

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-5-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) This question paper contains **three** Sections- **A, B** and **C**.
- (ii) **Each** section is compulsory.
- (iii) **Section-A** has 6 short-answer type-I questions of 2 marks each.
- (iv) **Section-B** has 4 short-answer type-II questions of 3 marks each.
- (v) **Section-C** has 4 long-answer type questions of 4 marks each.
- (vi) There is an internal choice in some questions.
- (vii) Question 14 is a case study based question with **two** subparts of 2 marks each.

Section A (12 marks)

Question numbers 1 to 6 carry 2 marks each.

1. Find: $\int \frac{dx}{\sqrt{4x - x^2}}.$

[2 marks]

Solution:

Concept:

- Completing the square gives $4x - x^2 = 4 - (x - 2)^2$, so the integrand has the standard inverse-sine form.
- The standard formula is $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}(u/a) + C$, valid for $|u| < a$.

Step 1: Complete the square Let $I = \int \frac{dx}{\sqrt{4x - x^2}}$. Since $4x - x^2 = 4 - (x - 2)^2$, we obtain

$$I = \int \frac{dx}{\sqrt{2^2 - (x - 2)^2}}.$$

For a real integrand with a nonzero denominator, $4x - x^2 > 0$, hence $0 < x < 4$.

Step 2: Apply the standard integral Put $u = x - 2$, so $du = dx$. Therefore

$$I = \int \frac{du}{\sqrt{4 - u^2}} = \sin^{-1}\left(\frac{u}{2}\right) + C.$$

The substitution is a translation, so no additional factor is required. As a check, differentiating gives

$$\frac{d}{dx} \sin^{-1}\left(\frac{x-2}{2}\right) = \frac{1/2}{\sqrt{1 - (x-2)^2/4}} = \frac{1}{\sqrt{4x - x^2}}.$$

Step 3: Conclusion

$$\int \frac{dx}{\sqrt{4x - x^2}} = \sin^{-1}\left(\frac{x-2}{2}\right) + C, \quad 0 < x < 4.$$

Quick Tip to Solve: For a quadratic under a square root, complete the square before choosing a substitution; here the centre is $x = 2$ and the radius is 2.

2. Find the general solution of the following differential equation: $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}.$

[2 marks]

Solution:

Concept:

- Multiply by e^y to separate the variables, using $e^{x-y} = e^x e^{-y}$.
- The chain rule gives $\frac{d}{dx}(e^y) = e^y \frac{dy}{dx}$; thus the left side becomes an exact derivative.

Quick Tip to Solve: When both terms contain e^{-y} , multiply by e^y and recognise $e^y y'$ as the derivative of e^y .

Step 1: Separate the variables The given equation is

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y} = e^{-y}(e^x + x^2).$$

Multiplying by e^y , which is positive for every real y , gives

$$e^y \frac{dy}{dx} = e^x + x^2, \quad e^y dy = (e^x + x^2) dx.$$

No solution is lost in this multiplication because e^y cannot vanish.

Step 2: Integrate and check Integrating both sides,

$$\int e^y dy = \int (e^x + x^2) dx \implies e^y = e^x + \frac{x^3}{3} + C.$$

The arbitrary constants from the two integrations have been combined into one constant C . Differentiating this relation gives $e^y y' = e^x + x^2$; division by e^y recovers the original equation exactly. For a real explicit solution, the expression inside the logarithm must be positive.

Step 3: Conclusion The general solution is

$$e^y = e^x + \frac{x^3}{3} + C.$$

Equivalently, $y = \ln(e^x + x^3/3 + C)$ on any interval where $e^x + x^3/3 + C > 0$.

Quick Tip to Solve: When both terms contain e^{-y} , multiply by e^y and recognise $e^y y'$ as the derivative of e^y .

3. Let X be a random variable which assumes values x_1, x_2, x_3, x_4 such that $2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4)$. Find the probability distribution of X . [2 marks]

Solution:

Concept:

- A probability distribution assigns nonnegative probabilities to all possible values and their sum must equal 1.
- Use one common parameter for the equal quantities; the probability ratios are reciprocals of the given multipliers.

Step 1: Express the probabilities using one parameter Let the common value be k :

$$2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4) = k.$$

Consequently,

$$P(X = x_1) = \frac{k}{2}, \quad P(X = x_2) = \frac{k}{3}, \quad P(X = x_3) = k, \quad P(X = x_4) = \frac{k}{5}.$$

Quick Tip to Solve: Convert equal weighted probabilities into reciprocal weights, then multiply by a common denominator: here the integer ratio is 15 : 10 : 30 : 6.

Thus the probabilities are not in the ratio $2 : 3 : 1 : 5$; they are in the reciprocal ratio $1/2 : 1/3 : 1 : 1/5$.

Step 2: Use total probability one Since these are all the values assumed by X ,

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1.$$

Taking denominator 30,

$$\frac{15k + 10k + 30k + 6k}{30} = 1 \implies k = \frac{30}{61}.$$

Substitution gives probabilities $15/61, 10/61, 30/61, 6/61$. Each is positive, and their sum is $(15 + 10 + 30 + 6)/61 = 1$, verifying the distribution.

