \frac{\pi}{2} = 90^\circ$.

Quick Tip to Solve: Square both magnitudes: their constant terms both equal 13, while the dot-product terms have opposite signs.

3(a). Find: $\int \frac{\sin 3x}{\sin x} dx$

[2 marks]

Solution:

Concept:

- The triple-angle identity is $\sin 3x = 3 \sin x - 4 \sin^3 x$.
- The power-reduction identity $\sin^2 x = (1 - \cos 2x)/2$ converts the quotient into readily integrable terms.

Step 1: Simplify the integrand Write $I = \int (\sin 3x / \sin x) dx$. On any interval where $\sin x \neq 0$, the triple-angle identity gives

$$\frac{\sin 3x}{\sin x} = \frac{3 \sin x - 4 \sin^3 x}{\sin x} = 3 - 4 \sin^2 x.$$

Now reduce the square using $\sin^2 x = (1 - \cos 2x)/2$:

$$3 - 4 \sin^2 x = 3 - 2(1 - \cos 2x) = 1 + 2 \cos 2x.$$

This avoids attempting to integrate the numerator and denominator separately, which is not a valid integration rule.

Step 2: Integrate and verify Therefore, by integrating each term,

$$\begin{aligned} I &= \int (1 + 2 \cos 2x) dx \\ &= x + 2 \left(\frac{\sin 2x}{2} \right) + C = x + \sin 2x + C. \end{aligned}$$

Quick Tip to Solve: Use the triple-angle identity first, then replace $\sin^2 x$ by $(1 - \cos 2x)/2$ before integrating.

The factor $1/2$ is required by the chain rule for $2x$. Differentiating the result gives $1 + 2 \cos 2x = 3 - 4 \sin^2 x = \sin 3x / \sin x$, which verifies the antiderivative wherever the given integrand is defined.

Step 3: Conclusion $\int \frac{\sin 3x}{\sin x} dx = x + \sin 2x + C$, on each interval where $\sin x \neq 0$.

Quick Tip to Solve: Use the triple-angle identity first, then replace $\sin^2 x$ by $(1 - \cos 2x)/2$ before integrating.

OR

3(b). Evaluate: $\int_0^{\frac{1}{2} \log 3} \frac{e^x}{e^{2x} + 1} dx$

[2 marks]

Solution:

Concept:

- A substitution must transform both the differential and the limits. Here $t = e^x$ gives $dt = e^x dx$ and $e^{2x} = t^2$.
- The standard formula is $\int \frac{dt}{1+t^2} = \tan^{-1} t + C$, where the principal inverse tangent takes values in $(-\pi/2, \pi/2)$.

Step 1: Transform the variable and limits Denote the required integral by I . Set $t = e^x$, since the numerator supplies exactly the differential $dt = e^x dx$. At $x = 0$, $t = 1$, whereas at $x = \frac{1}{2} \log 3$, $t = e^{\log \sqrt{3}} = \sqrt{3}$. Therefore

$$I = \int_1^{\sqrt{3}} \frac{dt}{1+t^2}.$$

No extra factor of two is needed: we differentiated e^x ; the denominator's e^{2x} simply becomes t^2 .

Step 2: Evaluate using inverse tangent Integrating and applying the transformed endpoints yields

$$I = [\tan^{-1} t]_1^{\sqrt{3}} = \frac{\pi}{3} - \frac{\pi}{4}.$$

Indeed, $\tan(\pi/3) = \sqrt{3}$ and $\tan(\pi/4) = 1$, and both angles lie in the principal range. The positive difference is consistent with a positive integrand over the given interval.

Step 3: Conclusion The required definite integral equals $\boxed{\frac{\pi}{12}}$.

Quick Tip to Solve: Match the numerator to $d(e^x)$, and change the endpoints to 1 and $\sqrt{3}$ before integrating.

4. Probabilities of A and B solving a specific problem are $\frac{2}{3}$ and $\frac{3}{5}$ respectively. If both of them try independently to solve the problem, then find the probability that the problem is solved. [2 marks]

Solution:

Concept:

- A problem is solved if at least one person solves it. The complementary event is that both fail, so $P(\text{solved}) = 1 - P(\text{both fail})$.
- Independence of the attempts also gives independence of the failure events. Thus $P(A' \cap B') = P(A')P(B')$, with A and B denoting their respective successes.

Step 1: Calculate the failure probabilities Let A be the event that person A solves the problem and B the event that person B solves it. The given success probabilities are $P(A) = \frac{2}{3}$ and $P(B) = \frac{3}{5}$. Hence

$$P(A') = 1 - \frac{2}{3} = \frac{1}{3}, \quad P(B') = 1 - \frac{3}{5} = \frac{2}{5}.$$

The problem remains unsolved only when neither person succeeds; a success by either one is sufficient.

