

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-2-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) This question paper contains **three** sections – **Section A, B** and **C**.
- (ii) Each section is **compulsory**.
- (iii) **Section A** has **6** short answer type I questions of **2** marks each.
- (iv) **Section B** has **4** short answer type II questions of **3** marks each.
- (v) **Section C** has **4** long answer type questions of **4** marks each.
- (vi) There is an internal choice in some questions.
- (vii) Question no. **14** is a case-study based question with **2** sub-parts of **2** marks each.

Section A (12 marks)

Questions number 1 to 6 carry 2 marks each.

1. Write the cartesian equation of the line PQ passing through points $P(2, 2, 1)$ and $Q(5, 1, -2)$. Hence, find the y -coordinate of the point on the line PQ whose z -coordinate is -2 . [2 marks]

Solution:

Concept:

- A line through two points has a direction vector equal to the difference of their position vectors. Its components are direction ratios of the line.
- Through (x_1, y_1, z_1) with nonzero direction ratios a, b, c , the cartesian equation is $\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = t$, where t is a real parameter.

Step 1: Obtain the direction ratios and equation From $P(2, 2, 1)$ to $Q(5, 1, -2)$, the direction vector is

$$\overrightarrow{PQ} = (5 - 2, 1 - 2, -2 - 1) = (3, -1, -3).$$

Using P as the point on the line gives

$$\frac{x - 2}{3} = \frac{y - 2}{-1} = \frac{z - 1}{-3} = t.$$

These ratios also equal 1 at Q , confirming that both given points lie on this line.

Step 2: Use the prescribed z -coordinate For $z = -2$, the common parameter satisfies

$$t = \frac{-2 - 1}{-3} = 1.$$

Hence $\frac{y - 2}{-1} = 1$, so $y - 2 = -1$ and $y = 1$. The corresponding $x = 2 + 3(1) = 5$, so the required point is indeed Q .

Step 3: Conclusion The cartesian equation is $\boxed{\frac{x - 2}{3} = \frac{y - 2}{-1} = \frac{z - 1}{-3}}$. The required y -coordinate is $\boxed{1}$.

Quick Tip to Solve: Subtract coordinates in the same order for all three direction ratios; here $z = -2$ immediately gives the parameter $t = 1$.

2. Probabilities of A and B solving a specific problem are $\frac{2}{3}$ and $\frac{3}{5}$ respectively. If both of them try independently to solve the problem, then find the probability that the problem is solved. [2 marks]

Solution:

Concept:

- The probability that a problem is solved means the probability that at least one person succeeds. Its complementary event is that both people fail.
- For independent events A and B , their complements are independent, and $P(A' \cap B') = P(A')P(B')$, where a prime denotes the complementary event.

Step 1: Calculate the probabilities of failure Let A denote the event that person A solves the problem and B the event that person B solves it. The given probabilities are $P(A) = \frac{2}{3}$ and $P(B) = \frac{3}{5}$. Therefore,

$$P(A') = 1 - \frac{2}{3} = \frac{1}{3}, \quad P(B') = 1 - \frac{3}{5} = \frac{2}{5}.$$

Step 2: Use the complement of both failing Since they try independently, the probability that neither solves the problem is

$$P(A' \cap B') = \frac{1}{3} \times \frac{2}{5} = \frac{2}{15}.$$

The problem is solved whenever A succeeds, B succeeds, or both succeed. These possibilities together form $A \cup B$, so

$$P(A \cup B) = 1 - P(A' \cap B') = 1 - \frac{2}{15} = \frac{13}{15}.$$

Simply adding the two success probabilities would count their joint success twice; the complement method avoids this double counting.

Step 3: Conclusion The probability that the problem is solved is $\boxed{\frac{13}{15}}$.

Quick Tip to Solve: For independent attempts and the words "problem is solved", subtract the product of the failure probabilities from 1.

3. A pair of dice is thrown. It is given that the sum of numbers appearing on both dice is an even number. Find the probability that the number appearing on at least one die is 3. [2 marks]

Solution:

Concept:

- For an event A under a given condition E , conditional probability is $P(A | E) = \frac{P(A \cap E)}{P(E)}$, provided $P(E) > 0$.
- Two fair dice have 36 equally likely ordered outcomes (i, j) . An even sum occurs exactly when both numbers are odd or both are even.

Step 1: Count the outcomes satisfying the condition Let E be the event that the sum is even, and A the event that at least one die shows 3. Each die has three odd numbers 1, 3, 5 and three even

Quick Tip to Solve: Restrict the sample space to the 18 even-sum outcomes, then count pairs containing 3 without counting (3,3) twice.

numbers 2, 4, 6. Thus the number of outcomes in E is

$$n(E) = 3 \times 3 + 3 \times 3 = 18, \quad P(E) = \frac{18}{36}.$$

The dice are distinguished when counting: for example, $(1, 3)$ and $(3, 1)$ are separate equally likely outcomes.

