

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-4-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) This question paper contains **three** sections – **Section A, B and C**.
- (ii) Each section is **compulsory**.
- (iii) **Section A** has **6** short answer type I questions of **2 marks** each.
- (iv) **Section B** has **4** short answer type II questions of **3 marks** each.
- (v) **Section C** has **4** long answer type questions of **4 marks** each.
- (vi) There is an internal choice in some questions.
- (vii) Question no. **14** is a case-study based question with **2** sub-parts of **2 marks** each.

Section A (12 marks)

Questions number 1 to 6 carry 2 marks each.

1(a). If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 400$ and $|\vec{b}| = 5$, then find the value of $|\vec{a}|$.

[2 marks]

Solution:

Concept:

- For two vectors making an angle θ , $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ and $|\vec{a} \cdot \vec{b}| = |\vec{a}||\vec{b}|\cos\theta$.
- Squaring and adding eliminates the angle by $\sin^2\theta + \cos^2\theta = 1$. Hence $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$.

Step 1: Use the vector product identity The supplied expression is precisely the sum in this identity. Consequently, the relation becomes

$$|\vec{a}|^2|\vec{b}|^2 = 400.$$

We do not need to find the angle between the vectors: the trigonometric terms have already combined to give unity. This identity holds equally for parallel, perpendicular and oblique vectors.

Step 2: Substitute the known magnitude Using $|\vec{b}| = 5$, we obtain

$$25|\vec{a}|^2 = 400, \quad |\vec{a}|^2 = \frac{400}{25} = 16.$$

A vector magnitude is nonnegative, so we take the positive square root. A negative square root would solve the scalar squared equation but cannot be a length.

Step 3: Conclusion Hence, $|\vec{a}| = 4$.

Quick Tip to Solve: Replace the sum of squared dot and cross products by the product of squared magnitudes; the angle cancels immediately.

OR

1(b). Find all the possible vectors of magnitude $5\sqrt{3}$ which are equally inclined to the coordinate axes.

[2 marks]

Solution:

Concept:

- The direction cosines l, m, n of a vector are the cosines of its angles with the positive coordinate axes. They satisfy $l^2 + m^2 + n^2 = 1$.
- A vector of magnitude r has components rl, rm, rn , so $\vec{v} = r(\hat{i}l + \hat{j}m + \hat{k}n)$. Equal inclinations imply equal direction cosines, including their signs.

Step 1: Determine the common direction cosine Let the equal angles with the coordinate axes be α . Then $l = m = n = \cos\alpha$. Applying the direction-cosine identity gives

$$3\cos^2\alpha = 1, \quad \cos\alpha = \pm\frac{1}{\sqrt{3}}.$$

Quick Tip to Solve: Equal angles to the coordinate axes require a common sign for all direction cosines; use 3 times their common square equal to 1.

All three direction cosines must have the same sign because the angles themselves are equal. Choosing the three signs independently would give unequal inclinations to the positive axes.

Step 2: Find the corresponding components The given magnitude is $r = 5\sqrt{3}$. With the positive common direction cosine, each component is $5\sqrt{3}/\sqrt{3} = 5$; with the negative common direction cosine, each is -5 . Thus the two possible component triples are $(5, 5, 5)$ and $(-5, -5, -5)$. Each has magnitude $\sqrt{25 + 25 + 25} = 5\sqrt{3}$, confirming the required length.

Step 3: Conclusion The required vectors are $\boxed{5\hat{i} + 5\hat{j} + 5\hat{k}}$ and $\boxed{-5\hat{i} - 5\hat{j} - 5\hat{k}}$.

Quick Tip to Solve: Equal angles to the coordinate axes require a common sign for all direction cosines; use 3 times their common square equal to 1.

2. Evaluate: $\int_0^1 x^2 e^x dx$.

[2 marks]

Solution:

Concept:

- Integration by parts states $\int u dv = uv - \int v du$. Choose the polynomial as u so its degree decreases upon differentiation.
- For a definite integral, evaluate the resulting antiderivative at the upper and lower limits and subtract: $\int_a^b f(x) dx = F(b) - F(a)$.

Step 1: Apply integration by parts twice Take $u = x^2$ and $dv = e^x dx$. Then $du = 2x dx$ and $v = e^x$, giving

$$\int x^2 e^x dx = x^2 e^x - 2 \int x e^x dx.$$

For the remaining integral, choose $u = x$ and $dv = e^x dx$. Hence

$$\int x e^x dx = x e^x - e^x.$$

Substitution gives the antiderivative $F(x) = e^x(x^2 - 2x + 2)$.

Step 2: Evaluate at the specified limits We now use the bounds 0 and 1:

$$\begin{aligned} \int_0^1 x^2 e^x dx &= [e^x(x^2 - 2x + 2)]_0^1 \\ &= e(1 - 2 + 2) - 1(0 - 0 + 2) \\ &= e - 2. \end{aligned}$$

In particular, the lower-limit contribution is 2, not zero: although x^2 vanishes there, the complete antiderivative does not. Differentiating $F(x)$ yields $e^x[(x^2 - 2x + 2) + (2x - 2)] = x^2 e^x$, verifying the integration.