Step 2: Use independence and take the complement Since the attempts are independent, the probability that both fail is

$$P(A' \cap B') = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}.$$

Therefore the probability of at least one success is

$$P(A \cup B) = 1 - \frac{2}{15} = \frac{13}{15}.$$

Adding the two success probabilities directly would count their joint success twice. The complement method avoids that overlap and includes the case in which both solve the problem.

Step 3: Conclusion The probability that the problem is solved is $\frac{13}{15}$.

Quick Tip to Solve: For at least one independent success, multiply the failure probabilities first and subtract the product from one.

5. Write the cartesian equation of the line PQ passing through points $P(2, 2, 1)$ and $Q(5, 1, -2)$. Hence, find the y -coordinate of the point on the line PQ whose z -coordinate is -2 . [2 marks]

Solution:

Concept:

- A line through (x_0, y_0, z_0) with nonzero direction ratios a, b, c has cartesian equation $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$.
- Direction ratios for the line through two points are found by subtracting corresponding coordinates. Equating all three ratios to a parameter gives the coordinates of every point on the line.

Step 1: Obtain direction ratios and the line equation For the given points $P(2, 2, 1)$ and $Q(5, 1, -2)$,

Quick Tip to Solve: The specified z -coordinate already equals that of Q ; use Q 's y -coordinate as a check after forming the line equation.

the direction vector is

$$\overrightarrow{PQ} = (5 - 2, 1 - 2, -2 - 1) = (3, -1, -3).$$

Using P as the point on the line gives the cartesian equation

$$\frac{x - 2}{3} = \frac{y - 2}{-1} = \frac{z - 1}{-3}.$$

The negative direction ratios are retained because the y- and z-coordinates decrease from P to Q.

Step 2: Find the point with the given z-coordinate Set each ratio equal to t . The parametric coordinates are

$$x = 2 + 3t, \quad y = 2 - t, \quad z = 1 - 3t.$$

Substituting $z = -2$ yields $1 - 3t = -2$, so $t = 1$. Consequently $y = 2 - 1 = 1$. This point has $x = 5$, so it is precisely Q, which independently checks the calculation.

Step 3: Conclusion The line is $\boxed{\frac{x - 2}{3} = \frac{y - 2}{-1} = \frac{z - 1}{-3}}$, and the required y-coordinate is $\boxed{1}$.

Quick Tip to Solve: The specified z-coordinate already equals that of Q; use Q's y-coordinate as a check after forming the line equation.

6. Find the particular solution of the differential equation $\frac{dy}{dx} = 2y^2$, given $y = 1$ when $x = 1$. [2 marks]

Solution:

Concept:

- A separable differential equation allows all terms involving y to be placed with dy , and all terms involving x with dx .
- The integration rule $\int y^{-2} dy = -y^{-1}$ is used for $y \neq 0$. The given value $y(1) = 1$ determines the integration constant.

Step 1: Separate the variables and integrate Since the required solution passes through $(1, 1)$, it is nonzero near the initial point. Thus division by y^2 is valid for this solution:

$$\frac{dy}{y^2} = 2 dx.$$

Integrating both sides gives

$$-\frac{1}{y} = 2x + C.$$

The constant solution $y = 0$ is excluded by the initial condition.

Step 2: Apply the initial condition and check the result Substituting $x = 1$ and $y = 1$, we obtain $-1 = 2 + C$, so $C = -3$. Hence $-1/y = 2x - 3$, or $y = 1/(3 - 2x)$. At $x = 1$ this gives $y = 1$. Also,

$$\frac{dy}{dx} = \frac{2}{(3 - 2x)^2} = 2y^2,$$

Quick Tip to Solve: Integrate the reciprocal square with its minus sign, then substitute the initial values before rearranging for y .

verifying the differential equation. The denominator vanishes at $x = 3/2$; the solution interval containing the initial point is $x < 3/2$.

Step 3: Conclusion The particular solution is $y = \frac{1}{3-2x}$, on the interval $x < \frac{3}{2}$ containing $x = 1$.

Quick Tip to Solve: Integrate the reciprocal square with its minus sign, then substitute the initial values before rearranging for y .

Section B (12 marks)

Questions number 7 to 10 carry 3 marks each.

7(a). Find the equation of the plane passing through points $(2, 1, 0)$, $(3, -2, -2)$ and $(1, 1, -7)$. Also, obtain its distance from the origin. [3 marks]

Solution:

Concept:

- A plane has equation $ax + by + cz + d = 0$, where the nonzero vector (a, b, c) is normal to it. Coordinates of every point on the plane satisfy this equation.
- The perpendicular distance of (x_0, y_0, z_0) from this plane is $\frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$. At the origin, this becomes $|d|/\sqrt{a^2 + b^2 + c^2}$.

