

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-3-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) This question paper contains **three** Sections – Section A, B and C.
- (ii) Each Section is compulsory.
- (iii) Section–A has **6** short answer type-I questions of **2** marks each.
- (iv) Section–B has **4** short answer type-II questions of **3** marks each.
- (v) Section–C has **4** long answer type questions of **4** marks each.
- (vi) There is an internal choice in some questions.
- (vii) Q. 14 is a case study based problem with **2** sub-parts of **2** marks each.

Section A (12 marks)

Question Nos. 1 to 6 carry 2 marks each.

1. Find: $\int \frac{dx}{x^2 - 6x + 13}$.

[2 marks]

Solution:

Concept:

- Complete the square to reduce a quadratic denominator to a sum of two squares.
- The standard result is $\int du/(u^2 + a^2) = a^{-1} \tan^{-1}(u/a) + C$, where $a > 0$ and C is an arbitrary constant.

Step 1: Complete the square The coefficient of x is -6 , so half of it is -3 . Add and subtract 9 to obtain

$$x^2 - 6x + 13 = (x - 3)^2 + 4 = (x - 3)^2 + 2^2.$$

The denominator is positive for every real x , hence the integral has no real singularities.

Step 2: Apply the standard integral Put $u = x - 3$, so $du = dx$. With $a = 2$, the integral becomes

$$\int \frac{du}{u^2 + 2^2} = \frac{1}{2} \tan^{-1} \left(\frac{u}{2} \right) + C.$$

Substituting back $u = x - 3$ gives the required antiderivative. Differentiation gives $\frac{1}{4} / [1 + (x - 3)^2/4] = 1/[x^2 - 6x + 13]$, verifying the coefficient.

Step 3: Conclusion

$$\int \frac{dx}{x^2 - 6x + 13} = \frac{1}{2} \tan^{-1} \left(\frac{x - 3}{2} \right) + C.$$

Quick Tip to Solve: Complete the square first; the square root of the remaining constant, here 2, determines both factors in the arctangent formula.

2. Find the general solution of the differential equation: $e^{dy/dx} = x^2$.

[2 marks]

Solution:

Concept:

- Taking natural logarithms in $e^v = x^2$ gives $v = \ln(x^2) = 2 \ln |x|$ for $x \neq 0$.
- Integration of $dy/dx = f(x)$ yields $y = \int f(x) dx + C$. Integration by parts uses $\int u dv = uv - \int v du$.

Step 1: Take natural logarithms Because an exponential is strictly positive, $x = 0$ cannot satisfy the given equation. On either interval $x > 0$ or $x < 0$, natural logarithms give

$$\frac{dy}{dx} = \ln(x^2) = 2 \ln |x|.$$

Using the absolute value retains both possible real intervals; writing $2 \ln x$ alone would restrict the

Quick Tip to Solve: Take the logarithm before integrating, and use $\ln$ of the absolute value to retain solutions for negative x .

answer to positive x .

Step 2: Integrate by parts Integrate with respect to x . Choose $u = \ln |x|$ and $dv = dx$, so $du = dx/x$ and $v = x$. Hence

$$\begin{aligned}y &= 2 \int \ln |x| dx + C \\&= 2 \left(x \ln |x| - \int dx \right) + C \\&= 2x \ln |x| - 2x + C.\end{aligned}$$

Differentiating produces $dy/dx = 2 \ln |x|$; exponentiating returns $e^{dy/dx} = x^2$, as required.

Step 3: Conclusion

$$y = 2x \ln |x| - 2x + C, \quad x \neq 0.$$

This is the general solution on any interval lying wholly on one side of zero.

Quick Tip to Solve: Take the logarithm before integrating, and use $\ln$ of the absolute value to retain solutions for negative x .

3. Write the projection of the vector $(\vec{b} + \vec{c})$ on the vector $\vec{a}$, where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. [2 marks]

Solution:

Concept:

- The scalar projection of $\vec{v}$ on a nonzero vector $\vec{a}$ is $\vec{v} \cdot \vec{a} / |\vec{a}|$.
- For Cartesian components, add vectors componentwise and use the dot product $(v_1, v_2, v_3) \cdot (a_1, a_2, a_3) = v_1a_1 + v_2a_2 + v_3a_3$.

Step 1: Find the vector being projected Add the corresponding components of the given vectors:

$$\begin{aligned}\vec{b} + \vec{c} &= (1 + 2)\hat{i} + (2 - 1)\hat{j} + (-2 + 4)\hat{k} \\&= 3\hat{i} + \hat{j} + 2\hat{k}.\end{aligned}$$

The vector onto which this is projected remains $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$; its magnitude determines the divisor.

Step 2: Evaluate the scalar projection Compute the dot product and the magnitude separately:

$$\begin{aligned}(\vec{b} + \vec{c}) \cdot \vec{a} &= 3(2) + 1(-2) + 2(1) = 6, \\|\vec{a}| &= \sqrt{2^2 + (-2)^2 + 1^2} = 3.\end{aligned}$$

Dividing the dot product by $|\vec{a}|$, rather than its square, gives the scalar projection $6/3 = 2$. The positive sign means that the component points in the direction of $\vec{a}$.

