

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-4-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) This question paper contains **three** sections – **Section A, B** and **C**.
- (ii) Each section is **compulsory**.
- (iii) **Section A** has **6** short answer type I questions of **2** marks each.
- (iv) **Section B** has **4** short answer type II questions of **3** marks each.
- (v) **Section C** has **4** long answer type questions of **4** marks each.
- (vi) There is an internal choice in some questions.
- (vii) Question no. 14 is a case-study based question with 2 sub-parts of **2** marks each.

Section A (12 marks)

Questions number 1 to 6 carry 2 marks each.

1. A bag contains 3 red and 4 white balls. Three balls are drawn at random, one-by-one without replacement from the bag. If the first ball drawn is red in colour, then find the probability that the remaining two balls drawn are also red in colour. [2 marks]

Solution:

Concept:

- Conditional probability uses the information already given. Once the first ball is known to be red, the remaining bag has two red balls and four white balls.
- For successive draws without replacement, multiply the probability of the second draw by the conditional probability of the third draw. The composition of the bag changes after each draw.

Step 1: Use the condition on the first draw Let R_i denote the event that the ball on draw i is red. We need $P(R_2 \cap R_3 | R_1)$. The occurrence of R_1 has already been specified, so we do not multiply by the probability $3/7$ of the first draw being red. After that red ball is removed, six balls remain, of which two are red. Hence $P(R_2 | R_1) = 2/6$.

Step 2: Apply the multiplication rule If the second ball is also red, five balls remain and only one of them is red. Thus $P(R_3 | R_1 \cap R_2) = 1/5$. Therefore,

$$P(R_2 \cap R_3 | R_1) = \frac{2}{6} \times \frac{1}{5} = \frac{1}{15}.$$

Equivalently, selecting both remaining red balls from the six remaining balls gives $\binom{2}{2} / \binom{6}{2} = 1/15$, confirming the result.

Step 3: Conclusion The required probability that both remaining draws are red, given that the first is red, is $\boxed{\frac{1}{15}}$.

Quick Tip to Solve: Remove the known first red ball before counting: the two successive red probabilities are $2/6$ and $1/5$.

2. A coin is tossed twice. The following table shows the probability distribution of number of tails:

X	0	1	2
$P(X)$	K	$6K$	$9K$

(a) Find the value of K .

(b) Is the coin tossed biased or unbiased? Justify your answer. [2 marks]

Solution:

Concept:

- A probability distribution has nonnegative probabilities whose sum over all possible values of the random variable is one.

Quick Tip to Solve: For a fair coin, zero tails and two tails must have equal probabilities; here they are in the ratio $1 : 9$.

- For an unbiased coin tossed twice independently, the number of tails has probabilities $1/4, 1/2, 1/4$ for zero, one and two tails, respectively.

Step 1: (a) Normalize the given distribution The random variable X counts tails and takes the mutually exclusive values 0, 1, 2. These exhaust all outcomes of two tosses. Hence

$$K + 6K + 9K = 1 \implies 16K = 1 \implies K = \frac{1}{16}.$$

Substituting gives $P(X = 0) = 1/16$, $P(X = 1) = 6/16 = 3/8$, and $P(X = 2) = 9/16$. They are nonnegative and sum to one.

Step 2: (b) Compare with an unbiased coin An unbiased coin has four equally likely ordered outcomes HH, HT, TH, TT . Therefore, the probabilities for zero, one and two tails would be $1/4, 1/2, 1/4$. The given distribution instead has $P(TT) = 9/16$ and $P(HH) = 1/16$, which are unequal. Thus it cannot come from an unbiased coin. In the usual independent-toss model, $P(T)^2 = 9/16$ gives $P(T) = 3/4$, and $P(H) = 1/4$; these also give $2(3/4)(1/4) = 6/16$ for exactly one tail.

Step 3: Conclusion (a) $K = \frac{1}{16}$. (b) The coin is **biased**, since the distribution differs from that of two tosses of an unbiased coin.

Quick Tip to Solve: For a fair coin, zero tails and two tails must have equal probabilities; here they are in the ratio 1 : 9.

3. The foot of a perpendicular drawn from the point $(-2, -1, -3)$ on a plane is $(1, -3, 3)$. Find the equation of the plane. [2 marks]

Solution:

Concept:

- The line joining a point to its perpendicular foot on a plane is normal to that plane. Its direction vector supplies the coefficients in the plane equation.
- A plane through (x_0, y_0, z_0) with normal vector (a, b, c) has equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Step 1: Find a normal vector to the plane Let the given point be $P = (-2, -1, -3)$ and its perpendicular foot be $H = (1, -3, 3)$. Because PH is perpendicular to the plane, a normal vector is

$$\overrightarrow{PH} = H - P = (1 + 2, -3 + 1, 3 + 3) = (3, -2, 6).$$

Its components therefore give the coefficients of x, y, z in the required plane equation. Reversing the vector would give an equivalent equation.

