

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-3-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) This question paper contains **three** Sections – Section **A**, **B** and **C**.
- (ii) Each Section is compulsory.
- (iii) Section–**A** has **6** short answer type-**I** questions of **2** marks each.
- (iv) Section–**B** has **4** short answer type-**II** questions of **3** marks each.
- (v) Section–**C** has **4** long answer type questions of **4** marks each.
- (vi) There is an internal choice in some questions.
- (vii) Q. **14** is a case study based problem with **2** sub-parts of **2** marks each.

Section A (12 marks)

Question Nos. 1 to 6 carry 2 marks each.

1. If the distance of the point $(1, 1, 1)$ from the plane $x - y + z + \lambda = 0$ is $\frac{5}{\sqrt{3}}$, find the value(s) of λ . [2 marks]

Solution:

Concept:

- The perpendicular distance of a point (x_0, y_0, z_0) from the plane $Ax + By + Cz + D = 0$ is $d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$. The absolute value makes this distance non-negative.
- For the given plane, the normal vector is $(1, -1, 1)$, whose magnitude is $\sqrt{3}$. An absolute-value equation must be solved for both signs.

Step 1: Substitute the point in the distance formula Here $A = 1$, $B = -1$, $C = 1$ and $D = \lambda$. At the point $(1, 1, 1)$, the expression in the numerator is $1 - 1 + 1 + \lambda = 1 + \lambda$. Therefore the given distance gives

$$\frac{|1 + \lambda|}{\sqrt{1^2 + (-1)^2 + 1^2}} = \frac{5}{\sqrt{3}}.$$

Multiplying by the positive number $\sqrt{3}$, we obtain $|1 + \lambda| = 5$.

Step 2: Solve both possible signs and check The absolute-value equation has two cases:

$$\begin{aligned} 1 + \lambda = 5 &\Rightarrow \lambda = 4, \\ 1 + \lambda = -5 &\Rightarrow \lambda = -6. \end{aligned}$$

For these two values, the numerator of the distance is respectively $|5|$ and $|-5|$, so both reproduce the required distance $5/\sqrt{3}$.

Step 3: Conclusion The required values are $\lambda = 4$ or $\lambda = -6$.

Quick Tip to Solve: Keep the absolute value until the distance equation is formed; then solve both the positive and negative cases.

2. Write the projection of the vector $(\vec{b} + \vec{c})$ on the vector $\vec{a}$, where $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. [2 marks]

Solution:

Concept:

- The scalar projection of a vector $\vec{u}$ on a non-zero vector $\vec{a}$ is $\frac{\vec{u} \cdot \vec{a}}{|\vec{a}|}$. It is the component measured along the direction of $\vec{a}$.
- Add vectors component by component, and evaluate their dot product by multiplying corresponding components and adding. The magnitude is the square root of the sum of squared components.

Quick Tip to Solve: For a scalar projection, divide the dot product by the magnitude of the vector on which you project, not by its square.

Step 1: Find the vector being projected Adding the components of the two given vectors yields

$$\begin{aligned}\vec{b} + \vec{c} &= (1 + 2)\hat{i} + (2 - 1)\hat{j} + (-2 + 4)\hat{k} \\ &= 3\hat{i} + \hat{j} + 2\hat{k}.\end{aligned}$$

This is the vector to project; $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$ supplies the direction of projection.

Step 2: Calculate the dot product and the magnitude The dot product is

$$(\vec{b} + \vec{c}) \cdot \vec{a} = 3(2) + 1(-2) + 2(1) = 6.$$

The magnitude of the direction vector is

$$|\vec{a}| = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3.$$

Thus the scalar projection is $6/3 = 2$. Its positive sign means that the component is along the direction of $\vec{a}$, rather than opposite to it.

Step 3: Conclusion The projection of $\vec{b} + \vec{c}$ on $\vec{a}$ is $\boxed{2}$.

Quick Tip to Solve: For a scalar projection, divide the dot product by the magnitude of the vector on which you project, not by its square.

3. Find the general solution of the differential equation: $\frac{dy}{dx} = \frac{3e^{2x} + 3e^{4x}}{e^x + e^{-x}}$. [2 marks]

Solution:

Concept:

- When $\frac{dy}{dx} = f(x)$, integrating with respect to x gives the general solution $y = \int f(x) dx + C$, where C is an arbitrary real constant.
- Use $e^x + e^{-x} = e^{-x}(1 + e^{2x})$ to expose a common factor. Also, $\int e^{kx} dx = \frac{e^{kx}}{k}$ for a non-zero constant k .

Step 1: Simplify the right-hand side Factor the numerator and denominator before integrating:

$$\begin{aligned}\frac{dy}{dx} &= \frac{3e^{2x}(1 + e^{2x})}{e^{-x}(1 + e^{2x})} \\ &= 3e^{3x}.\end{aligned}$$

Since $1 + e^{2x} > 0$ for every real x , cancellation is valid throughout the real domain. The original denominator is positive as well, so this simplification excludes no real value of x .

Step 2: Integrate and verify Integrating both sides with respect to x gives

$$y = \int 3e^{3x} dx = e^{3x} + C, \quad C \in \mathbb{R}.$$

Quick Tip to Solve: Factor out the lowest exponential power in the numerator and denominator before attempting the integration.