Step 3: Conclusion

$$\text{proj}_{\vec{a}}(\vec{b} + \vec{c}) = 2$$

(scalar projection).

Quick Tip to Solve: Add b and c before taking the dot product, then divide by the magnitude of a for the scalar projection.

4. If the distance of the point $(1, 1, 1)$ from the plane $x - y + z + \lambda = 0$ is $\frac{5}{\sqrt{3}}$, find the value(s) of λ . [2 marks]

Solution:

Concept:

- The perpendicular distance from (x_0, y_0, z_0) to $Ax + By + Cz + D = 0$ is $|Ax_0 + By_0 + Cz_0 + D|/\sqrt{A^2 + B^2 + C^2}$.
- If $|t| = k$ with $k > 0$, both equations $t = k$ and $t = -k$ must be solved.

Step 1: Substitute the point and plane coefficients The plane has coefficients $A = 1$, $B = -1$, $C = 1$, and $D = \lambda$. At the given point $(1, 1, 1)$, the distance is

$$d = \frac{|1 - 1 + 1 + \lambda|}{\sqrt{1^2 + (-1)^2 + 1^2}} = \frac{|1 + \lambda|}{\sqrt{3}}.$$

The absolute value is necessary because a distance cannot be negative.

Step 2: Solve both absolute-value cases Equating this expression to the given distance $5/\sqrt{3}$ and multiplying by the positive number $\sqrt{3}$ gives

$$|1 + \lambda| = 5.$$

Therefore $1 + \lambda = 5$ or $1 + \lambda = -5$, giving $\lambda = 4$ or $\lambda = -6$. Substitution checks both values: the plane expression at the point is respectively 5 and -5 , each having absolute value 5.

Step 3: Conclusion

$$\lambda = 4 \text{ or } \lambda = -6.$$

Quick Tip to Solve: Keep the absolute value in the point-to-plane distance formula and solve both signs to obtain both parallel planes.

5. Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of spade cards. [2 marks]

Solution:

Concept:

- With replacement, the two draws are independent and each draw has spade probability $p = 13/52 = 1/4$.
- If X counts successes in two independent trials, $P(X = r) = \binom{2}{r} p^r (1 - p)^{2-r}$, for $r = 0, 1, 2$.

Step 1: Identify the possible values and trial probabilities Let X be the number of spade cards in the two draws. Its possible values are 0, 1, 2. A standard pack contains 13 spades and 39 non-spades.

Quick Tip to Solve: With replacement, use a binomial model with $n = 2$ and $p = 1/4$; exactly one spade can occur in two orders.

Returning the first card before the second draw keeps

$$P(\text{spade}) = \frac{1}{4}, \quad P(\text{non-spade}) = \frac{3}{4}$$

on both draws and makes their outcomes independent.

Step 2: Compute the probability distribution For no spades, both cards must be non-spades. For exactly one spade, there are two disjoint orders. For two spades, both draws must give spades. Thus

$$\begin{aligned} P(X = 0) &= \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}, \\ P(X = 1) &= \frac{1}{4} \cdot \frac{3}{4} + \frac{3}{4} \cdot \frac{1}{4} = \frac{6}{16}, \\ P(X = 2) &= \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}. \end{aligned}$$

All three probabilities are nonnegative and their sum is $(9 + 6 + 1)/16 = 1$, verifying the distribution.

Step 3: Conclusion

X	0	1	2
$P(X)$	9/16	6/16	1/16

Quick Tip to Solve: With replacement, use a binomial model with $n = 2$ and $p = 1/4$; exactly one spade can occur in two orders.

6(a). A pair of dice is thrown and the sum of the numbers appearing on the dice is observed to be 7. Find the probability that the number 5 has appeared on at least one die. [2 marks]

Solution:

Concept:

- Conditional probability is $P(A | B) = P(A \cap B)/P(B)$, provided $P(B) > 0$.
- For two fair dice, ordered pairs are equally likely; after conditioning, count only the pairs consistent with the known sum.

Step 1: List the outcomes satisfying the observed sum Let B be the event that the sum is 7, and let A be the event that at least one die shows 5. The six ordered outcomes in B are

$$B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}.$$

Each originally has probability $1/36$, so $P(B) = 6/36$. Distinguishing the first die from the second ensures that $(2, 5)$ and $(5, 2)$ are counted separately.

Step 2: Count the favourable conditional outcomes Among these six outcomes, only $(2, 5)$ and $(5, 2)$ contain a 5. Thus

$$A \cap B = \{(2, 5), (5, 2)\}, \quad P(A \cap B) = \frac{2}{36}.$$

Quick Tip to Solve: Once the sum is known to be 7, restrict the sample space to its six ordered pairs before counting those containing 5.

Applying the conditional probability formula gives

$$P(A | B) = \frac{2/36}{6/36} = \frac{2}{6} = \frac{1}{3}.$$

The pair (5, 5) is excluded because its sum is 10, not the observed sum 7.

Step 3: Conclusion

$$P(\text{at least one 5} \mid \text{sum 7}) = \frac{1}{3}.$$

Quick Tip to Solve: Once the sum is known to be 7, restrict the sample space to its six ordered pairs before counting those containing 5.