Step 2: Use the foot as a point on the plane The foot H lies on the plane. Using the point-normal form,

$$\begin{aligned} 3(x - 1) - 2(y + 3) + 6(z - 3) &= 0, \\ 3x - 3 - 2y - 6 + 6z - 18 &= 0, \\ 3x - 2y + 6z - 27 &= 0. \end{aligned}$$

Quick Tip to Solve: Use the perpendicular foot as the point on the plane, and subtract the original point from the foot to obtain a normal vector.

For a check, substituting $H = (1, -3, 3)$ gives $3 + 6 + 18 - 27 = 0$. The normal $(3, -2, 6)$ is also exactly the direction of PH , verifying the perpendicular condition.

Step 3: Conclusion The required equation of the plane is $\boxed{3x - 2y + 6z - 27 = 0}$.

Quick Tip to Solve: Use the perpendicular foot as the point on the plane, and subtract the original point from the foot to obtain a normal vector.

4(a). If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 400$ and $|\vec{b}| = 5$, then find the value of $|\vec{a}|$.

[2 marks]

Solution:

Concept:

- For vectors making an angle θ , $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$.
- The identity $\sin^2 \theta + \cos^2 \theta = 1$ gives $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$.

Step 1: Combine the cross and dot product terms Let θ be the angle between the vectors. Squaring the two product formulae and adding eliminates the unknown angle:

$$\begin{aligned} |\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 &= |\vec{a}|^2 |\vec{b}|^2 (\sin^2 \theta + \cos^2 \theta) \\ &= |\vec{a}|^2 |\vec{b}|^2. \end{aligned}$$

The absolute value on the dot product does not affect its square, so this applies even when the dot product is negative.

Step 2: Substitute the given magnitude Using the given sum 400 and $|\vec{b}| = 5$, we get

$$25|\vec{a}|^2 = 400, \quad |\vec{a}|^2 = 16.$$

A magnitude is non-negative; hence only the positive square root is admissible. There is no need to determine the angle between the vectors.

Step 3: Conclusion Therefore, $\boxed{|\vec{a}| = 4}$.

Quick Tip to Solve: When squared cross and dot products are added, replace their sum directly by $|\vec{a}|^2 |\vec{b}|^2$.

OR

4(b). Find all the possible vectors of magnitude $5\sqrt{3}$ which are equally inclined to the coordinate axes.

[2 marks]

Solution:

Concept:

- The direction cosines l, m, n of a non-zero vector satisfy $l^2 + m^2 + n^2 = 1$.
- For a vector of magnitude r , its Cartesian form is $r(\hat{l} + m\hat{j} + n\hat{k})$. Equal inclinations to the

Quick Tip to Solve: Equal inclinations mean equal direction cosines, so first set $l = m = n$, retaining both common signs.

coordinate axes imply $l = m = n$.

Step 1: Find the common direction cosine Let the equal angles with the positive coordinate axes be $\alpha = \beta = \gamma$. Their cosines are therefore equal; write $l = m = n = t$. From the direction-cosine identity,

$$3t^2 = 1, \quad t = \frac{1}{\sqrt{3}} \text{ or } t = -\frac{1}{\sqrt{3}}.$$

The same sign must be used for all three direction cosines. Mixed signs would give unequal angles with the positive coordinate axes.

Step 2: Use the given magnitude The required magnitude is $r = 5\sqrt{3}$. Thus

$$\vec{a} = 5\sqrt{3}t(\hat{i} + \hat{j} + \hat{k}).$$

For $t = 1/\sqrt{3}$, all three components are 5; for $t = -1/\sqrt{3}$, all three are -5 . In either case the magnitude checks as $\sqrt{25 + 25 + 25} = 5\sqrt{3}$. The corresponding equal angles have cosine $1/\sqrt{3}$ or $-1/\sqrt{3}$, respectively.

Step 3: Conclusion The two possible vectors are $\boxed{5\hat{i} + 5\hat{j} + 5\hat{k}}$ and $\boxed{-5\hat{i} - 5\hat{j} - 5\hat{k}}$.

Quick Tip to Solve: Equal inclinations mean equal direction cosines, so first set $l = m = n$, retaining both common signs.

5. Find the general solution of the differential equation $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$. [2 marks]

Solution:

Concept:

- The product differential is $d(uv) = v \, du + u \, dv$.
- Since $d(\tan x) = \sec^2 x \, dx$ and $d(\tan y) = \sec^2 y \, dy$, the given left-hand side is an exact product differential.

Step 1: Recognise the product differential Take $u = \tan x$ and $v = \tan y$. By the product rule,

$$d(\tan x \tan y) = \tan y \sec^2 x \, dx + \tan x \sec^2 y \, dy.$$

This expression is exactly the left-hand side of the given equation. Therefore the equation can be written as

$$d(\tan x \tan y) = 0.$$

This recognition avoids division by $\tan x$ or $\tan y$, so zero values of these functions are not unnecessarily discarded.

Step 2: Integrate and check Integrating the zero differential gives

$$\tan x \tan y = C,$$

where C is an arbitrary real constant. Differentiating this relation reproduces the original differential

Quick Tip to Solve: Look for the product rule before separating variables: each secant-squared term differentiates the accompanying tangent.

equation, which verifies the result. The relation is understood on intervals where $\tan x$ and $\tan y$ are defined, namely where $\cos x \neq 0$ and $\cos y \neq 0$. The choice $C = 0$ includes the constant solutions $y = k\pi$, with integer k , wherever the original equation is defined.