Step 3: Conclusion Therefore, $\boxed{\int_0^1 x^2 e^x dx = e - 2}$.

Quick Tip to Solve: For a polynomial times an exponential, integrate by parts until the polynomial disappears, and retain the full lower-limit contribution.

3. Three friends A , B and C got their photograph clicked. Find the probability that B is standing at the central position, given that A is standing at the left corner. [2 marks]

Solution:

Concept:

- Conditional probability is $P(E | F) = P(E \cap F)/P(F)$, where F is the given event and $P(F) > 0$.
- All arrangements of three distinct friends are equally likely. Under a given condition, count only arrangements satisfying that condition.

Step 1: Use the given position of A Let E be the event that B stands in the centre and F the event that A stands at the left corner. Write arrangements in the order left, centre, right. Once A occupies the left corner, the two remaining friends can stand in the orders BC or CB . Thus the conditional sample space is $\{ABC, ACB\}$, with two equally likely outcomes.

Step 2: Count the favourable arrangements Only ABC has B in the central position. There is therefore one favourable arrangement among the two allowed arrangements. Alternatively, all $3! = 6$ arrangements give $P(F) = 2/6$ and $P(E \cap F) = 1/6$. Applying the conditional probability formula gives

$$P(E | F) = \frac{1/6}{2/6} = \frac{1}{2}.$$

The condition changes the denominator from six arrangements to the two with A on the left.

Step 3: Conclusion The required probability is $\boxed{\frac{1}{2}}$.

Quick Tip to Solve: Fix A at the left first, then count the two possible orders of B and C in the remaining positions.

4. A coin is tossed twice. The following table shows the probability distribution of number of tails:

X	0	1	2
$P(X)$	K	$6K$	$9K$

(a) Find the value of K .

(b) Is the coin tossed biased or unbiased? Justify your answer. [2 marks]

Solution:

Concept:

- The probabilities in a distribution sum to one: $\sum_x P(X = x) = 1$. Here X denotes the number of tails in two tosses.
- For an unbiased coin, heads and tails each have probability $1/2$; hence two heads and two tails each have probability $1/4$.

Step 1: Find the normalising constant The three possible values of X are 0, 1, 2, and the listed

Quick Tip to Solve: After normalising, compare the probabilities of zero and two tails with $1/4$, the unbiased values.

probabilities are $K, 6K, 9K$. Since these are all possible outcomes for the number of tails,

$$K + 6K + 9K = 1 \Rightarrow 16K = 1 \Rightarrow K = \frac{1}{16}.$$

Thus the distribution is $P(X = 0) = 1/16$, $P(X = 1) = 6/16$, and $P(X = 2) = 9/16$. These values are non-negative and sum to one.

Step 2: Check whether the coin is unbiased If the coin were unbiased, the four ordered outcomes HH, HT, TH, TT would each have probability $1/4$. Consequently, the distribution of the number of tails would be $1/4, 1/2, 1/4$. The given distribution instead has $P(X = 0) = 1/16$ and $P(X = 2) = 9/16$, both different from $1/4$. In fact it is consistent with independent tosses having $P(T) = 3/4$ and $P(H) = 1/4$, since their squares are $9/16$ and $1/16$, while $2(3/4)(1/4) = 6/16$.

Step 3: Conclusion (a) $K = \frac{1}{16}$. (b) The coin is **biased**, since its distribution differs from that of an unbiased coin.

Quick Tip to Solve: After normalising, compare the probabilities of zero and two tails with $1/4$, the unbiased values.

5. The foot of a perpendicular drawn from the point $(-2, -1, -3)$ on a plane is $(1, -3, 3)$. Find the equation of the plane. [2 marks]

Solution:

Concept:

- A line perpendicular to a plane gives a normal vector to that plane. The vector joining a point to its perpendicular foot is therefore a normal vector.
- A plane through (x_0, y_0, z_0) with normal vector (a, b, c) has equation $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.

Step 1: Find a normal vector Let the given point be $P = (-2, -1, -3)$ and its perpendicular foot on the plane be $H = (1, -3, 3)$. Since PH is perpendicular to the plane,

$$\overrightarrow{PH} = (1 - (-2), -3 - (-1), 3 - (-3)) = (3, -2, 6)$$

is a normal vector. The opposite vector would define the same plane, because multiplying a plane equation by a non-zero constant does not change its set of points.

Step 2: Use the foot as a point on the plane The foot H lies on the required plane. For any point $Q = (x, y, z)$ on that plane, $\overrightarrow{HQ}$ must be perpendicular to its normal vector. Hence

$$\begin{aligned}(3, -2, 6) \cdot (x - 1, y + 3, z - 3) &= 0, \\ 3(x - 1) - 2(y + 3) + 6(z - 3) &= 0, \\ 3x - 2y + 6z - 27 &= 0.\end{aligned}$$

As a check, substituting $H = (1, -3, 3)$ gives $3 + 6 + 18 - 27 = 0$, so the equation indeed contains the perpendicular foot.

Quick Tip to Solve: Use the vector from the given point to the perpendicular foot as the normal, and substitute the foot.