Step 2: Count the favourable outcomes within the even-sum event If one die shows 3, an odd number, the other must also be odd for the sum to be even. The outcomes in $A \cap E$ are

$$(3, 1), (3, 3), (3, 5), (1, 3), (5, 3).$$

There are five such outcomes; $(3, 3)$ is included once, even though both dice show 3. Therefore,

$$P(A | E) = \frac{P(A \cap E)}{P(E)} = \frac{5/36}{18/36} = \frac{5}{18}.$$

Step 3: Conclusion Given that the sum is even, the probability that at least one die shows 3 is $\boxed{\frac{5}{18}}$.

Quick Tip to Solve: Restrict the sample space to the 18 even-sum outcomes, then count pairs containing 3 without counting $(3,3)$ twice.

4. $\vec{a}$ and $\vec{b}$ are two unit vectors such that $|2\vec{a} + 3\vec{b}| = |3\vec{a} - 2\vec{b}|$. Find the angle between $\vec{a}$ and $\vec{b}$.

[2 marks]

Solution:

Concept:

- For any vector $\vec{v}$, $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$. Expanding a squared magnitude converts the given equality into a scalar equation.
- For unit vectors $\vec{a}$ and $\vec{b}$, $\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} = 1$ and $\vec{a} \cdot \vec{b} = \cos \theta$, where θ is their angle in $[0, \pi]$.

Step 1: Square and expand both magnitudes Both magnitudes are nonnegative, so squaring the given equality preserves it. Using distributivity of the scalar product gives

$$\begin{aligned} |2\vec{a} + 3\vec{b}|^2 &= 4|\vec{a}|^2 + 9|\vec{b}|^2 + 12\vec{a} \cdot \vec{b}, \\ |3\vec{a} - 2\vec{b}|^2 &= 9|\vec{a}|^2 + 4|\vec{b}|^2 - 12\vec{a} \cdot \vec{b}. \end{aligned}$$

Notice that the mixed terms have opposite signs, reflecting the plus and minus signs in the original vectors.

Step 2: Use the unit-vector condition Substituting $|\vec{a}| = |\vec{b}| = 1$ and equating the expansions, we obtain

$$13 + 12\vec{a} \cdot \vec{b} = 13 - 12\vec{a} \cdot \vec{b}, \quad 24\vec{a} \cdot \vec{b} = 0.$$

Thus $\cos \theta = 0$. In the permitted interval $0 \leq \theta \leq \pi$, this specifies $\theta = \pi/2$.

Quick Tip to Solve: Square both magnitudes first: the equal unit-vector square terms cancel, leaving only the mixed scalar products.

Step 3: Conclusion The angle between the two unit vectors is $\pi/2$ radians $= 90^\circ$; hence the vectors are perpendicular.

Quick Tip to Solve: Square both magnitudes first: the equal unit-vector square terms cancel, leaving only the mixed scalar products.

5. Find the product of the order and the degree of the differential equation

$$\left[\frac{d}{dx}(xy^2) \right] \cdot \frac{dy}{dx} + y = 0.$$

[2 marks]

Solution:

Concept:

- The order of a differential equation is the order of the highest derivative appearing in it after the indicated differentiations are performed.
- The degree is the power of the highest-order derivative when the equation is polynomial in its derivatives. Coefficients may depend on x and y .

Step 1: Expand the derivative by the product rule Treat y as a function of x . The product rule and the chain rule give

$$\frac{d}{dx}(xy^2) = y^2 + x \frac{d}{dx}(y^2) = y^2 + 2xy \frac{dy}{dx}.$$

Substituting this into the printed equation, rather than treating the unexpanded bracket as a separate higher derivative, yields

$$\left(y^2 + 2xy \frac{dy}{dx} \right) \frac{dy}{dx} + y = 0.$$

Step 2: Identify the highest derivative and its power Multiplying out the bracket puts the equation in polynomial form:

$$2xy \left(\frac{dy}{dx} \right)^2 + y^2 \frac{dy}{dx} + y = 0.$$

The only derivative present is dy/dx , which is a first derivative; hence the order is 1. Its highest power in this polynomial is 2, so the degree is 2. The exponent on y itself does not determine the degree, and the derivative of xy^2 introduces no second derivative.

Step 3: Conclusion Therefore, the required product is $\text{order} \times \text{degree} = 1 \times 2 = 2$.

Quick Tip to Solve: Perform every indicated differentiation before counting order; then count the power of the derivative, not the power of y .

6(a). Find: $\int \frac{\sqrt{\cot x}}{\sin x \cos x} dx$

[2 marks]

Solution:

Concept:

- Choose $t = \cot x$, whose derivative is $dt/dx = -\csc^2 x$. The identity $\cot x = \cos x / \sin x$ allows the given denominator to be expressed in terms of this substitution.
- The power rule is $\int t^n dt = t^{n+1}/(n+1) + C$ for $n \neq -1$, where C is an arbitrary constant of integration.

Step 1: Rewrite the integrand for substitution On an interval where the printed integrand is real and defined, $\sin x \cos x \neq 0$ and $\cot x > 0$. Since

$$\frac{1}{\sin x \cos x} = \frac{\csc^2 x}{\cot x},$$

the integral becomes

$$I = \int \frac{\sqrt{\cot x}}{\cot x} \csc^2 x dx = \int (\cot x)^{-1/2} \csc^2 x dx.$$

This form separates a power of $\cot x$ from its derivative, so the substitution accounts for every factor.

