

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-4-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) This question paper contains **three** sections – **Section A, B** and **C**.
- (ii) Each section is **compulsory**.
- (iii) **Section A** has **6** short answer type I questions of **2** marks each.
- (iv) **Section B** has **4** short answer type II questions of **3** marks each.
- (v) **Section C** has **4** long answer type questions of **4** marks each.
- (vi) There is an internal choice in some questions.
- (vii) Question no. **14** is a case-study based question with **2** sub-parts of **2** marks each.

Section A (12 marks)

Questions number 1 to 6 carry 2 marks each.

1. The foot of a perpendicular drawn from the point $(-2, -1, -3)$ on a plane is $(1, -3, 3)$. Find the equation of the plane. [2 marks]

Solution:

Concept:

- The line joining a point to its perpendicular foot on a plane is normal to the plane. The vector joining the given points therefore gives a normal vector.
- A plane through $H(x_0, y_0, z_0)$ with normal vector (a, b, c) has equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Step 1: Find a normal vector Let the given point be $P(-2, -1, -3)$ and its perpendicular foot be $H(1, -3, 3)$. Since PH is perpendicular to the plane, a normal vector is

$$\overrightarrow{PH} = (1 + 2, -3 + 1, 3 + 3) = (3, -2, 6).$$

Any nonzero scalar multiple of this vector gives the same plane, so reversing the subtraction would not change the result.

Step 2: Use the foot on the plane The foot H belongs to the required plane. Using this point and the computed normal vector gives

$$\begin{aligned} 3(x - 1) - 2(y + 3) + 6(z - 3) &= 0, \\ 3x - 2y + 6z - 3 - 6 - 18 &= 0. \end{aligned}$$

The constant term is obtained by expanding all three brackets. As a check, at H the expanded expression is $3 + 6 + 18 - 27 = 0$, confirming that the foot lies on the plane.

Step 3: Conclusion The required plane is $3x - 2y + 6z - 27 = 0$.

Quick Tip to Solve: Subtract the original point from its perpendicular foot for the normal, then use the foot in the point-normal equation.

2. A coin is tossed twice. The following table shows the probability distribution of number of tails:

X	0	1	2
$P(X)$	K	$6K$	$9K$

(a) Find the value of K .

(b) Is the coin tossed biased or unbiased? Justify your answer.

[2 marks]

Solution:

Concept:

- For a probability distribution, the probabilities of all possible values add to 1. Here X denotes the number of tails in two tosses.

Quick Tip to Solve: First add the three probabilities to find K ; then compare the distribution with the unbiased ratio $1 : 2 : 1$.

- For an unbiased coin, each toss has tail probability $1/2$, and two independent tosses give $P(X = 0) = 1/4$, $P(X = 1) = 1/2$, $P(X = 2) = 1/4$.

Step 1: Normalize the given distribution The possible values 0, 1, 2 account for all outcomes. Therefore

$$\begin{aligned}K + 6K + 9K &= 1, \\16K &= 1, \\K &= \frac{1}{16}.\end{aligned}$$

The corresponding probabilities are $1/16$, $6/16 = 3/8$, and $9/16$; they are all nonnegative and have sum 1.

Step 2: Compare with an unbiased coin An unbiased coin would give the equally likely ordered outcomes HH, HT, TH and TT. Consequently its distribution of tails would be $1/4, 1/2, 1/4$, in that order. The given distribution is different: for example, $P(X = 2) = 9/16$, rather than $1/4$. Hence the coin is biased. In the usual independent-toss model, if $p = P(T)$, then $p^2 = 9/16$ gives $p = 3/4$, which also yields $2p(1 - p) = 3/8$ as required.

Step 3: Conclusion (a) $K = 1/16$. (b) The coin is **biased**, since its distribution differs from that of an unbiased coin.

Quick Tip to Solve: First add the three probabilities to find K ; then compare the distribution with the unbiased ratio $1 : 2 : 1$.

3(a). If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 400$ and $|\vec{b}| = 5$, then find the value of $|\vec{a}|$.

[2 marks]

Solution:

Concept:

- If θ is the angle between two nonzero vectors, then $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$.
- The identity $\sin^2 \theta + \cos^2 \theta = 1$ gives $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$.

Step 1: Combine the two products Write both squared quantities using the angle between the vectors:

$$\begin{aligned}|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 &= |\vec{a}|^2 |\vec{b}|^2 (\sin^2 \theta + \cos^2 \theta) \\&= |\vec{a}|^2 |\vec{b}|^2.\end{aligned}$$

The modulus around the dot product causes no difficulty because squaring removes its possible negative sign. We therefore do not need to determine the angle.

Step 2: Substitute the given magnitude The sum is 400 and $|\vec{b}| = 5$. Hence

$$\begin{aligned}|\vec{a}|^2 (5)^2 &= 400, \\|\vec{a}|^2 &= \frac{400}{25} = 16.\end{aligned}$$

A vector magnitude is nonnegative, so we take the positive square root of 16. The negative root of a scalar equation cannot represent a magnitude. The resulting product of squared magnitudes is

Quick Tip to Solve: When squared dot and cross products are added, combine them into $|\vec{a}|^2 |\vec{b}|^2$ before substituting values.