Step 3: Conclusion The required plane is $\boxed{3x - 2y + 6z = 27}$.

Quick Tip to Solve: Use the vector from the given point to the perpendicular foot as the normal, and substitute the foot.

6. Find the general solution of the differential equation $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$. [2 marks]

Solution:

Concept:

- The product rule gives $d(uv) = v \, du + u \, dv$. A differential equation that is the total differential of a function can be integrated directly.
- Here $d(\tan x) = \sec^2 x \, dx$ and $d(\tan y) = \sec^2 y \, dy$, wherever the tangent functions are defined.

Step 1: Identify the total differential The equation is

$$\tan y \sec^2 x \, dx + \tan x \sec^2 y \, dy = 0.$$

Take $u = \tan x$ and $v = \tan y$. The first term is $v \, du$, while the second is $u \, dv$. Thus the entire left-hand side is the differential of their product:

$$d(\tan x \tan y) = 0.$$

This avoids division by either tangent and so also retains solutions for which the product is zero.

Step 2: Integrate and verify the constant Integrating the total differential gives

$$\tan x \tan y = C,$$

where C is an arbitrary real constant on a connected interval where the original equation is defined. Differentiating this relation gives

$$\sec^2 x \tan y \, dx + \tan x \sec^2 y \, dy = 0,$$

exactly the given differential equation. In particular, $C = 0$ includes the constant solutions $y = k\pi$, with integer k . Points where $\cos x = 0$ or $\cos y = 0$ are excluded because the original coefficients are undefined there.

Step 3: Conclusion The general solution is $\boxed{\tan x \tan y = C}$, where C is an arbitrary constant.

Quick Tip to Solve: Look for the product rule before separating variables: the two terms are precisely the differentials of the two tangent factors.

Section B (12 marks)

Questions number 7 to 10 carry 3 marks each.

7. If the area of the region bounded by the line $y = mx$ and the curve $x^2 = y$ is $\frac{32}{3}$ sq. units, then find the positive value of m , using integration. [3 marks]

Solution:

Concept:

- The area between two curves is the integral of the upper ordinate minus the lower ordinate over the interval bounded by their intersection points.
- For $y = mx$ and $y = x^2$, the intersections satisfy $mx = x^2$. Since the question asks for positive m , the bounded interval is $[0, m]$.

Step 1: Find the intersection points At an intersection the two values of y must agree. Therefore

$$x^2 = mx \implies x(x - m) = 0.$$

Hence $x = 0$ or $x = m$, and the intersection points are $(0, 0)$ and (m, m^2) . Because $m > 0$, the origin is the left endpoint and $x = m$ is the right endpoint of the bounded region. These endpoints supply the limits for integration with respect to x .

Step 2: Determine the upper curve and integrate For $0 \leq x \leq m$, the difference between the line and the parabola is

$$mx - x^2 = x(m - x) \geq 0.$$

Thus the line lies above the parabola throughout this interval. The required area A , in square units, is

$$\begin{aligned} A &= \int_0^m (mx - x^2) dx \\ &= \left[\frac{mx^2}{2} - \frac{x^3}{3} \right]_0^m \\ &= \frac{m^3}{2} - \frac{m^3}{3} = \frac{m^3}{6}. \end{aligned}$$

Here m is constant during integration; only x varies. The integrand is nonnegative, so this integral already equals the geometric area and requires no sign reversal.

Step 3: Use the given area to determine the slope Equating the integral to the given area yields

$$\frac{m^3}{6} = \frac{32}{3} \implies m^3 = 64.$$

Taking the real cube root gives $m = 4$, which satisfies the required positivity condition. As a check, with $m = 4$, the bounds become 0 and 4, and the computed area is $4^3/6 = 64/6 = 32/3$ square units, matching the question.

Step 4: Conclusion The positive value of the slope is $m = 4$.

Quick Tip to Solve: For a positive slope, $mx - x^2 = x(m - x)$ immediately gives both the integration limits and the upper-minus-lower order.

8. If $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\vec{b} = \hat{i} + \hat{j} - 2\hat{k}$ and $\vec{c} = \hat{i} + 3\hat{j} - \hat{k}$ and the projection of vector $\vec{c} + \lambda\vec{b}$ on vector $\vec{a}$ is $2\sqrt{6}$, then find the value of λ . [3 marks]

Solution:

Concept:

- The scalar projection of $\vec{u}$ on a nonzero vector $\vec{a}$ is $\vec{u} \cdot \vec{a} / |\vec{a}|$, the signed component along the direction of $\vec{a}$.
- The dot product distributes over addition: $(\vec{c} + \lambda\vec{b}) \cdot \vec{a} = \vec{c} \cdot \vec{a} + \lambda(\vec{b} \cdot \vec{a})$. Multiply corresponding Cartesian components and add.

Step 1: Evaluate the magnitude and dot products From $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$,

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}.$$

The given vectors yield

$$\vec{c} \cdot \vec{a} = (1)(2) + (3)(-1) + (-1)(1) = -2,$$

$$\vec{b} \cdot \vec{a} = (1)(2) + (1)(-1) + (-2)(1) = -1.$$

Thus $(\vec{c} + \lambda\vec{b}) \cdot \vec{a} = -2 - \lambda$. Retain both negative signs when forming the equation. The denominator is the magnitude of the vector onto which the projection is taken, namely $\vec{a}$.