Step 1: Substitute the three given points Let the plane be $ax + by + cz + d = 0$. For $A(2, 1, 0)$, $B(3, -2, -2)$ and $C(1, 1, -7)$, respectively, we have

$$\begin{aligned}2a + b + d &= 0, \\3a - 2b - 2c + d &= 0, \\a + b - 7c + d &= 0.\end{aligned}$$

Subtracting the first equation from the second and third eliminates d , giving $a - 3b - 2c = 0$ and $-a - 7c = 0$.

Step 2: Determine a normal vector and the plane equation The second relation gives $a = -7c$. Substitution into the first gives $-9c - 3b = 0$, so $b = -3c$. Choose $c = 1$; then $a = -7$, $b = -3$ and the first point equation gives $d = 17$. Thus the plane is $-7x - 3y + z + 17 = 0$, equivalently $7x + 3y - z - 17 = 0$. To check, the values of $7x + 3y - z$ at the three points are $14 + 3 = 17$, $21 - 6 + 2 = 17$ and $7 + 3 + 7 = 17$, respectively. All three points lie on this plane.

Step 3: Calculate the perpendicular distance from the origin Using coefficients $a = 7$, $b = 3$, $c = -1$, $d = -17$ and the origin $(0, 0, 0)$, the distance is

$$D = \frac{|7(0) + 3(0) - 0 - 17|}{\sqrt{7^2 + 3^2 + (-1)^2}} = \frac{17}{\sqrt{49 + 9 + 1}} = \frac{17}{\sqrt{59}}.$$

The absolute value makes the perpendicular distance positive, independent of which equivalent sign

Quick Tip to Solve: Subtract the equation for the first point from the other two to eliminate the constant term before finding the normal coefficients.

convention is used for the plane equation.

Step 4: Conclusion The required plane is $7x + 3y - z - 17 = 0$, and its distance from the origin is $\frac{17}{\sqrt{59}}$ units.

Quick Tip to Solve: Subtract the equation for the first point from the other two to eliminate the constant term before finding the normal coefficients.

OR

7(b). Find the distance between the lines $x = \frac{y-1}{2} = \frac{z-2}{3}$ **and** $x+1 = \frac{y+2}{2} = \frac{z-1}{3}$. [3 marks]

Solution:

Concept:

- A line in symmetric form determines a point on it and its direction vector. Lines whose direction vectors are proportional are parallel.
- For parallel lines through points A and B , with common direction vector $\mathbf{d}$, the perpendicular distance is $D = \frac{|\vec{AB} \times \mathbf{d}|}{|\mathbf{d}|}$. The cross product measures the area of the parallelogram with base $|\mathbf{d}|$ and height D .

Step 1: Identify points and the common direction Put the common ratio in the first line equal to t , and that in the second equal to s . Their parametric forms are

$$L_1 : (x, y, z) = (t, 1 + 2t, 2 + 3t),$$

$$L_2 : (x, y, z) = (-1 + s, -2 + 2s, 1 + 3s).$$

Thus $A = (0, 1, 2)$ belongs to the first line, $B = (-1, -2, 1)$ belongs to the second, and both have direction $\mathbf{d} = (1, 2, 3)$. The required distance is therefore the perpendicular separation of these parallel lines, rather than the length of an arbitrary joining segment.

Step 2: Calculate the perpendicular separation The vector joining the chosen points is $\vec{AB} = (-1, -3, -1)$. Hence

$$\vec{AB} \times \mathbf{d} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -3 & -1 \\ 1 & 2 & 3 \end{vmatrix} = -7\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

Its magnitude is $\sqrt{49 + 4 + 1} = \sqrt{54}$, while $|\mathbf{d}| = \sqrt{1 + 4 + 9} = \sqrt{14}$. Applying the parallel-line distance formula gives

$$D = \frac{\sqrt{54}}{\sqrt{14}} = \sqrt{\frac{27}{7}} = \frac{3\sqrt{21}}{7}.$$

The nonzero cross product also confirms that the two parallel lines are distinct.

Step 3: Conclusion The distance between the lines is $\frac{3\sqrt{21}}{7}$ units.

Quick Tip to Solve: First compare the direction ratios: for parallel lines use a joining vector crossed with the common direction, since the skew-line scalar triple-product formula would have a zero denominator.

8. Find the general solution of the differential equation $x \frac{dy}{dx} - y + x \sin\left(\frac{y}{x}\right) = 0$.

[3 marks]

Solution:

Concept:

- A homogeneous differential equation whose right-hand side depends on y/x is simplified by putting $v = y/x$, so that $y = vx$ and $dy/dx = v + x dv/dx$.
- Separation of variables uses $\int \csc v dv = \log |\tan(v/2)| + C$. Before dividing by $\sin v$, its zero values must be checked for solutions that division would omit.