Step 3: Conclusion The required probability distribution is

X	x_1	x_2	x_3	x_4
$P(X)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

Quick Tip to Solve: Convert equal weighted probabilities into reciprocal weights, then multiply by a common denominator: here the integer ratio is $15 : 10 : 30 : 6$.

4. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{a} \cdot \vec{b} = 1$ and $\vec{a} \times \vec{b} = \hat{j} - \hat{k}$, then find $|\vec{b}|$.

[2 marks]

Solution:

Concept:

- For any vectors $\vec{a}$ and $\vec{b}$, the dot and cross products satisfy $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$. This follows from the identities involving $\sin^2 \theta + \cos^2 \theta = 1$, where θ is the angle between the vectors.
- The unit vectors $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular. Thus a vector's squared magnitude is the sum of the squares of its Cartesian components.

Step 1: Evaluate the known squared magnitudes From $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, we get $|\vec{a}|^2 = 1^2 + 1^2 + 1^2 = 3$. Similarly, the given cross product yields $|\vec{a} \times \vec{b}|^2 = |\hat{j} - \hat{k}|^2 = 1^2 + (-1)^2 = 2$. Also, $(\vec{a} \cdot \vec{b})^2 = 1^2 = 1$.

Step 2: Substitute into the product identity Therefore

$$3|\vec{b}|^2 = 2 + 1 = 3.$$

Dividing by the nonzero value $|\vec{a}|^2 = 3$ gives $|\vec{b}|^2 = 1$. A magnitude is nonnegative, so we take the positive square root, rather than retaining both algebraic signs.

Step 3: Conclusion Hence, $|\vec{b}| = 1$.

Quick Tip to Solve: Combine the squared dot and cross products to obtain the magnitude directly, without solving for the three components of the unknown vector.

5. If a line makes an angle α, β, γ with the coordinate axes, then find the value of $\cos 2\alpha + \cos 2\beta + \cos 2\gamma$.

[2 marks]

Solution:

Concept:

- The direction cosines of a line are $l = \cos \alpha$, $m = \cos \beta$ and $n = \cos \gamma$, where the angles are measured with the three coordinate axes. They obey $l^2 + m^2 + n^2 = 1$.
- The double-angle identity $\cos 2\theta = 2 \cos^2 \theta - 1$ expresses each requested term in the squared direction cosines.

Step 1: Express the sum using squared cosines Applying the same double-angle identity to each of the three angles gives

$$\begin{aligned}\cos 2\alpha + \cos 2\beta + \cos 2\gamma &= (2 \cos^2 \alpha - 1) + (2 \cos^2 \beta - 1) \\ &\quad + (2 \cos^2 \gamma - 1) \\ &= 2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) - 3.\end{aligned}$$

Step 2: Use the direction-cosine relation Since these are the angles made by one line with the coordinate axes, their squared cosines sum to unity. Substituting this value into the expression above gives $2(1) - 3$. The relation applies to any direction of the line, so no individual angle needs to be determined.

Step 3: Conclusion Therefore, $\boxed{\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1}$.

Quick Tip to Solve: Convert each double-angle cosine to twice a squared direction cosine minus one; the three squared direction cosines always sum to one.

6(a). Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\overline{A} \cup \overline{B}) = \frac{1}{4}$. Find whether the events A and B are independent or not.

[2 marks]

Solution:

Concept:

- De Morgan's law gives $\overline{A} \cup \overline{B} = \overline{A \cap B}$. Therefore $P(A \cap B) = 1 - P(\overline{A} \cup \overline{B})$.
- Two events are independent precisely when $P(A \cap B) = P(A)P(B)$. Also, because $A \cap B \subseteq A$, every valid probability model must satisfy $P(A \cap B) \leq P(A)$.

Step 1: Apply the complement rule to the printed expression The printed expression is the union of the separate complements. Thus

$$P(A \cap B) = 1 - P(\overline{A} \cup \overline{B}) = 1 - \frac{1}{4} = \frac{3}{4}.$$

For comparison, the product of the specified individual probabilities is

$$P(A)P(B) = \frac{1}{2} \cdot \frac{7}{12} = \frac{7}{24}.$$

Quick Tip to Solve: Check the positions of the complement bars before applying De Morgan's law, then verify that an intersection probability cannot exceed either event's probability.

Step 2: Check independence and consistency The two values $3/4$ and $7/24$ are unequal, so the independence criterion is not satisfied by the stated data. In fact, there is a stronger inconsistency: $P(A \cap B) = 3/4 > P(A) = 1/2$, although an intersection is a subset of each event. Equivalently, $\overline{A} \subseteq \overline{A} \cup \overline{B}$ would require the latter probability to be at least $P(\overline{A}) = 1/2$, contrary to the given $1/4$.

Step 3: Conclusion The stated values fail the independence test. Furthermore, the printed probabilities are inconsistent, so no such pair of events exists.

Quick Tip to Solve: Check the positions of the complement bars before applying De Morgan's law, then verify that an intersection probability cannot exceed either event's probability.

OR

6(b). A box B_1 contains 1 white ball and 3 red balls. Another box B_2 contains 2 white balls and 3 red balls. If one ball is drawn at random from each of the boxes B_1 and B_2 , then find the probability that the two balls drawn are of the same colour. [2 marks]

Solution:

Concept:

- For an equally likely random draw, the probability of a colour is its number of balls divided by the total number of balls in that box.
- Draws from separate boxes are independent. Thus $P(E \cap F) = P(E)P(F)$, while probabilities of mutually exclusive cases are added.