Differentiating this expression yields $dy/dx = 3e^{3x}$, which is exactly the simplified right-hand side, and hence satisfies the original equation. The arbitrary constant provides the required family of solutions.

Step 3: Conclusion The general solution is $y = e^{3x} + C, \quad C \in \mathbb{R}.$

Quick Tip to Solve: Factor out the lowest exponential power in the numerator and denominator before attempting the integration.

4. Two cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the probability distribution of the number of spade cards. [2 marks]

Solution:

Concept:

- Let X be the number of spade cards in two draws. Its possible values are 0, 1, 2.
- There are 13 spades and 39 non-spades in a standard pack. Replacement makes the draws independent, with $P(S) = 1/4$ and $P(N) = 3/4$ on each draw.

Step 1: Compute the probabilities of zero and two spades For no spades both cards must be non-spades, while for two spades both must be spades. Multiplying the independent probabilities gives

$$P(X = 0) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}, \quad P(X = 2) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}.$$

Replacement restores the original pack before the second draw, so the probability of a spade stays $1/4$.

Step 2: Include both orders for exactly one spade Exactly one spade occurs in either order: spade then non-spade, or non-spade then spade. Since these events are disjoint, add their probabilities:

$$P(X = 1) = \frac{1}{4} \times \frac{3}{4} + \frac{3}{4} \times \frac{1}{4} = \frac{6}{16} = \frac{3}{8}.$$

As a check, $9/16 + 6/16 + 1/16 = 1$, so all possible values are covered.

Step 3: Conclusion The required probability distribution is

x	0	1	2
$P(X = x)$	$\frac{9}{16}$	$\frac{3}{8}$	$\frac{1}{16}$

Quick Tip to Solve: With replacement multiply independent probabilities; exactly one spade has two possible orders.

5. Find: $\int \frac{dx}{x^2 - 6x + 13}$.

[2 marks]

Solution:

Concept:

- Complete the square to express a quadratic denominator as a sum of squares: $x^2 - 6x + 13 = (x - 3)^2 + 4$.
- The standard antiderivative is $\int du/(u^2 + a^2) = (1/a) \tan^{-1}(u/a) + C$, where $a > 0$ and C is an arbitrary constant.

Step 1: Rewrite the denominator and substitute Adding and subtracting 9 in the denominator gives

$$x^2 - 6x + 13 = x^2 - 6x + 9 + 4 = (x - 3)^2 + 2^2.$$

Set $u = x - 3$. Since $du = dx$, the integral becomes

$$I = \int \frac{du}{u^2 + 2^2}.$$

The denominator is strictly positive for every real x , so this substitution introduces no excluded real values.

Step 2: Apply the inverse tangent formula and verify Using the standard formula with $a = 2$, we obtain

$$I = \frac{1}{2} \tan^{-1} \left(\frac{u}{2} \right) + C = \frac{1}{2} \tan^{-1} \left(\frac{x - 3}{2} \right) + C.$$

The chain rule checks the factor:

$$\frac{d}{dx} \left[\frac{1}{2} \tan^{-1} \left(\frac{x - 3}{2} \right) \right] = \frac{1/4}{1 + (x - 3)^2/4} = \frac{1}{x^2 - 6x + 13}.$$

Step 3: Conclusion

$$\boxed{\int \frac{dx}{x^2 - 6x + 13} = \frac{1}{2} \tan^{-1} \left(\frac{x - 3}{2} \right) + C.}$$

Quick Tip to Solve: Complete the square first; the remaining constant is 4, so use a equal to 2, not 4, in the arctangent formula.

6(a). A pair of dice is thrown and the sum of the numbers appearing on the dice is observed to be 7. Find the probability that the number 5 has appeared on at least one die.

[2 marks]

Solution:

Concept:

- Conditional probability restricts attention to outcomes consistent with the observed event. For events A, B with $P(B) > 0$, $P(A | B) = P(A \cap B)/P(B)$.
- For two fair dice, ordered pairs (i, j) , where $i, j \in \{1, 2, 3, 4, 5, 6\}$, are equally likely. The order

Quick Tip to Solve: Once the sum is given, list only ordered pairs with that sum and count favourable pairs within that list.

distinguishes which die shows each number.

Step 1: Restrict the sample space to a sum of seven Let B be the event that the observed sum is 7. The possible ordered pairs in this event are

$$B = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}.$$

These six outcomes have equal probability. Thus $P(B) = 6/36$; after the sum is known, these six outcomes form the relevant conditional sample space.

Step 2: Count the outcomes with at least one five Let A be the event that at least one die shows 5. Among the six outcomes listed above, only (2, 5) and (5, 2) satisfy this condition. The pair (5, 5) is excluded because its sum is 10. Therefore

$$P(A \cap B) = \frac{2}{36}, \quad P(A | B) = \frac{2/36}{6/36} = \frac{2}{6} = \frac{1}{3}.$$

Counting favourable outcomes out of all 36 without conditioning would ignore the observed information.

Step 3: Conclusion Given that the sum is 7, the probability that at least one die shows 5 is $\boxed{\frac{1}{3}}$.

Quick Tip to Solve: Once the sum is given, list only ordered pairs with that sum and count favourable pairs within that list.