$16 \times 25 = 400$, checking the original condition.

Step 3: Conclusion The required magnitude is $|\vec{a}| = 4$.

Quick Tip to Solve: When squared dot and cross products are added, combine them into $|\vec{a}|^2|\vec{b}|^2$ before substituting values.

OR

3(b). Find all the possible vectors of magnitude $5\sqrt{3}$ which are equally inclined to the coordinate axes. [2 marks]

Solution:

Concept:

- For a nonzero vector, direction cosines are $l = \cos \alpha$, $m = \cos \beta$, $n = \cos \gamma$, where the angles are measured with the positive coordinate axes.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$. A vector of magnitude r equals $r(\hat{i}l + \hat{j}m + \hat{k}n)$.

Step 1: Use equal inclination Equal inclination gives $\alpha = \beta = \gamma$, hence $l = m = n$. The direction-cosine identity then gives

$$3l^2 = 1 \Rightarrow l = \pm \frac{1}{\sqrt{3}}.$$

All three direction cosines are equal, so their signs must agree. Mixed signs give unequal angles with the positive coordinate axes.

Step 2: Recover and check the components Multiply each direction cosine by the magnitude $5\sqrt{3}$. The components are all 5, or all -5 , giving $5\hat{i} + 5\hat{j} + 5\hat{k}$ and $-5\hat{i} - 5\hat{j} - 5\hat{k}$. Each has magnitude $\sqrt{25 + 25 + 25} = 5\sqrt{3}$. For the first vector all direction cosines equal $1/\sqrt{3}$; for the second all equal $-1/\sqrt{3}$, so both satisfy equal inclination. These exhaust the two roots of the direction-cosine equation.

Step 3: Conclusion All possible vectors are $\vec{a} = \pm 5(\hat{i} + \hat{j} + \hat{k})$, with a common sign for all three components.

Quick Tip to Solve: Equal angles with the positive axes require equal direction cosines, including their signs; do not choose the three signs independently.

4. Find the general solution of the differential equation $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$. [2 marks]

Solution:

Concept:

- The product rule in differential form is $d(uv) = v \, du + u \, dv$.
- Since $d(\tan x) = \sec^2 x \, dx$ and $d(\tan y) = \sec^2 y \, dy$, the given expression can be recognised as an exact product differential.

Quick Tip to Solve: Look for a product derivative before separating variables; here it preserves the zero-tangent solutions without extra divisions.

Step 1: Recognise the product differential Start with the printed equation

$$\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0.$$

Set $u = \tan x$ and $v = \tan y$. The left side is $v \, du + u \, dv = d(uv)$. Therefore the equation becomes $d(\tan x \tan y) = 0$. This recognition avoids dividing by either tangent, which could discard solutions where a tangent is zero.

Step 2: Integrate and verify A function whose differential vanishes is constant along a solution curve. Integration therefore gives $\tan x \tan y = C$, where C is an arbitrary real constant. Differentiating this relation produces

$$\tan y \sec^2 x \, dx + \tan x \sec^2 y \, dy = 0,$$

exactly the original equation. The relation is used on intervals where both tangents are defined, namely $\cos x \neq 0$ and $\cos y \neq 0$. In particular, constant solutions $y = k\pi$, for integer k , are included by $C = 0$.

Step 3: Conclusion The general solution is $\boxed{\tan x \tan y = C}$, on intervals where $\cos x \cos y \neq 0$.

Quick Tip to Solve: Look for a product derivative before separating variables; here it preserves the zero-tangent solutions without extra divisions.

5. Evaluate: $\int_0^1 x^2 e^x \, dx$.

[2 marks]

Solution:

Concept:

- Integration by parts gives $\int u \, dv = uv - \int v \, du$. A polynomial times an exponential is simplified by differentiating the polynomial.
- For a definite integral, first obtain an antiderivative $F(x)$, then use $\int_a^b f(x) \, dx = F(b) - F(a)$.

Step 1: Integrate by parts twice Choose $u = x^2$ and $dv = e^x \, dx$, so $du = 2x \, dx$ and $v = e^x$. Then

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

In the remaining integral choose $u = x$ and $dv = e^x \, dx$. It gives $\int x e^x \, dx = x e^x - e^x$. Consequently an antiderivative is $F(x) = e^x(x^2 - 2x + 2)$. Differentiating gives $F'(x) = e^x(x^2 - 2x + 2) + e^x(2x - 2) = x^2 e^x$, checking the calculation.

Step 2: Substitute the two limits Apply the upper limit and then subtract the lower-limit value:

$$\begin{aligned} \int_0^1 x^2 e^x \, dx &= [e^x(x^2 - 2x + 2)]_0^1 \\ &= e(1 - 2 + 2) - 1(0 - 0 + 2) \\ &= e - 2. \end{aligned}$$

Quick Tip to Solve: Differentiate the polynomial in successive integrations by parts, and remember to subtract the full antiderivative value at zero.

The lower endpoint contributes 2, even though the integrand itself is zero there. The result is positive, consistent with the nonnegative integrand throughout $[0, 1]$.

Step 3: Conclusion The required value is $\boxed{e - 2}$.

Quick Tip to Solve: Differentiate the polynomial in successive integrations by parts, and remember to subtract the full antiderivative value at zero.