OR

6(b). The probability that A hits the target is $\frac{1}{3}$ and the probability that B hits it, is $\frac{2}{5}$. If both try to hit the target independently, find the probability that the target is hit. [2 marks]

Solution:

Concept:

- The target is hit when at least one person hits it; the complementary event is that both people miss.
- For independent attempts, multiply miss probabilities: $P(A^c \cap B^c) = P(A^c)P(B^c)$.

Step 1: Calculate the miss probabilities Let A denote the event that person A hits the target and B the event that person B hits it. The given probabilities are $P(A) = 1/3$ and $P(B) = 2/5$. Their complements are

$$P(A^c) = 1 - \frac{1}{3} = \frac{2}{3}, \quad P(B^c) = 1 - \frac{2}{5} = \frac{3}{5}.$$

Since the attempts are independent, the events that they miss are also independent.

Step 2: Use the complement of both missing The target remains unhit only if both people miss. Hence

$$P(A^c \cap B^c) = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}.$$

Taking its complement yields

$$P(A \cup B) = 1 - \frac{2}{5} = \frac{3}{5}.$$

As a check, inclusion-exclusion gives $1/3 + 2/5 - (1/3)(2/5) = 5/15 + 6/15 - 2/15 = 9/15$. Subtracting the intersection prevents counting the case in which both people hit twice.

Step 3: Conclusion

$$P(\text{target is hit}) = \frac{3}{5}.$$

Quick Tip to Solve: For at least one hit, subtract the probability that both miss from 1; multiplying hit probabilities gives only both hitting.

Section B (12 marks)

Question Nos. 7 to 10 carry 3 marks each.

7. Evaluate: $\int_0^{2\pi} \frac{dx}{1 + e^{\sin x}}.$

[3 marks]

Solution:

Concept:

- For a continuous function on $[0, a]$, reflection gives $\int_0^a f(x) dx = \int_0^a f(a - x) dx$.
- Here $\sin(2\pi - x) = -\sin x$. For every real t , $\frac{1}{1 + e^t} + \frac{1}{1 + e^{-t}} = 1$.

Step 1: Reflect the interval Denote the required integral by I . Since the denominator is positive, the integrand is continuous on the entire interval. Put $u = 2\pi - x$, so $dx = -du$. The limits $x = 0, 2\pi$ become $u = 2\pi, 0$, respectively. Hence

$$I = \int_0^{2\pi} \frac{dx}{1 + e^{\sin x}} = \int_0^{2\pi} \frac{du}{1 + e^{-\sin u}}.$$

Rename the dummy variable u as x . This produces a second expression for the same integral, with the sign of sine reversed.

Step 2: Add the two expressions Multiply numerator and denominator of the reflected fraction by $e^{\sin x}$. Adding the equal expressions for I gives

$$\begin{aligned} 2I &= \int_0^{2\pi} \left(\frac{1}{1 + e^{\sin x}} + \frac{e^{\sin x}}{1 + e^{\sin x}} \right) dx \\ &= \int_0^{2\pi} 1 dx = [x]_0^{2\pi} = 2\pi. \end{aligned}$$

Thus the two complicated fractions combine into a constant integrand. An antiderivative of the original exponential fraction is unnecessary.

Step 3: Conclusion Dividing the last equality by 2, the required integral is $\boxed{\pi}$.

Quick Tip to Solve: Reflect about the midpoint of the interval: the sign change in sine makes the two exponential fractions add to one.

8(a). Find the particular solution of the differential equation $x \frac{dy}{dx} - y = x^2 e^x$, given $y(1) = 0$. [3 marks]

Solution:

Concept:

- A linear differential equation has the form $y' + P(x)y = Q(x)$, with integrating factor $IF = e^{\int P(x) dx}$.
- Multiplication by the integrating factor turns the left side into a product derivative. An initial condition then determines the integration constant.

Quick Tip to Solve: After division by x , recognize the derivative of y/x ; the initial value fixes the constant immediately.

Step 1: Write the linear form and find the integrating factor The initial point is $x = 1$, so work on the interval $x > 0$, where division by x is valid. Dividing the equation by x gives

$$\frac{dy}{dx} - \frac{y}{x} = xe^x.$$

Thus $P(x) = -1/x$ and $Q(x) = xe^x$. The integrating factor is

$$\text{IF} = e^{\int -1/x dx} = e^{-\log x} = \frac{1}{x}.$$

Step 2: Integrate the product derivative Multiplying the linear equation by $1/x$, we obtain

$$\frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = e^x, \quad \frac{d}{dx} \left(\frac{y}{x} \right) = e^x.$$

Integrating with respect to x therefore yields $y/x = e^x + C$, or $y = x(e^x + C)$, where C is a constant.

Step 3: Apply the initial condition and check Substituting $x = 1$ and $y = 0$ gives $0 = e + C$, so $C = -e$. For this value, $y' = e^x + xe^x - e$. Therefore

$$xy' - y = x(e^x + xe^x - e) - x(e^x - e) = x^2 e^x.$$

Also, substitution of $x = 1$ gives zero, as required by the initial condition.