OR

6(b). The probability that A hits the target is $\frac{1}{3}$ and the probability that B hits it, is $\frac{2}{5}$. If both try to hit the target independently, find the probability that the target is hit. [2 marks]

Solution:

Concept:

- Let A and B denote the events that the respective players hit the target. The target is hit when at least one of these events occurs, namely $A \cup B$.
- For independent trials the miss events are also independent, so $P(A' \cap B') = P(A')P(B')$. The complement rule gives $P(A \cup B) = 1 - P(A' \cap B')$.

Step 1: Find each player's probability of missing The given hit probabilities are $P(A) = \frac{1}{3}$ and $P(B) = \frac{2}{5}$. Therefore,

$$P(A') = 1 - \frac{1}{3} = \frac{2}{3}, \quad P(B') = 1 - \frac{2}{5} = \frac{3}{5}.$$

Here the prime denotes the complementary event. A miss by only one player does not mean that the target is missed, because the other player may still hit it.

Step 2: Use the complement of both players missing By the stated independence,

$$P(A' \cap B') = \frac{2}{3} \times \frac{3}{5} = \frac{2}{5}.$$

Quick Tip to Solve: For two independent attempts, subtract the product of their miss probabilities from 1; this includes both hitting without double counting.

The event that the target is hit includes A alone hitting, B alone hitting and both hitting. These possibilities are exactly the complement of both missing. Thus,

$$P(A \cup B) = 1 - \frac{2}{5} = \frac{3}{5}.$$

Step 3: Conclusion The probability that the target is hit is $\boxed{\frac{3}{5}}$.

Quick Tip to Solve: For two independent attempts, subtract the product of their miss probabilities from 1; this includes both hitting without double counting.

Section B (12 marks)

Question Nos. 7 to 10 carry 3 marks each.

7. Find the shortest distance between the following lines: $\vec{r} = 3\hat{i} + 5\hat{j} + 7\hat{k} + \lambda(\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r} = (-\hat{i} - \hat{j} - \hat{k}) + \mu(7\hat{i} - 6\hat{j} + \hat{k})$. [3 marks]

Solution:

Concept:

- A line in vector form $\vec{r} = \vec{a} + \lambda\vec{b}$ passes through the point with position vector $\vec{a}$ and has direction vector $\vec{b}$.
- For two non-parallel lines, their shortest distance is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$. The cross product supplies a direction perpendicular to both lines; the scalar projection onto its unit vector gives the separation.

Step 1: Identify points and direction vectors From the given equations,

$$\begin{aligned}\vec{a}_1 &= 3\hat{i} + 5\hat{j} + 7\hat{k}, & \vec{b}_1 &= \hat{i} - 2\hat{j} + \hat{k}, \\ \vec{a}_2 &= -\hat{i} - \hat{j} - \hat{k}, & \vec{b}_2 &= 7\hat{i} - 6\hat{j} + \hat{k}.\end{aligned}$$

Hence the displacement from the chosen point on the first line to the chosen point on the second is

$$\vec{a}_2 - \vec{a}_1 = -4\hat{i} - 6\hat{j} - 8\hat{k}.$$

Step 2: Calculate the common perpendicular direction Expanding the cross product along the first row gives

$$\begin{aligned}\vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix} \\ &= \hat{i}(-2 + 6) - \hat{j}(1 - 7) + \hat{k}(-6 + 14) \\ &= 4\hat{i} + 6\hat{j} + 8\hat{k}.\end{aligned}$$

This is nonzero, so the two direction vectors are not parallel and the shortest-distance formula applies.

Quick Tip to Solve: Here the point displacement is the negative of the cross product, so the chosen points already give a common perpendicular; its length checks the formula.

Its magnitude is

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{16 + 36 + 64} = \sqrt{116} = 2\sqrt{29}.$$

Step 3: Project the displacement onto the common perpendicular The scalar triple product is

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = (-4)(4) + (-6)(6) + (-8)(8) = -116.$$

Taking its absolute value ensures a non-negative distance:

$$d = \frac{116}{2\sqrt{29}} = 2\sqrt{29}.$$

As a check, the displacement $(-4, -6, -8)$ is the negative of the cross product, so it is already perpendicular to both lines. The two chosen points therefore are endpoints of the shortest connecting segment, whose length is $\sqrt{116}$.

Step 4: Conclusion The shortest distance between the given lines is $2\sqrt{29}$ units.

Quick Tip to Solve: Here the point displacement is the negative of the cross product, so the chosen points already give a common perpendicular; its length checks the formula.

8(a). The two adjacent sides of a parallelogram are represented by vectors $2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\hat{i} - 2\hat{j} - 3\hat{k}$. Find the unit vector parallel to one of its diagonals. Also, find the area of the parallelogram. [3 marks]

Solution:

Concept:

- If adjacent sides drawn from the same vertex are represented by $\vec{a}$ and $\vec{b}$, the diagonal from that vertex is $\vec{a} + \vec{b}$. The other diagonal has direction $\vec{a} - \vec{b}$.
- A unit vector along a nonzero vector $\vec{d}$ is $\vec{d}/|\vec{d}|$. The area of the parallelogram with adjacent sides $\vec{a}, \vec{b}$ is $|\vec{a} \times \vec{b}|$, since the cross product has magnitude $|\vec{a}||\vec{b}|\sin\theta$.

