CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-3-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 2 hours **Maximum Marks: 40**

General Instructions

(i) This question paper contains **three** Sections – Section **A**, **B** and **C**.
(ii) Each Section is compulsory.
(iii) Section–**A** has **6** short answer type-**I** questions of **2** marks each.
(iv) Section–**B** has **4** short answer type-**II** questions of **3** marks each.
(v) Section–**C** has **4** long answer type questions of **4** marks each.
(vi) There is an internal choice in some questions.
(vii) Q. **14** is a case study based problem with **2** sub-parts of **2** marks each.

Section A (12 marks)

Question Nos. ***1*** *to* ***6*** *carry* ***2*** *marks each.*

1. Write the projection of the vector $(\vec{b}+\vec{c})$ on the vector $\vec{a}$, where $\vec{a}=2\hat{i}-2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-2\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. [2 marks]

Solution:

Concept:

- The scalar projection of a vector $\vec{v}$ on a nonzero vector $\vec{a}$ is $\vec{v}\cdot\vec{a}/|\vec{a}|$. It gives the signed component of $\vec{v}$ along the direction of $\vec{a}$.
- For vectors in Cartesian form, add corresponding components and calculate the dot product by multiplying corresponding components and adding the products.

Step 1: Add the two vectors First combine the coefficients of each unit vector:

$$\begin{aligned}\vec{b}+\vec{c}&=(1+2)\hat{i}+(2-1)\hat{j}+(-2+4)\hat{k}\\&=3\hat{i}+\hat{j}+2\hat{k}.\end{aligned}$$

This resultant is the vector whose projection is required.

Step 2: Evaluate the component along the given direction The magnitude of the direction vector is

$$|\vec{a}|=\sqrt{2^2+(-2)^2+1^2}=3.$$

Its dot product with the resultant is

$$(\vec{b}+\vec{c})\cdot\vec{a}=3(2)+1(-2)+2(1)=6.$$

Thus the scalar projection is $6/3=2$. The positive sign means the component points along $\vec{a}$, rather than opposite to it.

Step 3: Conclusion The projection of $\vec{b}+\vec{c}$ on $\vec{a}$ is $\boxed{2}$.

Quick Tip to Solve: Add the vectors first, then divide their dot product with the direction vector by its magnitude, not its squared magnitude.

2. Find the general solution of the differential equation: $\log\left(\dfrac{dy}{dx}\right)=ax+by$. [2 marks]

Solution:

Concept:

- Exponentiation removes the natural logarithm: $\log(dy/dx)=ax+by$ implies $dy/dx=e^{ax+by}>0$.
- A separable differential equation is solved by moving all factors involving y beside dy, and all factors involving x beside dx, before integrating.

Quick Tip to Solve: Exponentiate before separating variables; when integrating, check whether either parameter is zero before dividing by it.

Step 1: Separate the variables Using $e^{ax+by} = e^{ax}e^{by}$, the equation becomes

$$\frac{dy}{dx} = e^{ax}e^{by}, \qquad e^{-by}\,dy = e^{ax}\,dx.$$

The exponential factors never vanish, so this rearrangement loses no solution.

Step 2: Integrate and obtain the arbitrary constant For $a \neq 0$ and $b \neq 0$, integrate both sides:

$$-\frac{e^{-by}}{b} = \frac{e^{ax}}{a} + C_1.$$

Multiplying by ab and absorbing the constant gives

$$ae^{-by} + be^{ax} = C.$$

Differentiation gives $-abe^{-by}y' + abe^{ax} = 0$, which recovers $y' = e^{ax+by}$. Real solutions are taken on intervals where $(C - be^{ax})/a > 0$.

Step 3: Include zero values of the parameters If $a = 0$, $b \neq 0$, integration instead gives $e^{-by} + bx = C$, with $C - bx > 0$. If $b = 0$, $a \neq 0$, it gives $y = e^{ax}/a + C$. If both parameters vanish, $y' = 1$, so $y = x + C$.

Step 4: Conclusion For nonzero a, b, the general solution is $\boxed{ae^{-by} + be^{ax} = C}$. The zero-parameter cases are as stated above.

Quick Tip to Solve: Exponentiate before separating variables; when integrating, check whether either parameter is zero before dividing by it.

3. Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of spade cards. [2 marks]

Solution:

Concept:

- A probability distribution lists each possible value of a random variable and its probability. The probabilities must be nonnegative and sum to one.
- Replacement restores the original pack before the second draw. Consequently the draws are independent, and on each draw the probability of a spade is $13/52 = 1/4$, while that of a non-spade is $3/4$.

Step 1: Define the random variable and possible outcomes Let X denote the number of spade cards in the two draws. Its possible values are $0, 1, 2$. Write S for a spade and N for a non-spade. The four type sequences are NN, SN, NS, SS. Since the first card is replaced, probabilities of successive types are multiplied.

Step 2: Compute each probability No spade corresponds to NN, one spade to the two mutually

Quick Tip to Solve: For exactly one spade, include both orders: spade then non-spade, and non-spade then spade.

exclusive sequences SN and NS, and two spades to SS. Therefore

$$P(X=0)=\frac{3}{4}\frac{3}{4}=\frac{9}{16},$$
$$P(X=1)=\frac{1}{4}\frac{3}{4}+\frac{3}{4}\frac{1}{4}=\frac{6}{16},$$
$$P(X=2)=\frac{1}{4}\frac{1}{4}=\frac{1}{16}.$$

The total is $(9+6+1)/16=1$, confirming that all possibilities have been included.