Step 1: Calculate the colour probabilities in each box Box B_1 contains $1 + 3 = 4$ balls and box B_2 contains $2 + 3 = 5$ balls. Therefore,

$$P(W_1) = \frac{1}{4}, \quad P(R_1) = \frac{3}{4},$$

$$P(W_2) = \frac{2}{5}, \quad P(R_2) = \frac{3}{5}.$$

Here W_i and R_i denote drawing a white ball and a red ball, respectively, from box B_i . Taking a ball from one box does not change the contents of the other, so the colour outcomes in the two boxes are independent.

Step 2: Add the two cases of matching colours The two balls have the same colour precisely when both are white or both are red. These two cases cannot occur together in a single pair of draws. Independence gives

$$P(W_1 \cap W_2) = \frac{1}{4} \cdot \frac{2}{5} = \frac{2}{20}, \quad P(R_1 \cap R_2) = \frac{3}{4} \cdot \frac{3}{5} = \frac{9}{20}.$$

Adding these mutually exclusive probabilities gives

$$P(\text{same colour}) = \frac{2}{20} + \frac{9}{20}.$$

Quick Tip to Solve: For matching colours from different boxes, multiply the two white probabilities and the two red probabilities separately, then add.

Step 3: Conclusion The probability that the two balls drawn are of the same colour is $\boxed{\frac{11}{20}}$.

Quick Tip to Solve: For matching colours from different boxes, multiply the two white probabilities and the two red probabilities separately, then add.

Section B (12 marks)

Question numbers 7 to 10 carry 3 marks each.

7. Evaluate: $\int_0^{\pi/4} \frac{dx}{1 + \tan x}$.

[3 marks]

Solution:

Concept:

- The substitution $x = a - t$ in a definite integral gives $\int_0^a f(x) dx = \int_0^a f(a - t) dt$.
- The complementary-angle identity is $\tan(\pi/4 - t) = \frac{1 - \tan t}{1 + \tan t}$. Pairing an integrand with its reflected form can simplify their sum.

Step 1: Reflect the variable about the interval endpoints Denote the required integral by

$$I = \int_0^{\pi/4} \frac{dx}{1 + \tan x}.$$

Throughout the interval, $0 \leq \tan x \leq 1$, so the denominator is positive and the integrand is continuous. Put $x = \pi/4 - t$, giving $dx = -dt$. The new limits are $t = \pi/4$ at $x = 0$ and $t = 0$ at $x = \pi/4$. Reversing the limits removes the minus sign:

$$I = \int_0^{\pi/4} \frac{dt}{1 + \tan(\pi/4 - t)}.$$

Step 2: Simplify the reflected integrand Using the tangent subtraction identity,

$$\begin{aligned} \frac{1}{1 + \tan(\pi/4 - t)} &= \frac{1}{1 + \frac{1 - \tan t}{1 + \tan t}} \\ &= \frac{1 + \tan t}{2}. \end{aligned}$$

Thus the original integral equals

$$I = \frac{1}{2} \int_0^{\pi/4} (1 + \tan t) dt.$$

This substitution transforms the rational tangent expression into the sum of two standard integrands; no symmetry assumption about the original denominator is needed.

Quick Tip to Solve: Reflect the variable using $x = \pi/4 - t$: the tangent subtraction identity changes the integrand directly to $(1 + \tan t)/2$.

Step 3: Integrate and apply both limits Since $\int \tan t \, dt = -\ln |\cos t|$ and $\cos t > 0$ on the interval,

$$\begin{aligned} I &= \frac{1}{2} [t - \ln(\cos t)]_0^{\pi/4} \\ &= \frac{1}{2} \left(\frac{\pi}{4} - \ln \frac{1}{\sqrt{2}} \right) \\ &= \frac{1}{2} \left(\frac{\pi}{4} + \frac{1}{2} \ln 2 \right). \end{aligned}$$

The lower-limit contribution is zero because $\ln(\cos 0) = \ln 1 = 0$.

Step 4: Conclusion Hence, $\int_0^{\pi/4} \frac{dx}{1 + \tan x} = \frac{\pi}{8} + \frac{1}{4} \ln 2$.

Quick Tip to Solve: Reflect the variable using $x = \pi/4 - t$: the tangent subtraction identity changes the integrand directly to $(1 + \tan t)/2$.

8(a). If $\vec{a}$ and $\vec{b}$ are two vectors such that $|\vec{a} + \vec{b}| = |\vec{b}|$, then prove that $(\vec{a} + 2\vec{b})$ is perpendicular to $\vec{a}$. [3 marks]

Solution:

Concept:

- The square of a vector's magnitude equals its scalar product with itself: $|\vec{v}|^2 = \vec{v} \cdot \vec{v}$. This converts an equality of lengths into an algebraic relation between scalar products.
- The scalar product is distributive and symmetric: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. A zero scalar product is the criterion for orthogonality; for nonzero vectors this means their angle is 90° .