6. There are two bags. Bag I contains 1 red and 3 white balls, and Bag II contains 3 red and 5 white balls. A bag is selected at random and a ball is drawn from it. Find the probability that the ball so drawn is red in colour. [2 marks]

Solution:

Concept:

- The law of total probability combines mutually exclusive and exhaustive ways in which an event can occur. If B_1 and B_2 denote selection of Bags I and II, and R denotes drawing a red ball, then $P(R) = P(B_1)P(R | B_1) + P(B_2)P(R | B_2)$.
- For a random draw from a bag, the conditional probability of a red ball equals the number of red balls divided by the total number of balls in that bag.

Step 1: Find the probabilities for each bag Since the bag is selected at random from the two bags, $P(B_1) = P(B_2) = \frac{1}{2}$. Bag I has $1 + 3 = 4$ balls, while Bag II has $3 + 5 = 8$ balls. Hence,

$$P(R | B_1) = \frac{1}{4}, \quad P(R | B_2) = \frac{3}{8}.$$

Step 2: Apply the law of total probability A red ball can be drawn by selecting Bag I and drawing its red ball, or by selecting Bag II and drawing one of its red balls. These two routes are mutually exclusive. Therefore,

$$\begin{aligned} P(R) &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{8} \\ &= \frac{1}{8} + \frac{3}{16} = \frac{5}{16}. \end{aligned}$$

The bags have equal selection probabilities, even though their numbers of balls differ; consequently, the balls in the combined collection are not all equally likely to be selected.

Step 3: Conclusion The probability that the ball drawn is red is $\boxed{\frac{5}{16}}$.

Quick Tip to Solve: Average the two within-bag red probabilities with weights $\frac{1}{2}$; do not pool the balls and use $\frac{4}{12}$.

Section B (12 marks)

Questions number 7 to 10 carry 3 marks each.

7. Using integration, find the area of the region $\{(x, y) : y^2 \leq x \leq y\}$.

[3 marks]

Solution:

Concept:

- The area between curves described as $x = f(y)$ and $x = g(y)$ is found using horizontal strips:
 $A = \int_a^b [x_{\text{right}} - x_{\text{left}}] dy$, where a, b are the bounding values of y .
- The region $y^2 \leq x \leq y$ is bounded on the left by the parabola $x = y^2$ and on the right by the line $x = y$. A strip has non-negative width only where $y^2 \leq y$.

Step 1: Determine the intersection points and limits The two boundary curves intersect when their values of x are equal. Thus,

$$y^2 = y \implies y(y - 1) = 0,$$

giving $y = 0$ and $y = 1$. The corresponding points are $(0, 0)$ and $(1, 1)$. Moreover,

$$y^2 \leq y \iff y(y - 1) \leq 0 \iff 0 \leq y \leq 1.$$

This also proves that the inequalities give no additional region outside these limits.

Step 2: Express the area as an integral At each fixed y between 0 and 1, x runs from y^2 to y . Therefore the horizontal strip has width $y - y^2$, thickness dy , and area $(y - y^2) dy$. Adding these strips gives

$$A = \int_0^1 (y - y^2) dy.$$

The integrand is non-negative throughout the interval, so no absolute value or splitting into smaller intervals is necessary.

Step 3: Evaluate the definite integral Integrating each power of y and applying the limits,

$$\begin{aligned} A &= \left[\frac{y^2}{2} - \frac{y^3}{3} \right]_0^1 \\ &= \left(\frac{1}{2} - \frac{1}{3} \right) - (0 - 0) = \frac{1}{6}. \end{aligned}$$

Step 4: Conclusion The required area is $\boxed{\frac{1}{6} \text{ square units}}$.

Quick Tip to Solve: Since both bounds are already expressed as x in terms of y , integrate right minus left with respect to y between 0 and 1.

8(a). If a line makes 60° and 45° angles with the positive directions of x -axis and z -axis respectively, then find the angle that it makes with the positive direction of y -axis. Hence, write the direction cosines of the line. [3 marks]

Solution:

Concept:

- Direction cosines are $l = \cos \alpha$, $m = \cos \beta$, $n = \cos \gamma$, where the angles are measured with the positive coordinate axes.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$. Each direction angle is between 0° and 180° , so a direction cosine can be negative.

Step 1: Identify the known direction cosines From the given angles $\alpha = 60^\circ$ and $\gamma = 45^\circ$, we obtain

$$l = \cos 60^\circ = \frac{1}{2}, \quad n = \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

Let β be the angle with the positive y -axis. Then the remaining direction cosine is $m = \cos \beta$. Although the known cosines are positive, this does not force the remaining cosine to be positive: a direction can have positive x - and z -components and a negative y -component.

Step 2: Apply the sum-of-squares identity Substituting the known values gives

$$\begin{aligned} \frac{1}{4} + \cos^2 \beta + \frac{1}{2} &= 1, \\ \cos^2 \beta &= \frac{1}{4}, \\ \cos \beta &= \pm \frac{1}{2}. \end{aligned}$$

In the allowed interval, the positive value gives $\beta = 60^\circ$ and the negative value gives $\beta = 120^\circ$. Both choices satisfy the identity because $1/4 + 1/4 + 1/2 = 1$. The question imposes no additional condition that excludes the obtuse angle, so both possibilities must be retained.