Step 3: Conclusion The general solution is $\tan x \tan y = C$, where C is an arbitrary constant.

Quick Tip to Solve: Look for the product rule before separating variables: each secant-squared term differentiates the accompanying tangent.

6. Evaluate: $\int_0^1 x^2 e^x dx$.

[2 marks]

Solution:

Concept:

- Integration by parts gives $\int u dv = uv - \int v du$. Taking the polynomial as u reduces its degree at each application.
- The definite integral is $F(1) - F(0)$, where F is an antiderivative of the integrand. Both endpoint values must be included.

Step 1: Integrate by parts twice Take $u = x^2$, $dv = e^x dx$, so that $du = 2x dx$ and $v = e^x$. Thus

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx.$$

For the remaining integral, take $u = x$, $dv = e^x dx$. Then

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x.$$

Substituting this expression gives the antiderivative $F(x) = e^x(x^2 - 2x + 2)$. The polynomial has decreased in degree twice, leaving an elementary exponential integral.

Step 2: Evaluate at the endpoints Applying the limits carefully gives

$$\begin{aligned} \int_0^1 x^2 e^x dx &= [e^x(x^2 - 2x + 2)]_0^1 \\ &= e(1 - 2 + 2) - 1(0 - 0 + 2) \\ &= e - 2. \end{aligned}$$

Although the integrand is zero at the lower endpoint, its antiderivative there is 2, which must be subtracted. Differentiation checks the working: $F'(x) = e^x[(x^2 - 2x + 2) + (2x - 2)] = x^2 e^x$.

Step 3: Conclusion The required value is $e - 2$.

Quick Tip to Solve: Differentiate the quadratic in successive integrations by parts, then remember to subtract the nonzero antiderivative at zero.

Section B (12 marks)

Questions number 7 to 10 carry 3 marks each.

7. Find the area of the region $\{(x, y) : x^2 \leq y \leq x + 2\}$, using integration.

[3 marks]

Solution:

Concept:

- For a region between $y = f(x)$ and $y = g(x)$, with $g(x) \geq f(x)$ on $[a, b]$, its area is $A = \int_a^b [g(x) - f(x)] dx$.
- The limits are the abscissae of intersection of the bounding curves. A vertical strip has height equal to the upper ordinate minus the lower ordinate.

Step 1: Find the intersections and the interval The bounding curves are the parabola $y = x^2$ and the straight line $y = x + 2$. At their intersections their ordinates are equal, so

$$x^2 = x + 2 \implies x^2 - x - 2 = 0 \implies (x - 2)(x + 1) = 0.$$

Thus the intersection abscissae are $x = -1$ and $x = 2$, with intersection points $(-1, 1)$ and $(2, 4)$. The given simultaneous inequalities require $x^2 \leq x + 2$, or $(x - 2)(x + 1) \leq 0$. This holds precisely for $-1 \leq x \leq 2$, establishing the complete interval of the region.

Step 2: Express the area using vertical strips Throughout this interval, the line is the upper boundary and the parabola is the lower boundary. Their ordinate difference is

$$(x + 2) - x^2 = (2 - x)(x + 1) \geq 0.$$

A vertical strip of width dx therefore has area $[(x + 2) - x^2] dx$. Adding all such strips between the two intersection abscissae yields

$$A = \int_{-1}^2 (x + 2 - x^2) dx.$$

No reversal of the two boundaries is needed within these limits.

Step 3: Integrate and evaluate both endpoints Integrating each polynomial term gives

$$\begin{aligned} A &= \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2 \\ &= \left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \\ &= \frac{10}{3} - \left(-\frac{7}{6} \right) = \frac{27}{6} = \frac{9}{2}. \end{aligned}$$

In particular the negative abscissa at the lower endpoint makes the cubic contribution positive in the antiderivative there; retaining this sign gives the correct endpoint subtraction.

Step 4: Conclusion The area of the given region is $\boxed{\frac{9}{2}}$ square units.

Quick Tip to Solve: Solve the two boundary equations first, then factor their difference to identify the upper curve over the entire integration interval.

8(a). Find: $\int \frac{1}{e^x + 1} dx$.

[3 marks]

Solution:

Concept:

- The substitution rule replaces both the variable and its differential. With $t = e^x$, we have $dt = e^x dx = t dx$, so $dx = dt/t$.
- Partial fractions give $\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$. Use $\int du/u = \ln|u| + C$, where C is an arbitrary constant.

Step 1: Substitute the exponential variable Denote the required integral by I . Put $t = e^x$, so $t > 0$ for every real x . Since $dt = t dx$, substitution gives

$$I = \int \frac{1}{t+1} \frac{dt}{t} = \int \frac{1}{t(t+1)} dt.$$

The extra factor $1/t$ arises from the differential. It is essential: merely changing e^x to t in the denominator without changing dx would produce a different integral.