Step 1: Reduce the equation using the homogeneous substitution The printed equation is defined for $x \neq 0$. Dividing it by x gives

$$\frac{dy}{dx} = \frac{y}{x} - \sin\left(\frac{y}{x}\right).$$

Put $y = vx$. Substitution of $dy/dx = v + x dv/dx$ into this equation yields

$$v + x \frac{dv}{dx} = v - \sin v, \quad x \frac{dv}{dx} = -\sin v.$$

The terms v cancel, leaving a separable equation for v and x .

Step 2: Separate and integrate On an interval where $\sin v \neq 0$, separate the variables:

$$\frac{dv}{\sin v} = -\frac{dx}{x}.$$

Integration gives

$$\log \left| \tan \frac{v}{2} \right| = -\log |x| + C_1.$$

Combine the logarithms and absorb the fixed sign into an arbitrary real constant C . Substituting $v = y/x$ then gives

$$x \tan\left(\frac{y}{2x}\right) = C.$$

This relation describes the solutions locally on intervals avoiding $x = 0$, with a continuous branch chosen for y/x .

Step 3: Check the solutions excluded during division If $\sin v = 0$, then $v = n\pi$, where n is any integer. The reduced equation gives $dv/dx = 0$, so each straight line $y = n\pi x$ solves the original equation: its first two terms cancel and its sine term vanishes. The even values of n are included in the tangent relation with $C = 0$. The odd values must be recorded separately because the tangent is undefined there.

Step 4: Conclusion The general integral is $x \tan\left(\frac{y}{2x}\right) = C$, on intervals with $x \neq 0$. Also included in the complete solution set are $y = (2k+1)\pi x, k \in \mathbb{Z}$; the even multiples of πx occur when $C = 0$.

Quick Tip to Solve: After putting $y = vx$, cancel the two terms containing v ; check $\sin v = 0$ before dividing so that the straight-line solutions are retained.

9(a). Evaluate:

$$\int_0^1 \tan^{-1} x \, dx$$

[3 marks]

Solution:

Concept:

- Integration by parts states $\int u \, dv = uv - \int v \, du$. Choose the inverse trigonometric factor as u , because its derivative is a rational function.
- The derivative of $\tan^{-1} x$ is $\frac{1}{1+x^2}$. Also, $\int \frac{x}{1+x^2} \, dx = \frac{1}{2} \ln(1+x^2) + C$.

Step 1: Apply integration by parts Let $I = \int_0^1 \tan^{-1} x \, dx$. Take $u = \tan^{-1} x$ and $dv = dx$, so that $du = \frac{dx}{1+x^2}$ and $v = x$. The integrand is continuous throughout $[0, 1]$, so the definite integration-by-parts formula applies directly:

$$I = [x \tan^{-1} x]_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx.$$

The boundary term is $1 \cdot \tan^{-1} 1 - 0 \cdot \tan^{-1} 0 = \pi/4$, since the principal inverse tangent is measured in radians.

Step 2: Evaluate the remaining integral In the second integral, put $t = 1 + x^2$, giving $dt = 2x \, dx$. Its limits become $t = 1$ at $x = 0$ and $t = 2$ at $x = 1$. Hence

$$\int_0^1 \frac{x}{1+x^2} \, dx = \frac{1}{2} \int_1^2 \frac{dt}{t} = \frac{1}{2} [\ln t]_1^2 = \frac{1}{2} \ln 2.$$

Here $t > 0$ on the entire interval, so the logarithm is well-defined. Substituting this value in the integration-by-parts expression completes the evaluation without an arbitrary constant, because the original integral has fixed limits.

Step 3: Conclusion Thus, $\int_0^1 \tan^{-1} x \, dx = \boxed{\frac{\pi}{4} - \frac{1}{2} \ln 2}$.

Quick Tip to Solve: Differentiate the inverse tangent and integrate 1; the remaining numerator x matches the derivative of the denominator $1+x$ squared.

OR

9(b). Find:

$$\int \frac{2x}{x^2 + 3x + 2} \, dx$$

[3 marks]

Solution:

Concept:

- A proper rational function whose denominator has distinct linear factors can be expressed as a sum of partial fractions with constant numerators.

Quick Tip to Solve: After clearing denominators, substitute $x=-1$ and $x=-2$ to obtain $A=-2$ and $B=4$ immediately, then retain absolute values in the logarithms.

- For a linear factor $x + a$, the standard integral is $\int \frac{dx}{x + a} = \ln |x + a| + C$. Absolute values allow either sign of the factor, away from its zero.