Step 3: Conclusion The angle is 60° or 120° . The corresponding direction cosines are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ or $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$, respectively.

Quick Tip to Solve: Keep both square-root signs when finding a direction cosine unless the remaining direction angle is explicitly acute.

OR

8(b). Check whether the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ are skew or not. [3 marks]

Solution:

Concept:

- Skew lines in space are non-parallel lines that do not intersect. Therefore both parallelism and intersection must be checked.

Quick Tip to Solve: After solving two coordinate equations for the parameters, always verify the third equation before declaring that two spatial lines intersect.

- In symmetric form, the denominators give direction ratios. Introducing a separate parameter for each line gives coordinates of every point on that line.

Step 1: Compare the direction ratios The direction ratios of the first line are $(2, 3, 4)$, while those of the second are $(5, 2, 1)$, since its last expression is $z = z/1$. These triples are not proportional; for example $2/5 \neq 3/2$. Hence the two lines are not parallel. This check alone does not establish that they are skew, since non-parallel lines can still intersect.

Step 2: Test whether a common point exists Let the common ratio on the first line be t , and that on the second line be s . Their parametric equations are

$$\begin{aligned} L_1 : (x, y, z) &= (1 + 2t, 2 + 3t, 3 + 4t), \\ L_2 : (x, y, z) &= (4 + 5s, 1 + 2s, s). \end{aligned}$$

A common point would require equality of all three coordinates. The x - and y -coordinates give

$$2t - 5s = 3, \quad 3t - 2s = -1.$$

Multiplying these equations by 3 and 2, respectively, and subtracting yields $-11s = 11$, so $s = -1$. Then $3t + 2 = -1$, giving $t = -1$.

Step 3: Check the third coordinate For these parameters the first line has $z = 3 + 4(-1) = -1$, and the second has $z = s = -1$. Thus the third equation also holds. Substituting into the other coordinates gives $x = 1 + 2(-1) = -1$ and $y = 2 + 3(-1) = -1$; the second line gives the same values. This verifies an actual common point rather than merely a solution of two coordinate equations.

Step 4: Conclusion The lines intersect at $(-1, -1, -1)$. Hence they are not skew.

Quick Tip to Solve: After solving two coordinate equations for the parameters, always verify the third equation before declaring that two spatial lines intersect.

9(a). Find: $\int \frac{1}{e^x + 1} dx$.

[3 marks]

Solution:

Concept:

- Substitution converts an integral involving e^x into a rational integral: if $t = e^x$, then $dt = e^x dx = t dx$ and $dx = dt/t$.
- Partial fractions give $1/[t(t + 1)] = 1/t - 1/(t + 1)$. The basic rule is $\int du/u = \ln |u| + C$, where C is an arbitrary constant.

Step 1: Substitute the exponential expression Let $I = \int dx/(e^x + 1)$. Put $t = e^x$. Since $e^x > 0$ for every real x , we have $t > 0$. Differentiating the substitution gives $dt = t dx$, so it is essential to replace dx by dt/t , rather than just dt . Thus

$$I = \int \frac{1}{t + 1} \frac{dt}{t} = \int \frac{dt}{t(t + 1)}.$$

Quick Tip to Solve: With $t = e^x$, replace dx by dt/t ; the extra factor $1/t$ makes the partial fractions immediate.

Step 2: Resolve into partial fractions and integrate Write $1/[t(t+1)] = A/t + B/(t+1)$. On multiplying by $t(t+1)$, we obtain $1 = A(t+1) + Bt$. Comparing the constant and linear coefficients gives $A = 1$ and $B = -1$. Therefore

$$\begin{aligned} I &= \int \left(\frac{1}{t} - \frac{1}{t+1} \right) dt \\ &= \ln |t| - \ln |t+1| + C \\ &= x - \ln(1 + e^x) + C. \end{aligned}$$

Both t and $t+1$ are positive, so the absolute-value signs disappear after substituting back.

Step 3: Verify by differentiation The derivative of the expression obtained is

$$\frac{d}{dx} [x - \ln(1 + e^x)] = 1 - \frac{e^x}{1 + e^x} = \frac{1}{1 + e^x},$$

which equals the given integrand for all real x .

Step 4: Conclusion Hence, $\int \frac{dx}{e^x + 1} = x - \ln(1 + e^x) + C$, where C is an arbitrary constant.

Quick Tip to Solve: With $t = e^x$, replace dx by dt/t ; the extra factor $1/t$ makes the partial fractions immediate.

OR

9(b). Evaluate: $\int_1^4 (|x| + |3 - x|) dx$.

[3 marks]

Solution:

Concept:

- For a real number u , $|u| = u$ when $u \geq 0$, and $|u| = -u$ when $u < 0$. Determine the sign of each expression before removing its modulus.
- A definite integral can be split at a point where the integrand changes its formula: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, with $a \leq c \leq b$.