Step 4: Conclusion The particular solution is $y = x(e^x - e)$, on the interval $x > 0$ containing the initial point.

Quick Tip to Solve: After division by x , recognize the derivative of y/x ; the initial value fixes the constant immediately.

OR

8(b). Find the general solution of the differential equation $x \frac{dy}{dx} = y(\log y - \log x + 1)$. [3 marks]

Solution:

Concept:

- For a homogeneous first-order equation, substitute $y = vx$, so that $\frac{dy}{dx} = v + x \frac{dv}{dx}$. Here $v = y/x$ is the ratio of the dependent and independent variables.
- The real logarithms in the printed equation require $x > 0$ and $y > 0$; hence $v > 0$. On this domain, $\log y - \log x = \log(y/x) = \log v$.

Step 1: Reduce the equation using the ratio of the variables Substitute $y = vx$ into the given equation. Then

$$x \left(v + x \frac{dv}{dx} \right) = vx(\log v + 1).$$

Quick Tip to Solve: Cancel the extra term v after substituting $y=vx$, and restore $v=1$ if division by $\log v$ excludes it.

Since $x > 0$, division by x , followed by subtraction of v , gives

$$x \frac{dv}{dx} = v \log v.$$

Thus the equation becomes separable in the variables v and x .

Step 2: Integrate and retain the constant-ratio solution For $v \neq 1$, separate variables and integrate:

$$\int \frac{dv}{v \log v} = \int \frac{dx}{x}, \quad \log |\log v| = \log x + C_1.$$

Absorbing the sign and the exponential of the integration constant into a real constant C yields $\log v = Cx$. The value $v = 1$, excluded during separation, also solves the reduced equation and is included by $C = 0$. Therefore $\log(y/x) = Cx$, or $y = xe^{Cx}$.

Step 3: Verify the family in the original equation For $y = xe^{Cx}$, we have $y' = e^{Cx}(1 + Cx)$ and $\log y - \log x = Cx$. Hence $xy' = xe^{Cx}(1 + Cx) = y(\log y - \log x + 1)$, confirming that the whole family satisfies the printed equation on $x > 0$.

Step 4: Conclusion The general solution is $y = xe^{Cx}$, $x > 0$, $C \in \mathbb{R}$, equivalently $\log(y/x) = Cx$.

Quick Tip to Solve: Cancel the extra term v after substituting $y=vx$, and restore $v=1$ if division by $\log v$ excludes it.

9(a). The two adjacent sides of a parallelogram are represented by vectors $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to one of its diagonals. Also, find the area of the parallelogram. [3 marks]

Solution:

Concept:

- If adjacent sides are represented by $\vec{a}$ and $\vec{b}$, the diagonal from their common vertex has vector $\vec{d} = \vec{a} + \vec{b}$. A unit vector in its direction is $\vec{d}/|\vec{d}|$.
- The area of the parallelogram is $|\vec{a} \times \vec{b}|$, where the cross product is perpendicular to both adjacent sides.

Step 1: Find a diagonal and its unit vector Let $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$. Choose the diagonal joining the common vertex of these sides to the opposite vertex. Its vector is

$$\vec{d} = \vec{a} + \vec{b} = 3\hat{i} - 6\hat{j} + 2\hat{k}.$$

Its magnitude is $|\vec{d}| = \sqrt{3^2 + (-6)^2 + 2^2} = 7$. Thus a unit vector parallel to this diagonal is $(3\hat{i} - 6\hat{j} + 2\hat{k})/7$; its negative is also parallel.

Quick Tip to Solve: Use the sum of the adjacent side vectors for a diagonal, but use their cross product for area.

Step 2: Evaluate the cross product of the sides Using the determinant formula,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix} \\ &= (12 + 10)\hat{i} - (-6 - 5)\hat{j} + (-4 + 4)\hat{k} \\ &= 22\hat{i} + 11\hat{j}.\end{aligned}$$

In particular, the middle component is positive because the cofactor sign reverses the value -11 .

Step 3: Obtain the magnitude and check the direction The area is therefore

$$|\vec{a} \times \vec{b}| = \sqrt{22^2 + 11^2} = \sqrt{605} = 11\sqrt{5}.$$

As a check, $(22, 11, 0) \cdot (2, -4, 5) = 44 - 44 = 0$ and $(22, 11, 0) \cdot (1, -2, -3) = 22 - 22 = 0$, so the computed cross product is perpendicular to each side, as required.

Step 4: Conclusion A unit vector parallel to one diagonal is $\frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{7}$. The area of the parallelogram is $11\sqrt{5}$ square units.

Quick Tip to Solve: Use the sum of the adjacent side vectors for a diagonal, but use their cross product for area.

OR

9(b). If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that the vector $(\vec{a} + \lambda\vec{b})$ is perpendicular to vector $\vec{c}$, then find the value of λ . [3 marks]

Solution:

Concept:

- Two nonzero vectors $\vec{u}$ and $\vec{v}$ are perpendicular precisely when $\vec{u} \cdot \vec{v} = 0$. In Cartesian components, $(u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = u_1v_1 + u_2v_2 + u_3v_3$.
- The dot product distributes over vector addition: $(\vec{a} + \lambda\vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \lambda(\vec{b} \cdot \vec{c})$. Here λ is the scalar to be determined.