Step 1: Factor the denominator and obtain partial fractions Factor $x^2 + 3x + 2 = (x + 1)(x + 2)$. Since the numerator has lower degree than the denominator, write

$$\frac{2x}{(x + 1)(x + 2)} = \frac{A}{x + 1} + \frac{B}{x + 2}.$$

Multiplication by $(x + 1)(x + 2)$ gives the polynomial identity

$$2x = A(x + 2) + B(x + 1) = (A + B)x + (2A + B).$$

Equating coefficients yields $A + B = 2$ and $2A + B = 0$. Subtracting the first equation from the second gives $A = -2$, and then $B = 4$. Thus the integrand splits into $-2/(x + 1) + 4/(x + 2)$.

Step 2: Integrate the two linear fractions Apply linearity and integrate each term separately:

$$\begin{aligned} \int \frac{2x}{x^2 + 3x + 2} dx &= -2 \int \frac{dx}{x + 1} + 4 \int \frac{dx}{x + 2} \\ &= -2 \ln |x + 1| + 4 \ln |x + 2| + C. \end{aligned}$$

To check the coefficients, differentiate the expression:

$$-\frac{2}{x + 1} + \frac{4}{x + 2} = \frac{-2(x + 2) + 4(x + 1)}{(x + 1)(x + 2)} = \frac{2x}{x^2 + 3x + 2}.$$

This verification also confirms both signs. The antiderivative is valid on each interval excluding $x = -1$ and $x = -2$; the integration constant may differ between these intervals.

Step 3: Conclusion Hence, $\int \frac{2x}{x^2 + 3x + 2} dx = \boxed{4 \ln |x + 2| - 2 \ln |x + 1| + C}$.

Quick Tip to Solve: After clearing denominators, substitute $x = -1$ and $x = -2$ to obtain $A = -2$ and $B = 4$ immediately, then retain absolute values in the logarithms.

10. Using vectors, find the value of 'b' if the points $A(-1, -1, 2)$, $B(2, b, 5)$ and $C(3, 11, 6)$ are collinear. Also, determine the ratio in which the point B divides the line-segment AC internally. [3 marks]

Solution:

Concept:

- For points A and B with position vectors $\vec{a}$ and $\vec{b}$, the displacement vector is $\vec{AB} = \vec{b} - \vec{a}$. Subtract corresponding coordinates to find its components.
- Three distinct points are collinear if their displacement vectors are parallel. Thus $\vec{AB} = t\vec{AC}$ for a scalar t . If $0 < t < 1$, then B lies inside the segment AC , and $AB : BC = t : (1 - t)$.

Step 1: Form the displacement vectors Let $\hat{i}, \hat{j}, \hat{k}$ be the unit vectors along the coordinate axes. From

Quick Tip to Solve: Use the known x-components to find the fraction of AC reached by B ; then use the y-components to find b . Convert that fraction to $t:(1-t)$ for the division ratio.

the given coordinates,

$$\begin{aligned}\overrightarrow{AB} &= (2+1)\hat{i} + (b+1)\hat{j} + (5-2)\hat{k} \\ &= 3\hat{i} + (b+1)\hat{j} + 3\hat{k}, \\ \overrightarrow{AC} &= (3+1)\hat{i} + (11+1)\hat{j} + (6-2)\hat{k} \\ &= 4\hat{i} + 12\hat{j} + 4\hat{k}.\end{aligned}$$

These are nonzero vectors, so their parallelism can be tested by comparing corresponding components.

Step 2: Apply the condition for collinearity Since the points are collinear, write $\overrightarrow{AB} = t\overrightarrow{AC}$. The coefficient of $\hat{i}$ gives $3 = 4t$, hence $t = \frac{3}{4}$. The coefficient of $\hat{k}$ gives the same equation, so the two known coordinates are consistent. Comparing the coefficients of $\hat{j}$,

$$b+1 = 12\left(\frac{3}{4}\right) = 9, \quad b = 8.$$

Thus, with this value, all three components satisfy $\overrightarrow{AB} = \frac{3}{4}\overrightarrow{AC}$.

Step 3: Determine the internal division ratio Since $0 < \frac{3}{4} < 1$, the point B is between A and C . Also,

$$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB} = \frac{1}{4}\overrightarrow{AC}.$$

Therefore, taking magnitudes along the same line,

$$AB : BC = \frac{3}{4}|\overrightarrow{AC}| : \frac{1}{4}|\overrightarrow{AC}| = 3 : 1.$$

As a coordinate check, the internal section formula for $AB : BC = 3 : 1$ gives

$$B = \frac{1A + 3C}{4} = \left(\frac{-1+9}{4}, \frac{-1+33}{4}, \frac{2+18}{4}\right) = (2, 8, 5),$$

agreeing with the given coordinates of B and the computed value of b .

Step 4: Conclusion The required value is $b = 8$, and B divides AC internally in the ratio $AB : BC = 3 : 1$.