Step 1: Determine the formulas on the two intervals Throughout $[1, 4]$, $x > 0$, hence $|x| = x$. The expression $3 - x$ is non-negative on $[1, 3]$ and non-positive on $[3, 4]$. Consequently,

$$|x| + |3 - x| = \begin{cases} x + (3 - x) = 3, & 1 \leq x \leq 3, \\ x + (x - 3) = 2x - 3, & 3 \leq x \leq 4. \end{cases}$$

At $x = 3$, both formulas give 3, so either expression can be used at the splitting point. There is no need to split at zero, since zero lies outside the integration interval.

Quick Tip to Solve: On $[1, 4]$, only $3 - x$ changes sign; split at 3 and keep $|x| = x$ throughout.

Step 2: Split and evaluate the definite integral Using the change of sign at $x = 3$,

$$\begin{aligned} I &= \int_1^3 3 \, dx + \int_3^4 (2x - 3) \, dx \\ &= [3x]_1^3 + [x^2 - 3x]_3^4 \\ &= (9 - 3) + [(16 - 12) - (9 - 9)] \\ &= 6 + 4. \end{aligned}$$

The first contribution comes from a constant integrand over an interval of length two. On the second interval, the integrand increases linearly from 3 to 5, so its average is 4; over an interval of length one this independently confirms the second contribution as 4.

Step 3: Conclusion Therefore, $\int_1^4 (|x| + |3 - x|) \, dx = 10$.

Quick Tip to Solve: On $[1, 4]$, only $3 - x$ changes sign; split at 3 and keep $|x| = x$ throughout.

10. If $\vec{a}$ and $\vec{b}$ are two vectors of equal magnitude and α is the angle between them, then prove that $\frac{|\vec{a} + \vec{b}|}{|\vec{a} - \vec{b}|} = \cot\left(\frac{\alpha}{2}\right)$. [3 marks]

Solution:

Concept:

- For vectors $\vec{u}$ and $\vec{v}$, $|\vec{u} \pm \vec{v}|^2 = |\vec{u}|^2 + |\vec{v}|^2 \pm 2\vec{u} \cdot \vec{v}$. The dot product equals $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \alpha$, where α is their included angle.
- The half-angle identities are $1 + \cos \alpha = 2 \cos^2(\alpha/2)$ and $1 - \cos \alpha = 2 \sin^2(\alpha/2)$. Since an included angle belongs to $[0, \pi]$, both half-angle sine and cosine are non-negative.

Step 1: Use the common magnitude in the dot product Let $|\vec{a}| = |\vec{b}| = r$. The angle between the vectors is defined for nonzero vectors, so $r > 0$. Hence

$$\vec{a} \cdot \vec{b} = r^2 \cos \alpha.$$

Step 2: Calculate the magnitudes of the sum and difference Expanding the dot products gives

$$\begin{aligned} |\vec{a} + \vec{b}|^2 &= (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) \\ &= r^2 + r^2 + 2r^2 \cos \alpha \\ &= 2r^2(1 + \cos \alpha), \\ |\vec{a} - \vec{b}|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\ &= r^2 + r^2 - 2r^2 \cos \alpha \\ &= 2r^2(1 - \cos \alpha). \end{aligned}$$

Step 3: Apply the half-angle identities and take non-negative roots Substituting the half-angle

Quick Tip to Solve: Expand the squared magnitudes first, cancel the common magnitude factor, and then use the two half-angle identities.

identities into these expressions yields

$$\begin{aligned} |\vec{a} + \vec{b}|^2 &= 4r^2 \cos^2(\alpha/2), \\ |\vec{a} - \vec{b}|^2 &= 4r^2 \sin^2(\alpha/2). \end{aligned}$$

Since magnitudes are non-negative and $0 \leq \alpha/2 \leq \pi/2$, their square roots are $2r \cos(\alpha/2)$ and $2r \sin(\alpha/2)$, respectively. For the ratio in the question to be defined we require $\vec{a} \neq \vec{b}$, or $\alpha > 0$. If $\alpha = 0$, both the given ratio and the cotangent expression are undefined; at $\alpha = \pi$, the ratio is well defined and equals zero.

Step 4: Conclusion Therefore, wherever the stated ratio is defined,

$$\frac{|\vec{a} + \vec{b}|}{|\vec{a} - \vec{b}|} = \frac{2r \cos(\alpha/2)}{2r \sin(\alpha/2)} = \boxed{\cot\left(\frac{\alpha}{2}\right)}.$$

Quick Tip to Solve: Expand the squared magnitudes first, cancel the common magnitude factor, and then use the two half-angle identities.

Section C (16 marks)

Questions number 11 to 14 carry 4 marks each.

11(a). Find the particular solution of the differential equation $x \frac{dy}{dx} + y + \frac{1}{1+x^2} = 0$, given that $y(1) = 0$. [4 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$. Its integrating factor is $I(x) = e^{\int P(x) dx}$; multiplying by this factor makes the left side the derivative of $I(x)y$.
- The product rule gives $\frac{d}{dx}(xy) = x \frac{dy}{dx} + y$. Also, $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$, where $\tan^{-1} x$ denotes the principal inverse tangent in radians.