Step 1: Compute the two scalar products Write the given vectors in component form:

$$\vec{a} = (2, 2, 3), \quad \vec{b} = (-1, 2, 1), \quad \vec{c} = (3, 1, 0).$$

The missing $\hat{k}$ component in $\vec{c}$ is zero. Therefore

$$\vec{a} \cdot \vec{c} = 2(3) + 2(1) + 3(0) = 8, \quad \vec{b} \cdot \vec{c} = (-1)(3) + 2(1) + 1(0) = -1.$$

The negative first component of $\vec{b}$ is essential for the second result.

Quick Tip to Solve: Compute $\vec{a} \cdot \vec{c}$ and $\vec{b} \cdot \vec{c}$ first; perpendicularity then gives one linear equation in λ .

Step 2: Apply perpendicularity to obtain the scalar The given condition requires

$$0 = (\vec{a} + \lambda \vec{b}) \cdot \vec{c} = 8 + \lambda(-1) = 8 - \lambda.$$

Thus the unique scalar satisfying the condition is $\lambda = 8$, since the coefficient $\vec{b} \cdot \vec{c} = -1$ is nonzero.

Step 3: Verify perpendicularity by direct substitution Substituting $\lambda = 8$ into the vector combination gives

$$\vec{a} + 8\vec{b} = (2 - 8)\hat{i} + (2 + 16)\hat{j} + (3 + 8)\hat{k} = -6\hat{i} + 18\hat{j} + 11\hat{k}.$$

Its dot product with $\vec{c}$ is $(-6)(3) + 18(1) + 11(0) = 0$. Both this vector and $\vec{c}$ are nonzero, so the zero dot product verifies that they are perpendicular.

Step 4: Conclusion The required value is $\boxed{\lambda = 8}$.

Quick Tip to Solve: Compute $\vec{a} \cdot \vec{c}$ and $\vec{b} \cdot \vec{c}$ first; perpendicularity then gives one linear equation in λ .

10. Show that the lines: $\frac{1-x}{2} = \frac{y-3}{4} = \frac{z}{-1}$ and $\frac{x-4}{3} = \frac{2y-2}{-4} = z-1$ are coplanar. [3 marks]

Solution:

Concept:

- Lines through points with position vectors $\mathbf{a}_1, \mathbf{a}_2$ and direction vectors $\mathbf{b}_1, \mathbf{b}_2$ are coplanar when $(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = 0$. This scalar triple product tests whether the joining vector and the two directions lie in one plane.
- A symmetric line must first be written with coefficients of x, y, z equal to one in the numerators. In particular, $1 - x = -(x - 1)$ and $2y - 2 = 2(y - 1)$; these change the corresponding direction components.

Step 1: Identify points and direction vectors Writing the first line in standard symmetric form gives

$$\frac{x-1}{-2} = \frac{y-3}{4} = \frac{z-0}{-1}.$$

Hence it passes through $A = (1, 3, 0)$ and has direction vector $\mathbf{b}_1 = (-2, 4, -1)$. For the second line, divide the middle numerator and denominator by 2:

$$\frac{x-4}{3} = \frac{y-1}{-2} = \frac{z-1}{1}.$$

It passes through $B = (4, 1, 1)$, with direction vector $\mathbf{b}_2 = (3, -2, 1)$. Therefore $\overrightarrow{AB} = (3, -2, 1)$.

Quick Tip to Solve: Normalize the numerators first: the directions are $(-2, 4, -1)$ and $(3, -2, 1)$, and the joining vector equals the second direction, making the determinant zero immediately.

Step 2: Evaluate the coplanarity test The required scalar triple product is

$$\begin{aligned}\overrightarrow{AB} \cdot (\mathbf{b}_1 \times \mathbf{b}_2) &= \begin{vmatrix} 3 & -2 & 1 \\ -2 & 4 & -1 \\ 3 & -2 & 1 \end{vmatrix} \\ &= 3(4 - 2) + 2(-2 + 3) + 1(4 - 12) \\ &= 6 + 2 - 8 = 0.\end{aligned}$$

The first and third rows are identical, so the determinant also vanishes directly. As a geometric check, the second line is $(x, y, z) = (4, 1, 1) + t(3, -2, 1)$. At $t = -1$, it gives $(1, 3, 0) = A$, a point on the first line. Their directions are not proportional, so these are two distinct intersecting lines and determine a plane.

Step 3: Conclusion The scalar triple product is zero; equivalently, the two lines intersect at $(1, 3, 0)$. Hence the given lines are coplanar.

Quick Tip to Solve: Normalize the numerators first: the directions are $(-2, 4, -1)$ and $(3, -2, 1)$, and the joining vector equals the second direction, making the determinant zero immediately.

Section C (16 marks)

Question Nos. 11 to 14 carry 4 marks each.