Step 3: Conclusion The required probability distribution is

x	0	1	2
$P(X=x)$	9/16	6/16	1/16

Quick Tip to Solve: For exactly one spade, include both orders: spade then non-spade, and non-spade then spade.

4(a). A pair of dice is thrown and the sum of the numbers appearing on the dice is observed to be 7. Find the probability that the number 5 has appeared on at least one die. [2 marks]

Solution:

Concept:

- Conditional probability restricts the sample space to the event already known to have occurred. For events E and F, $P(E \mid F)=P(E\cap F)/P(F)$, provided $P(F)>0$.
- For two fair dice, the 36 ordered pairs are equally likely. The order distinguishes the first die from the second die, so $(2,5)$ and $(5,2)$ are separate outcomes.

Step 1: Restrict the outcomes to the observed sum Let F be the event that the sum is 7. Its possible ordered outcomes are

$$F=\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}.$$

Thus $P(F)=6/36$. Since all these pairs had equal probability before the observation, they remain equally likely within this restricted sample space.

Step 2: Count the outcomes containing a 5 Let E be the event that at least one die shows 5. Among the six outcomes in F, precisely $(2,5)$ and $(5,2)$ belong to E. Consequently,

$$P(E\cap F)=\frac{2}{36},\qquad P(E\mid F)=\frac{2/36}{6/36}=\frac{2}{6}.$$

The pair $(5,5)$ is excluded because its sum is 10, so it cannot occur under the given observation.

Step 3: Conclusion The required probability that at least one die shows 5, given that the sum is 7, is $\boxed{\frac{1}{3}}$.

Quick Tip to Solve: Once a sum is given, list only ordered pairs with that sum; use their count as the denominator, rather than all 36 pairs.

OR

4(b). The probability that A hits the target is $\frac{1}{3}$ and the probability that B hits it, is $\frac{2}{5}$. If both try to hit the target independently, find the probability that the target is hit. [2 marks]

Solution:

Concept:

- The target is hit when at least one person hits it. The complementary event is that both people miss, so $P(\text{target hit}) = 1 - P(\text{both miss})$.
- For independent trials, the probabilities multiply. Independence of the hit events also gives independence of the corresponding miss events.

Step 1: Calculate each probability of missing Let H_A and H_B denote the events that A and B hit the target, respectively. The given probabilities yield

$$P(H_A^c) = 1 - \frac{1}{3} = \frac{2}{3}, \qquad P(H_B^c) = 1 - \frac{2}{5} = \frac{3}{5}.$$

Here the superscript c denotes the complementary event. A miss by one person does not prevent the other person from hitting.

Step 2: Use the complement of both missing Since the two attempts are independent,

$$P(H_A^c \cap H_B^c) = \frac{2}{3} \times \frac{3}{5} = \frac{2}{5}.$$

Therefore the required probability is calculated as

$$P(H_A \cup H_B) = 1 - \frac{2}{5} = \frac{3}{5}.$$

As a check, the addition rule gives $\frac{1}{3} + \frac{2}{5} - \frac{1}{3} \times \frac{2}{5} = \frac{9}{15}$. The intersection must be subtracted because simultaneous hits are otherwise counted twice.

Step 3: Conclusion The probability that the target is hit is $\boxed{\frac{3}{5}}$.

Quick Tip to Solve: For independent attempts and an "at least one hit" event, subtract the product of the miss probabilities from 1.

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5. If the distance of the point $(1, 1, 1)$ from the plane $x - y + z + \lambda = 0$ is $\frac{5}{\sqrt{3}}$, find the value(s) of λ. [2 marks]

Solution:

Concept:

- The perpendicular distance of (x_1, y_1, z_1) from the plane $ax + by + cz + d = 0$ is $D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$.

Quick Tip to Solve: Cancel the common square-root denominator first, then retain both signs when solving the absolute-value equation for the plane constant.

- Distance is non-negative, so the numerator uses an absolute value. An equation $|u| = k$, with $k > 0$, has the two possibilities $u = k$ and $u = -k$.

Step 1: Substitute the point and plane coefficients For the plane $x - y + z + \lambda = 0$, the coefficients are $a = 1$, $b = -1$, $c = 1$, and $d = \lambda$. Substituting the point $(1, 1, 1)$ into the distance formula gives

$$D = \frac{|1 - 1 + 1 + \lambda|}{\sqrt{1^2 + (-1)^2 + 1^2}} = \frac{|1 + \lambda|}{\sqrt{3}}.$$

The denominator is the length of the plane's normal vector $(1, -1, 1)$.

Step 2: Solve the absolute-value equation Equating this expression to the given distance,

$$\frac{|1 + \lambda|}{\sqrt{3}} = \frac{5}{\sqrt{3}} \quad \Longrightarrow \quad |1 + \lambda| = 5.$$

Hence $1 + \lambda = 5$ or $1 + \lambda = -5$, giving $\lambda = 4$ or $\lambda = -6$, respectively. For these values the numerator becomes $|5|$ and $|-5|$, so both yield the required distance $5/\sqrt{3}$.

Step 3: Conclusion The possible values are $\boxed{\lambda = 4 \text{ or } \lambda = -6}$.

Quick Tip to Solve: Cancel the common square-root denominator first, then retain both signs when solving the absolute-value equation for the plane constant.