Quick Tip to Solve: Use the known x-components to find the fraction of AC reached by B ; then use the y-components to find b . Convert that fraction to $t:(1-t)$ for the division ratio.

Section C (16 marks)

Questions number 11 to 14 carry 4 marks each.

11(a). Using integration, find the area of the smaller region enclosed by the curve $4x^2 + 4y^2 = 9$ and the line $2x + 2y = 3$. [4 marks]

Solution:

Concept:

- The circle $4x^2 + 4y^2 = 9$ has centre at the origin and radius $r = \frac{3}{2}$. Its upper boundary is $y = \sqrt{\frac{9}{4} - x^2}$.
- For a region between an upper curve $y = f(x)$ and a lower curve $y = g(x)$, its area is $\int_a^b [f(x) - g(x)] dx$, where a, b are the intersection abscissae.
- The integral of $\sqrt{r^2 - x^2}$ may be found using $x = r \sin \theta$, which converts it to $r^2 \int \cos^2 \theta d\theta$.

Step 1: Locate the intersection points The line can be written as $y = \frac{3}{2} - x$. Substituting into the equation of the circle gives

$$\begin{aligned} x^2 + \left(\frac{3}{2} - x\right)^2 &= \frac{9}{4}, \\ 2x^2 - 3x &= 0, \\ x(2x - 3) &= 0. \end{aligned}$$

Thus the two intersection points are $(0, \frac{3}{2})$ and $(\frac{3}{2}, 0)$. Between these points, both boundaries have nonnegative ordinates and the upper circular arc lies above the line.

Step 2: Identify the smaller enclosed region The desired region is the circular segment in the first quadrant between the upper arc and the chord $x + y = \frac{3}{2}$. It is contained in the first-quadrant quarter-circle, so its area is smaller than one quarter of the full circle and therefore smaller than the region on the other side of the chord. Its vertical strips extend from the line up to the circular arc, giving

$$A = \int_0^{3/2} \left[\sqrt{\frac{9}{4} - x^2} - \left(\frac{3}{2} - x\right) \right] dx.$$

Step 3: Evaluate the circular-arc integral Put $x = \frac{3}{2} \sin \theta$. Then $dx = \frac{3}{2} \cos \theta d\theta$, and the limits $x = 0, \frac{3}{2}$ become $\theta = 0, \frac{\pi}{2}$. Hence

$$\begin{aligned} \int_0^{3/2} \sqrt{\frac{9}{4} - x^2} dx &= \frac{9}{4} \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= \frac{9}{8} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} \\ &= \frac{9\pi}{16}. \end{aligned}$$

Step 4: Subtract the area under the chord The second integral is

$$\int_0^{3/2} \left(\frac{3}{2} - x\right) dx = \left[\frac{3x}{2} - \frac{x^2}{2} \right]_0^{3/2} = \frac{9}{4} - \frac{9}{8} = \frac{9}{8}.$$

Quick Tip to Solve: Find the intercepts of the chord first; here they make the integral a quarter-circle area minus the right triangle under the chord.

Step 5: Conclusion The area of the smaller enclosed region is $\boxed{\frac{9\pi}{16} - \frac{9}{8} = \frac{9}{16}(\pi - 2) \text{ square units}}$.

Quick Tip to Solve: Find the intercepts of the chord first; here they make the integral a quarter-circle area minus the right triangle under the chord.

OR

11(b). If the area of the region bounded by the curve $y^2 = 4ax$ and the line $x = 4a$ is $\frac{256}{3}$ sq. units, then using integration, find the value of a , where $a > 0$. [4 marks]

Solution:

Concept:

- For a region between a right boundary $x_R(y)$ and a left boundary $x_L(y)$, its area is $A = \int_{y_1}^{y_2} [x_R(y) - x_L(y)] dy$.
- The parabola $y^2 = 4ax$, with $a > 0$, opens to the right and is symmetric about the x -axis. Its equation in terms of y is $x = \frac{y^2}{4a}$.

Step 1: Locate the intersections and choose the boundaries Substitute the equation of the vertical line, $x = 4a$, into the parabola:

$$y^2 = 4a(4a) = 16a^2.$$

Since $a > 0$, the intersection ordinates are $y = -4a$ and $y = 4a$. Thus the two intersection points are $(4a, -4a)$ and $(4a, 4a)$. For every ordinate between these limits, the parabola is the left boundary and the line is the right boundary, since $y^2 \leq 16a^2$ implies $\frac{y^2}{4a} \leq 4a$. A horizontal strip therefore has length $4a - \frac{y^2}{4a}$ and thickness dy .