11. Find the area of the region bounded by curve $4x^2 = y$ and the line $y = 8x + 12$, using integration. [4 marks]

Solution:

Concept:

- The area between curves $y = f(x)$ and $y = g(x)$, from $x = a$ to $x = b$, is $\int_a^b [f(x) - g(x)] dx$ when $f(x) \geq g(x)$ throughout that interval. The limits are obtained from the intersections of the bounding curves.
- Integrate the vertical height of a strip, namely upper ordinate minus lower ordinate. This gives geometric area even when the region is not bounded by the coordinate axes.

Step 1: Find the points of intersection At an intersection the two expressions for y agree, so

$$\begin{aligned}4x^2 &= 8x + 12, \\ x^2 - 2x - 3 &= 0, \\ (x - 3)(x + 1) &= 0.\end{aligned}$$

Thus $x = -1$ and $x = 3$. Substituting into $y = 4x^2$ gives the intersection points $(-1, 4)$ and $(3, 36)$. These abscissae are the limits of integration for the bounded region.

Step 2: Identify the upper curve and set up the area The difference between the line and the parabola is

$$(8x + 12) - 4x^2 = -4(x - 3)(x + 1) = 4(3 - x)(x + 1).$$

Quick Tip to Solve: Factor the height as $4(3 - x)(x + 1)$: it gives the limits $-1, 3$ and confirms the line stays above the parabola before you integrate.

For $-1 \leq x \leq 3$, both $3 - x$ and $x + 1$ are nonnegative. Therefore the line is above the parabola throughout the interval, and their difference is the nonnegative height of each vertical strip. Hence the required area is

$$A = \int_{-1}^3 (8x + 12 - 4x^2) dx.$$

Step 3: Evaluate the definite integral Integrating term by term and evaluating at both limits gives

$$\begin{aligned} A &= \left[4x^2 + 12x - \frac{4x^3}{3} \right]_{-1}^3 \\ &= (36 + 36 - 36) - \left(4 - 12 + \frac{4}{3} \right) \\ &= 36 - \left(-\frac{20}{3} \right) \\ &= \frac{108 + 20}{3} = \frac{128}{3}. \end{aligned}$$

The lower-limit value is negative, so subtracting it increases the result. Since the height was non-negative on the entire interval, this integral is already the geometric area and requires no further absolute-value adjustment.

Step 4: Conclusion The area bounded by $y = 4x^2$ and $y = 8x + 12$ is $\boxed{\frac{128}{3} \text{ square units}}$.

Quick Tip to Solve: Factor the height as $4(3 - x)(x + 1)$: it gives the limits $-1, 3$ and confirms the line stays above the parabola before you integrate.

12(a). Find: $\int \frac{x^2}{(x^2 + 1)(3x^2 + 4)} dx.$

[4 marks]

Solution:

Concept:

- For a rational integrand whose denominator is a product of distinct quadratic factors, use partial fractions with constants when both the numerator and denominator contain only even powers of x .
- The standard inverse tangent integral is $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$, where $a > 0$ and C is an arbitrary constant.

Step 1: Resolve the integrand into partial fractions Let the required integral be I . Since the numerator has lower degree than the denominator, polynomial division is unnecessary. Write

$$\frac{x^2}{(x^2 + 1)(3x^2 + 4)} = \frac{A}{x^2 + 1} + \frac{B}{3x^2 + 4}.$$

Multiplying by the common denominator gives an identity valid for every real x :

$$x^2 = A(3x^2 + 4) + B(x^2 + 1) = (3A + B)x^2 + (4A + B).$$

Quick Tip to Solve: Compare the constant terms first: $4A + B = 0$. Combining this with $3A + B = 1$ immediately gives $A = -1$; remember the substitution scale in the second inverse tangent integral.

Equating the coefficients of x^2 and the constant terms, we obtain $3A + B = 1$ and $4A + B = 0$. Subtracting the first equation from the second gives $A = -1$, and substitution then gives $B = 4$. Thus the integrand is $-1/(x^2 + 1) + 4/(3x^2 + 4)$. The constant term cancels, while the coefficient of x^2 is $-3 + 4 = 1$, confirming the decomposition.

Step 2: Integrate the two quadratic fractions By linearity of integration,

$$I = - \int \frac{dx}{x^2 + 1} + 4 \int \frac{dx}{3x^2 + 4}.$$

The first integral is $\tan^{-1} x$. For the second, set $u = \sqrt{3}x/2$, so $dx = (2/\sqrt{3}) du$ and $3x^2 + 4 = 4(1 + u^2)$. Therefore

$$4 \int \frac{dx}{3x^2 + 4} = \frac{2}{\sqrt{3}} \int \frac{du}{1 + u^2} = \frac{2}{\sqrt{3}} \tan^{-1} u.$$

Returning to x yields the inverse tangent term with argument $\sqrt{3}x/2$. Both denominators are strictly positive for real x , so this antiderivative is valid on the entire real line with one arbitrary integration constant.

Step 3: Verify the coefficients by differentiation The derivative of the two obtained terms is

$$-\frac{1}{1+x^2} + \frac{2}{\sqrt{3}} \frac{\sqrt{3}/2}{1+3x^2/4} = -\frac{1}{x^2+1} + \frac{4}{3x^2+4} = \frac{x^2}{(x^2+1)(3x^2+4)}.$$

This reproduces the printed integrand exactly and checks both the minus sign and the scale factor introduced by substitution.