6. Find: $\displaystyle\int \frac{dx}{x^2 - 6x + 13}$. [2 marks]

Solution:

Concept:

- Complete the square in the quadratic denominator to reduce it to a standard inverse-tangent integral.
- For a positive constant a, $\displaystyle\int \frac{du}{u^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$, where C is an arbitrary constant.

Step 1: Complete the square Write the denominator as

$$x^2 - 6x + 13 = x^2 - 6x + 9 + 4 = (x - 3)^2 + 2^2.$$

Thus it is positive for every real x, and the integral has the standard form with the shifted variable $x - 3$.

Step 2: Substitute and integrate Put $u = x - 3$. Then $du = dx$, so

$$\begin{aligned}\int \frac{dx}{x^2 - 6x + 13} &= \int \frac{du}{u^2 + 2^2} \\ &= \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C.\end{aligned}$$

Replacing u by $x - 3$ gives the required antiderivative. The factor $1/2$ is essential: differentiating the inverse tangent introduces the additional factor $1/2$ from its inner argument, yielding exactly

Quick Tip to Solve: Complete the square first; the square root of the remaining constant supplies both the argument denominator and the outside factor.

$1/((x-3)^2+4)$.

Step 3: Conclusion $\int \frac{dx}{x^2-6x+13} = \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+C.$

Quick Tip to Solve: Complete the square first; the square root of the remaining constant supplies both the argument denominator and the outside factor.

Section B (12 marks)

Question Nos. ***7*** *to* ***10*** *carry* ***3*** *marks each.*

7(a). If $\vec{a},\vec{b},\vec{c}$ are three vectors such that $\vec{a}\cdot\vec{b}=\vec{a}\cdot\vec{c}$ and $\vec{a}\times\vec{b}=\vec{a}\times\vec{c}$, $\vec{a}\neq\vec{0}$, then show that $\vec{b}=\vec{c}$.

[3 marks]

Solution:

Concept:

- Dot and cross products distribute over subtraction: $\vec{a}\cdot(\vec{b}-\vec{c})=\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}$ and similarly for the cross product.
- The identity $|\vec{a}\times\vec{d}|^2+(\vec{a}\cdot\vec{d})^2=|\vec{a}|^2|\vec{d}|^2$ relates the two products to the magnitudes of the vectors.

Step 1: Introduce the difference vector Let $\vec{d}=\vec{b}-\vec{c}$. Subtracting the given dot-product equality gives

$$\vec{a}\cdot\vec{d}=\vec{a}\cdot\vec{b}-\vec{a}\cdot\vec{c}=0.$$

Similarly, subtracting the given cross-product equality gives

$$\vec{a}\times\vec{d}=\vec{a}\times\vec{b}-\vec{a}\times\vec{c}=\vec{0}.$$

Both conditions concern the same difference vector, so they must hold simultaneously.

Step 2: Apply the magnitude identity Using the two vanishing products in the standard identity, we obtain

$$\begin{aligned}|\vec{a}|^2|\vec{d}|^2 &= |\vec{a}\times\vec{d}|^2+(\vec{a}\cdot\vec{d})^2\\ &= 0+0=0.\end{aligned}$$

Since $\vec{a}\neq\vec{0}$, its squared magnitude is strictly positive. Hence $|\vec{d}|^2=0$, which implies $\vec{d}=\vec{0}$. The nonzero hypothesis is necessary here: if $\vec{a}$ were zero, both original product equalities would hold for arbitrary $\vec{b}$ and $\vec{c}$.

Step 3: Interpret the difference The equation $\vec{d}=\vec{0}$ is precisely $\vec{b}-\vec{c}=\vec{0}$. Adding $\vec{c}$ to both sides establishes the required equality. Geometrically, a nonzero difference could not be both parallel to $\vec{a}$, as the zero cross product requires, and perpendicular to it, as the zero dot product requires.

Step 4: Conclusion Therefore, $\vec{b}=\vec{c}$, as required.

Quick Tip to Solve: Set the difference equal to a single vector and combine its zero dot and cross products using the squared-magnitude identity.

OR

7(b). **If $|\vec{a}| = 3$, $|\vec{b}| = 5$, $|\vec{c}| = 4$ and $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then find the value of $(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a})$.** [3 marks]

Solution:

Concept:

- The dot product of a vector with itself is the square of its magnitude: $\vec{u} \cdot \vec{u} = |\vec{u}|^2$.
- The dot product distributes over vector addition and is commutative: $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$. Thus each distinct pair occurs twice when a vector sum is dotted with itself.

Step 1: Take the dot product of the given sum with itself Since $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, its dot product with itself is zero. Expanding both factors gives

$$\begin{aligned} 0 &= (\vec{a} + \vec{b} + \vec{c}) \cdot (\vec{a} + \vec{b} + \vec{c}) \\ &= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 \\ &\quad + 2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}). \end{aligned}$$

The factor 2 appears because, for example, $\vec{a} \cdot \vec{b}$ and $\vec{b} \cdot \vec{a}$ are equal. The same applies to the other two pairs.

Step 2: Substitute the magnitudes and isolate the required sum Let $S = \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}$. The given magnitudes determine the three self-products, so the expanded identity becomes

$$\begin{aligned} 0 &= 3^2 + 5^2 + 4^2 + 2S \\ &= 9 + 25 + 16 + 2S \\ &= 50 + 2S. \end{aligned}$$

Hence $2S = -50$. This method finds the combined dot products directly; the individual angles between the vectors are unnecessary.