Step 1: Square the given equality of magnitudes Both magnitudes are nonnegative, so squaring the given condition preserves the equality. Thus

$$|\vec{a} + \vec{b}|^2 = |\vec{b}|^2 \implies (\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{b} \cdot \vec{b}.$$

Expand the left-hand side using distributivity:

$$\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = \vec{b} \cdot \vec{b}.$$

Step 2: Isolate the scalar-product relation Cancel the common term $\vec{b} \cdot \vec{b}$ from the two sides and combine the equal cross terms. We obtain

$$\vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} = 0.$$

Use distributivity in reverse to evaluate the scalar product required for the proof:

$$\begin{aligned} \vec{a} \cdot (\vec{a} + 2\vec{b}) &= \vec{a} \cdot \vec{a} + 2\vec{a} \cdot \vec{b} \\ &= 0. \end{aligned}$$

Quick Tip to Solve: Square the magnitude equality first; the two cross terms combine into the coefficient 2 needed in the target scalar product.

The given magnitude condition therefore yields exactly the vanishing scalar product of the two vectors in the question.

Step 3: Conclusion Therefore, $\vec{a} \cdot (\vec{a} + 2\vec{b}) = 0$, so $\vec{a} + 2\vec{b}$ is perpendicular to $\vec{a}$, as required. If either vector is zero, the same orthogonality identity still holds.

Quick Tip to Solve: Square the magnitude equality first; the two cross terms combine into the coefficient 2 needed in the target scalar product.

OR

8(b). If $\vec{a}$ and $\vec{b}$ are unit vectors and θ is the angle between them, then prove that $\sin \frac{\theta}{2} = \frac{1}{2} |\vec{a} - \vec{b}|$. [3 marks]

Solution:

Concept:

- For vectors with angle θ between them, $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$. A unit vector has magnitude 1, so here $\vec{a} \cdot \vec{b} = \cos \theta$.
- The half-angle identity is $1 - \cos \theta = 2 \sin^2(\theta/2)$. Since the angle between vectors lies in $[0, \pi]$, its half lies in $[0, \pi/2]$, where sine is nonnegative.

Step 1: Calculate the squared magnitude of the difference Express the squared magnitude as a scalar product and expand:

$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) \\ &= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - 2\vec{a} \cdot \vec{b}. \end{aligned}$$

Because both vectors are unit vectors, $\vec{a} \cdot \vec{a} = \vec{b} \cdot \vec{b} = 1$, and their scalar product is $\cos \theta$. Substitution gives

$$|\vec{a} - \vec{b}|^2 = 1 + 1 - 2 \cos \theta = 2(1 - \cos \theta).$$

Step 2: Apply the half-angle identity and take the square root Replace $1 - \cos \theta$ by $2 \sin^2(\theta/2)$:

$$|\vec{a} - \vec{b}|^2 = 4 \sin^2 \frac{\theta}{2}.$$

The magnitude on the left is nonnegative. Also, $0 \leq \theta \leq \pi$ implies $\sin(\theta/2) \geq 0$, so the nonnegative square root is

$$|\vec{a} - \vec{b}| = 2 \sin \frac{\theta}{2}.$$

This sign check is necessary: the square root of a square is an absolute value unless nonnegativity has been established.

Step 3: Conclusion Dividing by 2 gives $\sin \frac{\theta}{2} = \frac{1}{2} |\vec{a} - \vec{b}|$, which proves the required identity, including the endpoint angles 0 and π .

Quick Tip to Solve: Expand the squared magnitude of the difference, then use $1 - \cos \theta = 2 \sin^2(\theta/2)$ and check the square-root sign.

9. Find the equation of the plane passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 10$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and passing through the point $(-2, 3, 1)$. [3 marks]

Solution:

Concept:

- For $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, a vector plane equation becomes a linear equation in x, y, z ; its coefficients form a normal vector.
- If two distinct planes have equations $F_1 = 0$ and $F_2 = 0$, the plane $F_1 + \lambda F_2 = 0$ contains their common line, since both expressions vanish at every point of that line.

Step 1: Write the given planes in Cartesian form Expanding the dot products gives

$$F_1 = x + y + z - 10 = 0, \quad F_2 = 2x + 3y - z + 4 = 0.$$

Their normal vectors $(1, 1, 1)$ and $(2, 3, -1)$ are not proportional, so the planes intersect in a line. A plane through that line can therefore be written as

$$(x + y + z - 10) + \lambda(2x + 3y - z + 4) = 0.$$

The given point does not lie in the second plane, so the required member of the family is represented by a finite value of λ .

Step 2: Impose the condition on the given point Substitute $x = -2, y = 3, z = 1$ into the family. Keeping the constant +4 in the second plane gives

$$\begin{aligned} (-2 + 3 + 1 - 10) + \lambda\{2(-2) + 3(3) - 1 + 4\} &= 0, \\ -8 + 8\lambda &= 0, \\ \lambda &= 1. \end{aligned}$$

Thus the plane is obtained by adding the two original left-hand sides:

$$x + y + z - 10 + 2x + 3y - z + 4 = 0.$$

The z -terms cancel. As a check, its left-hand side at $(-2, 3, 1)$ is $3(-2) + 4(3) - 6 = 0$. Also, every point of the original intersection satisfies this equation because it is the sum of two zero expressions.

Step 3: Conclusion The required plane is $3x + 4y - 6 = 0$, or equivalently $\vec{r} \cdot (3\hat{i} + 4\hat{j}) = 6$.