Step 2: Resolve the fraction and integrate Write $1/[t(t+1)] = A/t + B/(t+1)$. Clearing denominators gives $1 = A(t+1) + Bt$. Comparing coefficients yields $A = 1$ and $B = -1$. Therefore,

$$\begin{aligned} I &= \int \left(\frac{1}{t} - \frac{1}{t+1} \right) dt \\ &= \ln|t| - \ln|t+1| + C \\ &= x - \ln(1 + e^x) + C. \end{aligned}$$

Because $t = e^x > 0$ and $t+1 > 0$, both logarithms have positive arguments after substitution. The identity $\ln(e^x) = x$ is valid for all real x . To verify, differentiate: $1 - e^x/(1 + e^x) = 1/(1 + e^x)$, which reproduces the original integrand exactly.

Step 3: Conclusion Thus $\int \frac{1}{e^x + 1} dx = x - \ln(1 + e^x) + C$, for all real x .

Quick Tip to Solve: When substituting $t = e^x$, immediately write $dx = dt/t$; the rational integrand then separates into two logarithmic terms.

OR

8(b). Evaluate: $\int_1^4 (|x| + |3-x|) dx$.

[3 marks]

Solution:

Concept:

- The absolute value satisfies $|u| = u$ if $u \geq 0$ and $|u| = -u$ if $u < 0$. Determine the sign on each interval before integrating.
- A definite integral can be split at an interior point: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$. Here

Quick Tip to Solve: Locate the zeros inside the limits first: only $x = 3$ matters here, while $|x| = x$ throughout the interval.

$3 - x$ changes sign at $x = 3$.

Step 1: Remove the absolute values on suitable intervals Throughout $[1, 4]$, $x > 0$, so $|x| = x$. For $1 \leq x \leq 3$, $3 - x \geq 0$, hence $|3 - x| = 3 - x$ and the integrand is $x + 3 - x = 3$. For $3 \leq x \leq 4$, $3 - x \leq 0$, hence $|3 - x| = x - 3$ and the integrand is $x + x - 3 = 2x - 3$. Both expressions agree at $x = 3$, so the common endpoint introduces no ambiguity. The zero of x lies outside the integration interval and requires no extra split.

Step 2: Integrate the two expressions and combine Using the sign change at $x = 3$, the required integral I becomes

$$\begin{aligned} I &= \int_1^3 3 \, dx + \int_3^4 (2x - 3) \, dx \\ &= [3x]_1^3 + [x^2 - 3x]_3^4 \\ &= (9 - 3) + \{(16 - 12) - (9 - 9)\} \\ &= 6 + 4. \end{aligned}$$

The first interval contributes its constant height 3 multiplied by its length 2. On the second interval, the integrand increases linearly from 3 to 5; its average value is 4 over an interval of length 1, confirming the second contribution. Evaluating each antiderivative at both endpoints prevents a sign error in the second term.

Step 3: Conclusion Therefore, $\int_1^4 (|x| + |3 - x|) \, dx = 10$.

Quick Tip to Solve: Locate the zeros inside the limits first: only $x = 3$ matters here, while $|x| = x$ throughout the interval.

9. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors of equal magnitude, then prove that the vector $(2\vec{a} + \vec{b} + 2\vec{c})$ is equally inclined to both $\vec{a}$ and $\vec{c}$. Also, find the angle between $\vec{a}$ and $(2\vec{a} + \vec{b} + 2\vec{c})$. [3 marks]

Solution:

Concept:

- For nonzero vectors, the angle satisfies $\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$, with $0 \leq \theta \leq \pi$.
- Mutually perpendicular vectors have zero pairwise dot products. Let their common magnitude be $k > 0$, so each vector's dot product with itself is k^2 .

Step 1: Find the resultant magnitude Write $\vec{v} = 2\vec{a} + \vec{b} + 2\vec{c}$. By mutual perpendicularity, $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = 0$. Consequently, all cross terms vanish when calculating the squared magnitude:

$$\begin{aligned} |\vec{v}|^2 &= (2\vec{a} + \vec{b} + 2\vec{c}) \cdot (2\vec{a} + \vec{b} + 2\vec{c}) \\ &= 4k^2 + k^2 + 4k^2 = 9k^2. \end{aligned}$$

Hence $|\vec{v}| = 3k$. The common magnitude is positive, as the angle with a zero vector would be undefined.

Step 2: Compare the two inclination angles Let α and γ be the angles made by $\vec{v}$ with $\vec{a}$ and $\vec{c}$,

Quick Tip to Solve: For equal orthogonal vectors, divide a coefficient by the square root of the sum of all squared coefficients to obtain the corresponding angle's cosine.

respectively. Taking dot products and using the vanishing mixed products gives

$$\begin{aligned}\vec{a} \cdot \vec{v} &= 2k^2, & \vec{c} \cdot \vec{v} &= 2k^2, \\ \cos \alpha &= \frac{2k^2}{k(3k)} = \frac{2}{3}, & \cos \gamma &= \frac{2k^2}{k(3k)} = \frac{2}{3}.\end{aligned}$$

Cosine is one-to-one for angles in $[0, \pi]$, so these equal cosines imply $\alpha = \gamma$. The positive dot products show that both angles are acute, consistent with the positive coefficients of $\vec{a}$ and $\vec{c}$ in the resultant.