Step 2: Use the given projection Substituting the stated scalar projection gives

$$\frac{-2 - \lambda}{\sqrt{6}} = 2\sqrt{6},$$

$$-2 - \lambda = 12,$$

$$-\lambda = 14.$$

Check the resulting value directly: $\vec{c} + \lambda\vec{b} = -13\hat{i} - 11\hat{j} + 27\hat{k}$. Its dot product with $\vec{a}$ is $-26 + 11 + 27 = 12$. Dividing by $\sqrt{6}$ gives $12/\sqrt{6} = 2\sqrt{6}$, which agrees with the stated projection. The positive projection means the component along $\vec{a}$ is positive, even though the parameter is negative.

Step 3: Conclusion Therefore $\lambda = -14$.

Quick Tip to Solve: Compute the two dot products first; the projection condition reduces directly to $-2 - \lambda = 12$.

9(a). If a line makes 60° and 45° angles with the positive directions of x -axis and z -axis respectively, then find the angle that it makes with the positive direction of y -axis. Hence, write the direction cosines of the line. [3 marks]

Solution:

Concept:

- If a directed line makes angles α, β, γ with the positive coordinate axes, its direction cosines are $l = \cos \alpha, m = \cos \beta, n = \cos \gamma$.
- Direction cosines satisfy $l^2 + m^2 + n^2 = 1$. A squared equation determines the magnitude of a

Quick Tip to Solve: After using the sum of squared direction cosines, keep both signs of the missing cosine unless its direction angle is specified as acute.

missing direction cosine, so both possible signs must be considered.

Step 1: Substitute the known direction cosines Here $\alpha = 60^\circ$ and $\gamma = 45^\circ$, so

$$l = \cos 60^\circ = \frac{1}{2}, \quad n = \cos 45^\circ = \frac{1}{\sqrt{2}}.$$

Let β be the angle with the positive direction of the y -axis. Using the sum of squares identity,

$$\begin{aligned} \frac{1}{4} + \cos^2 \beta + \frac{1}{2} &= 1, \\ \cos^2 \beta &= \frac{1}{4}. \end{aligned}$$

This identity follows from normalising a direction vector to unit length and is applicable to the three angles measured from the same directed line.

Step 2: Determine both permitted angles Taking square roots gives $\cos \beta = \pm \frac{1}{2}$. Direction angles lie between 0° and 180° ; therefore β may be 60° or 120° . No condition in the question fixes whether the line points towards positive or negative y . For $\beta = 60^\circ$, the direction cosines are $(1/2, 1/2, 1/\sqrt{2})$. For $\beta = 120^\circ$, they are $(1/2, -1/2, 1/\sqrt{2})$. In either case their squared values sum to $1/4 + 1/4 + 1/2 = 1$, while the given positive x and z components are preserved. The two choices represent different permitted directions, rather than reversal of the same direction.

Step 3: Conclusion The angle is 60° or 120° , with respective direction cosines $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$ or $\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{\sqrt{2}}\right)$.

Quick Tip to Solve: After using the sum of squared direction cosines, keep both signs of the missing cosine unless its direction angle is specified as acute.

OR

9(b). Check whether the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ are skew or not. [3 marks]

Solution:

Concept:

- Two lines are skew when they are neither parallel nor intersecting. Both conditions must be checked; a failed intersection test alone is insufficient when the lines are parallel.
- In symmetric equations, denominators give direction ratios. Equating the common ratio to a parameter describes every point on the line.

Step 1: Identify the directions Set the common ratios for the first and second lines equal to s and t ,

Quick Tip to Solve: Read the final expression z as $z/1$; solve two coordinate equations and verify the third using the same parameter pair.

respectively. The final expression z for the second line means $z/1$, so

$$L_1 : x = 1 + 2s, y = 2 + 3s, z = 3 + 4s,$$

$$L_2 : x = 4 + 5t, y = 1 + 2t, z = t.$$

Their direction vectors are $(2, 3, 4)$ and $(5, 2, 1)$. Since $2/5 \neq 3/2$, these vectors are not proportional. Thus the lines are not parallel. We must next check whether they intersect in three-dimensional space.

Step 2: Find and verify a common point Equality of the first two coordinates at an intersection requires

$$2s - 5t = 3, \quad 3s - 2t = -1.$$

Multiply the first equation by 3 and the second by 2. Subtracting the resulting equations gives $-11t = 11$, hence $t = -1$. Substitution into $3s - 2t = -1$ gives $s = -1$. Now check the remaining coordinate: $3 + 4s = -1$, which equals $t = -1$. Thus all three coordinate equations hold for the same parameter pair. Substituting into either line gives $x = -1, y = -1, z = -1$. Checking the third coordinate is essential, since agreement in only two coordinates would not establish an intersection in space.

Step 3: Conclusion The lines intersect at $(-1, -1, -1)$, so they are **not skew**.

Quick Tip to Solve: Read the final expression z as $z/1$; solve two coordinate equations and verify the third using the same parameter pair.