Step 2: Integrate the resulting power Put $t = \cot x$. Then $dt = -\csc^2 x dx$, and hence

$$I = - \int t^{-1/2} dt = -2t^{1/2} + C.$$

The minus sign is essential because cotangent has negative derivative. As a check, differentiating after substitution back gives

$$\frac{d}{dx}(-2\sqrt{\cot x}) = \frac{\csc^2 x}{\sqrt{\cot x}} = \frac{\sqrt{\cot x}}{\sin x \cos x},$$

which reproduces the original integrand.

Step 3: Conclusion Therefore, $\boxed{\int \frac{\sqrt{\cot x}}{\sin x \cos x} dx = -2\sqrt{\cot x} + C}$, on each interval where the integrand is defined.

Quick Tip to Solve: Replace $1/(\sin x \cos x)$ by $\csc^2 x$ divided by $\cot x$, then substitute $\cot x$ and retain the negative sign.

6(b). Find: $\int \frac{1}{x(x^2 + 4)} dx$

[2 marks]

Solution:

Concept:

- Partial fractions rewrite a rational integrand as simpler terms whose antiderivatives are known. Since the denominator contains the factors x and $x^2 + 4$, use a constant numerator over x and a linear numerator over $x^2 + 4$.
- The logarithmic rules are $\int dx/x = \ln|x| + C$ and $\int f'(x)/f(x) dx = \ln|f(x)| + C$. Here $x^2 + 4 > 0$ for every real x , while the original integrand requires $x \neq 0$.

Step 1: Decompose the integrand Write

$$\frac{1}{x(x^2 + 4)} = \frac{A}{x} + \frac{Bx + D}{x^2 + 4}.$$

Multiplying by $x(x^2 + 4)$ gives

$$1 = A(x^2 + 4) + x(Bx + D) = (A + B)x^2 + Dx + 4A.$$

Comparing coefficients, $4A = 1$, $A + B = 0$ and $D = 0$. Thus $A = 1/4$, $B = -1/4$, and

$$\frac{1}{x(x^2 + 4)} = \frac{1}{4x} - \frac{x}{4(x^2 + 4)}.$$

Step 2: Integrate the two logarithmic terms Therefore,

$$\begin{aligned} \int \frac{dx}{x(x^2 + 4)} &= \frac{1}{4} \int \frac{dx}{x} - \frac{1}{4} \int \frac{x dx}{x^2 + 4} \\ &= \frac{1}{4} \ln|x| - \frac{1}{8} \ln(x^2 + 4) + C. \end{aligned}$$

In the second integral, put $u = x^2 + 4$, so $du = 2x dx$; the extra factor $1/2$ accounts for the coefficient $1/8$. As a check, differentiating this expression gives

$$\frac{1}{4x} - \frac{x}{4(x^2 + 4)} = \frac{x^2 + 4 - x^2}{4x(x^2 + 4)} = \frac{1}{x(x^2 + 4)}.$$

Step 3: Conclusion Hence, $\int \frac{dx}{x(x^2 + 4)} = \frac{1}{4} \ln|x| - \frac{1}{8} \ln(x^2 + 4) + C$, valid on any interval excluding $x = 0$.

Quick Tip to Solve: Split this integrand as $\frac{1}{4}(1/x - x/(x^2 + 4))$; both terms then integrate immediately to logarithms.

Section B (12 marks)

Questions number 7 to 10 carry 3 marks each.

7(a). Evaluate:

$$\int_0^1 \tan^{-1} x \, dx$$

[3 marks]

Solution:

Concept:

- For a definite integral, integration by parts gives $\int_a^b u \, dv = [uv]_a^b - \int_a^b v \, du$. Choose the inverse trigonometric factor as u , since its derivative is a rational function.
- Here $\tan^{-1} x$ is the principal inverse tangent, with angles measured in radians. Its derivative is $1/(1+x^2)$, and the endpoint values are $\tan^{-1} 0 = 0$ and $\tan^{-1} 1 = \pi/4$.
- A logarithmic substitution applies to $\int x/(1+x^2) \, dx$: the derivative of the denominator is $2x$, so this integral is $\frac{1}{2} \ln(1+x^2) + C$.

Step 1: Apply integration by parts Let $I = \int_0^1 \tan^{-1} x \, dx$. Take

$$u = \tan^{-1} x, \quad dv = dx, \quad du = \frac{dx}{1+x^2}, \quad v = x.$$

Applying the definite integration-by-parts formula,

$$\begin{aligned} I &= [x \tan^{-1} x]_0^1 - \int_0^1 \frac{x}{1+x^2} \, dx \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2} \, dx. \end{aligned}$$

The lower endpoint of the boundary term is zero because $x \tan^{-1} x = 0$ at $x = 0$.

Step 2: Evaluate the remaining definite integral Put $t = 1 + x^2$; then $dt = 2x \, dx$. When $x = 0$, $t = 1$, and when $x = 1$, $t = 2$. Hence

$$\begin{aligned} \int_0^1 \frac{x}{1+x^2} \, dx &= \frac{1}{2} \int_1^2 \frac{dt}{t} \\ &= \frac{1}{2} [\ln t]_1^2 = \frac{1}{2} (\ln 2 - \ln 1) = \frac{1}{2} \ln 2. \end{aligned}$$

The substitution changes both limits, so no return to the variable x is needed. The logarithm has a positive argument throughout these limits.