Step 3: Conclusion The required value is $\boxed{\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a} = -25}$.

Quick Tip to Solve: When three vectors add to zero, dot the entire sum with itself: the sum of their pairwise dot products is minus half the sum of their squared magnitudes.

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8. Evaluate: $\int_{-1}^{2} |x^3 - x| \, dx$. [3 marks]

Solution:

Concept:

- Absolute value makes an expression non-negative: $|f(x)| = f(x)$ where $f(x) \geq 0$, and $|f(x)| = -f(x)$ where $f(x) < 0$.
- Split a definite integral at points where the expression inside the absolute value changes sign. Here $x^3 - x = x(x-1)(x+1)$ has zeros $-1, 0, 1$.

Quick Tip to Solve: Factor the cubic before removing the absolute value. Split at 0 and 1, and reverse the sign only on $(0, 1)$.

Step 1: Determine the signs on the integration interval Write $f(x) = x(x-1)(x+1)$. On $(-1, 0)$, the first two factors are negative and the last is positive, so $f(x) > 0$. On $(0, 1)$, only $x - 1$ is negative, so $f(x) < 0$. On $(1, 2)$, all three factors are positive, so $f(x) > 0$. At the boundary zeros the absolute value is zero. Therefore

$$I = \int_{-1}^{0} (x^3 - x)\,dx + \int_{0}^{1} (x - x^3)\,dx + \int_{1}^{2} (x^3 - x)\,dx.$$

Keeping the middle interval separate avoids cancellation of positive areas by the negative values of the original cubic.

Step 2: Integrate each polynomial on its interval An antiderivative of $x^3 - x$ is $F(x) = x^4/4 - x^2/2$; that of $x - x^3$ is $-F(x)$. Evaluate the endpoint differences:

$$\int_{-1}^{0} (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{-1}^{0} = 0 - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{1}{4},$$

$$\int_{0}^{1} (x - x^3)\,dx = \left[\frac{x^2}{2} - \frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4},$$

$$\int_{1}^{2} (x^3 - x)\,dx = \left[\frac{x^4}{4} - \frac{x^2}{2}\right]_{1}^{2} = (4 - 2) - \left(\frac{1}{4} - \frac{1}{2}\right) = \frac{9}{4}.$$

Each contribution is non-negative, as expected for an integral of an absolute value.

Step 3: Conclusion Adding the three contributions gives $\boxed{\int_{-1}^{2} |x^3 - x|\,dx = \frac{1}{4} + \frac{1}{4} + \frac{9}{4} = \frac{11}{4}}$.

Quick Tip to Solve: Factor the cubic before removing the absolute value. Split at 0 and 1, and reverse the sign only on $(0, 1)$.

9. Show that the lines: $\frac{1-x}{2} = \frac{y-3}{4} = \frac{z}{-1}$ and $\frac{x-4}{3} = \frac{2y-2}{-4} = z - 1$ are coplanar. [3 marks]

Solution:

Concept:

- A line written as $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ passes through (x_0, y_0, z_0) and has direction vector (a, b, c).
- Two lines through points A, B, with direction vectors $\mathbf{u}, \mathbf{v}$, are coplanar if $\overrightarrow{AB} \cdot (\mathbf{u} \times \mathbf{v}) = 0$. This scalar triple product tests whether their three vectors lie in one plane.

Step 1: Write both lines in standard form Reverse the numerator of the first ratio, changing the sign of its denominator. Also divide the numerator and denominator involving y in the second line by 2.

Quick Tip to Solve: First simplify a numerator such as $2y - 2$ before reading direction ratios. Here $\overrightarrow{AB}$ equals the second direction vector, so the coplanarity determinant has repeated rows.

Thus the given lines become

$$\frac{x-1}{-2}=\frac{y-3}{4}=\frac{z}{-1}, \qquad \frac{x-4}{3}=\frac{y-1}{-2}=\frac{z-1}{1}.$$

We can therefore choose $A=(1,3,0)$, $B=(4,1,1)$, with direction vectors $\mathbf{u}=(-2,4,-1)$ and $\mathbf{v}=(3,-2,1)$. In particular, the direction ratio corresponding to y in the second line is -2, since its printed numerator is $2y-2$.

Step 2: Evaluate the scalar triple product The vector joining the chosen points is $\overrightarrow{AB}=(3,-2,1)$. Substituting the three vectors into the coplanarity condition gives

$$\overrightarrow{AB}\cdot(\mathbf{u}\times\mathbf{v})=\begin{vmatrix}3 & -2 & 1\\ -2 & 4 & -1\\ 3 & -2 & 1\end{vmatrix}$$
$$=0.$$

The first and third rows are identical, so the determinant vanishes. Moreover, $\mathbf{u}\times\mathbf{v}=(2,-1,-8)\neq\mathbf{0}$, so the lines are not parallel. As a direct check, the second line at parameter -1 gives $(1,3,0)=A$, a point on the first line. The lines therefore intersect and determine a plane.

Step 3: Conclusion The scalar triple product is zero, hence the two given lines are coplanar. Their common plane can also be written as $2(x-1)-(y-3)-8z=0$, or $2x-y-8z+1=0$.