Step 1: Put the equation in linear form The given equation is

$$x \frac{dy}{dx} + y + \frac{1}{1+x^2} = 0.$$

Since the initial point is $x = 1$, we work on the interval $x > 0$, which contains it and avoids the singular point $x = 0$. Dividing by x , we obtain

$$\frac{dy}{dx} + \frac{1}{x}y = -\frac{1}{x(1+x^2)}.$$

Thus $P(x) = 1/x$ and $Q(x) = -1/[x(1+x^2)]$.

Quick Tip to Solve: Recognise $xy' + y$ as $(xy)'$ immediately; then integrate and use $\tan^{-1} 1 = \pi/4$ to fix the constant.

Step 2: Find the integrating factor and integrate On $x > 0$, the integrating factor is

$$I(x) = e^{\int dx/x} = e^{\log x} = x.$$

Multiplication by x therefore gives

$$\frac{d}{dx}(xy) = -\frac{1}{1+x^2}.$$

Integrating both sides with respect to x ,

$$xy = -\tan^{-1} x + C.$$

The constant C is determined by the prescribed initial value rather than left arbitrary in the particular solution.

Step 3: Apply the initial condition Substituting $x = 1$ and $y(1) = 0$ into the integrated equation gives

$$0 = -\tan^{-1} 1 + C = -\frac{\pi}{4} + C, \quad C = \frac{\pi}{4}.$$

Hence the relation satisfying the initial condition is $xy = \pi/4 - \tan^{-1} x$.

Step 4: Verify the equation and initial value For $y = (\pi/4 - \tan^{-1} x)/x$, the quotient rule gives

$$y' = -\frac{1}{x(1+x^2)} - \frac{\pi/4 - \tan^{-1} x}{x^2}.$$

Therefore $xy' + y = -1/(1+x^2)$, so the given differential equation is satisfied. At $x = 1$, the numerator is $\pi/4 - \pi/4 = 0$, giving the required value $y(1) = 0$.

Step 5: Conclusion The particular solution on the interval containing the initial point is

$$\boxed{y(x) = \frac{\pi/4 - \tan^{-1} x}{x}}, \quad x > 0.$$

Quick Tip to Solve: Recognise $xy' + y$ as $(xy)'$ immediately; then integrate and use $\tan^{-1} 1 = \pi/4$ to fix the constant.

OR

11(b). Find the general solution of the differential equation $x(y^3 + x^3) dy = (2y^4 + 5x^3y) dx$. [4 marks]

Solution:

Concept:

- A first-order equation is homogeneous when dy/dx is a function of y/x . Put $y = vx$, where v is a function of x , so that $dy/dx = v + x dv/dx$.
- After separating variables, integrate both sides. Use $\int dt/t = \ln |t| + C$, with the absolute value ensuring a real logarithm for either sign of a nonzero argument.

Quick Tip to Solve: After putting $y = vx$, split the separated fraction into $1/(4v)$ and $3v^2/[4(v^3 + 4)]$; both integrals are immediately logarithms.

- If separation divides by a function of v , check its zeros separately: they may give constant-ratio solutions of the original equation.

Step 1: Recognise the homogeneous equation From the printed equation, on an interval where the denominator is nonzero,

$$\frac{dy}{dx} = \frac{2y^4 + 5x^3y}{x(y^3 + x^3)} = \frac{2(y/x)^4 + 5(y/x)}{(y/x)^3 + 1}.$$

Both numerator and denominator in the original ratio have total degree four. Thus the substitution $v = y/x$ applies for $x \neq 0$. It reduces the equation to

$$v + x \frac{dv}{dx} = \frac{2v^4 + 5v}{v^3 + 1}.$$

Step 2: Separate the variables Subtract v , combining over the same denominator:

$$x \frac{dv}{dx} = \frac{2v^4 + 5v - v(v^3 + 1)}{v^3 + 1} = \frac{v(v^3 + 4)}{v^3 + 1}.$$

For $v(v^3 + 4) \neq 0$, separation gives

$$\frac{v^3 + 1}{v(v^3 + 4)} dv = \frac{dx}{x}.$$

The useful decomposition is

$$\frac{v^3 + 1}{v(v^3 + 4)} = \frac{1}{4v} + \frac{3v^2}{4(v^3 + 4)},$$

since the combined numerator is $(v^3 + 4 + 3v^3)/4 = v^3 + 1$.

Step 3: Integrate and return to the original variables Integrating and using $d(v^3 + 4) = 3v^2 dv$, we obtain

$$\frac{1}{4} \ln |v| + \frac{1}{4} \ln |v^3 + 4| = \ln |x| + C_1.$$

Hence $\ln |v(v^3 + 4)| = 4 \ln |x| + C_2$. Absorbing the magnitude and sign into an arbitrary constant gives $v(v^3 + 4) = Cx^4$. Substituting $v = y/x$ and multiplying by x^4 gives the implicit family in the original variables.

Step 4: Check excluded constant ratios and verify The factors divided out vanish at $v = 0$ and $v^3 = -4$. They give $y = 0$ and $y = -\sqrt[3]{4}x$, respectively; substitution confirms both satisfy the printed equation. Both are retained by allowing $C = 0$ in the implicit family. To verify the family directly, differentiate $y^4 + 4x^3y = Cx^8$:

$$4(y^3 + x^3) \frac{dy}{dx} + 12x^2y = 8Cx^7.$$

Substitute $Cx^8 = y^4 + 4x^3y$, multiply by $x/4$, and simplify to obtain $x(y^3 + x^3) dy/dx = 2y^4 + 5x^3y$, exactly the given equation. Solutions are taken on intervals where the differential equation defines the required branch.