Step 3: Conclusion Therefore, $\int_0^1 \tan^{-1} x \, dx = \frac{\pi}{4} - \frac{1}{2} \ln 2$.

Quick Tip to Solve: For this inverse-tangent integral, take $u = \tan^{-1} x$ and $dv = dx$; the leftover numerator x is half the derivative of $1+x^2$.

7(b). Find:

$$\int \frac{2x}{x^2 + 3x + 2} dx$$

[3 marks]

Solution:

Concept:

- Partial fractions convert a proper rational function with distinct linear factors into a sum of terms that can be integrated separately.
- For a nonzero real linear expression, $\int \frac{dx}{x+a} = \ln|x+a| + C$. Absolute values allow the formula on any interval avoiding the zero of the denominator.

Step 1: Factor the denominator and determine the coefficients The denominator factors as $x^2 + 3x + 2 = (x+1)(x+2)$. Since the numerator has smaller degree than the denominator, no polynomial division is needed. Write

$$\frac{2x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}.$$

Multiplying by $(x+1)(x+2)$ gives

$$2x = A(x+2) + B(x+1) = (A+B)x + (2A+B).$$

Comparing coefficients, $A+B=2$ and $2A+B=0$. Subtracting the first equation from the second gives $A=-2$, and then $B=4$.

Step 2: Integrate each partial fraction Substitute the coefficients and use the logarithmic integral:

$$\begin{aligned} \int \frac{2x}{x^2 + 3x + 2} dx &= -2 \int \frac{dx}{x+1} + 4 \int \frac{dx}{x+2} \\ &= -2 \ln|x+1| + 4 \ln|x+2| + C. \end{aligned}$$

The constants from the two separate integrations combine into one arbitrary constant C . The integrand and antiderivative are considered on any interval that excludes $x=-1$ and $x=-2$.

Step 3: Check by differentiation Differentiating the logarithmic expression gives

$$-\frac{2}{x+1} + \frac{4}{x+2} = \frac{-2(x+2) + 4(x+1)}{(x+1)(x+2)} = \frac{2x}{x^2 + 3x + 2},$$

which is the original integrand.

Step 4: Conclusion

$$\boxed{\int \frac{2x}{x^2 + 3x + 2} dx = 4 \ln|x+2| - 2 \ln|x+1| + C.}$$

Quick Tip to Solve: After clearing the denominators, put $x=-1$ and $x=-2$ in the polynomial identity to obtain $A=-2$ and $B=4$ directly.

8(a). Find the equation of the plane passing through points $(2, 1, 0)$, $(3, -2, -2)$ and $(1, 1, -7)$. Also, obtain its distance from the origin. [3 marks]

Solution:

Concept:

- A plane through a point A with a normal vector $\mathbf{n}$ has equation $\mathbf{n} \cdot (\mathbf{r} - \mathbf{a}) = 0$, where $\mathbf{a}$ is the position vector of A and $\mathbf{r} = (x, y, z)$.
- Two independent vectors lying in a plane determine its normal through their cross product. For the plane $ax + by + cz + d = 0$, the perpendicular distance from the origin is $\frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.

Step 1: Obtain two vectors lying in the plane Let the given points be $A = (2, 1, 0)$, $B = (3, -2, -2)$ and $C = (1, 1, -7)$. Subtract the coordinates of A from those of B and C :

$$\overrightarrow{AB} = (1, -3, -2), \quad \overrightarrow{AC} = (-1, 0, -7).$$

These vectors are not parallel, so the three points determine a unique plane.

Step 2: Find a normal and form the plane equation A normal to both vectors is their cross product:

$$\begin{aligned} \overrightarrow{AB} \times \overrightarrow{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & -2 \\ -1 & 0 & -7 \end{vmatrix} \\ &= 21\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}. \end{aligned}$$

Dividing the normal by 3, choose $\mathbf{n} = (7, 3, -1)$. Using the point $A = (2, 1, 0)$, the point-normal equation becomes

$$7(x - 2) + 3(y - 1) - (z - 0) = 0,$$

or $7x + 3y - z - 17 = 0$. As a check, substituting B gives $21 - 6 + 2 - 17 = 0$, and substituting C gives $7 + 3 + 7 - 17 = 0$; thus both remaining points lie on the plane.

Step 3: Calculate the perpendicular distance from the origin In the equation $7x + 3y - z - 17 = 0$, the coefficients are $a = 7$, $b = 3$, $c = -1$ and $d = -17$. Therefore the distance from $(0, 0, 0)$ is

$$D = \frac{|7(0) + 3(0) - 0 - 17|}{\sqrt{7^2 + 3^2 + (-1)^2}} = \frac{17}{\sqrt{49 + 9 + 1}} = \frac{17}{\sqrt{59}}.$$

The absolute value in the numerator ensures a nonnegative distance.