10(a). Find: $\int \frac{1}{e^x + 1} dx$.

[3 marks]

Solution:

Concept:

- An indefinite integral represents a family of antiderivatives, so an arbitrary constant C is included in the answer.
- A substitution must transform both the integrand and dx . With $t = e^x$, we have $dt = e^x dx = t dx$, hence $dx = dt/t$.
- Partial fractions separate a rational integrand into elementary logarithmic integrals: $\int du/u = \ln|u| + C$.

Step 1: Substitute for the exponential Write $I = \int dx/(e^x + 1)$. Put $t = e^x$, so $t > 0$ for every real x , and $dx = dt/t$. Thus

$$I = \int \frac{1}{t+1} \frac{dt}{t} = \int \frac{dt}{t(t+1)}.$$

The extra factor $1/t$ comes from transforming the differential; it cannot be omitted. The substitution has converted the exponential expression into a rational function.

Quick Tip to Solve: Split $1/(1 + e^x) = 1 - e^x/(1 + e^x)$ to recognize the logarithmic derivative of $1 + e^x$ immediately.

Step 2: Resolve into partial fractions and integrate Since

$$\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1},$$

integrating term by term gives

$$\begin{aligned} I &= \ln|t| - \ln|t+1| + C \\ &= x - \ln(1+e^x) + C. \end{aligned}$$

Here $t = e^x > 0$ and $t+1 > 0$, so the absolute-value signs may be removed. Also $\ln(e^x) = x$, which restores the original variable.

Step 3: Verify by differentiation The proposed antiderivative is defined for all real x , since $1+e^x > 0$. Its derivative is

$$\frac{d}{dx}[x - \ln(1+e^x)] = 1 - \frac{e^x}{1+e^x} = \frac{1}{1+e^x},$$

which agrees exactly with the given integrand.

Step 4: Conclusion $\int \frac{dx}{e^x+1} = x - \ln(1+e^x) + C$, where C is an arbitrary real constant.

Quick Tip to Solve: Split $1/(1+e^x) = 1 - e^x/(1+e^x)$ to recognize the logarithmic derivative of $1+e^x$ immediately.

OR

10(b). Evaluate: $\int_1^4 (|x| + |3-x|) dx$.

[3 marks]

Solution:

Concept:

- The modulus satisfies $|u| = u$ when $u \geq 0$, and $|u| = -u$ when $u < 0$. Thus its expression must be chosen according to the sign of its argument.
- A definite integral is additive over adjacent intervals: $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, where $a < c < b$. Split at a point where a modulus argument changes sign.

Step 1: Determine the signs on the interval of integration For every $x \in [1, 4]$, x is positive, so $|x| = x$ throughout. The other argument, $3-x$, is non-negative for $1 \leq x \leq 3$, and negative for $3 < x \leq 4$. Consequently,

$$|3-x| = \begin{cases} 3-x, & 1 \leq x \leq 3, \\ x-3, & 3 < x \leq 4. \end{cases}$$

Both expressions give zero at $x = 3$, so including this endpoint in either interval causes no discrepancy.

Step 2: Split the integral and simplify each integrand Denoting the required integral by I , use the

Quick Tip to Solve: First locate the zeros of the modulus arguments: only $x = 3$ lies inside $[1, 4]$, so split there and replace $|x|$ directly by x .

sign change at $x = 3$ to write

$$\begin{aligned} I &= \int_1^3 \{x + (3 - x)\} dx + \int_3^4 \{x + (x - 3)\} dx \\ &= \int_1^3 3 dx + \int_3^4 (2x - 3) dx. \end{aligned}$$

On the first interval the two terms involving x cancel. On the second interval the negative argument of the modulus must be reversed before adding it to x .

Step 3: Apply the limits to the antiderivatives The antiderivatives of 3 and $2x - 3$ are $3x$ and $x^2 - 3x$, respectively. Hence,

$$\begin{aligned} I &= [3x]_1^3 + [x^2 - 3x]_3^4 \\ &= (9 - 3) + \{(16 - 12) - (9 - 9)\} \\ &= 6 + 4. \end{aligned}$$

As a check on the working, the first interval has constant integrand 3 and length 2, giving 6; the second has positive linear integrand increasing from 3 to 5 over length 1, giving 4.

Step 4: Conclusion Therefore, $\int_1^4 (|x| + |3 - x|) dx = \boxed{10}$.

Quick Tip to Solve: First locate the zeros of the modulus arguments: only $x = 3$ lies inside $[1, 4]$, so split there and replace $|x|$ directly by x .

Section C (16 marks)

Questions number 11 to 14 carry 4 marks each.

11. Evaluate: $\int_{-3}^3 \frac{x^4}{1 + e^x} dx$.

[4 marks]

Solution:

Concept:

- Reflection property of a definite integral: $\int_{-a}^a f(x) dx = \int_{-a}^a f(-x) dx$. It follows by the substitution $x = -t$, reversing the limits and cancelling the negative differential.
- For the exponential denominators, $\frac{1}{1 + e^x} + \frac{1}{1 + e^{-x}} = 1$. The numerator x^4 is even, so it is unchanged by reflection.
- For an even function g , $\int_{-a}^a g(x) dx = 2 \int_0^a g(x) dx$; the power rule gives $\int x^4 dx = x^5/5 + C$.