Quick Tip to Solve: Substitute the point into each plane expression first: the values -8 and 8 immediately give the family parameter as 1.

10(a). Find: $\int e^x \sin 2x \, dx$.

[3 marks]

Solution:

Concept:

- Integration by parts uses $\int u \, dv = uv - \int v \, du$, where u is differentiated and dv is integrated.
- Repeating integration by parts for an exponential times a sine produces the original integral again. Collect that integral algebraically, remembering the factor 2 from differentiating $\sin 2x$ or $\cos 2x$.

Step 1: Apply integration by parts once Let $I = \int e^x \sin 2x \, dx$. Choose $u = \sin 2x$ and $dv = e^x \, dx$, so $du = 2 \cos 2x \, dx$ and $v = e^x$. Then

$$I = e^x \sin 2x - 2 \int e^x \cos 2x \, dx.$$

Denote the remaining integral by $J = \int e^x \cos 2x \, dx$. We keep the arbitrary constant until the final expression, since intermediate constants combine into a single constant.

Step 2: Apply integration by parts again and collect the original integral In J , choose $u = \cos 2x$ and $dv = e^x \, dx$. Here $du = -2 \sin 2x \, dx$, so the minus sign in integration by parts yields

$$J = e^x \cos 2x + 2I.$$

Substitute this into the first relation:

$$\begin{aligned} I &= e^x \sin 2x - 2(e^x \cos 2x + 2I), \\ 5I &= e^x (\sin 2x - 2 \cos 2x). \end{aligned}$$

Dividing by 5 determines an antiderivative. To verify its signs and denominator, differentiate the resulting expression using the product rule:

$$\begin{aligned} \frac{d}{dx} \left[\frac{e^x}{5} (\sin 2x - 2 \cos 2x) \right] &= \frac{e^x}{5} [\sin 2x - 2 \cos 2x \\ &\quad + 2 \cos 2x + 4 \sin 2x] \\ &= e^x \sin 2x. \end{aligned}$$

This agrees with the integrand for every real x .

Step 3: Conclusion

$$\boxed{\int e^x \sin 2x \, dx = \frac{e^x}{5} (\sin 2x - 2 \cos 2x) + C},$$

where C is an arbitrary real constant.

Quick Tip to Solve: After the second integration by parts, move the term $-4I$ to the left; this produces the denominator 5 and prevents a common sign error.

10(b). Find: $\int \frac{2x}{(x^2 + 1)(x^2 + 2)} dx$.

[3 marks]

Solution:

Concept:

- A substitution is useful when the numerator contains the derivative of a repeated expression in the denominator. Here, setting $t = x^2$ gives $dt = 2x dx$.
- Partial fractions express a rational function with distinct linear factors as a sum of simpler fractions. The standard integral is $\int \frac{dt}{t + a} = \ln |t + a| + C$, where C is an arbitrary constant.

Step 1: Substitute for the quadratic expression Let the required integral be I . The same expression x^2 appears in both denominator factors, while its derivative $2x$ appears in the numerator. Therefore put $t = x^2$, so that $dt = 2x dx$. This gives

$$I = \int \frac{dt}{(t + 1)(t + 2)}.$$

The substitution converts both quadratic factors into linear factors, allowing us to use partial fractions directly.

Step 2: Determine the partial fractions Write

$$\frac{1}{(t + 1)(t + 2)} = \frac{A}{t + 1} + \frac{B}{t + 2}.$$

Multiplying by $(t + 1)(t + 2)$ gives $1 = A(t + 2) + B(t + 1)$. At $t = -1$, this identity gives $A = 1$; at $t = -2$, it gives $B = -1$. Hence the transformed integrand is $\frac{1}{t + 1} - \frac{1}{t + 2}$. These values are used to determine the algebraic identity, even though the substitution gives $t \geq 0$ for real x .

Step 3: Integrate and substitute back Integrating the two terms separately,

$$\begin{aligned} I &= \int \left(\frac{1}{t + 1} - \frac{1}{t + 2} \right) dt \\ &= \ln |t + 1| - \ln |t + 2| + C \\ &= \ln(x^2 + 1) - \ln(x^2 + 2) + C. \end{aligned}$$

Both $x^2 + 1$ and $x^2 + 2$ are positive for every real x , so absolute-value signs can be omitted. As a check, differentiating the last expression gives

$$\frac{2x}{x^2 + 1} - \frac{2x}{x^2 + 2} = \frac{2x}{(x^2 + 1)(x^2 + 2)},$$

which agrees with the given integrand.

Step 4: Conclusion $\int \frac{2x}{(x^2 + 1)(x^2 + 2)} dx = \ln \left(\frac{x^2 + 1}{x^2 + 2} \right) + C$, where C is an arbitrary constant.

Quick Tip to Solve: When the numerator is $2x$ and both denominator factors contain x^2 , substitute $t = x^2$ first; then subtract the two simple reciprocal factors.

Section C (16 marks)

Question numbers 11 to 14 carry 4 marks each.