Step 4: Conclusion The required plane is $7x + 3y - z - 17 = 0$, and its distance from the origin is

$$\frac{17}{\sqrt{59}} \text{ units}.$$

Quick Tip to Solve: Reduce the cross-product normal to $(7, 3, -1)$ before forming the plane equation; then use the absolute constant term divided by the normal's length for the origin's distance.

8(b). Find the distance between the lines $x = \frac{y-1}{2} = \frac{z-2}{3}$ and $x+1 = \frac{y+2}{2} = \frac{z-1}{3}$. [3 marks]

Solution:

Concept:

- A symmetric line equation gives a point on the line and a direction vector. Two lines having proportional direction vectors are parallel.
- For parallel lines through points A and B , with common nonzero direction vector $\mathbf{b}$, their perpendicular distance is $d = \frac{|\overrightarrow{AB} \times \mathbf{b}|}{|\mathbf{b}|}$. The cross product removes the component of the joining vector along the lines.

Step 1: Identify the points and common direction Set each common ratio equal to a parameter. The first line is

$$(x, y, z) = (0, 1, 2) + t(1, 2, 3),$$

and the second line is

$$(x, y, z) = (-1, -2, 1) + s(1, 2, 3).$$

Thus choose $A = (0, 1, 2)$ and $B = (-1, -2, 1)$. Both lines have direction $\mathbf{b} = (1, 2, 3)$, so they are parallel. In particular, the denominator attached to x is implicitly 1 in both equations.

Step 2: Calculate the perpendicular separation The joining vector is $\overrightarrow{AB} = (-1, -3, -1)$. Hence

$$\overrightarrow{AB} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -3 & -1 \\ 1 & 2 & 3 \end{vmatrix} = -7\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

Therefore $|\overrightarrow{AB} \times \mathbf{b}| = \sqrt{49 + 4 + 1} = \sqrt{54}$ and $|\mathbf{b}| = \sqrt{1 + 4 + 9} = \sqrt{14}$. Substitution gives

$$d = \frac{\sqrt{54}}{\sqrt{14}} = \sqrt{\frac{27}{7}} = \frac{3\sqrt{21}}{7}.$$

The cross product is nonzero, confirming that the parallel lines are distinct. Reversing the joining vector changes only its sign and leaves the distance unchanged.

Step 3: Conclusion The distance between the two lines is $\boxed{\frac{3\sqrt{21}}{7}}$ units.

Quick Tip to Solve: Compare direction ratios first: identical ratios mean parallel lines, so use the cross-product distance formula rather than the skew-line scalar triple product formula.

9. Solve the following differential equation: $(1 + e^{y/x}) dy + e^{y/x} \left(1 - \frac{y}{x}\right) dx = 0$.

[3 marks]

Solution:

Concept:

- A first-order differential equation whose right-hand side depends only on y/x is homogeneous. Put $y = vx$, so $\frac{dy}{dx} = v + x \frac{dv}{dx}$, on an interval where $x \neq 0$.
- The logarithmic integration rule is $\int \frac{f'(v)}{f(v)} dv = \ln |f(v)| + C$, wherever $f(v) \neq 0$. Here the useful function is $f(v) = v + e^v$.

Step 1: Substitute the homogeneous variable From the printed equation,

$$(1 + e^{y/x}) \frac{dy}{dx} + e^{y/x} \left(1 - \frac{y}{x}\right) = 0.$$

Set $v = y/x$ and $y = vx$. Substituting the derivative gives

$$(1 + e^v) \left(v + x \frac{dv}{dx}\right) + e^v(1 - v) = 0,$$
$$x(1 + e^v) \frac{dv}{dx} + v + e^v = 0.$$

The terms ve^v cancel. The original equation contains y/x , so $x = 0$ is excluded throughout.

Step 2: Separate and integrate For $v + e^v \neq 0$, separation yields

$$\frac{1 + e^v}{v + e^v} dv = -\frac{dx}{x}.$$

Since $\frac{d}{dv}(v + e^v) = 1 + e^v$, integration gives

$$\ln |v + e^v| = -\ln |x| + C_1.$$

Combining the logarithms and absorbing the sign into the arbitrary constant gives $x(v + e^v) = C$. Replacing v by y/x produces the implicit relation $y + xe^{y/x} = C$. The constant-ratio solution satisfying $v + e^v = 0$, omitted when separating, is also included in this relation by taking $C = 0$.

Step 3: Verify the integrated relation Differentiate the left-hand side directly:

$$d(y + xe^{y/x}) = dy + e^{y/x} dx + xe^{y/x} d(y/x)$$
$$= (1 + e^{y/x}) dy + e^{y/x} \left(1 - \frac{y}{x}\right) dx.$$

Thus setting its differential equal to zero reproduces exactly the given equation. Also $1 + e^{y/x} > 0$, so the implicit relation locally defines a differentiable solution for each permitted initial point.

Step 4: Conclusion The general solution is $y + xe^{y/x} = C$, where C is an arbitrary real constant and $x \neq 0$.

Quick Tip to Solve: After substituting $y = vx$, look for a denominator whose derivative is the numerator: the derivative of $v + e^v$ is $1 + e^v$, so the integral is a logarithm.