Step 5: Conclusion The general solution is $y^4 + 4x^3y = Cx^8$, where C is an arbitrary real constant. The constant-ratio solutions lost during separation are included when $C = 0$.

Quick Tip to Solve: After putting $y = vx$, split the separated fraction into $1/(4v)$ and $3v^2/[4(v^3 + 4)]$; both integrals are immediately logarithms.

12. Evaluate: $\int_0^{\pi} \frac{x}{9 \sin^2 x + 16 \cos^2 x} dx.$

[4 marks]

Solution:

Concept:

- The reflection property of a definite integral is $\int_0^a f(x) dx = \int_0^a f(a-x) dx$. It is useful when the factor x is multiplied by a function symmetric about $a/2$.
- Here $\sin^2(\pi-x) = \sin^2 x$ and $\cos^2(\pi-x) = \cos^2 x$. The denominator therefore remains unchanged under reflection.
- On $0 \leq x < \pi/2$, the substitution $t = \tan x$ gives $dx = dt/(1+t^2)$, $\sin^2 x = t^2/(1+t^2)$ and $\cos^2 x = 1/(1+t^2)$.

Step 1: Use reflection to remove the factor x Let

$$I = \int_0^{\pi} \frac{x}{9 \sin^2 x + 16 \cos^2 x} dx.$$

Applying the reflection property with $a = \pi$, and using the squared trigonometric identities, gives

$$I = \int_0^{\pi} \frac{\pi - x}{9 \sin^2 x + 16 \cos^2 x} dx.$$

Add this expression to the original integral. The two numerators add to the constant π , so

$$2I = \pi \int_0^{\pi} \frac{dx}{9 \sin^2 x + 16 \cos^2 x}.$$

This addition avoids integration by parts and reduces the problem to an integral without a linear factor.

Step 2: Reduce the interval to the first quadrant Write $J = \int_0^{\pi} \frac{dx}{9 \sin^2 x + 16 \cos^2 x}$. Its integrand takes the same value at x and $\pi - x$, so the contributions from the two half-intervals are equal. Hence

$$J = 2 \int_0^{\pi/2} \frac{dx}{9 \sin^2 x + 16 \cos^2 x}.$$

The denominator is positive throughout the interval, including at $\pi/2$, so the original integral has no singularity. We use a limit only because the tangent substitution tends to infinity at the upper endpoint.

Step 3: Evaluate the remaining integral by tangent substitution Put $t = \tan x$. The limits become $t = 0$ and $t \rightarrow \infty$. Substituting the expressions for the squares and for dx , the common factor $1+t^2$ cancels:

$$\begin{aligned} J &= 2 \int_0^{\infty} \frac{dt}{9t^2 + 16} \\ &= \frac{1}{6} \left[\tan^{-1} \left(\frac{3t}{4} \right) \right]_0^{\infty} \\ &= \frac{1}{6} \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{12}. \end{aligned}$$

The antiderivative can be checked by differentiation: the derivative of $\frac{1}{12} \tan^{-1}(3t/4)$ is $1/(9t^2 + 16)$.

Quick Tip to Solve: When an integral from zero to pi contains x times a symmetric trigonometric function, reflect x to pi minus x and add before attempting a substitution.

Thus the factor of two from the half-interval is included correctly.

Step 4: Conclusion Since $2I = \pi J$ and $J = \pi/12$, the required value is

$$I = \frac{\pi^2}{24}.$$

Quick Tip to Solve: When an integral from zero to pi contains x times a symmetric trigonometric function, reflect x to pi minus x and add before attempting a substitution.

13. Find the equations of the planes passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j}) = 6$ and $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, which are at a distance of 1 unit from the origin. [4 marks]

Solution:

Concept:

- If two intersecting planes have equations $P_1 = 0$ and $P_2 = 0$, then $P_1 + \lambda P_2 = 0$, for a real parameter λ , passes through their common line. Points on that line satisfy both equations separately.
- The perpendicular distance of the origin from $ax + by + cz = d$ is $D = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$, where (a, b, c) is a normal vector to the plane.

Step 1: Express the given planes in Cartesian form Put $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. The two equations become

$$x + 3y - 6 = 0, \quad 3x - y - 4z = 0.$$

Their normals $(1, 3, 0)$ and $(3, -1, -4)$ are not proportional, so they intersect in a line. Every point of this line lies on each member of the family

$$(x + 3y - 6) + \lambda(3x - y - 4z) = 0.$$

Thus the required plane can be written as

$$(1 + 3\lambda)x + (3 - \lambda)y - 4\lambda z = 6.$$

The second given plane, excluded from this parameter form, passes through the origin and has distance zero; it cannot be a required plane.