11. Three persons A , B and C apply for a job of manager in a private company. Chances of their selection are in the ratio $1 : 2 : 4$. The probability that A , B and C can introduce changes to increase the profits of a company are 0.8 , 0.5 and 0.3 respectively. If increase in the profit does not take place, find the probability that it is due to the appointment of A . [4 marks]

Solution:

Concept:

- Mutually exclusive selection events: let E_A, E_B, E_C denote the appointment of A, B, C , respectively. Since exactly one applicant is selected, these events partition the sample space.
- Complement rule: if F is the event that profit does not increase, then $P(F | E_i) = 1 - P(F^c | E_i)$, where F^c denotes an increase in profit.
- Total probability and Bayes' theorem: $P(F) = \sum_i P(E_i)P(F | E_i)$, and $P(E_A | F) = \frac{P(E_A)P(F | E_A)}{P(F)}$.

Step 1: Convert the selection ratio into probabilities The given selection chances are in the ratio $1 : 2 : 4$. Their total is $1 + 2 + 4 = 7$, so

$$P(E_A) = \frac{1}{7}, \quad P(E_B) = \frac{2}{7}, \quad P(E_C) = \frac{4}{7}.$$

These probabilities add to one. They describe appointment before we receive any information about the company's profit.

Step 2: Find the conditional probabilities of no increase The probabilities of introducing changes that increase profit, conditional on the respective appointments, are $0.8, 0.5, 0.3$. Taking complements gives

$$\begin{aligned} P(F | E_A) &= 1 - 0.8 = 0.2 = \frac{1}{5}, \\ P(F | E_B) &= 1 - 0.5 = 0.5 = \frac{1}{2}, \\ P(F | E_C) &= 1 - 0.3 = 0.7 = \frac{7}{10}. \end{aligned}$$

We use the no-increase probabilities because the observed event in the question is that an increase does not take place.

Step 3: Obtain the probability of the observed event No increase can occur after any one of the three mutually exclusive appointments. Therefore the law of total probability gives

$$\begin{aligned} P(F) &= \frac{1}{7} \cdot \frac{1}{5} + \frac{2}{7} \cdot \frac{1}{2} + \frac{4}{7} \cdot \frac{7}{10} \\ &= \frac{1}{35} + \frac{1}{7} + \frac{2}{5} = \frac{1 + 5 + 14}{35} = \frac{4}{7}. \end{aligned}$$

The joint probability that A is appointed and profit does not increase is $P(E_A \cap F) = \frac{1}{7} \cdot \frac{1}{5} = \frac{1}{35}$.

Quick Tip to Solve: Multiply each selection weight by its no-increase probability: $1(0.2), 2(0.5), 4(0.7)$. Divide A 's weight 0.2 by their total 4 ; the common selection denominator cancels.

Step 4: Apply Bayes' theorem to the appointment of A The required probability is the probability of A's appointment conditional on the observed failure to increase profit. Thus

$$P(E_A | F) = \frac{P(E_A \cap F)}{P(F)} = \frac{1/35}{4/7} = \frac{1}{35} \cdot \frac{7}{4} = \frac{1}{20}.$$

The denominator includes no-increase outcomes from all three applicants, while the numerator includes only those following A's appointment. This conditioning distinguishes the requested probability from the prior selection chance $1/7$.

Step 5: Conclusion Given that profit does not increase, the probability that A was appointed is

$$\frac{1}{20} = 0.05.$$

Quick Tip to Solve: Multiply each selection weight by its no-increase probability: $1(0.2), 2(0.5), 4(0.7)$. Divide A's weight 0.2 by their total 4; the common selection denominator cancels.

12. Find the area bounded by the curves $y = |x - 1|$ and $y = 1$, using integration.

[4 marks]

Solution:

Concept:

- The area between two curves is the integral of the upper ordinate minus the lower ordinate over the interval enclosing the region: $A = \int_a^b [y_{\text{upper}} - y_{\text{lower}}] dx$.
- The absolute value changes its expression where its argument vanishes. Here $|x - 1| = 1 - x$ for $x \leq 1$, and $|x - 1| = x - 1$ for $x \geq 1$, so the area integral must be split at $x = 1$.

Step 1: Find the intersection points At an intersection, the two ordinates are equal. Hence

$$|x - 1| = 1 \implies x - 1 = \pm 1.$$

Thus $x = 0$ or $x = 2$, and the intersection points are $(0, 1)$ and $(2, 1)$. These limits enclose the bounded region. For every x between 0 and 2, we have $|x - 1| \leq 1$. Therefore the line $y = 1$ is the upper boundary and the absolute-value curve is the lower boundary throughout this interval.

Step 2: Split the area at the vertex The argument $x - 1$ changes sign at $x = 1$, where the curve has its vertex $(1, 0)$. On $[0, 1]$, the lower ordinate is $1 - x$, so the vertical height of the region is $1 - (1 - x) = x$. On $[1, 2]$, the lower ordinate is $x - 1$, giving height $1 - (x - 1) = 2 - x$. Both heights are nonnegative, as required for an area. Consequently,

$$A = \int_0^1 x dx + \int_1^2 (2 - x) dx.$$

Quick Tip to Solve: For $y = |x - 1|$, split at $x = 1$ before integrating. The heights x and $2 - x$ are symmetric, so either half-area can be doubled as a check.