Step 1: Reflect the integration variable Let the required integral be

$$I = \int_{-3}^3 \frac{x^4}{1 + e^x} dx.$$

The integrand is continuous throughout the interval because $1 + e^x > 0$. Put $x = -t$, so $dx = -dt$.

Quick Tip to Solve: On symmetric limits, pair the integrand at x and $-x$: an even numerator over $1 + e^x$ becomes half the corresponding polynomial integral.

The lower limit $x = -3$ becomes $t = 3$, and the upper limit $x = 3$ becomes $t = -3$. Thus

$$I = \int_3^{-3} \frac{t^4}{1 + e^{-t}} (-dt) = \int_{-3}^3 \frac{t^4}{1 + e^{-t}} dt.$$

Renaming the dummy variable t as x gives a second expression for exactly the same integral.

Step 2: Add the original and reflected expressions Adding these two expressions term by term,

$$\begin{aligned} 2I &= \int_{-3}^3 x^4 \left(\frac{1}{1 + e^x} + \frac{1}{1 + e^{-x}} \right) dx \\ &= \int_{-3}^3 x^4 \left(\frac{1}{1 + e^x} + \frac{e^x}{1 + e^x} \right) dx \\ &= \int_{-3}^3 x^4 dx. \end{aligned}$$

The exponential factors disappear because the two fractions have complementary numerators. Although x^4 is even, the original integrand itself is not even; the reflection-and-addition step is what permits this simplification.

Step 3: Evaluate the remaining polynomial integral Since x^4 is even, the last integral equals twice its integral over $[0, 3]$. Therefore

$$2I = 2 \int_0^3 x^4 dx = 2 \left[\frac{x^5}{5} \right]_0^3 = \frac{486}{5}.$$

Dividing the equation by 2 gives the required value. Equivalently, the reflected pair contributes exactly one copy of the polynomial integral, so each of the two equal integrals contributes half of it.

Step 4: Conclusion Hence, $\int_{-3}^3 \frac{x^4}{1 + e^x} dx = \boxed{\frac{243}{5}}.$

Quick Tip to Solve: On symmetric limits, pair the integrand at x and $-x$: an even numerator over $1 + e^x$ becomes half the corresponding polynomial integral.

12. Find the equations of the planes passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + 3\hat{j}) = 6$ and $\vec{r} \cdot (3\hat{i} - \hat{j} - 4\hat{k}) = 0$, which are at a distance of 1 unit from the origin. [4 marks]

Solution:

Concept:

- If two intersecting planes are written as $P_1 = 0$ and $P_2 = 0$, the family $P_1 + \lambda P_2 = 0$, with real parameter λ , contains every plane through their common line except $P_2 = 0$ itself.
- For a plane $ax + by + cz = d$, its perpendicular distance from the origin is $D = \frac{|d|}{\sqrt{a^2 + b^2 + c^2}}$. Here (a, b, c) is a normal vector to the plane.

Quick Tip to Solve: The given normals have dot product $3 - 3 = 0$, so the squared length of the family normal is immediately $10 + 26\lambda^2$; this avoids expanding every square.

Step 1: Form the family through the intersection Write $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. The given planes become

$$x + 3y - 6 = 0, \quad 3x - y - 4z = 0.$$

Since the given normals are not proportional, the planes intersect in a line. The required family is

$$(x + 3y - 6) + \lambda(3x - y - 4z) = 0,$$

or

$$(1 + 3\lambda)x + (3 - \lambda)y - 4\lambda z = 6.$$

The excluded member $3x - y - 4z = 0$ passes through the origin and therefore has distance zero. It cannot satisfy the required distance, so the displayed family is sufficient for this problem.

Step 2: Apply the prescribed distance from the origin Using the coefficients of the family, the condition $D = 1$ gives

$$\frac{6}{\sqrt{(1 + 3\lambda)^2 + (3 - \lambda)^2 + 16\lambda^2}} = 1.$$

The terms linear in λ cancel. Thus

$$\begin{aligned}(1 + 3\lambda)^2 + (3 - \lambda)^2 + 16\lambda^2 &= 10 + 26\lambda^2, \\ 36 &= 10 + 26\lambda^2, \\ \lambda^2 &= 1.\end{aligned}$$

Hence the two possible parameters are $\lambda = 1$ and $\lambda = -1$.

Step 3: Substitute and verify both members For $\lambda = 1$, substitution gives $4x + 2y - 4z = 6$, or $2x + y - 2z = 3$. For $\lambda = -1$, it gives $-2x + 4y + 4z = 6$, or $-x + 2y + 2z = 3$. Both contain the original intersection. Their distances from the origin are

$$\frac{3}{\sqrt{2^2 + 1^2 + (-2)^2}} = 1, \quad \frac{3}{\sqrt{(-1)^2 + 2^2 + 2^2}} = 1.$$

Step 4: Conclusion The required planes are $\boxed{\vec{r} \cdot (2\hat{i} + \hat{j} - 2\hat{k}) = 3}$ and $\boxed{\vec{r} \cdot (-\hat{i} + 2\hat{j} + 2\hat{k}) = 3}$.