Step 1: Choose a diagonal and find its direction Let $\vec{a} = 2\hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} - 3\hat{k}$. Adding corresponding components gives the diagonal through their common starting vertex:

$$\vec{d} = \vec{a} + \vec{b} = 3\hat{i} - 6\hat{j} + 2\hat{k}.$$

Its length is $|\vec{d}| = \sqrt{3^2 + (-6)^2 + 2^2} = \sqrt{49} = 7$. Thus normalising this direction means dividing each component by 7; the squared components of the resulting vector add to $49/49 = 1$. Either orientation along the chosen diagonal is parallel to it.

Step 2: Calculate the cross product for the area Use the adjacent sides, rather than the diagonal, in

Quick Tip to Solve: Add adjacent-side vectors to get one diagonal, but use their cross product to find the parallelogram's area.

the area formula. Expanding the determinant along the first row,

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -4 & 5 \\ 1 & -2 & -3 \end{vmatrix} \\ &= \hat{i}\{12 + 10\} - \hat{j}\{-6 - 5\} + \hat{k}\{-4 + 4\} \\ &= 22\hat{i} + 11\hat{j}.\end{aligned}$$

Its magnitude is $\sqrt{22^2 + 11^2} = \sqrt{605} = 11\sqrt{5}$. This gives the area directly, without needing to calculate the angle between the sides.

Step 3: Conclusion One required unit vector is $\frac{3\hat{i} - 6\hat{j} + 2\hat{k}}{7}$, and its negative is also parallel to the same diagonal. The area is $11\sqrt{5}$ square units.

Quick Tip to Solve: Add adjacent-side vectors to get one diagonal, but use their cross product to find the parallelogram's area.

OR

8(b). If $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k}$, $\vec{b} = -\hat{i} + 2\hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j}$ are such that the vector $(\vec{a} + \lambda\vec{b})$ is perpendicular to vector $\vec{c}$, then find the value of λ . [3 marks]

Solution:

Concept:

- Two nonzero vectors are perpendicular if and only if their scalar product is zero: $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos 90^\circ = 0$.
- The scalar product is distributive, so $(\vec{a} + \lambda\vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \lambda(\vec{b} \cdot \vec{c})$. In component form, multiply corresponding components and add them.

Step 1: Translate perpendicularity into a scalar equation The given vectors have components $\vec{a} = (2, 2, 3)$, $\vec{b} = (-1, 2, 1)$ and $\vec{c} = (3, 1, 0)$. In particular, the missing $\hat{k}$ term of $\vec{c}$ means its third component is zero. Evaluate the two required scalar products:

$$\begin{aligned}\vec{a} \cdot \vec{c} &= 2(3) + 2(1) + 3(0) = 8, \\ \vec{b} \cdot \vec{c} &= (-1)(3) + 2(1) + 1(0) = -1.\end{aligned}$$

The prescribed perpendicularity therefore gives $8 + \lambda(-1) = 0$. This equation is linear in λ , with a nonzero coefficient, so it determines a unique value.

Step 2: Solve the equation and verify the condition Rearranging $8 - \lambda = 0$ gives $\lambda = 8$. Substitution into the original vector expression is a useful check on the negative sign in $\vec{b}$:

$$\begin{aligned}\vec{a} + 8\vec{b} &= (2 - 8)\hat{i} + (2 + 16)\hat{j} + (3 + 8)\hat{k} \\ &= -6\hat{i} + 18\hat{j} + 11\hat{k}.\end{aligned}$$

Quick Tip to Solve: Compute the two dot products separately; the zero third component of $\vec{c}$ removes every $\hat{k}$ contribution.

This vector is nonzero, and its dot product with the nonzero vector $\vec{c}$ is $(-6)(3) + 18(1) + 11(0) = -18 + 18 = 0$. Thus the resulting vectors really do make a right angle, as required.

Step 3: Conclusion The required scalar is $\lambda = 8$.

Quick Tip to Solve: Compute the two dot products separately; the zero third component of $\vec{c}$ removes every $\hat{k}$ contribution.

9(a). Find the particular solution of the differential equation $x \frac{dy}{dx} - y = x^2 e^x$, given $y(1) = 0$. [3 marks]

Solution:

Concept:

- A linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$. Its integrating factor is $IF = e^{\int P(x) dx}$.
- Multiplication by the integrating factor makes the left side an exact derivative. The given initial value then determines the constant of integration.

Step 1: Put the equation in linear form The given equation is $x \frac{dy}{dx} - y = x^2 e^x$. Since the initial value is prescribed at $x = 1$, we work on the interval $x > 0$ containing that point. Dividing by x gives

$$\frac{dy}{dx} - \frac{y}{x} = x e^x.$$

Thus $P(x) = -1/x$ and $Q(x) = x e^x$. The integrating factor is

$$e^{\int -\frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}.$$

Step 2: Integrate the exact derivative Multiplying the linear equation by $1/x$, we obtain

$$\frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} = e^x.$$

By the quotient rule, the left side is $\frac{d}{dx}(y/x)$. Therefore

$$\frac{d}{dx} \left(\frac{y}{x} \right) = e^x, \quad \frac{y}{x} = e^x + C.$$

Multiplying through by x gives the family $y = x(e^x + C)$.