Quick Tip to Solve: First simplify a numerator such as $2y-2$ before reading direction ratios. Here $\overrightarrow{AB}$ equals the second direction vector, so the coplanarity determinant has repeated rows.

10(a). Find the particular solution of the differential equation $x\,\dfrac{dy}{dx}-y=x^2\,e^x$, given $y(1)=0$.

[3 marks]

Solution:

Concept:

- For a linear differential equation $\frac{dy}{dx}+P(x)y=Q(x)$, the integrating factor is $\text{IF}=e^{\int P(x)\,dx}$. Multiplication by this factor makes the left side the derivative of $y\,\text{IF}$.
- An initial condition fixes the arbitrary constant in the general solution. Work on an interval containing the initial point where the standard form is defined; here we use $x>0$, since $y(1)=0$.

Step 1: Put the equation in linear form The given differential equation is $x\frac{dy}{dx}-y=x^2e^x$. Since the specified initial point is $x=1$, divide by x on the positive interval containing this point:

$$\frac{dy}{dx}-\frac{1}{x}y=xe^x.$$

Comparing with the linear form gives $P(x)=-1/x$ and $Q(x)=xe^x$. The integrating factor is therefore

$$\text{IF}=e^{\int -\frac{1}{x}\,dx}=e^{-\log x}=\frac{1}{x}, \qquad x>0.$$

Quick Tip to Solve: Recognise $xy'-y=x^2(y/x)'$: dividing the printed equation by x^2 immediately gives $(y/x)'=e^x$, which leads to the same integration.

A nonzero constant multiplying the integrating factor would cancel from the solution, so it is enough to use $1/x$.

Step 2: Integrate after multiplying by the factor Multiplying the standard equation by $1/x$ gives

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = e^x.$$

By the quotient rule the left side is $\frac{d}{dx}(y/x)$. Integrating both sides therefore yields

$$\frac{d}{dx}\left(\frac{y}{x}\right) = e^x, \qquad \frac{y}{x} = e^x + C.$$

Thus the family of solutions on this interval is $y = x(e^x + C)$.

Step 3: Apply the given initial condition Substitute $x = 1$ and $y = 0$ into the integrated relation:

$$0 = e + C, \qquad C = -e.$$

To verify the selected solution, differentiate $y = x(e^x - e)$:

$$y' = e^x - e + xe^x,$$
$$xy' - y = x(e^x - e + xe^x) - x(e^x - e) = x^2e^x.$$

The substitution reproduces the differential equation, and at $x = 1$ the value of y is zero as required.

Step 4: Conclusion The required particular solution is $\boxed{y = x(e^x - e)}$, on the interval $x > 0$ containing the initial point.

Quick Tip to Solve: Recognise $xy' - y = x^2(y/x)'$: dividing the printed equation by x^2 immediately gives $(y/x)' = e^x$, which leads to the same integration.

OR

10(b). Find the general solution of the differential equation $x\dfrac{dy}{dx} = y(\log y - \log x + 1)$. [3 marks]

Solution:

Concept:

- A homogeneous first-order differential equation depends on the ratio y/x. Put $y = vx$, so that $dy/dx = v + x\,dv/dx$.
- For real logarithms the printed equation requires $x > 0$ and $y > 0$. Hence $v = y/x > 0$ and $\log y - \log x = \log v$.
- The substitution $u = \log v$ gives $du/dx = (1/v)\,dv/dx$, reducing the equation to a linear equation whose constant solution must also be retained.

Step 1: Substitute the ratio of the dependent and independent variables Dividing the given equation

Quick Tip to Solve: After substituting $y = vx$, use $u = \log v$ and remember that $u = 0$ must remain in the final family.

by x, its right-hand side becomes

$$\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right).$$

Write $y = vx$. Differentiating by the product rule and substituting gives

$$x\left(v + x\frac{dv}{dx}\right) = vx(\log v + 1).$$

The terms xv on the two sides cancel. Since $x > 0$, we obtain

$$x\frac{dv}{dx} = v\log v.$$

Step 2: Integrate after a logarithmic substitution Since $v > 0$, set $u = \log v$. Then the preceding equation becomes

$$x\frac{du}{dx} = u.$$

For $u \neq 0$, separation and integration give

$$\int\frac{du}{u} = \int\frac{dx}{x}, \qquad \log|u| = \log x + C_1.$$

Thus $u = Cx$, where C is a real constant incorporating the sign of u and the constant of integration. The solution $u = 0$, which would be lost on dividing by u, is included by taking $C = 0$.

Step 3: Return to the original variables and check Replacing u and v gives $\log(y/x) = Cx$. Equivalently, $y = xe^{Cx}$. For this family,

$$\frac{dy}{dx} = e^{Cx}(1 + Cx), \qquad \log y - \log x = Cx.$$

Therefore both sides of the printed differential equation equal $xe^{Cx}(1 + Cx)$. The family also satisfies the logarithmic domain because $x > 0$ implies $y > 0$.

Step 4: Conclusion The general solution is $\boxed{\log(y/x) = Cx}$, or equivalently $\boxed{y = xe^{Cx}}$, for $x > 0$, with $C \in \mathbb{R}$. In particular, $C = 0$ gives $y = x$.

Quick Tip to Solve: After substituting $y = vx$, use $u = \log v$ and remember that $u = 0$ must remain in the final family.