10. If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - \hat{j} - 3\hat{k}$, then find the vector $\vec{c}$, given that $\vec{a} \times \vec{c} = \vec{b}$ and $\vec{a} \cdot \vec{c} = 4$. [3 marks]

Solution:

Concept:

- For vectors expressed in the orthonormal basis $\hat{i}, \hat{j}, \hat{k}$, their cross product is evaluated by expanding a determinant. Equality of vectors means equality of their three corresponding components.
- The scalar product is $\vec{a} \cdot \vec{c} = a_x c_x + a_y c_y + a_z c_z$. The cross-product condition determines the component perpendicular to $\vec{a}$, while the dot-product condition fixes the component parallel to $\vec{a}$.

Step 1: Express the unknown vector and expand the cross product Let $\vec{c} = x\hat{i} + y\hat{j} + z\hat{k}$, where x, y, z are real numbers. Using the given vector $\vec{a} = \hat{i} - \hat{j} + \hat{k}$, we obtain

$$\begin{aligned}\vec{a} \times \vec{c} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ x & y & z \end{vmatrix} \\ &= (-z - y)\hat{i} + (x - z)\hat{j} + (y + x)\hat{k}.\end{aligned}$$

Equating this with $\vec{b} = 2\hat{i} - \hat{j} - 3\hat{k}$ gives

$$-y - z = 2, \quad x - z = -1, \quad x + y = -3.$$

Step 2: Use the dot product to determine all three components From $x - z = -1$, we have $x = z - 1$. The equation $x + y = -3$ then gives $y = -2 - z$. These also satisfy $-y - z = 2$, so the cross-product equations are consistent. The additional condition $\vec{a} \cdot \vec{c} = 4$ gives

$$\begin{aligned}x - y + z &= 4, \\ (z - 1) - (-2 - z) + z &= 4, \\ 3z + 1 &= 4.\end{aligned}$$

Thus $z = 1$, and substitution gives $x = 0$ and $y = -3$.

Step 3: Verify both prescribed products With the components just found, the cross product becomes

$$\vec{a} \times \vec{c} = (-1 + 3)\hat{i} + (0 - 1)\hat{j} + (-3 + 0)\hat{k} = 2\hat{i} - \hat{j} - 3\hat{k} = \vec{b}.$$

The dot product is $1(0) + (-1)(-3) + 1(1) = 4$. Hence both conditions hold, including the condition that resolves the otherwise undetermined parallel component.

Step 4: Conclusion The required vector is $\boxed{\vec{c} = -3\hat{j} + \hat{k}}$.

Quick Tip to Solve: After equating cross-product components, express two unknowns in terms of the third; use the dot product to fix that last unknown. Check the middle component carefully when expanding the determinant.

Section C (16 marks)

Questions number 11 to 14 carry 4 marks each.

11. Evaluate: $\int_0^1 x(1-x)^n dx$

[4 marks]

Solution:

Concept:

- A substitution must transform the variable, differential and limits together. Put $t = 1 - x$, so $x = 1 - t$ and $dx = -dt$.
- The power rule gives $\int_0^1 t^r dt = 1/(r+1)$ for $r > -1$. The result below applies to nonnegative integer n , and more generally to real $n > -1$.

Step 1: Transform the integral Let $I = \int_0^1 x(1-x)^n dx$. With $t = 1 - x$, the lower limit $x = 0$ becomes $t = 1$, and the upper limit $x = 1$ becomes $t = 0$. Thus

$$\begin{aligned} I &= \int_1^0 (1-t)t^n (-dt) \\ &= \int_0^1 (t^n - t^{n+1}) dt. \end{aligned}$$

Reversing the limits cancels the minus sign from $dx = -dt$. Expanding the product leaves two powers that can be integrated directly.

Step 2: Apply the power rule and evaluate Integrating the two terms separately,

$$\begin{aligned} I &= \left[\frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2} \right]_0^1 \\ &= \frac{1}{n+1} - \frac{1}{n+2}. \end{aligned}$$

Both powers equal one at the upper limit. At the lower limit both tend to zero, since $n+1 > 0$ and $n+2 > 0$. Hence the lower endpoint contributes zero.

Step 3: Combine the fractions Taking the common denominator gives

$$\frac{1}{n+1} - \frac{1}{n+2} = \frac{(n+2) - (n+1)}{(n+1)(n+2)}.$$

The numerator simplifies to one because the two terms containing the parameter cancel. For example, at $n = 0$ this gives $1/2$, agreeing with $\int_0^1 x dx$.

Step 4: Conclusion Hence,

$$\boxed{\int_0^1 x(1-x)^n dx = \frac{1}{(n+1)(n+2)}} \quad (n > -1).$$

Quick Tip to Solve: Use $t = 1 - x$ and reverse the limits; expand $(1-t)t^n$ to obtain the difference of two simple power integrals.

12(a). Using integration, find the area of the smaller region enclosed by the curve $4x^2 + 4y^2 = 9$ and the line $2x + 2y = 3$. [4 marks]

Solution:

Concept:

- The area between two curves is obtained by integrating the upper ordinate minus the lower ordinate over their common interval: $A = \int_a^b [y_{\text{upper}} - y_{\text{lower}}] dx$.
- The equation $x^2 + y^2 = r^2$ represents a circle of radius r , centred at the origin. Here $r = \frac{3}{2}$, and its upper semicircle is $y = \sqrt{\frac{9}{4} - x^2}$.