Step 3: Use the initial condition and verify Substituting $x = 1$ and $y = 0$ yields $0 = e + C$, so $C = -e$. For this value, differentiation gives $y' = e^x + x e^x - e$. Hence

$$xy' - y = x(e^x + x e^x - e) - x(e^x - e) = x^2 e^x.$$

The initial condition is also satisfied because $y(1) = e - e = 0$.

Quick Tip to Solve: In an equation of the form $xy' - y = x^2 f(x)$, divide by x^2 to recognise $(y/x)' = f(x)$ immediately.

Step 4: Conclusion The required particular solution is $y = x(e^x - e)$, on the interval $x > 0$ containing the initial point.

Quick Tip to Solve: In an equation of the form $xy' - y = x^2 f(x)$, divide by x^2 to recognise $(y/x)' = f(x)$ immediately.

OR

9(b). Find the general solution of the differential equation $x \frac{dy}{dx} = y(\log y - \log x + 1)$. [3 marks]

Solution:

Concept:

- For a homogeneous equation $dy/dx = F(y/x)$, substitute $y = vx$, where $v = y/x$. Then $dy/dx = v + x dv/dx$.
- The logarithms in this question require $x > 0$ and $y > 0$. Consequently $v > 0$, and $\log y - \log x = \log(y/x) = \log v$.

Step 1: Recognise the homogeneous equation Dividing the printed equation by x gives

$$\frac{dy}{dx} = \frac{y}{x} \left(\log \frac{y}{x} + 1 \right).$$

The right side depends only on y/x , so the substitution $y = vx$ is appropriate. Because the original equation contains real logarithms, we solve on $x > 0, y > 0$.

Step 2: Reduce to an equation in the ratio Substituting $y = vx$ and $y' = v + xv'$ in the equation gives

$$v + x \frac{dv}{dx} = v(\log v + 1).$$

Cancelling v from both sides leads to

$$x \frac{dv}{dx} = v \log v.$$

Set $u = \log v$. Since $du/dx = (1/v) dv/dx$, this becomes $x du/dx = u$. This form retains the constant-ratio solution $v = 1$, corresponding to $u = 0$.

Step 3: Integrate and check the family Rewrite the last equation as $u' - u/x = 0$. Its integrating factor on $x > 0$ is $1/x$, so

$$\frac{d}{dx} \left(\frac{u}{x} \right) = 0, \quad u = Cx.$$

Therefore $\log(y/x) = Cx$, or $y = xe^{Cx}$. To check, differentiate this expression to obtain $y' = e^{Cx}(1 + Cx)$. Also $\log y - \log x = Cx$; hence both sides of the original equation equal $xe^{Cx}(1 + Cx)$. The case $C = 0$ gives $y = x$, so it is included without dividing by $\log v$.

Step 4: Conclusion The general solution is $\log(y/x) = Cx$, equivalently $y = xe^{Cx}$, where C is an arbitrary real constant and $x > 0$.

Quick Tip to Solve: After setting $v = y/x$, use $u = \log v$ so that the solution $y = x$ is retained automatically.

10. Evaluate: $\int_{-\pi/2}^{\pi/2} (\sin |x| + \cos |x|) dx$.

[3 marks]

Solution:

Concept:

- The absolute value satisfies $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$; it must be applied before using trigonometric parity.
- If $f(-x) = f(x)$, then f is even and $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$, where $a > 0$.
- The antiderivatives are $\int \sin x dx = -\cos x + C$ and $\int \cos x dx = \sin x + C$, with C an arbitrary constant.

Step 1: Establish the symmetry of the actual integrand Write $f(x) = \sin |x| + \cos |x|$. Since $|-x| = |x|$,

$$f(-x) = \sin |-x| + \cos |-x| = \sin |x| + \cos |x| = f(x).$$

Thus both terms in the given integrand are even. Although $\sin x$ itself is odd, $\sin |x|$ is even: the absolute value makes its argument the same at x and $-x$. Hence the sine contribution does not cancel over these symmetric limits.

Step 2: Reduce the integral to the positive half of the interval Let I denote the required integral. The limits are $-\pi/2$ and $\pi/2$, so evenness gives

$$I = 2 \int_0^{\pi/2} (\sin |x| + \cos |x|) dx = 2 \int_0^{\pi/2} (\sin x + \cos x) dx.$$

In the second equality we used $|x| = x$ throughout $[0, \pi/2]$. This removes the absolute value without changing the integrand on the interval being integrated.

Step 3: Apply the antiderivative and the endpoint values Integrating the two terms separately and then substituting the limits,

$$\begin{aligned} I &= 2[-\cos x + \sin x]_0^{\pi/2} \\ &= 2[(-\cos(\pi/2) + \sin(\pi/2)) - (-\cos 0 + \sin 0)] \\ &= 2[(0 + 1) - (-1 + 0)]. \end{aligned}$$

The subtraction of the lower endpoint contributes positively, since the antiderivative equals -1 at zero.