Quick Tip to Solve: The given normals have dot product $3 - 3 = 0$, so the squared length of the family normal is immediately $10 + 26\lambda^2$; this avoids expanding every square.

13(a). Find the particular solution of the differential equation $x \frac{dy}{dx} + y + \frac{1}{1+x^2} = 0$, given that $y(1) = 0$. [4 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$. Its integrating factor is $\text{I.F.} = e^{\int P(x) dx}$.
- Multiplication by the integrating factor makes the left side an exact product derivative. Here the useful identity is $\frac{d}{dx}(xy) = x \frac{dy}{dx} + y$, and $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$.

Step 1: Write the linear equation and find the integrating factor Since the initial condition is at $x = 1$, work on the interval $x > 0$, where division by x is valid. Rearranging the given equation yields

$$\frac{dy}{dx} + \frac{1}{x}y = -\frac{1}{x(1+x^2)}.$$

Thus $P(x) = 1/x$ and $Q(x) = -1/[x(1+x^2)]$. The integrating factor on this interval is

$$\text{I.F.} = e^{\int dx/x} = e^{\log x} = x.$$

Step 2: Integrate the product derivative Multiplying the linear equation by x , we get

$$x \frac{dy}{dx} + y = -\frac{1}{1+x^2} \implies \frac{d}{dx}(xy) = -\frac{1}{1+x^2}.$$

Integrating both sides with respect to x gives

$$xy = -\tan^{-1} x + C, \quad xy + \tan^{-1} x = C.$$

The negative sign comes from moving the positive term $1/(1+x^2)$ to the right side of the original equation.

Step 3: Apply the initial condition and verify Substitute $x = 1$ and $y(1) = 0$ into the integrated relation:

$$1 \cdot 0 + \tan^{-1} 1 = C \implies C = \frac{\pi}{4}.$$

Hence $xy = \pi/4 - \tan^{-1} x$. Differentiating this relation gives $xy' + y = -1/(1+x^2)$, which reproduces the printed differential equation. At $x = 1$, its right side is $\pi/4 - \pi/4 = 0$, so the initial condition is also satisfied. The solution is considered on $x > 0$, the interval containing the initial point and avoiding $x = 0$.

Step 4: Conclusion The required particular solution is

$$y(x) = \frac{\pi/4 - \tan^{-1} x}{x}, \quad x > 0.$$

Quick Tip to Solve: Recognise $xy' + y$ immediately as $(xy)'$; integrate once, then use $\tan^{-1} 1 = \pi/4$ to determine the constant.

13(b). Find the general solution of the differential equation $x(y^3 + x^3) dy = (2y^4 + 5x^3y) dx$. [4 marks]

Solution:

Concept:

- A first-order differential equation is homogeneous when its derivative can be written as a function of y/x . Substitute $y = vx$, where $v = y/x$, so that $\frac{dy}{dx} = v + x \frac{dv}{dx}$.
- After separation, integrate each side and absorb any fixed multiplicative factor into the arbitrary constant. Solutions excluded by division must be checked in the original equation.

Step 1: Express the equation in homogeneous form Starting with the printed equation,

$$x(y^3 + x^3) \frac{dy}{dx} = 2y^4 + 5x^3y.$$

For $x \neq 0$, write $y = vx$. Since $y^3 + x^3 = x^3(v^3 + 1)$, cancellation of x^4 gives

$$(v^3 + 1) \left(v + x \frac{dv}{dx} \right) = 2v^4 + 5v.$$

Thus the powers of x cancel, as expected for a homogeneous equation. Subtracting $v(v^3 + 1) = v^4 + v$ from the right-hand side yields

$$x(v^3 + 1) \frac{dv}{dx} = v^4 + 4v = v(v^3 + 4).$$

Step 2: Separate variables and integrate For $v(v^3 + 4) \neq 0$, separate the variables:

$$\frac{v^3 + 1}{v(v^3 + 4)} dv = \frac{dx}{x}.$$

The useful decomposition is

$$\frac{v^3 + 1}{v(v^3 + 4)} = \frac{1}{4v} + \frac{3v^2}{4(v^3 + 4)}.$$

Indeed, combining the right-hand fractions gives numerator $(v^3 + 4 + 3v^3)/4 = v^3 + 1$. Therefore,

$$\begin{aligned} \frac{1}{4} \int \frac{dv}{v} + \frac{1}{4} \int \frac{3v^2 dv}{v^3 + 4} &= \int \frac{dx}{x}, \\ \frac{1}{4} \ln |v| + \frac{1}{4} \ln |v^3 + 4| &= \ln |x| + C_1. \end{aligned}$$

For the second integral, the substitution $u = v^3 + 4$ gives $du = 3v^2 dv$. Multiplying by four and combining logarithms gives

$$\ln |v(v^3 + 4)| = 4 \ln |x| + C_2.$$

On any interval where these factors do not vanish, exponentiation and absorption of the fixed sign into a constant give $v(v^3 + 4) = Cx^4$.