Step 3: Conclusion Hence the vector is equally inclined to $\vec{a}$ and $\vec{c}$. The required angle is $\cos^{-1}(2/3)$, approximately 48.19° .

Quick Tip to Solve: For equal orthogonal vectors, divide a coefficient by the square root of the sum of all squared coefficients to obtain the corresponding angle's cosine.

10(a). If a line makes 60° and 45° angles with the positive directions of x -axis and z -axis respectively, then find the angle that it makes with the positive direction of y -axis. Hence, write the direction cosines of the line. [3 marks]

Solution:

Concept:

- Direction cosines (l, m, n) are the cosines of the angles (α, β, γ) with the positive coordinate axes, and satisfy $l^2 + m^2 + n^2 = 1$.
- Axis angles lie in $[0^\circ, 180^\circ]$. A known squared direction cosine determines its magnitude but does not, by itself, determine its sign.

Step 1: Substitute the two given axis angles The line makes $\alpha = 60^\circ$ with the positive x -axis and $\gamma = 45^\circ$ with the positive z -axis. Therefore

$$l = \cos 60^\circ = \frac{1}{2}, \quad n = \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

These are positive because the two given angles are acute. The angle with the positive y -axis has not been specified as acute, so its cosine must be found without presuming a sign.

Step 2: Find both admissible values of the remaining cosine Using the direction-cosine identity,

$$\begin{aligned}\left(\frac{1}{2}\right)^2 + m^2 + \left(\frac{1}{\sqrt{2}}\right)^2 &= 1, \\ m^2 &= 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}, \\ m &= \pm \frac{1}{2}.\end{aligned}$$

Thus $\cos \beta = 1/2$ gives $\beta = 60^\circ$, whereas $\cos \beta = -1/2$ gives $\beta = 120^\circ$. Both triples have squared entries summing to one and retain the stipulated positive x - and z -direction cosines. The printed data allow both orientations relative to the positive y -axis; choosing only the acute value would require an additional restriction.

Quick Tip to Solve: Retain both signs after taking a square root of a squared direction cosine unless the question fixes the remaining axis angle as acute or obtuse.

Step 3: Conclusion The angle is 60° or 120° . The corresponding direction cosines are $(1/2, 1/2, 1/\sqrt{2})$ or $(1/2, -1/2, 1/\sqrt{2})$, respectively.

Quick Tip to Solve: Retain both signs after taking a square root of a squared direction cosine unless the question fixes the remaining axis angle as acute or obtuse.

OR

10(b). Check whether the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ are skew or not.

[3 marks]

Solution:

Concept:

- Skew lines are lines in three-dimensional space which are neither parallel nor intersecting. Thus, both parallelism and intersection must be checked before concluding that two lines are skew.
- A symmetric line equation determines a point and a direction vector. Introducing an independent parameter for each line converts the intersection test into three simultaneous coordinate equations.

Step 1: Express each line in parametric form and check parallelism Let the common value in the first line be λ and that in the second line be μ . The coordinate forms are

$$L_1: x = 1 + 2\lambda, \quad y = 2 + 3\lambda, \quad z = 3 + 4\lambda,$$

$$L_2: x = 4 + 5\mu, \quad y = 1 + 2\mu, \quad z = \mu.$$

In particular, the final equality in the second equation is $z = \mu$, so its direction vector is $(5, 2, 1)$, while the first has direction vector $(2, 3, 4)$. These vectors are not proportional, since $2/5 \neq 3/2$. Therefore, the lines are not parallel.

Step 2: Test whether a common point exists At an intersection, all three coordinates must agree for the same pair (λ, μ) . Equating the first two coordinates gives

$$2\lambda - 5\mu = 3,$$

$$3\lambda - 2\mu = -1.$$

Multiplying the first equation by 3 and the second by 2, then subtracting, gives $-11\mu = 11$, hence $\mu = -1$. Substitution into the first equation gives $\lambda = -1$. These values make both lines have $x = -1$ and $y = -1$. However, their third coordinates are respectively

$$z_1 = 3 + 4(-1) = -1, \quad z_2 = \mu = -1.$$

Thus the third coordinate also agrees. The coordinate equations are consistent, and the point $(-1, -1, -1)$ lies on both lines.

Step 3: Conclusion The lines intersect at $(-1, -1, -1)$; therefore, they are **not skew**.

Quick Tip to Solve: Use independent parameters for the two lines and verify the third coordinate after solving the first two coordinate equations; here it confirms the intersection.

Section C (16 marks)

Questions number 11 to 14 carry 4 marks each.