Step 4: Conclusion

$$\int \frac{x^2}{(x^2+1)(3x^2+4)} dx = -\tan^{-1} x + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{\sqrt{3}x}{2} \right) + C.$$

Quick Tip to Solve: Compare the constant terms first: $4A + B = 0$. Combining this with $3A + B = 1$ immediately gives $A = -1$; remember the substitution scale in the second inverse tangent integral.

OR

12(b). Evaluate: $\int_{-2}^1 \sqrt{5 - 4x - x^2} dx.$

[4 marks]

Solution:

Concept:

- Completing the square converts a quadratic radical to the standard form $\sqrt{a^2 - u^2}$. Here $5 - 4x - x^2 = 9 - (x + 2)^2$, so the constant a is 3.
- For $\sqrt{a^2 - u^2}$, use $u = a \sin \theta$. On $0 \leq \theta \leq \pi/2$, the cosine is nonnegative, hence $\sqrt{a^2 - a^2 \sin^2 \theta} = a \cos \theta$ without a sign ambiguity.

Quick Tip to Solve: After completing the square, shift the limits immediately: here they become 0 and 3, revealing a quarter-circle integral of radius 3.

- A definite integral requires changing the limits with the variable. The identity $\cos^2 \theta = (1 + \cos 2\theta)/2$ then makes the resulting trigonometric integral elementary.

Step 1: Complete the square and shift the variable Write the requested integral as I . Since

$$5 - 4x - x^2 = 9 - (x^2 + 4x + 4) = 9 - (x + 2)^2,$$

set $u = x + 2$, so $du = dx$. The lower limit $x = -2$ gives $u = 0$, while the upper limit $x = 1$ gives $u = 3$. Therefore

$$I = \int_0^3 \sqrt{9 - u^2} du.$$

The radicand is nonnegative on this entire interval, including its zero at the upper endpoint.

Step 2: Remove the radical by a trigonometric substitution Put $u = 3 \sin \theta$, choosing $0 \leq \theta \leq \pi/2$. Then $du = 3 \cos \theta d\theta$. The transformed endpoints are $\theta = 0$ for $u = 0$ and $\theta = \pi/2$ for $u = 3$. On this interval,

$$\sqrt{9 - u^2} = \sqrt{9(1 - \sin^2 \theta)} = 3 \cos \theta.$$

Substituting both the radical and the differential gives

$$I = 9 \int_0^{\pi/2} \cos^2 \theta d\theta = \frac{9}{2} \int_0^{\pi/2} (1 + \cos 2\theta) d\theta.$$

The factor 9 comes from multiplying the two factors $3 \cos \theta$, one from the radical and one from du .

Step 3: Evaluate the transformed endpoints Integrating the constant and cosine terms separately,

$$\begin{aligned} I &= \frac{9}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} \\ &= \frac{9}{2} \left(\frac{\pi}{2} + \frac{\sin \pi}{2} - 0 - \frac{\sin 0}{2} \right). \end{aligned}$$

Both sine terms vanish. As a check, $\sqrt{9 - u^2}$ is the upper ordinate of the circle $u^2 + y^2 = 9$. The interval from 0 to 3 covers one quarter of this circle, so its area must be one quarter of $\pi \cdot 3^2$, consistent with the endpoint evaluation.

Step 4: Conclusion Hence, $\int_{-2}^1 \sqrt{5 - 4x - x^2} dx = \boxed{\frac{9\pi}{4}}.$

Quick Tip to Solve: After completing the square, shift the limits immediately: here they become 0 and 3, revealing a quarter-circle integral of radius 3.

13. Find the distance of the point $(1, -2, 9)$ from the point of intersection of the line $\vec{r} = 4\hat{i} + 2\hat{j} + 7\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 10$. [4 marks]

Solution:

Concept:

- A point on a line in vector form is obtained by giving its parameter a value. For the given line, its Cartesian coordinates are $x = 4 + 3\lambda$, $y = 2 + 4\lambda$ and $z = 7 + 2\lambda$.
- The plane equation $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 10$ means $x - y + z = 10$. The intersection point must satisfy both the line and the plane equations.
- The distance between $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Step 1: Express the intersection point using the line parameter Let Q be the point at which the given line meets the plane, and let $P = (1, -2, 9)$ be the given point. Since Q lies on the line, write

$$Q = (4 + 3\lambda, 2 + 4\lambda, 7 + 2\lambda).$$

The direction vector of the line is $(3, 4, 2)$, while a normal vector to the plane is $(1, -1, 1)$. Their scalar product is $3 - 4 + 2 = 1 \neq 0$, so the line has a unique intersection with this plane.

Step 2: Substitute in the plane equation At the intersection, $x - y + z = 10$. Hence

$$\begin{aligned}(4 + 3\lambda) - (2 + 4\lambda) + (7 + 2\lambda) &= 10, \\ 9 + \lambda &= 10, \\ \lambda &= 1.\end{aligned}$$

Therefore $Q = (4 + 3, 2 + 4, 7 + 2) = (7, 6, 9)$. As a check, $7 - 6 + 9 = 10$, so this point satisfies the plane equation as required.