Quick Tip to Solve: After setting $y = vx$, split $(v^3 + 1)/[v(v^3 + 4)]$ into $1/(4v) + 3v^2/[4(v^3 + 4)]$; both terms then integrate directly to logarithms.

Step 3: Restore the original variables and check excluded solutions Substitution of $v = y/x$ gives

$$\frac{y}{x} \left(\frac{y^3}{x^3} + 4 \right) = Cx^4, \quad y(y^3 + 4x^3) = Cx^8.$$

The separated calculation excluded $v = 0$ and $v^3 = -4$. They correspond to $y = 0$ and $y = -\sqrt[3]{4}x$, respectively. Both satisfy the original differential equation, and both are included by taking $C = 0$ in this implicit family. To verify the family directly, differentiate $y^4 + 4x^3y = Cx^8$:

$$4(y^3 + x^3) \frac{dy}{dx} + 12x^2y = 8Cx^7.$$

Multiply by $x/4$ and use $Cx^8 = y^4 + 4x^3y$. This gives

$$x(y^3 + x^3) \frac{dy}{dx} = 2(y^4 + 4x^3y) - 3x^3y = 2y^4 + 5x^3y,$$

exactly the given equation. The solution is understood on intervals where the corresponding branch is differentiable.

Step 4: Conclusion The general solution is $y^4 + 4x^3y = Cx^8$, where C is an arbitrary real constant. The straight-line solutions $y = 0$ and $y = -\sqrt[3]{4}x$ are included when $C = 0$.

Quick Tip to Solve: After setting $y = vx$, split $(v^3 + 1)/[v(v^3 + 4)]$ into $1/(4v) + 3v^2/[4(v^3 + 4)]$; both terms then integrate directly to logarithms.

14. Case-Study Based Question

In a game of Archery, each ring of the Archery target is valued. The centremost ring is worth 10 points and rest of the rings are allotted points 9 to 1 in sequential order moving outwards.

Archer A is likely to earn 10 points with a probability of 0.8 and Archer B is likely to earn 10 points with a probability of 0.9.

Based on the above information, answer the following questions:

If both of them hit the Archery target, then find the probability that

[4 marks]

14(a). Exactly one of them earns 10 points.

[2 marks]

Solution:

Concept:

- Let E denote the event that Archer A earns 10 points and F the event that Archer B earns 10 points. The given probabilities are $P(E) = 0.8$ and $P(F) = 0.9$.
- For the separate independent shots of the two archers, probabilities multiply. Complementary events satisfy $P(E') = 1 - P(E)$. For mutually exclusive cases, probabilities add.

Step 1: Find the probabilities of not earning 10 points We use the given probabilities for the shots in which both archers hit the target. Not earning 10 points means earning a point value other than 10; it

Quick Tip to Solve: For exactly one success in two independent trials, add success-failure and failure-success; exclude both-success.

does not mean missing the target. Thus,

$$P(E') = 1 - 0.8 = 0.2, \quad P(F') = 1 - 0.9 = 0.1.$$

Step 2: Separate the two possible cases Exactly one archer earns 10 points if A earns 10 and B does not, or if B earns 10 and A does not. These cases cannot occur together. Using independence of the two shots,

$$P(E \cap F') = 0.8 \times 0.1 = 0.08,$$

$$P(E' \cap F) = 0.2 \times 0.9 = 0.18.$$

Therefore the required probability is obtained by adding these two mutually exclusive probabilities: $0.08 + 0.18$.

Step 3: Conclusion The probability that exactly one of the two archers earns 10 points is 0.26.

Quick Tip to Solve: For exactly one success in two independent trials, add success-failure and failure-success; exclude both-success.

14(b). Both of them earn 10 points.

[2 marks]

Solution:

Concept:

- If E is the event that Archer A earns 10 points and F the event that Archer B earns 10 points, then both earning 10 points is the intersection $E \cap F$.
- For independent events, the multiplication rule is $P(E \cap F) = P(E)P(F)$. Independence means the outcome of one archer's shot does not change the other archer's success probability.

Step 1: Identify the given events and probabilities The centremost ring gives 10 points, so earning 10 points means hitting this ring. For the shots described in the question, let E and F denote this success for A and B respectively. The information supplied gives

$$P(E) = 0.8, \quad P(F) = 0.9.$$

We use these probabilities for the stipulated situation in which both archers hit the target; no additional probability of missing the target is supplied or needed.

Step 2: Apply the multiplication rule The two archers take separate independent shots. Both must succeed, so we require their intersection, rather than the union that would describe at least one success. Consequently,

$$\begin{aligned} P(E \cap F) &= P(E)P(F) \\ &= 0.8 \times 0.9 \\ &= \frac{8}{10} \times \frac{9}{10} = \frac{72}{100}. \end{aligned}$$

Adding the two individual probabilities would count different cases and would even exceed 1, so it cannot represent the required joint success.

Quick Tip to Solve: Translate both independent successes into an intersection and multiply the two individual success probabilities.

Step 3: Conclusion The probability that both archers earn 10 points is $\boxed{0.72}$.

Quick Tip to Solve: Translate both independent successes into an intersection and multiply the two individual success probabilities.