Section C (16 marks)

Question Nos. ***11*** *to* ***14*** *carry* ***4*** *marks each.*

11(a). Find: $\int \frac{x^2}{(x^2+1)(3x^2+4)}\,dx$. [4 marks]

Solution:

Concept:

- Partial fractions split a rational function with distinct quadratic factors into simpler fractions whose antiderivatives are standard inverse tangent functions.
- For positive constants a, b, $\int \frac{dx}{a^2+b^2x^2} = \frac{1}{ab}\tan^{-1}\left(\frac{bx}{a}\right) + C$, where C is an arbitrary real constant.

Step 1: Decompose the rational function Both quadratic factors are even in x, so take constant numerators and write

$$\frac{x^2}{(x^2+1)(3x^2+4)} = \frac{A}{x^2+1} + \frac{B}{3x^2+4}.$$

Multiplying by the denominator gives the polynomial identity

$$x^2 = A(3x^2+4) + B(x^2+1) = (3A+B)x^2 + (4A+B).$$

Equating the coefficient of x^2 and the constant term yields $3A+B=1$ and $4A+B=0$. Subtracting the first equation from the second gives $A=-1$, and then $B=4$. Thus the numerator reconstructed from these values is $-(3x^2+4)+4(x^2+1)=x^2$, verifying the decomposition.

Step 2: Integrate the simpler fractions Using the decomposition, the required integral becomes

$$I = -\int \frac{dx}{1+x^2} + 4\int \frac{dx}{4+3x^2}.$$

The first antiderivative is $-\tan^{-1}x$. For the second integral, let $u = \frac{\sqrt{3}x}{2}$, so $dx = \frac{2}{\sqrt{3}}du$ and $4+3x^2 = 4(1+u^2)$. Hence

$$4\int \frac{dx}{4+3x^2} = \frac{2}{\sqrt{3}}\int \frac{du}{1+u^2} = \frac{2}{\sqrt{3}}\tan^{-1}u.$$

Replace u by its expression in x and combine the two antiderivatives, using one constant of integration.

Step 3: Check by differentiation Differentiating the combined expression gives

$$-\frac{1}{1+x^2} + \frac{1}{1+3x^2/4} = -\frac{1}{1+x^2} + \frac{4}{4+3x^2} = \frac{x^2}{(x^2+1)(3x^2+4)}.$$

This is exactly the given integrand. Since both denominator factors are positive for every real x, the antiderivative is valid on the entire real line.

Quick Tip to Solve: Equate the constant terms first: $4A+B=0$. Then compare the x^2 coefficients to obtain $A=-1$ immediately; remember the scale factor when integrating $1/(4+3x^2)$.

Step 4: Conclusion

$$\boxed{\int \frac{x^2}{(x^2+1)(3x^2+4)}\,dx = -\tan^{-1}x + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right) + C.}$$

Quick Tip to Solve: Equate the constant terms first: $4A + B = 0$. Then compare the x^2 coefficients to obtain $A = -1$ immediately; remember the scale factor when integrating $1/(4+3x^2)$.

OR

11(b). Evaluate: $\int_{-2}^{1} \sqrt{5-4x-x^2}\,dx$. [4 marks]

Solution:

Concept:

- Completing the square converts a quadratic radicand to the standard form a^2-u^2: here $5-4x-x^2 = 9-(x+2)^2$.
- For $-a \le u \le a$, the substitution $u = a\sin\theta$, with $-\pi/2 \le \theta \le \pi/2$, makes $\sqrt{a^2-u^2} = a\cos\theta$. The chosen interval ensures that the cosine is non-negative.
- Use $\cos^2\theta = (1+\cos 2\theta)/2$ to integrate the square of the cosine. In a definite integral, transform both limits along with the variable.

Step 1: Complete the square and shift the variable Let the required integral be I. Rewriting its radicand gives

$$5-4x-x^2 = 9-(x^2+4x+4) = 9-(x+2)^2.$$

Put $u = x+2$, so $du = dx$. When $x = -2$, $u = 0$, and when $x = 1$, $u = 3$. Thus

$$I = \int_0^3 \sqrt{9-u^2}\,du.$$

On $0 \le u \le 3$, the radicand is non-negative.

Step 2: Use a trigonometric substitution with the transformed limits Set $u = 3\sin\theta$ and choose $0 \le \theta \le \pi/2$. Then

$$du = 3\cos\theta\,d\theta, \qquad \sqrt{9-u^2} = 3\sqrt{1-\sin^2\theta} = 3\cos\theta.$$

There is no sign ambiguity because $\cos\theta \ge 0$ on the chosen interval. The lower limit $u = 0$ corresponds to $\theta = 0$, and the upper limit $u = 3$ corresponds to $\theta = \pi/2$. Therefore

$$I = 9\int_0^{\pi/2} \cos^2\theta\,d\theta.$$

Quick Tip to Solve: Complete the square first: the shift $u = x+2$ changes the limits to 0 and 3, immediately revealing the quarter-circle integral of radius 3.

Step 3: Apply the double-angle identity and evaluate Using the identity for $\cos^2\theta$, we obtain

$$\begin{aligned} I &= \frac{9}{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta \\ &= \frac{9}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2} \\ &= \frac{9}{2}\left(\frac{\pi}{2}+\frac{\sin\pi}{2}-0-\frac{\sin 0}{2}\right). \end{aligned}$$

The sine terms vanish at both endpoints. This also agrees with the area of a quarter-circle of radius 3.