Step 4: Conclusion Therefore, $\int_{-\pi/2}^{\pi/2} (\sin |x| + \cos |x|) dx = \boxed{4}$.

Quick Tip to Solve: Check parity after applying the absolute value: $\sin |x|$ is even, so double the integral over $[0, \pi/2]$ instead of cancelling the sine term.

Section C (16 marks)

Question Nos. 11 to 14 carry 4 marks each.

11. Find the distance of the point $(1, -2, 9)$ from the point of intersection of the line $\vec{r} = 4\hat{i} + 2\hat{j} + 7\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 10$. [4 marks]

Solution:

Concept:

- A point on a line with vector equation $\vec{r} = \vec{a} + \lambda\vec{b}$ has coordinates obtained by adding the corresponding components of $\vec{a}$ and $\lambda\vec{b}$. Here λ is a real parameter.
- The plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 10$ has Cartesian equation $x - y + z = 10$. The intersection point must satisfy both the line and this plane.
- The distance between $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Step 1: Express a general point on the line Let Q be the point where the given line meets the plane. Comparing the coefficients of $\hat{i}, \hat{j}, \hat{k}$ in the vector equation gives

$$x = 4 + 3\lambda, \quad y = 2 + 4\lambda, \quad z = 7 + 2\lambda.$$

Thus $Q = (4 + 3\lambda, 2 + 4\lambda, 7 + 2\lambda)$. We must determine the parameter for the point of intersection before calculating its distance from the given point.

Step 2: Use the plane equation to determine the intersection Substituting these coordinates in $x - y + z = 10$, we obtain

$$\begin{aligned}(4 + 3\lambda) - (2 + 4\lambda) + (7 + 2\lambda) &= 10, \\ 9 + \lambda &= 10, \\ \lambda &= 1.\end{aligned}$$

Hence the intersection point is $Q = (7, 6, 9)$. As a check, its coordinates satisfy $7 - 6 + 9 = 10$. Also, the coefficient of λ is nonzero, so the intersection is unique.

Step 3: Calculate the distance from the given point The given point is $P = (1, -2, 9)$. The coordinate differences from P to Q are $7 - 1 = 6$, $6 - (-2) = 8$, and $9 - 9 = 0$. Therefore

$$\begin{aligned}PQ &= \sqrt{(7 - 1)^2 + (6 - (-2))^2 + (9 - 9)^2} \\ &= \sqrt{6^2 + 8^2 + 0^2} \\ &= \sqrt{100}.\end{aligned}$$

The equal third coordinates make the third contribution zero; the negative second coordinate of P must be subtracted with its sign preserved.

Step 4: Conclusion The point of intersection is $(7, 6, 9)$, and its distance from $(1, -2, 9)$ is 10 units.

Quick Tip to Solve: Substitute the line's coordinates into the plane first; here the parameter coefficients combine to $3 - 4 + 2 = 1$, and the final distance uses the 6, 8, 10 triple.

12(a). Find: $\int \frac{x^2}{(x^2 + 1)(3x^2 + 4)} dx.$

[4 marks]

Solution:

Concept:

- Partial fractions express a rational function with distinct quadratic factors as a sum of simpler fractions. Since the integrand is even in x , constant numerators suffice here.
- The inverse-tangent formula is $\int \frac{du}{1+u^2} = \tan^{-1} u + C$, where C is an arbitrary real constant.
- In a substitution $u = kx$, the differential changes to $du = k dx$. This scale factor must be included when integrating a quadratic denominator.

Step 1: Resolve the integrand into partial fractions Write

$$\frac{x^2}{(x^2 + 1)(3x^2 + 4)} = \frac{A}{x^2 + 1} + \frac{B}{3x^2 + 4}.$$

Multiplying by the common denominator gives the polynomial identity

$$x^2 = A(3x^2 + 4) + B(x^2 + 1) = (3A + B)x^2 + (4A + B).$$

Equality of the coefficients of x^2 and the constant terms requires

$$3A + B = 1, \quad 4A + B = 0.$$

Subtracting the first equation from the second gives $A = -1$; substituting this into either equation gives $B = 4$. Thus the numerator of the combined fractions is $-(3x^2 + 4) + 4(x^2 + 1) = x^2$, which verifies the decomposition.

Step 2: Integrate the two simpler terms The required integral becomes

$$I = - \int \frac{dx}{x^2 + 1} + 4 \int \frac{dx}{3x^2 + 4}.$$

The first term is $-\tan^{-1} x$. For the second term, factor out 4 from the denominator and set $u = \frac{\sqrt{3}x}{2}$, so that $dx = \frac{2}{\sqrt{3}} du$. Hence

$$\begin{aligned} 4 \int \frac{dx}{3x^2 + 4} &= \int \frac{dx}{1 + (\sqrt{3}x/2)^2} \\ &= \frac{2}{\sqrt{3}} \int \frac{du}{1 + u^2} \\ &= \frac{2}{\sqrt{3}} \tan^{-1} u. \end{aligned}$$

Both quadratic denominators are positive for every real x , so this antiderivative is valid on the whole real line.