Step 3: Evaluate the two definite integrals Apply the power rule separately on the two intervals:

$$\begin{aligned} A &= \left[\frac{x^2}{2} \right]_0^1 + \left[2x - \frac{x^2}{2} \right]_1^2 \\ &= \left(\frac{1}{2} - 0 \right) + \left[\left(4 - \frac{4}{2} \right) - \left(2 - \frac{1}{2} \right) \right] \\ &= \frac{1}{2} + \left(2 - \frac{3}{2} \right). \end{aligned}$$

Each integral gives the area on its own side of the vertex; adding them covers the entire bounded region without overlap.

Step 4: Conclusion Therefore the required bounded area is 1 square unit.

Quick Tip to Solve: For $y = |x - 1|$, split at $x = 1$ before integrating. The heights x and $2 - x$ are symmetric, so either half-area can be doubled as a check.

13(a). Solve the following differential equation: $(y - \sin^2 x) dx + \tan x dy = 0$.

[4 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$. Its integrating factor is $\text{I.F.} = e^{\int P(x) dx}$.
- Multiplication by the integrating factor makes the left side a product derivative: $\frac{d}{dx}(y \text{ I.F.}) = Q(x) \text{ I.F.}$. Integrating introduces one arbitrary constant C .
- Here the identity $\sin^2 x \cot x = \sin x \cos x$ simplifies the right side, and the substitution $u = \sin x$ evaluates the resulting integral.

Step 1: Put the equation in linear form The given differential equation is

$$(y - \sin^2 x) dx + \tan x dy = 0.$$

Rearranging and dividing by $\tan x$ on an interval where it is nonzero gives

$$\begin{aligned} \tan x \frac{dy}{dx} + y &= \sin^2 x, \\ \frac{dy}{dx} + \cot x y &= \sin x \cos x. \end{aligned}$$

Thus $P(x) = \cot x$ and $Q(x) = \sin x \cos x$; the equation is linear in the dependent variable y .

Step 2: Find and apply the integrating factor Since $\int \cot x dx = \log |\sin x|$, an integrating factor on such an interval is $\sin x$. A nonzero constant multiple of an integrating factor is equally valid, so the sign of $\sin x$ does not affect the method. Multiplying the linear equation throughout by $\sin x$, we obtain

$$\sin x \frac{dy}{dx} + y \cos x = \sin^2 x \cos x.$$

Quick Tip to Solve: Divide first by $\tan x$: the coefficient of y becomes $\cot x$, so the integrating factor is $\sin x$ and the final integral is a direct substitution in $\sin x$.

By the product rule, the left side is $\frac{d}{dx}(y \sin x)$. Hence

$$\frac{d}{dx}(y \sin x) = \sin^2 x \cos x.$$

Step 3: Integrate and check the relation Integrating both sides with respect to x gives

$$y \sin x = \int \sin^2 x \cos x \, dx.$$

Put $u = \sin x$, so that $du = \cos x \, dx$. The integral becomes

$$y \sin x = \int u^2 \, du = \frac{\sin^3 x}{3} + C.$$

To check, differentiate this relation:

$$\sin x \frac{dy}{dx} + y \cos x = \sin^2 x \cos x.$$

Dividing by $\cos x$, wherever the original $\tan x$ is defined, recovers $\tan x \frac{dy}{dx} + y = \sin^2 x$, exactly the given equation. The explicit solution applies on intervals with $\sin x \neq 0$; the implicit relation avoids introducing a denominator in the final answer.

Step 4: Conclusion The general solution is $y \sin x = \frac{\sin^3 x}{3} + C$, or equivalently $y = \frac{\sin^2 x}{3} + C \csc x$ for $\sin x \neq 0$, where C is an arbitrary constant.

Quick Tip to Solve: Divide first by $\tan x$: the coefficient of y becomes $\cot x$, so the integrating factor is $\sin x$ and the final integral is a direct substitution in $\sin x$.

OR

13(b). Find the general solution of the differential equation: $(x^3 + y^3) dy = x^2 y \, dx$.

[4 marks]

Solution:

Concept:

- A differential equation is homogeneous when the coefficient functions have the same degree. Here $x^3 + y^3$ and $x^2 y$ both have degree three, so a ratio substitution reduces the equation to one with separable variables.
- Treating x as a function of y , put $x = vy$, where $v = x/y$ on an interval with $y \neq 0$. Then $dx/dy = v + y \, dv/dy$. We use $\int dy/y = \ln |y|$, valid on either side of zero.

Step 1: Express the equation using the ratio substitution The given equation is

$$(x^3 + y^3) dy = x^2 y \, dx.$$

Quick Tip to Solve: Use $x = vy$ here: after substitution the two v^3 terms cancel immediately, leaving $v^2 \, dv = dy/y$.

Its equal-degree terms suggest a homogeneous substitution. Using $x = vy$ gives $x^3 = v^3y^3$, $x^2 = v^2y^2$, and $dx/dy = v + y dv/dy$. For $y \neq 0$, substitute into the differential equation and cancel y^3 :

$$\begin{aligned}(v^3 + 1)y^3 &= v^2y^3 \left(v + y \frac{dv}{dy} \right), \\ v^3 + 1 &= v^3 + v^2y \frac{dv}{dy}, \\ 1 &= v^2y \frac{dv}{dy}.\end{aligned}$$

The cancellation of the two v^3 terms is what makes the resulting equation separable.