Step 1: Find the points of intersection Dividing the curve equation by 4 and the line equation by 2 gives

$$x^2 + y^2 = \frac{9}{4}, \quad y = \frac{3}{2} - x.$$

Substitute the line in the circle:

$$\begin{aligned} x^2 + \left(\frac{3}{2} - x\right)^2 &= \frac{9}{4}, \\ 2x^2 - 3x &= 0, \\ x(2x - 3) &= 0. \end{aligned}$$

Thus the intersection points are $P(0, \frac{3}{2})$ and $Q(\frac{3}{2}, 0)$.

Step 2: Identify the smaller region and set up its area The smaller region lies between the first-quadrant minor arc PQ and its chord. For $0 \leq x \leq \frac{3}{2}$, the arc is above the line, since both ordinates are nonnegative and

$$\left(\frac{9}{4} - x^2\right) - \left(\frac{3}{2} - x\right)^2 = x(3 - 2x) \geq 0.$$

Hence its area is

$$A = \int_0^{3/2} \left[\sqrt{\frac{9}{4} - x^2} - \left(\frac{3}{2} - x\right) \right] dx.$$

Step 3: Evaluate the definite integral Applying the standard integral with $r = \frac{3}{2}$,

$$\begin{aligned} A &= \left[\frac{x}{2} \sqrt{\frac{9}{4} - x^2} + \frac{9}{8} \sin^{-1}\left(\frac{2x}{3}\right) - \frac{3x}{2} + \frac{x^2}{2} \right]_0^{3/2} \\ &= \frac{9}{8} \cdot \frac{\pi}{2} - \frac{9}{4} + \frac{9}{8} \\ &= \frac{9\pi}{16} - \frac{9}{8}. \end{aligned}$$

We used $\sin^{-1}(1) = \frac{\pi}{2}$; the square-root term vanishes at the upper endpoint, and all terms vanish at the lower endpoint.

Step 4: Conclusion The required area of the smaller region is $\boxed{\frac{9(\pi - 2)}{16} \text{ square units}}.$

Quick Tip to Solve: Find the chord's two intercepts first; integrate the upper circular arc minus the straight line only between those intersection points.

12(b). If the area of the region bounded by the curve $y^2 = 4ax$ and the line $x = 4a$ is $\frac{256}{3}$ sq. units, then using integration, find the value of a , where $a > 0$. [4 marks]

Solution:

Concept:

- For horizontal strips, the area between a right boundary $x_R(y)$ and a left boundary $x_L(y)$ is $A = \int_{y_1}^{y_2} [x_R(y) - x_L(y)] dy$, where y_1, y_2 are the intersection ordinates.
- The parabola $y^2 = 4ax$, with $a > 0$, opens to the right and is symmetric about the x -axis. Its boundary can be written as $x = y^2/(4a)$.
- The condition $a > 0$ fixes the order of the integration limits and selects the positive value of a after the area equation is solved.

Step 1: Find the points of intersection Substitute the line $x = 4a$ into the parabola:

$$y^2 = 4a(4a) = 16a^2.$$

Since $a > 0$, the ordinates are $y = -4a$ and $y = 4a$. Thus the two intersection points are $(4a, -4a)$ and $(4a, 4a)$, and these ordinates are the limits for horizontal strips.

Step 2: Set up the integral for the enclosed area At any ordinate between $-4a$ and $4a$, the left boundary is the parabola $x = y^2/(4a)$, while the right boundary is the line $x = 4a$. The width of a horizontal strip is therefore $4a - y^2/(4a)$, which is non-negative throughout this interval. Hence

$$A = \int_{-4a}^{4a} \left(4a - \frac{y^2}{4a} \right) dy = 2 \int_0^{4a} \left(4a - \frac{y^2}{4a} \right) dy.$$

The factor 2 includes both halves of the region, above and below the x -axis.

Step 3: Evaluate the integral and use the given area Treating a as a constant while integrating with respect to y , we obtain

$$\begin{aligned} A &= 2 \left[4ay - \frac{y^3}{12a} \right]_0^{4a} \\ &= 2 \left(16a^2 - \frac{64a^3}{12a} \right) \\ &= 2 \left(16a^2 - \frac{16a^2}{3} \right) = \frac{64a^2}{3}. \end{aligned}$$

The stated area is $256/3$ square units. Equating the two expressions gives

$$\frac{64a^2}{3} = \frac{256}{3} \implies a^2 = 4.$$

Of the two algebraic roots, only the positive one satisfies the printed restriction $a > 0$. Substitution of this value into $64a^2/3$ reproduces the given area.

Step 4: Conclusion The required positive value is $a = 2$.

Quick Tip to Solve: Use horizontal strips here: line minus parabola gives the polynomial integrand $4a - y^2/(4a)$. Integrate from 0 to $4a$ and double to include both symmetric halves.