Step 4: Conclusion Hence, $\int_{-2}^{1}\sqrt{5-4x-x^2}\,dx = \boxed{\frac{9\pi}{4}}$.

Quick Tip to Solve: Complete the square first: the shift $u = x+2$ changes the limits to 0 and 3, immediately revealing the quarter-circle integral of radius 3.

12. Using integration, find the area of the region bounded by the curves $x^2+y^2=4$, $x=\sqrt{3}\,y$ and x-axis lying in the first quadrant. [4 marks]

Solution:

Concept:

- The area between a right-hand boundary $x=f(y)$ and a left-hand boundary $x=g(y)$, from $y=a$ to $y=b$, is $A=\int_a^b[f(y)-g(y)]\,dy$.
- In the first quadrant, the circle $x^2+y^2=4$ has right-hand branch $x=\sqrt{4-y^2}$. The line $x=\sqrt{3}\,y$, together with the positive x-axis, bounds the required region below the line and inside the circle.
- The standard integral is $\int\sqrt{a^2-y^2}\,dy=\frac{y}{2}\sqrt{a^2-y^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{y}{a}\right)+C$, where $|y|\le a$.

Step 1: Find the intersection and the integration limits Substitute $x=\sqrt{3}\,y$ into the equation of the circle:

$$3y^2+y^2=4 \quad\Rightarrow\quad y^2=1.$$

Since the region lies in the first quadrant, take $y=1$, giving $x=\sqrt{3}$. Thus the line and circle meet at $(\sqrt{3},1)$. The circle meets the positive x-axis at $(2,0)$, while the line meets that axis at the origin. Hence the required region extends from $y=0$ to $y=1$.

Step 2: Express the area using horizontal strips For any fixed height y between 0 and 1, the horizontal strip begins on the line at $x=\sqrt{3}\,y$ and ends on the circle at $x=\sqrt{4-y^2}$. Its width is therefore the circle's x-coordinate minus the line's x-coordinate. This width is nonnegative because $4-y^2\ge 3y^2$ throughout this interval. Thus

$$A=\int_0^1\left(\sqrt{4-y^2}-\sqrt{3}\,y\right)dy.$$

Quick Tip to Solve: Integrate with respect to y: one horizontal strip runs from $x=\sqrt{3}\,y$ to $x=\sqrt{4-y^2}$ for the entire interval $0\le y\le 1$, so no split integral is needed.

Step 3: Evaluate the definite integral Applying the standard integral with $a = 2$, we obtain

$$A = \left[\frac{y}{2}\sqrt{4-y^2} + 2\sin^{-1}\left(\frac{y}{2}\right) - \frac{\sqrt{3}}{2}y^2\right]_0^1.$$

At the upper limit, $\sqrt{4-1} = \sqrt{3}$ and $\sin^{-1}(1/2) = \pi/6$; at the lower limit, every term is zero. Consequently,

$$A = \frac{\sqrt{3}}{2} + 2\left(\frac{\pi}{6}\right) - \frac{\sqrt{3}}{2} - 0.$$

The two algebraic terms cancel. As a check on the region, the line makes an angle $\pi/6$ with the positive x-axis, so the boundary encloses a sector of radius 2; its sector-area formula agrees with the integral.

Step 4: Conclusion The area of the required region is $\boxed{\frac{\pi}{3} \text{ square units}}$.

Quick Tip to Solve: Integrate with respect to y: one horizontal strip runs from $x = \sqrt{3}\,y$ to $x = \sqrt{4-y^2}$ for the entire interval $0 \le y \le 1$, so no split integral is needed.

13. Find the distance of the point $(1, -2, 9)$ from the point of intersection of the line $\vec{r} = 4\hat{i} + 2\hat{j} + 7\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 10$. [4 marks]

Solution:

Concept:

- A point on the line $\vec{r} = \vec{a} + \lambda\vec{b}$ has position vector determined by the same parameter λ in all three coordinates. Here $\vec{a} = (4, 2, 7)$ and the direction vector is $\vec{b} = (3, 4, 2)$.
- In the plane equation $\vec{r} \cdot \vec{n} = d$, $\vec{n}$ is the normal vector. The intersection point must satisfy both the line equation and the plane equation.
- The distance between $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Step 1: Express the point on the line in coordinates Let Q be the point where the given line meets the plane. Comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in the line equation gives

$$Q = (4 + 3\lambda,\ 2 + 4\lambda,\ 7 + 2\lambda).$$

The normal vector of the plane is $\hat{i} - \hat{j} + \hat{k}$, so its Cartesian equation is $x - y + z = 10$. We must substitute all three coordinates of Q into this equation; the required distance is from $P = (1, -2, 9)$ to this particular point, rather than the perpendicular distance from P to the plane.

Step 2: Determine the intersection parameter Since Q lies in the plane,

$$\begin{aligned}(4 + 3\lambda) - (2 + 4\lambda) + (7 + 2\lambda) &= 10,\\ 9 + (3 - 4 + 2)\lambda &= 10,\\ 9 + \lambda &= 10,\\ \lambda &= 1.\end{aligned}$$

Quick Tip to Solve: Substitute the line coordinates into $x - y + z = 10$ first; once the intersection is known, use the point-to-point distance formula, keeping $6 - (-2) = 8$.

There is a unique intersection because the direction vector has nonzero dot product with the plane normal: $(3,4,2)\cdot(1,-1,1)=1$.