11. Find the equations of the planes passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j}) = 6$ and $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, which are at a distance of 1 unit from the origin. [4 marks]

Solution:

Concept:

- If two intersecting planes have equations $P_1 = 0$ and $P_2 = 0$, then $P_1 + \lambda P_2 = 0$, where λ is real, represents the planes through their common line, apart from the second plane itself.
- For a plane $Ax + By + Cz + D = 0$, its perpendicular distance from the origin is $d = |D|/\sqrt{A^2 + B^2 + C^2}$. Here (A, B, C) are the components of its normal vector.
- Write the position vector as $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ to convert a vector plane equation into Cartesian form.

Step 1: Form the family through the given intersection The two given planes become

$$P_1 = x + 3y - 6 = 0, \quad P_2 = 3x - y - 4z = 0.$$

Every point on their common line satisfies both equations. Therefore it also satisfies their linear combination

$$(x + 3y - 6) + \lambda(3x - y - 4z) = 0,$$

or

$$(1 + 3\lambda)x + (3 - \lambda)y - 4\lambda z - 6 = 0.$$

The normals of the original planes are not parallel, so the line of intersection exists. The omitted member $P_2 = 0$ passes through the origin and has distance zero; hence it cannot be a required plane.

Step 2: Impose the specified perpendicular distance Apply the point-to-plane distance formula at $(0, 0, 0)$:

$$1 = \frac{6}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + 16\lambda^2}}.$$

The denominator is positive. Squaring and expanding gives

$$\begin{aligned}(1 + 6\lambda + 9\lambda^2) + (9 - 6\lambda + \lambda^2) + 16\lambda^2 &= 36, \\ 10 + 26\lambda^2 &= 36, \\ \lambda^2 &= 1.\end{aligned}$$

Thus the two admissible parameter values are $\lambda = 1$ and $\lambda = -1$. Squaring introduces no extraneous values because the original distance is nonnegative.

Step 3: Substitute the parameter values and verify the distances For $\lambda = 1$, the family gives $4x + 2y - 4z - 6 = 0$, which simplifies to $2x + y - 2z = 3$. For $\lambda = -1$, it gives $-2x + 4y + 4z - 6 = 0$,

Quick Tip to Solve: Keep the constant term fixed at -6 in the plane family; the linear terms in the squared normal length cancel, leaving a simple equation in λ^2 .

which simplifies to $-x + 2y + 2z = 3$. Their distances from the origin are respectively

$$\frac{3}{\sqrt{2^2 + 1^2 + (-2)^2}} = 1, \quad \frac{3}{\sqrt{(-1)^2 + 2^2 + 2^2}} = 1.$$

Both came from the plane family, so both contain the required intersection line as well as satisfying the distance condition.

Step 4: Conclusion The required planes are $2x + y - 2z = 3$ and $-x + 2y + 2z = 3$. In vector form they are $\vec{r} \cdot (2\hat{i} + \hat{j} - 2\hat{k}) = 3$ and $\vec{r} \cdot (-\hat{i} + 2\hat{j} + 2\hat{k}) = 3$.

Quick Tip to Solve: Keep the constant term fixed at -6 in the plane family; the linear terms in the squared normal length cancel, leaving a simple equation in λ^2 .

12(a). Find the particular solution of the differential equation $x \frac{dy}{dx} + y + \frac{1}{1+x^2} = 0$, given that $y(1) = 0$. [4 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$. Multiplication by its integrating factor $I(x) = e^{\int P(x) dx}$ makes its left-hand side the derivative of $I(x)y$.
- Here $x \frac{dy}{dx} + y = \frac{d}{dx}(xy)$ by the product rule. Also, $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$, where $\tan^{-1} x$ denotes the principal inverse tangent.
- An initial condition fixes the arbitrary constant. Since the initial point is $x = 1$, the solution is considered on the interval $x > 0$, where division by x is valid.

Step 1: Identify the linear equation and its integrating factor Rearrange the given equation as

$$x \frac{dy}{dx} + y = -\frac{1}{1+x^2}.$$

For $x > 0$, divide by x to obtain

$$\frac{dy}{dx} + \frac{1}{x}y = -\frac{1}{x(1+x^2)}.$$

Thus $P(x) = 1/x$ and $Q(x) = -1/[x(1+x^2)]$. The integrating factor is

$$I(x) = e^{\int dx/x} = e^{\log x} = x.$$

Step 2: Integrate the resulting product derivative Multiplying the linear equation by x gives

$$\frac{d}{dx}(xy) = -\frac{1}{1+x^2}.$$

Quick Tip to Solve: Recognise $xy' + y$ as $(xy)'$ immediately; integrate the remaining rational term and use $y(1) = 0$ to fix the constant.

Integrating both sides with respect to x ,

$$xy = -\tan^{-1} x + C.$$

There is only one arbitrary constant because this is a first-order differential equation. The negative sign follows directly from moving the given positive rational term to the right-hand side.

Step 3: Apply the initial condition and verify the constant Substitute $x = 1$ and $y(1) = 0$ into the integrated relation:

$$0 = -\tan^{-1} 1 + C = -\frac{\pi}{4} + C, \quad C = \frac{\pi}{4}.$$

Hence $xy = \pi/4 - \tan^{-1} x$. Differentiating this relation gives $xy' + y = -1/(1 + x^2)$, exactly the rearranged original equation. At $x = 1$, its right-hand side is $\pi/4 - \pi/4 = 0$, so the initial condition also holds.