13. Find the coordinates of the point where the line through $(5, 1, 6)$ and $(3, 4, 1)$ crosses the zx -plane.

[4 marks]

Solution:

Concept:

- A line through A and B has equation $\mathbf{r} = \mathbf{a} + t(\mathbf{b} - \mathbf{a})$, where $\mathbf{a}, \mathbf{b}$ are their position vectors and $t \in \mathbb{R}$.
- The zx -plane has equation $y = 0$. Impose this condition on the line and use the resulting parameter to find x and z .

Step 1: Find the line's direction vector Let $A = (5, 1, 6)$ and $B = (3, 4, 1)$. Then

$$\overrightarrow{AB} = (3 - 5, 4 - 1, 1 - 6) = (-2, 3, -5).$$

These are direction ratios of the line. Since the second component is nonzero, the line has a unique intersection with the zx -plane.

Step 2: Write the parametric equation A general point on the line through A is

$$(x, y, z) = (5, 1, 6) + t(-2, 3, -5), \quad t \in \mathbb{R}.$$

Hence

$$x = 5 - 2t, \quad y = 1 + 3t, \quad z = 6 - 5t.$$

At $t = 0$ these equations give A , and at $t = 1$ they give B , verifying both given points.

Step 3: Set the second coordinate equal to zero A point in the zx -plane has $y = 0$. Therefore

$$1 + 3t = 0 \implies t = -\frac{1}{3}.$$

A negative parameter is valid: the question concerns the whole line, so the intersection may lie beyond the segment joining the given points.

Step 4: Substitute and verify the coordinates Using the same value $t = -\frac{1}{3}$ in all three equations,

$$x = 5 - 2\left(-\frac{1}{3}\right) = \frac{17}{3},$$

$$y = 1 + 3\left(-\frac{1}{3}\right) = 0,$$

$$z = 6 - 5\left(-\frac{1}{3}\right) = \frac{23}{3}.$$

As a check, the displacement from A is $\left(\frac{2}{3}, -1, \frac{5}{3}\right) = -\frac{1}{3}(-2, 3, -5)$, parallel to the line's direction vector. The zero second coordinate also confirms that the point is in the required plane.

Step 5: Conclusion The required point is $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$.

Quick Tip to Solve: For the zx -plane, set $y = 0$ first; substitute the resulting parameter into both remaining coordinate equations.

14. Case-Study Based Question

At the start of a cricket match, a coin is tossed and the team winning the toss has the opportunity to choose to bat or bowl. Such a coin is unbiased with equal probabilities of getting head and tail.

Based on the above information, answer the following questions:

[4 marks]

14(a). If such a coin is tossed 2 times, then find the probability distribution of number of tails.

[2 marks]

Solution:

Concept:

- A random variable assigns a numerical value to each outcome. Here let X denote the number of tails in two tosses, so its possible values are 0, 1, 2.
- For an unbiased coin, $P(H) = P(T) = \frac{1}{2}$. Independent tosses give the probability of each ordered pair as $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Step 1: List the equally likely outcomes Writing H for a head and T for a tail, the sample space for two tosses is

$$S = \{HH, HT, TH, TT\}.$$

All four outcomes are equally likely. The order of the tosses matters: HT and TH are two distinct outcomes, although both contain one tail.

Step 2: Assign the number of tails to each outcome The outcome HH gives $X = 0$, the two outcomes HT, TH give $X = 1$, and TT gives $X = 2$. Hence

$$P(X = 0) = P(HH) = \frac{1}{4},$$

$$P(X = 1) = P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2},$$

$$P(X = 2) = P(TT) = \frac{1}{4}.$$

These probabilities are nonnegative and their sum is $\frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$, as required for a probability distribution.

Step 3: Conclusion The probability distribution of the number of tails is

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Quick Tip to Solve: For two tosses, write HH, HT, TH and TT first; count HT and TH separately when finding the probability of exactly one tail.

14(b). Find the probability of getting at least one head in three tosses of such a coin.

[2 marks]

Solution:

Concept:

- For any event A , its complement A^c contains all outcomes outside A , and $P(A) = 1 - P(A^c)$.
- An unbiased coin has $P(H) = P(T) = \frac{1}{2}$. Since successive tosses are independent, the probability of a specified sequence is the product of the probabilities for its tosses.

Step 1: Identify the complement of the required event Let A be the event of getting at least one head in three tosses. The phrase “at least one” includes getting one, two or three heads. Its complement is getting no head at all, which means getting a tail on every toss. Thus

$$A^c = \{TTT\}.$$

Using the complement avoids separately counting three groups of outcomes.

Step 2: Calculate the probability of three tails The probability of a tail on each toss is $\frac{1}{2}$. By independence,

$$P(A^c) = P(TTT) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}.$$

There are $2^3 = 8$ equally likely ordered sequences in three tosses, and only TTT has no head. Therefore the other seven sequences satisfy the required condition, which also checks the complement calculation.

Step 3: Conclusion The probability of getting at least one head in three tosses is

$$P(A) = 1 - P(A^c) = 1 - \frac{1}{8} = \boxed{\frac{7}{8}}.$$

Quick Tip to Solve: For “at least one head”, subtract the probability of all tails from 1; in three tosses only TTT needs to be excluded.