Step 3: Check by differentiation Differentiating the obtained expression yields

$$-\frac{1}{1+x^2} + \frac{1}{1+3x^2/4} = -\frac{1}{x^2+1} + \frac{4}{3x^2+4} = \frac{x^2}{(x^2+1)(3x^2+4)}.$$

Quick Tip to Solve: Compare the constant and quadratic coefficients to find the partial fractions; then factor each quadratic denominator into the form $1 + u^2$ before integrating.

This reproduces the given integrand and checks both the sign and the substitution factor.

Step 4: Conclusion Therefore,

$$\int \frac{x^2}{(x^2 + 1)(3x^2 + 4)} dx = -\tan^{-1} x + \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{\sqrt{3}x}{2} \right) + C.$$

Quick Tip to Solve: Compare the constant and quadratic coefficients to find the partial fractions; then factor each quadratic denominator into the form $1 + u^2$ before integrating.

OR

12(b). Evaluate: $\int_{-2}^1 \sqrt{5 - 4x - x^2} dx.$

[4 marks]

Solution:

Concept:

- Completing the square converts the quadratic under the radical to $a^2 - t^2$, for which the substitution $t = a \sin \theta$ removes the square root.
- Under a change of variable, both limits must be changed. On $0 \leq \theta \leq \pi/2$, $\cos \theta \geq 0$, so $\sqrt{a^2 \cos^2 \theta} = a \cos \theta$.
- The double-angle identity $\cos^2 \theta = (1 + \cos 2\theta)/2$ gives an elementary integral after substitution.

Step 1: Complete the square and shift the variable Let the required integral be I . Rewrite the expression inside the square root as

$$5 - 4x - x^2 = 9 - (x + 2)^2.$$

Put $t = x + 2$, so that $dt = dx$. When $x = -2$, $t = 0$, and when $x = 1$, $t = 3$. Thus

$$I = \int_0^3 \sqrt{9 - t^2} dt.$$

Throughout these limits, $0 \leq t \leq 3$, so the radicand is nonnegative and the integral is real.

Step 2: Apply the trigonometric substitution Take $t = 3 \sin \theta$, with $0 \leq \theta \leq \pi/2$. Then $dt = 3 \cos \theta d\theta$. The lower limit $t = 0$ gives $\theta = 0$, while the upper limit $t = 3$ gives $\theta = \pi/2$. Since cosine is nonnegative on this interval,

$$\sqrt{9 - t^2} = \sqrt{9(1 - \sin^2 \theta)} = 3 \cos \theta.$$

Consequently,

$$I = 9 \int_0^{\pi/2} \cos^2 \theta d\theta.$$

Quick Tip to Solve: Write the radicand as $9 - (x + 2)^2$ first; the shifted limits 0 and 3 immediately reveal a quarter-circle integral of radius 3.

Step 3: Integrate and evaluate the endpoint terms Using the double-angle identity, we obtain

$$\begin{aligned} I &= \frac{9}{2} \int_0^{\pi/2} (1 + \cos 2\theta) d\theta \\ &= \frac{9}{2} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} \\ &= \frac{9}{2} \left(\frac{\pi}{2} + \frac{\sin \pi}{2} - 0 - \frac{\sin 0}{2} \right). \end{aligned}$$

Both sine terms vanish. As a check, $\sqrt{9 - t^2}$ describes the upper semicircle of radius 3; integrating from 0 to 3 gives the area of its first-quadrant quarter, namely $\pi(3)^2/4$, consistent with this evaluation.

Step 4: Conclusion Therefore, $\int_{-2}^1 \sqrt{5 - 4x - x^2} dx = \boxed{\frac{9\pi}{4}}$.

Quick Tip to Solve: Write the radicand as $9 - (x + 2)^2$ first; the shifted limits 0 and 3 immediately reveal a quarter-circle integral of radius 3.

13. Find the area of the region enclosed by the curves $y^2 = x$, $x = \frac{1}{4}$, $y = 0$ and $x = 1$, using integration. [4 marks]

Solution:

Concept:

- For a region between an upper curve $y = f(x)$ and a lower curve $y = g(x)$, bounded by $x = a$ and $x = b$, its area is $A = \int_a^b [f(x) - g(x)] dx$.
- The parabola $y^2 = x$ has branches $y = \sqrt{x}$ and $y = -\sqrt{x}$. The line $y = 0$ is the x -axis; the upper bounded region has height $\sqrt{x}$.
- The power rule gives $\int x^{1/2} dx = \frac{2}{3}x^{3/2} + C$, where C is the constant of integration. Area is nonnegative and is measured in square units.

Step 1: Identify the boundary and integration limits The two vertical lines are $x = \frac{1}{4}$ and $x = 1$. For $\frac{1}{4} \leq x \leq 1$, the upper branch of the parabola is $y = \sqrt{x}$, while the lower boundary of the region above the axis is $y = 0$. A thin vertical strip of width dx therefore has area

$$dA = (\sqrt{x} - 0) dx.$$

The parabola meets the left vertical boundary at $(\frac{1}{4}, \frac{1}{2})$ and the right boundary at $(1, 1)$. Thus the height stays positive throughout the interval and no splitting of the integral is required. The lower branch encloses an equal reflected region below the axis; choosing that branch would give the same individual area. We evaluate the region bounded by the axis and the upper branch.