Step 3: Apply the three-dimensional distance formula The displacement from P to Q is

$$\overrightarrow{PQ} = (7 - 1)\hat{i} + (6 - (-2))\hat{j} + (9 - 9)\hat{k} = 6\hat{i} + 8\hat{j}.$$

Its magnitude is

$$PQ = \sqrt{6^2 + 8^2 + 0^2} = \sqrt{36 + 64} = \sqrt{100}.$$

The two points have the same z -coordinate, which explains why the third squared coordinate difference vanishes.

Step 4: Conclusion The line meets the plane at $(7, 6, 9)$, and its distance from $(1, -2, 9)$ is 10 units.

Quick Tip to Solve: Substitute all three parameter coordinates into the plane first; then use the distance formula from the given point to that intersection.

14. Case Study Problem:

A shopkeeper sells three types of flower seeds A_1, A_2, A_3 . They are sold in the form of a mixture, where the proportions of these seeds are 4 : 4 : 2, respectively. The germination rates of the three types of seeds are 45%, 60% and 35% respectively.

Based on the above information, answer the following questions:

[4 marks]

14(a). Calculate the probability that a randomly chosen seed will germinate.

[2 marks]

Solution:

Concept:

- Let A_i denote the event that the chosen seed is of type i , for $i = 1, 2, 3$, and let G be the event that it germinates. The three types form mutually exclusive and exhaustive events.
- The law of total probability gives $P(G) = \sum_{i=1}^3 P(A_i)P(G | A_i)$. Each germination rate must be weighted by the proportion of its seed type in the mixture.

Step 1: Convert the proportions and rates to probabilities The total number of ratio parts is $4 + 4 + 2 = 10$. Thus

$$P(A_1) = \frac{4}{10}, \quad P(A_2) = \frac{4}{10}, \quad P(A_3) = \frac{2}{10}.$$

The stated germination rates are conditional probabilities:

$$P(G | A_1) = \frac{45}{100}, \quad P(G | A_2) = \frac{60}{100}, \quad P(G | A_3) = \frac{35}{100}.$$

These rates describe germination within each type, so they cannot be added without accounting for the type proportions.

Step 2: Use total probability for germination A germinating seed belongs to exactly one of the three types. Add the probabilities of these disjoint possibilities:

$$\begin{aligned} P(G) &= \frac{4}{10} \cdot \frac{45}{100} + \frac{4}{10} \cdot \frac{60}{100} + \frac{2}{10} \cdot \frac{35}{100} \\ &= \frac{180 + 240 + 70}{1000} = \frac{490}{1000} = \frac{49}{100}. \end{aligned}$$

Step 3: Conclusion The probability that a randomly chosen seed germinates is $\boxed{\frac{49}{100} = 0.49}$.

Quick Tip to Solve: Normalize the ratio first, then multiply each germination rate by its matching mixture proportion before adding.

14(b). Calculate the probability that the seed is of type A_2 , given that a randomly chosen seed germinates. [2 marks]

Solution:

Concept:

- Let A_i be the event that a seed is of type i , and let G be the event that it germinates. The events A_1, A_2, A_3 are mutually exclusive and exhaustive.
- The total probability of germination is $P(G) = \sum_{i=1}^3 P(A_i)P(G | A_i)$. Bayes' theorem gives $P(A_2 | G) = \frac{P(A_2)P(G | A_2)}{P(G)}$.

Step 1: Calculate the total probability of germination The mixture ratio 4 : 4 : 2 gives

$$P(A_1) = \frac{2}{5}, \quad P(A_2) = \frac{2}{5}, \quad P(A_3) = \frac{1}{5}.$$

The stated germination rates mean

$$P(G | A_1) = \frac{45}{100}, \quad P(G | A_2) = \frac{60}{100}, \quad P(G | A_3) = \frac{35}{100}.$$

Weight each rate by the proportion of its seed type:

$$\begin{aligned} P(G) &= \frac{2}{5} \cdot \frac{45}{100} + \frac{2}{5} \cdot \frac{60}{100} + \frac{1}{5} \cdot \frac{35}{100} \\ &= \frac{18 + 24 + 7}{100} = \frac{49}{100}. \end{aligned}$$

Step 2: Apply Bayes' theorem to the observed germination The probability that a seed is both of type A_2 and germinates is

$$P(A_2 \cap G) = P(A_2)P(G | A_2) = \frac{2}{5} \cdot \frac{60}{100} = \frac{24}{100}.$$

Since the chosen seed has germinated, the sample space is restricted to germinated seeds. Hence

$$P(A_2 | G) = \frac{P(A_2 \cap G)}{P(G)} = \frac{24/100}{49/100} = \frac{24}{49}.$$

Step 3: Conclusion The required probability that the germinated seed is of type A_2 is $\boxed{\frac{24}{49}}$.

Quick Tip to Solve: Multiply each mixture proportion by its germination rate; divide the type A_2 contribution by the sum of all three contributions.