Step 2: Integrate the horizontal strips The required region includes both the upper and lower halves. Its area is

$$A = \int_{-4a}^{4a} \left(4a - \frac{y^2}{4a} \right) dy.$$

The integrand is even, so symmetry gives

$$\begin{aligned} A &= 2 \int_0^{4a} \left(4a - \frac{y^2}{4a} \right) dy \\ &= 2 \left[4ay - \frac{y^3}{12a} \right]_0^{4a} \\ &= 2 \left(16a^2 - \frac{64a^3}{12a} \right) \\ &= 2 \left(16a^2 - \frac{16a^2}{3} \right) \\ &= \frac{64a^2}{3}. \end{aligned}$$

Quick Tip to Solve: Integrate with respect to y for this vertical chord: use line minus parabola, and double the integral from 0 to $4a$ to include both halves.

The subtraction is in the order right boundary minus left boundary, ensuring a non-negative strip length throughout the interval.

Step 3: Use the given area and the sign restriction Equating the integrated area to the given area yields

$$\frac{64a^2}{3} = \frac{256}{3} \implies a^2 = 4.$$

The algebraic roots are $a = 2$ and $a = -2$. The condition $a > 0$ selects the positive root. As a check, substituting this positive value into the area expression gives $\frac{64(2)^2}{3} = \frac{256}{3}$, exactly the area specified in the question.

Step 4: Conclusion Hence, the required value is $a = 2$.

Quick Tip to Solve: Integrate with respect to y for this vertical chord: use line minus parabola, and double the integral from 0 to $4a$ to include both halves.

12. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$. [4 marks]

Solution:

Concept:

- In the symmetric equation of a line, equate each ratio to a common parameter t . This gives the coordinates of every point on the line in parametric form.
- A point at which a line meets a plane must satisfy both equations. Substitute the parametric coordinates of the line into the plane equation to determine the parameter of the intersection.
- For coordinate differences $\Delta x, \Delta y, \Delta z$, the distance is $d = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$.

Step 1: Express the coordinates on the line using a parameter Let the common value of the three ratios be t :

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = t.$$

Therefore, a general point on the given line has coordinates

$$x = 2 + 3t, \quad y = -1 + 4t, \quad z = 2 + 12t.$$

Step 2: Find the intersection with the plane Substitute these coordinates into $x - y + z = 5$:

$$\begin{aligned} (2 + 3t) - (-1 + 4t) + (2 + 12t) &= 5, \\ 2 + 3t + 1 - 4t + 2 + 12t &= 5, \\ 5 + 11t &= 5, \\ t &= 0. \end{aligned}$$

Thus the intersection point is $P = (2, -1, 2)$. This point lies on the plane because $2 - (-1) + 2 = 5$,

Quick Tip to Solve: First test the line's base point $(2, -1, 2)$ in the plane. It already satisfies the plane equation, so the intersection parameter is zero; then use the coordinate differences 3, 4, 12.

and each ratio in the line equation is zero. The coefficient 11 confirms a unique intersection.

Step 3: Calculate the distance from the given point Write the given point as $A = (-1, -5, -10)$. The coordinate differences from A to P are

$$2 - (-1) = 3, \quad -1 - (-5) = 4, \quad 2 - (-10) = 12.$$

Hence the three-dimensional distance formula gives

$$\begin{aligned} AP &= \sqrt{3^2 + 4^2 + 12^2} \\ &= \sqrt{9 + 16 + 144} \\ &= \sqrt{169}. \end{aligned}$$

All three coordinate differences enter the distance calculation.

Step 4: Conclusion The distance of $(-1, -5, -10)$ from the point of intersection is 13 units.

Quick Tip to Solve: First test the line's base point $(2, -1, 2)$ in the plane. It already satisfies the plane equation, so the intersection parameter is zero; then use the coordinate differences 3, 4, 12.

13. Evaluate: $\int_0^{\pi/2} \frac{x}{\sin x + \cos x} dx$

[4 marks]

Solution:

Concept:

- For a definite integral, reflection of the variable gives $\int_0^a f(x) dx = \int_0^a f(a-x) dx$. Here take $a = \pi/2$; the reflection interchanges sine and cosine while preserving their sum.
- The identity $\sin x + \cos x = \sqrt{2} \cos(x - \pi/4)$ converts the remaining integral into a secant integral. Use $\int \sec t dt = \log |\sec t + \tan t| + C$.

Step 1: Reflect the variable about the midpoint Denote the required integral by I . Put $x = \pi/2 - t$, so that $dx = -dt$. The limits $x = 0, \pi/2$ become $t = \pi/2, 0$, respectively. Reversing these limits removes the negative sign. Since $\sin(\pi/2 - t) = \cos t$ and $\cos(\pi/2 - t) = \sin t$, we obtain

$$I = \int_0^{\pi/2} \frac{\pi/2 - t}{\sin t + \cos t} dt = \int_0^{\pi/2} \frac{\pi/2 - x}{\sin x + \cos x} dx.$$

In the last equality, the dummy integration variable has simply been renamed.