Step 4: Conclusion The required particular solution is

$$y(x) = \frac{\pi/4 - \tan^{-1} x}{x}, \quad x > 0.$$

Quick Tip to Solve: Recognise $xy' + y$ as $(xy)'$ immediately; integrate the remaining rational term and use $y(1) = 0$ to fix the constant.

OR

12(b). Find the general solution of the differential equation $x(y^3 + x^3) dy = (2y^4 + 5x^3y) dx$. [4 marks]

Solution:

Concept:

- A first-order differential equation is homogeneous when its slope can be expressed as a function of y/x . The substitution $y = vx$, where v depends on x , gives $dy/dx = v + x dv/dx$.
- After substitution, separate the variables and integrate. If a factor involving v is divided out, check its zero values separately so that constant-ratio solutions are not lost.

Step 1: Express the slope in homogeneous form On an interval where the slope expression is defined and $x \neq 0$, the given equation becomes

$$\frac{dy}{dx} = \frac{2y^4 + 5x^3y}{x(y^3 + x^3)} = \frac{2(y/x)^4 + 5(y/x)}{(y/x)^3 + 1}.$$

Thus the right-hand side depends only on the ratio y/x , which identifies the equation as homogeneous. Put $y = vx$; differentiating by the product rule gives $dy/dx = v + x dv/dx$.

Step 2: Substitute and separate the variables Substituting the new variable into the slope equation

Quick Tip to Solve: After setting $v = y/x$, split the separated fraction as $1/(4v) + 3v^2/[4(v^3 + 4)]$; both integrals then become logarithms immediately.

gives

$$v + x \frac{dv}{dx} = \frac{2v^4 + 5v}{v^3 + 1},$$
$$x \frac{dv}{dx} = \frac{2v^4 + 5v - v(v^3 + 1)}{v^3 + 1} = \frac{v(v^3 + 4)}{v^3 + 1}.$$

For $v \neq 0$ and $v^3 + 4 \neq 0$, separation therefore yields

$$\frac{v^3 + 1}{v(v^3 + 4)} dv = \frac{dx}{x}.$$

The numerator after subtraction is $v^4 + 4v$; retaining both terms is essential for the subsequent integration.

Step 3: Decompose the fraction and integrate Use the identity

$$\frac{v^3 + 1}{v(v^3 + 4)} = \frac{1}{4v} + \frac{3v^2}{4(v^3 + 4)}.$$

Indeed, combining the two terms gives the numerator $(v^3 + 4 + 3v^3)/4 = v^3 + 1$. Hence integration gives

$$\frac{1}{4} \ln |v| + \frac{1}{4} \ln |v^3 + 4| = \ln |x| + C_1,$$
$$\ln |v(v^3 + 4)| = 4 \ln |x| + C_2.$$

In the second integral, $d(v^3 + 4) = 3v^2 dv$. Exponentiating and absorbing the sign and exponential constant into an arbitrary real constant gives $v(v^3 + 4) = Cx^4$.

Step 4: Restore the original variables and check excluded ratios Since $v = y/x$, substitution gives

$$\frac{y}{x} \left(\frac{y^3}{x^3} + 4 \right) = Cx^4, \quad y(y^3 + 4x^3) = Cx^8.$$

Separation excluded $v = 0$ and $v^3 = -4$. These give $y = 0$ and $y = -\sqrt[3]{4}x$, respectively. Substitution into the original differential equation verifies both, and both are included in the implicit family by taking $C = 0$. As an additional check, differentiating the implicit relation gives

$$4(y^3 + x^3) \frac{dy}{dx} + 12x^2y = 8Cx^7.$$

Replacing Cx^7 by $y(y^3 + 4x^3)/x$ recovers the given equation on an interval with $x \neq 0$.

Step 5: Conclusion The general solution is $y^4 + 4x^3y = Cx^8$, where C is an arbitrary real constant. The constant-ratio solutions $y = 0$ and $y = -\sqrt[3]{4}x$ are included when $C = 0$.

Quick Tip to Solve: After setting $v = y/x$, split the separated fraction as $1/(4v) + 3v^2/[4(v^3 + 4)]$; both integrals then become logarithms immediately.

13. Evaluate: $\int_0^{\pi/2} (2 \log \cos x - \log \sin 2x) dx$.

[4 marks]

Solution:

Concept:

- The reflection property of a definite integral is $\int_0^a f(x) dx = \int_0^a f(a-x) dx$. For $a = \pi/2$, it interchanges $\sin x$ and $\cos x$.
- Use $\sin 2x = 2 \sin x \cos x$ and the logarithm laws $\log(ab) = \log a + \log b$ and $\log(a/b) = \log a - \log b$, for positive arguments.