Step 2: Separate the variables and integrate Transfer the factor y to the other side, keeping the variable v with its differential:

$$v^2 dv = \frac{dy}{y}.$$

Integrating both sides gives

$$\frac{v^3}{3} = \ln |y| + C_1.$$

Now substitute $v = x/y$ and absorb the factor three into the arbitrary constant $C = 3C_1$. This produces an implicit one-parameter family in the original variables. The absolute value in the logarithm allows both positive and negative nonzero values of y .

Step 3: Check the relation and the excluded case Differentiating the resulting relation verifies the algebra:

$$\begin{aligned}d \left(\frac{x^3}{y^3} \right) &= 3 \frac{dy}{y}, \\ \frac{3x^2}{y^3} dx - \frac{3x^3}{y^4} dy &= \frac{3}{y} dy, \\ x^2y dx &= (x^3 + y^3) dy.\end{aligned}$$

Thus differentiation recovers the printed equation exactly. The substitution excluded $y = 0$. Substituting the constant function $y = 0$, for which $dy = 0$, directly into the original equation makes both sides zero; it is an additional constant solution.

Step 4: Conclusion The general solution for $y \neq 0$ is

$$\boxed{\frac{x^3}{y^3} - 3 \ln |y| = C},$$

where C is an arbitrary real constant. The additional constant solution is $\boxed{y = 0}$.

Quick Tip to Solve: Use $x = vy$ here: after substitution the two v^3 terms cancel immediately, leaving $v^2 dv = dy/y$.

14. Case Study Problem:

Two motorcycles A and B are running at the speed more than the allowed speed on the roads represented by the lines $\vec{r} = \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and $\vec{r} = (3\hat{i} + 3\hat{j}) + \mu(2\hat{i} + \hat{j} + \hat{k})$ respectively.

Based on the above information, answer the following questions:

[4 marks]

14(a). Find the shortest distance between the given lines.

[2 marks]

Solution:

Concept:

- For non-parallel lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, their shortest distance is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$.
- The cross product is perpendicular to both line directions. A zero scalar triple product means that the two non-parallel lines are coplanar and therefore intersect.

Step 1: Identify the position and direction vectors From the given equations, take

$$\vec{a}_1 = (0, 0, 0), \quad \vec{a}_2 = (3, 3, 0), \quad \vec{b}_1 = (1, 2, -1), \quad \vec{b}_2 = (2, 1, 1).$$

Thus the displacement between the chosen points on the two roads is $\vec{a}_2 - \vec{a}_1 = (3, 3, 0)$.

Step 2: Calculate the common normal and substitute The normal vector is

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 3\hat{i} - 3\hat{j} - 3\hat{k}.$$

Its magnitude is $\sqrt{9 + 9 + 9} = 3\sqrt{3}$, which is nonzero, so the lines are not parallel. Their scalar triple product is

$$(3, 3, 0) \cdot (3, -3, -3) = 9 - 9 + 0 = 0.$$

Consequently the shortest-distance formula gives $d = |0|/(3\sqrt{3})$. This zero separation is consistent with intersecting roads; the formula does not require assuming that the roads are skew.

Step 3: Conclusion The shortest distance between the given lines is $\boxed{0}$.

Quick Tip to Solve: Compute the scalar triple product first: if it vanishes and the directions are not parallel, the shortest distance is zero.

14(b). Find the point at which the motorcycles may collide.

[2 marks]

Solution:

Concept:

- A possible collision point must lie on both road lines. At an intersection, the three corresponding coordinates of their vector equations are equal.
- The parameters λ and μ specify positions on the two lines. Solving two coordinate equations gives candidate values; the third equation must also hold.

Quick Tip to Solve: Use the simplest coordinate equation, here $\mu = -\lambda$, first and check the unused equation before accepting the intersection.

Step 1: Equate the coordinates of the two positions The position on the first road is $(\lambda, 2\lambda, -\lambda)$, while that on the second road is $(3 + 2\mu, 3 + \mu, \mu)$. A common point therefore satisfies

$$\lambda = 3 + 2\mu, \quad 2\lambda = 3 + \mu, \quad -\lambda = \mu.$$

The last equation gives $\mu = -\lambda$. Substituting it in the first equation gives $\lambda = 3 - 2\lambda$, hence $3\lambda = 3$ and $\lambda = 1$. Thus $\mu = -1$.

Step 2: Verify the remaining coordinate and find the point For these values, the second coordinate equation gives $2\lambda = 2$ and $3 + \mu = 2$, so it is satisfied as well. Substitution in both original lines yields

$$\begin{aligned}\vec{r}_A &= 1(\hat{i} + 2\hat{j} - \hat{k}) = \hat{i} + 2\hat{j} - \hat{k}, \\ \vec{r}_B &= 3\hat{i} + 3\hat{j} - (2\hat{i} + \hat{j} + \hat{k}) = \hat{i} + 2\hat{j} - \hat{k}.\end{aligned}$$

Both motorcycles' roads therefore pass through the same point. They may collide there if they reach it simultaneously; the line equations identify the location of the possible collision.

Step 3: Conclusion The point at which the motorcycles may collide is $(1, 2, -1)$, with position vector $\hat{i} + 2\hat{j} - \hat{k}$.

Quick Tip to Solve: Use the simplest coordinate equation, here $\mu = -\lambda$, first and check the unused equation before accepting the intersection.