Step 2: Add the two expressions to remove the numerator variable Adding the original expression for I and its reflected expression gives

$$2I = \int_0^{\pi/2} \frac{x + (\pi/2 - x)}{\sin x + \cos x} dx = \frac{\pi}{2} \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}.$$

Thus, if $J = \int_0^{\pi/2} (\sin x + \cos x)^{-1} dx$, then $I = \pi J/4$. The denominator is positive throughout this interval, so these integrals are finite and the transformations are valid.

Quick Tip to Solve: Reflect x to $\pi/2 - x$ first: the denominator stays the same, while adding the numerators produces the constant $\pi/2$.

Step 3: Evaluate the remaining elementary integral Put $t = x - \pi/4$, with limits $-\pi/4$ and $\pi/4$. Then

$$\begin{aligned} J &= \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} \sec t \, dt \\ &= \frac{1}{\sqrt{2}} [\log |\sec t + \tan t|]_{-\pi/4}^{\pi/4} \\ &= \frac{1}{\sqrt{2}} \log \left(\frac{\sqrt{2} + 1}{\sqrt{2} - 1} \right). \end{aligned}$$

Both logarithm arguments are positive. Since $(\sqrt{2}-1)(\sqrt{2}+1) = 1$, the ratio is $(\sqrt{2}+1)^2$. Consequently $J = \sqrt{2} \log(1 + \sqrt{2})$. Substitution into $I = \pi J/4$ now evaluates the requested integral.

Step 4: Conclusion

$$\boxed{\int_0^{\pi/2} \frac{x}{\sin x + \cos x} \, dx = \frac{\pi}{2\sqrt{2}} \log(1 + \sqrt{2})}.$$

Here log denotes the natural logarithm.

Quick Tip to Solve: Reflect x to $\pi/2 - x$ first: the denominator stays the same, while adding the numerators produces the constant $\pi/2$.

14. Case-Study Based Question

At the start of a cricket match, a coin is tossed and the team winning the toss has the opportunity to choose to bat or bowl. Such a coin is unbiased with equal probabilities of getting head and tail.

Based on the above information, answer the following questions:

[4 marks]

14(a). If such a coin is tossed 2 times, then find the probability distribution of number of tails.

[2 marks]

Solution:

Concept:

- An unbiased coin has $P(H) = P(T) = \frac{1}{2}$, where H denotes a head and T a tail.
- Tosses are independent, so the probability of an ordered pair is the product of the probabilities of its two results. A probability distribution lists every possible value of a random variable with its probability.

Step 1: Define the random variable and sample space Let X be the number of tails in two tosses. The ordered outcomes are

$$S = \{HH, HT, TH, TT\}.$$

Each outcome has probability $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. The possible values of X are 0, 1, 2; there can be no negative number of tails or more than two tails.

Step 2: Group the outcomes by the number of tails The outcome HH gives $X = 0$, the distinct

Quick Tip to Solve: Keep HT and TH as separate outcomes: both give one tail, so that probability is twice the probability of HH or TT.

outcomes HT and TH give $X = 1$, and TT gives $X = 2$. Thus

$$P(X = 0) = \frac{1}{4},$$

$$P(X = 1) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$P(X = 2) = \frac{1}{4}.$$

These probabilities are nonnegative and their sum is $\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$, verifying that all outcomes have been included.

Step 3: Conclusion The required probability distribution is

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Quick Tip to Solve: Keep HT and TH as separate outcomes: both give one tail, so that probability is twice the probability of HH or TT .

14(b). Find the probability of getting at least one head in three tosses of such a coin.

[2 marks]

Solution:

Concept:

- For an unbiased coin, $P(H) = P(T) = \frac{1}{2}$, and the results of different tosses are independent.
- Complement rule: $P(A) = 1 - P(A^c)$. The complement of getting at least one head is getting no heads, which means that every toss produces a tail.

Step 1: Identify the complementary event Let A denote the event of getting at least one head in three tosses. This includes one head, two heads and three heads. Its complement A^c contains just the ordered outcome TTT . Using this complement avoids separately counting all three possible positive numbers of heads.

Step 2: Calculate the probability of three tails Independence allows us to multiply the individual probabilities:

$$P(A^c) = P(TTT) = P(T)P(T)P(T) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

Hence the complement rule gives

$$P(A) = 1 - \frac{1}{8} = \frac{7}{8}.$$

As a check, three tosses have $2^3 = 8$ equally likely ordered outcomes. Exactly one, TTT , has no head; the remaining seven each have at least one head. This gives the same probability.

Step 3: Conclusion The probability of getting at least one head in three tosses is $\boxed{\frac{7}{8}}$.

Quick Tip to Solve: For at least one head, subtract the probability of all tails from 1; do not count only the outcomes with exactly one head.