Step 1: Simplify the integrand using the double-angle identity Let the required integral be I . On the open interval $0 < x < \pi/2$, both $\sin x$ and $\cos x$ are positive, so the logarithm laws apply without absolute values. Thus

$$\begin{aligned} 2 \log \cos x - \log \sin 2x &= 2 \log \cos x - \log(2 \sin x \cos x) \\ &= \log \cos x - \log \sin x - \log 2. \end{aligned}$$

By linearity of the convergent improper integrals,

$$I = \int_0^{\pi/2} \log \cos x dx - \int_0^{\pi/2} \log \sin x dx - \int_0^{\pi/2} \log 2 dx.$$

Step 2: Show that the two variable logarithmic integrals are equal Put $x = \pi/2 - t$ in the first integral. Then $dx = -dt$, and the limits $x = 0$ and $x = \pi/2$ become $t = \pi/2$ and $t = 0$, respectively. Hence

$$\begin{aligned} \int_0^{\pi/2} \log \cos x dx &= \int_{\pi/2}^0 \log \cos(\pi/2 - t)(-dt) \\ &= \int_0^{\pi/2} \log \sin t dt. \end{aligned}$$

Since a definite integral is independent of the name of its integration variable, this equals $\int_0^{\pi/2} \log \sin x dx$. Therefore their difference in the expression for I is zero; neither integral needs to be evaluated separately.

Step 3: Integrate the remaining constant Substituting the equality established above gives

$$\begin{aligned} I &= -\log 2 \int_0^{\pi/2} 1 dx \\ &= -\log 2 [x]_0^{\pi/2} \\ &= -\log 2 \left(\frac{\pi}{2} - 0 \right). \end{aligned}$$

Step 4: Conclusion The required value is $\boxed{-\frac{\pi}{2} \log 2}$.

Quick Tip to Solve: Expand $\log \sin 2x$ first; the integrals of $\log \sin x$ and $\log \cos x$ over $[0, \pi/2]$ cancel by reflection, saving their separate evaluation.

14. Case-Study Based Question

In a game of Archery, each ring of the Archery target is valued. The centremost ring is worth 10 points and rest of the rings are allotted points 9 to 1 in sequential order moving outwards.

Archer A is likely to earn 10 points with a probability of 0.8 and Archer B is likely to earn 10 points with a probability of 0.9.

Based on the above information, answer the following questions:

If both of them hit the Archery target, then find the probability that

[4 marks]

14(a). Exactly one of them earns 10 points.

[2 marks]

Solution:

Concept:

- For independent performances, the probability that two specified events occur together is the product of their individual probabilities: $P(E \cap F) = P(E)P(F)$.
- The complement rule gives $P(E') = 1 - P(E)$. Exactly one success means either the first succeeds and the second fails, or the first fails and the second succeeds; these cases are mutually exclusive.

Step 1: Define the scoring events and their complements Let E be the event that Archer A earns 10 points and F the event that Archer B earns 10 points. The given probabilities are $P(E) = 0.8$ and $P(F) = 0.9$. Thus,

$$P(E') = 1 - 0.8 = 0.2, \quad P(F') = 1 - 0.9 = 0.1.$$

Here a complement means that the archer earns a score other than 10 while hitting the target. Treat the two archers' performances as independent.

Step 2: Add the two mutually exclusive possibilities Exactly one archer earns 10 points in the disjoint cases $E \cap F'$ and $E' \cap F$. Independence permits multiplication within each case, and mutual exclusivity permits addition between cases. Therefore,

$$\begin{aligned} P(\text{exactly one earns 10}) &= P(E)P(F') + P(E')P(F) \\ &= (0.8)(0.1) + (0.2)(0.9) \\ &= 0.08 + 0.18 = 0.26. \end{aligned}$$

The event that both earn 10 points is excluded because it contains two successes rather than exactly one.

Step 3: Conclusion The probability that exactly one of the two archers earns 10 points is 0.26.

Quick Tip to Solve: For exactly one success in two independent trials, use $p(1 - q) + (1 - p)q$; include both possible choices of the successful archer.

Solution:**Concept:**

- When two events are independent, the occurrence of one does not change the probability of the other. Their joint probability is $P(E \cap F) = P(E)P(F)$.
- The word "both" specifies the intersection of the two scoring events. It requires each archer to earn 10 points in the same pair of performances.

Step 1: Identify the required intersection Let E denote Archer A earning 10 points and F denote Archer B earning 10 points. From the case study,

$$P(E) = 0.8, \quad P(F) = 0.9.$$

The question asks for both of these events to occur, so the required event is $E \cap F$. Treating the two archers' performances as independent gives $P(F | E) = P(F) = 0.9$: A's score does not affect B's chance of scoring 10.

Step 2: Apply the multiplication rule The multiplication rule for the joint event gives

$$\begin{aligned} P(E \cap F) &= P(E)P(F | E) \\ &= P(E)P(F) \\ &= (0.8)(0.9) = 0.72. \end{aligned}$$

Addition would describe a union and would count outcomes in which only one archer earns 10, so it does not express the requirement "both". The product is smaller than each individual probability, as expected because satisfying both scoring conditions is more restrictive than satisfying either one alone.

Step 3: Conclusion The probability that both archers earn 10 points is 0.72.

Quick Tip to Solve: Translate "both" into an intersection; for independent archers multiply their probabilities of earning 10 points.