Quick Tip to Solve: Use vertical strips: the height is $\sqrt{x}$, and the lower-limit value $(1/4)^{3/2}$ is $1/8$, not $1/4$.

Step 2: Integrate the strip areas Adding all vertical strips between the specified limits gives

$$\begin{aligned} A &= \int_{1/4}^1 \sqrt{x} \, dx \\ &= \int_{1/4}^1 x^{1/2} \, dx \\ &= \left[\frac{2}{3} x^{3/2} \right]_{1/4}^1 \\ &= \frac{2}{3} \left(1^{3/2} - \left(\frac{1}{4} \right)^{3/2} \right). \end{aligned}$$

To substitute the lower limit correctly, use

$$\left(\frac{1}{4} \right)^{3/2} = \left(\sqrt{\frac{1}{4}} \right)^3 = \left(\frac{1}{2} \right)^3 = \frac{1}{8}.$$

Hence the definite integral simplifies to $\frac{2}{3} \left(1 - \frac{1}{8} \right) = \frac{2}{3} \cdot \frac{7}{8}$. Both the integrand and the interval width are positive, consistent with the geometric area.

Step 3: Conclusion The required area is $\boxed{\frac{7}{12} \text{ square units}}$.

Quick Tip to Solve: Use vertical strips: the height is $\sqrt{x}$, and the lower-limit value $(1/4)^{3/2}$ is $1/8$, not $1/4$.

14. Case Study Problem:

A shopkeeper sells three types of flower seeds A_1, A_2, A_3 . They are sold in the form of a mixture, where the proportions of these seeds are $4 : 4 : 2$, respectively. The germination rates of the three types of seeds are 45%, 60% and 35% respectively.

Based on the above information, answer the following questions:

[4 marks]

14(a). Calculate the probability that a randomly chosen seed will germinate.

[2 marks]

Solution:

Concept:

- Let G be the event that a randomly chosen seed germinates, and let A_i mean that it is of type i , for $i = 1, 2, 3$.
- The seed types form a mutually exclusive and exhaustive partition. The law of total probability gives $P(G) = \sum_{i=1}^3 P(A_i)P(G | A_i)$.

Step 1: Convert proportions and germination rates into probabilities The total number of proportion units is $4 + 4 + 2 = 10$. Hence

$$P(A_1) = \frac{4}{10}, \quad P(A_2) = \frac{4}{10}, \quad P(A_3) = \frac{2}{10}.$$

Quick Tip to Solve: Multiply each germination rate by its mixture proportion before adding the three contributions.

The given rates apply within each type, so

$$P(G | A_1) = \frac{45}{100}, \quad P(G | A_2) = \frac{60}{100}, \quad P(G | A_3) = \frac{35}{100}.$$

Step 2: Add the weighted germination probabilities Germination can occur through any of the three disjoint seed types. Therefore,

$$\begin{aligned} P(G) &= \frac{4}{10} \frac{45}{100} + \frac{4}{10} \frac{60}{100} + \frac{2}{10} \frac{35}{100} \\ &= \frac{180 + 240 + 70}{1000} = \frac{490}{1000} = \frac{49}{100}. \end{aligned}$$

Weighting is essential because the three types do not occur in equal proportions; a simple average of the three germination rates would not describe this mixture.

Step 3: Conclusion The probability that a randomly chosen seed germinates is $\frac{49}{100} = 0.49$.

Quick Tip to Solve: Multiply each germination rate by its mixture proportion before adding the three contributions.

14(b). Calculate the probability that the seed is of type A_2 , given that a randomly chosen seed germinates. [2 marks]

Solution:

Concept:

- Let G denote germination and A_i denote the seed being of type i . The required probability is $P(A_2 | G)$, since germination has already been observed.
- Bayes' theorem gives $P(A_2 | G) = P(A_2)P(G | A_2)/P(G)$, where the denominator includes germinated seeds from all three types.

Step 1: Find the probability of germination in the mixture From the ratio 4 : 4 : 2, the probabilities of the three types are 4/10, 4/10 and 2/10, respectively. Applying total probability to the given germination rates,

$$\begin{aligned} P(G) &= \frac{4}{10} \frac{45}{100} + \frac{4}{10} \frac{60}{100} + \frac{2}{10} \frac{35}{100} \\ &= \frac{180 + 240 + 70}{1000} = \frac{49}{100}. \end{aligned}$$

Step 2: Apply Bayes' theorem for type A_2 The probability of choosing a type A_2 seed which germinates is

$$P(A_2 \cap G) = P(A_2)P(G | A_2) = \frac{4}{10} \frac{60}{100} = \frac{24}{100}.$$

Dividing this joint probability by the probability of the conditioning event gives

$$P(A_2 | G) = \frac{24/100}{49/100} = \frac{24}{49}.$$

Quick Tip to Solve: For a reversed conditional probability, divide the relevant weighted germination contribution by the total germination probability.

This ratio measures the fraction of all germinated seeds belonging to type A_2 . It is different from $P(G \mid A_2) = 0.60$, because the condition and the event have been interchanged.

Step 3: Conclusion Given that the seed germinates, the probability that it is of type A_2 is $\boxed{\frac{24}{49}}$.

Quick Tip to Solve: For a reversed conditional probability, divide the relevant weighted germination contribution by the total germination probability.