Step 2: Apply the distance condition to determine the parameter The constant on the right is 6, so the distance formula gives

$$\frac{6}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + 16\lambda^2}} = 1.$$

Expanding the squared normal length, the linear terms cancel:

$$\begin{aligned}(1 + 3\lambda)^2 + (3 - \lambda)^2 + 16\lambda^2 &= 1 + 6\lambda + 9\lambda^2 + 9 - 6\lambda + \lambda^2 + 16\lambda^2 \\ &= 10 + 26\lambda^2.\end{aligned}$$

Quick Tip to Solve: Use a plane pencil first; here the given normals are perpendicular, so the mixed parameter terms in the squared normal length cancel.

Both sides of the distance equation are positive, so squaring is valid. Hence

$$10 + 26\lambda^2 = 36 \implies \lambda^2 = 1 \implies \lambda = 1 \text{ or } -1.$$

Step 3: Substitute both values and check the resulting planes For $\lambda = 1$, the family gives $4x + 2y - 4z = 6$, or $2x + y - 2z = 3$. For $\lambda = -1$, it gives $-2x + 4y + 4z = 6$, or $-x + 2y + 2z = 3$. Each equation was obtained as a linear combination of the given plane equations, so each contains their common line. Their perpendicular distances from the origin are respectively

$$\frac{3}{\sqrt{2^2 + 1^2 + (-2)^2}} = 1, \quad \frac{3}{\sqrt{(-1)^2 + 2^2 + 2^2}} = 1.$$

The two parameter values yield distinct planes and exhaust the distance condition.

Step 4: Conclusion The required planes are

$$\vec{r} \cdot (2\hat{i} + \hat{j} - 2\hat{k}) = 3 \quad \text{and} \quad \vec{r} \cdot (-\hat{i} + 2\hat{j} + 2\hat{k}) = 3.$$

Quick Tip to Solve: Use a plane pencil first; here the given normals are perpendicular, so the mixed parameter terms in the squared normal length cancel.

14. Case-Study Based Question

In a game of Archery, each ring of the Archery target is valued. The centremost ring is worth 10 points and rest of the rings are allotted points 9 to 1 in sequential order moving outwards.

Archer A is likely to earn 10 points with a probability of 0.8 and Archer B is likely to earn 10 points with a probability of 0.9.

Based on the above information, answer the following questions:

If both of them hit the Archery target, then find the probability that

[4 marks]

14(a). Exactly one of them earns 10 points.

[2 marks]

Solution:

Concept:

- Let E and F denote the events that Archer A and Archer B earn 10 points, respectively. Their probabilities are $P(E) = 0.8$ and $P(F) = 0.9$.
- For the independent shots of the two archers, multiply probabilities for simultaneous outcomes. The complement rule gives $P(E') = 1 - P(E)$ and $P(F') = 1 - P(F)$.
- Exactly one success means $(E \cap F') \cup (E' \cap F)$. These two cases are mutually exclusive, so their probabilities add.

Step 1: Calculate the probabilities of missing the 10-point ring Both archers hit the target; here a

Quick Tip to Solve: For exactly one success in two independent trials, use $p(1 - q) + (1 - p)q$, counting both possible archers.

failure means earning a score other than 10, not missing the target. Therefore,

$$P(E') = 1 - 0.8 = 0.2, \quad P(F') = 1 - 0.9 = 0.1.$$

Step 2: Add the two exclusive ways to earn exactly one 10-point score Archer A may earn 10 points while Archer B does not, or Archer B may earn 10 points while Archer A does not. Using independence,

$$\begin{aligned} P(E \cap F') &= 0.8 \times 0.1 = 0.08, \\ P(E' \cap F) &= 0.2 \times 0.9 = 0.18, \\ P((E \cap F') \cup (E' \cap F)) &= 0.08 + 0.18 = 0.26. \end{aligned}$$

The two cases cannot occur together, since each requires a different archer to be the only one scoring 10.

Step 3: Conclusion The probability that exactly one of the two archers earns 10 points is $\boxed{0.26}$.

Quick Tip to Solve: For exactly one success in two independent trials, use $p(1 - q) + (1 - p)q$, counting both possible archers.

14(b). Both of them earn 10 points.

[2 marks]

Solution:

Concept:

- Let E be the event that Archer A earns 10 points and F the event that Archer B earns 10 points. The given probabilities are $P(E) = 0.8$ and $P(F) = 0.9$.
- Both archers earning 10 points is the intersection event $E \cap F$, because the two required outcomes must occur together.
- The shots of the two archers are independent. Hence the multiplication rule is $P(E \cap F) = P(E)P(F)$.

Step 1: Identify the simultaneous outcome required Both archers have hit the target, and the required outcome is that each earns the centremost ring's score of 10 points. We therefore require Archer A's event E and Archer B's event F simultaneously. This is $E \cap F$, not the event that at least one archer earns 10 points.

Step 2: Apply the multiplication rule for independent shots One archer's score does not affect the other archer's score, so the probability of the joint outcome is the product of the two given probabilities:

$$\begin{aligned} P(E \cap F) &= P(E)P(F) \\ &= 0.8 \times 0.9 \\ &= 0.72. \end{aligned}$$

As a check, the joint probability is smaller than each individual probability, as it must be: every outcome in $E \cap F$ is also an outcome in both E and F .

Step 3: Conclusion The probability that both archers earn 10 points is $\boxed{0.72}$.

Quick Tip to Solve: Translate the word "both" into an intersection and multiply the probabilities when the two trials are independent.