Step 3: Obtain and check the intersection point Substituting $\lambda=1$ into the line coordinates gives

$$Q=(4+3,\ 2+4,\ 7+2)=(7,6,9).$$

This point satisfies the plane equation, since $7-6+9=10$. Its displacement from the line's fixed point is $(7,6,9)-(4,2,7)=(3,4,2)$, so it also satisfies the line equation with the same parameter in each coordinate.

Step 4: Calculate the distance from the given point With $P=(1,-2,9)$ and $Q=(7,6,9)$, the displacement vector is

$$\overrightarrow{PQ}=(7-1,\ 6-(-2),\ 9-9)=(6,8,0).$$

Therefore,

$$PQ=\sqrt{6^2+8^2+0^2}=\sqrt{36+64}=\sqrt{100}.$$

In particular, the equal third coordinates make the third squared difference zero; the negative second coordinate of P makes the second difference 8.

Step 5: Conclusion The point of intersection is $(7,6,9)$, and its distance from $(1,-2,9)$ is $\boxed{10\text{ units}}$.

Quick Tip to Solve: Substitute the line coordinates into $x-y+z=10$ first; once the intersection is known, use the point-to-point distance formula, keeping $6-(-2)=8$.

14. Case Study Problem:

A shopkeeper sells three types of flower seeds A_1, A_2, A_3. They are sold in the form of a mixture, where the proportions of these seeds are $4:4:2$, respectively. The germination rates of the three types of seeds are 45%, 60% and 35% respectively.

Based on the above information, answer the following questions: [4 marks]

14(a). Calculate the probability that a randomly chosen seed will germinate. [2 marks]

Solution:

Concept:

- Let A_i denote the event that the selected seed is of type A_i, for $i=1,2,3$, and let G denote germination. The three seed types form mutually exclusive and exhaustive events.
- The law of total probability gives $P(G)=\sum_{i=1}^{3}P(A_i)P(G\mid A_i)$. The proportions determine the probabilities of selecting each type, while germination rates are conditional probabilities.

Step 1: Convert proportions and germination rates to probabilities The total number of proportion units is $4+4+2=10$. Therefore,

$$P(A_1)=\frac{4}{10},\qquad P(A_2)=\frac{4}{10},\qquad P(A_3)=\frac{2}{10}.$$

Quick Tip to Solve: Normalize the ratio first, then multiply each seed-type probability by its own germination rate and add the three products.

The given germination rates imply

$$P(G \mid A_1) = \frac{45}{100}, \quad P(G \mid A_2) = \frac{60}{100}, \quad P(G \mid A_3) = \frac{35}{100}.$$

Step 2: Add the weighted germination probabilities A seed can germinate after being selected from any one of the three types. Since these possibilities are disjoint, their joint probabilities add:

$$\begin{aligned} P(G) &= \frac{4}{10}\frac{45}{100} + \frac{4}{10}\frac{60}{100} + \frac{2}{10}\frac{35}{100} \\ &= \frac{180 + 240 + 70}{1000} = \frac{490}{1000} = \frac{49}{100}. \end{aligned}$$

Each rate must be weighted by its share of the mixture; an unweighted average would incorrectly treat all three types as equally likely.

Step 3: Conclusion The probability that a randomly chosen seed germinates is $\boxed{\frac{49}{100} = 0.49}$.

Quick Tip to Solve: Normalize the ratio first, then multiply each seed-type probability by its own germination rate and add the three products.

14(b). Calculate the probability that the seed is of type A_2, given that a randomly chosen seed germinates. [2 marks]

Solution:

Concept:

- Let A_i denote selection of a type A_i seed and G denote germination. Bayes' theorem gives $P(A_2 \mid G) = P(A_2)P(G \mid A_2)/P(G)$, provided $P(G) > 0$.
- The denominator is the total germination probability, obtained by adding the contributions from all three mutually exclusive seed types. Conditioning on germination changes the sample space to germinated seeds.

Step 1: Find the probability of germination The mixture proportions $4 : 4 : 2$ give $P(A_1) = 4/10$, $P(A_2) = 4/10$ and $P(A_3) = 2/10$. Using the germination rates,

$$\begin{aligned} P(G) &= \frac{4}{10}\frac{45}{100} + \frac{4}{10}\frac{60}{100} + \frac{2}{10}\frac{35}{100} \\ &= \frac{180 + 240 + 70}{1000} = \frac{49}{100}. \end{aligned}$$

Step 2: Apply Bayes' theorem for type A2 The probability that a seed is both of type A_2 and germinates is

$$P(A_2 \cap G) = P(A_2)P(G \mid A_2) = \frac{4}{10}\frac{60}{100} = \frac{24}{100}.$$

Quick Tip to Solve: For a reverse conditional probability, put the requested type's weighted germination contribution over the total germination probability.

Divide this contribution by the probability of the given event:

$$P(A_2 \mid G) = \frac{P(A_2 \cap G)}{P(G)} = \frac{24/100}{49/100} = \frac{24}{49}.$$

The given 60% rate is $P(G \mid A_2)$; it is not the required reverse conditional probability $P(A_2 \mid G)$.

Step 3: Conclusion Given that the seed germinates, the probability that it is of type A_2 is $\boxed{\frac{24}{49}}$.

Quick Tip to Solve: For a reverse conditional probability, put the requested type's weighted germination contribution over the total germination probability.