

CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-5-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

General Instructions

- (i) This question paper contains **three** Sections- **A, B** and **C**.
- (ii) **Each** section is compulsory.
- (iii) **Section-A** has 6 short-answer type-I questions of 2 marks each.
- (iv) **Section-B** has 4 short-answer type-II questions of 3 marks each.
- (v) **Section-C** has 4 long-answer type questions of 4 marks each.
- (vi) There is an internal choice in some questions.
- (vii) Question 14 is a case study based question with two subparts of 2 marks each.

Section A (12 marks)

Question numbers 1 to 6 carry 2 marks each.

1. Find the vector equation of a line passing through a point with position vector $2\hat{i} - \hat{j} + \hat{k}$ and parallel to the line joining the points $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$. [2 marks]

Solution:

Concept:

- A line through position vector $\vec{a}$, parallel to a nonzero vector $\vec{b}$, has equation $\vec{r} = \vec{a} + \lambda\vec{b}$, where $\lambda \in \mathbb{R}$.
- For points with position vectors $\vec{p}$ and $\vec{q}$, the vector $\vec{PQ} = \vec{q} - \vec{p}$ gives the direction of their joining line.

Step 1: Calculate the direction vector Let P and Q have position vectors $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$, respectively. Subtract corresponding components:

$$\vec{PQ} = (\hat{i} + 2\hat{j} + 2\hat{k}) - (-\hat{i} + 4\hat{j} + \hat{k}) = 2\hat{i} - 2\hat{j} + \hat{k}.$$

This vector is nonzero, so it is a valid direction vector for the required parallel line.

Step 2: Use the given point and check the conditions The required line passes through the point with position vector $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$. Insert this point and the calculated direction vector into $\vec{r} = \vec{a} + \lambda\vec{b}$. At $\lambda = 0$, the position vector is exactly $\vec{a}$. Varying λ moves in the direction $\vec{PQ}$, so the line is parallel to the given joining line. Both conditions are satisfied.

Step 3: Conclusion The vector equation is

$$\vec{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k}), \quad \lambda \in \mathbb{R}.$$

Quick Tip to Solve: Subtract the endpoint vectors for the direction, and use the given passing point as the fixed vector.

2. Find: $\int \frac{dx}{\sqrt{4x - x^2}}.$

[2 marks]

Solution:

Concept:

- Completing the square converts a quadratic expression into the standard form $a^2 - u^2$.
- The standard integral is $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}(u/a) + C$, for $a > 0$ and $|u| < a$. Here C is an arbitrary constant.

Step 1: Complete the square Write the expression under the radical as

$$4x - x^2 = 4 - (x - 2)^2.$$

Quick Tip to Solve: Complete the square first: $4x$ minus x squared becomes 4 minus the square of x minus 2 .

Consequently the integral becomes $\int \frac{dx}{\sqrt{4 - (x - 2)^2}}$. For a real integrand its denominator must be nonzero and real, giving $0 < x < 4$.

Step 2: Apply the inverse-sine formula Set $u = x - 2$. Then $du = dx$, so

$$\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{du}{\sqrt{2^2 - u^2}} = \sin^{-1} \left(\frac{u}{2} \right) + C.$$

Substitute $u = x - 2$. As a check, differentiation gives

$$\frac{d}{dx} \sin^{-1} \left(\frac{x - 2}{2} \right) = \frac{1/2}{\sqrt{1 - (x - 2)^2/4}} = \frac{1}{\sqrt{4x - x^2}},$$

which reproduces the original integrand on its real domain.

Step 3: Conclusion

$$\boxed{\int \frac{dx}{\sqrt{4x - x^2}} = \sin^{-1} \left(\frac{x - 2}{2} \right) + C, \quad 0 < x < 4.}$$

Quick Tip to Solve: Complete the square first: $4x$ minus x squared becomes 4 minus the square of x minus 2 .

3. Find the general solution of the following differential equation: $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$. [2 marks]

Solution:

Concept:

- An equation $dy/dx = f(x)e^{-y}$ is separable: multiply by e^y and integrate with respect to x .
- The chain rule gives $\frac{d}{dx}(e^y) = e^y \frac{dy}{dx}$. Thus the left side after multiplication is an exact derivative.

Step 1: Separate the variables Factor e^{-y} on the right side of the given equation:

$$\frac{dy}{dx} = e^{-y}(e^x + x^2).$$

Multiplying by e^y , which is positive for every real y , gives

$$e^y \frac{dy}{dx} = e^x + x^2, \quad e^y dy = (e^x + x^2) dx.$$

No real solution is lost by this multiplication.

Step 2: Integrate and verify Integrate both sides, absorbing the integration constants into one arbitrary constant C :

$$\int e^y dy = \int (e^x + x^2) dx, \quad e^y = e^x + \frac{x^3}{3} + C.$$

Differentiating this relation yields $e^y(dy/dx) = e^x + x^2$, and division by e^y recovers the printed equation. If written explicitly as a logarithm, the solution is defined on any interval where $e^x + x^3/3 + C > 0$.

Quick Tip to Solve: Factor out e to the power minus y , then multiply by e to the power y to expose the derivative of e to the power y .

Step 3: Conclusion The general solution is

$$e^y = e^x + \frac{x^3}{3} + C.$$

Equivalently, $y = \ln(e^x + x^3/3 + C)$ wherever the logarithm argument is positive.

Quick Tip to Solve: Factor out e to the power minus y , then multiply by e to the power y to expose the derivative of e to the power y .

4(a). Events A and B are such that $P(A) = \frac{1}{2}$, $P(B) = \frac{7}{12}$ and $P(\overline{A \cup B}) = \frac{1}{4}$. Find whether the events A and B are independent or not. [2 marks]

Solution:

Concept:

- The complement rule gives $P(E) = 1 - P(\overline{E})$ for any event E .
- For two events, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Independence requires $P(A \cap B) = P(A)P(B)$.

Step 1: Find the probability of the union The given probability refers to the complement of the entire union. Therefore, applying the complement rule,

$$P(A \cup B) = 1 - P(\overline{A \cup B}) = 1 - \frac{1}{4} = \frac{3}{4}.$$

This is the probability that at least one of the two events occurs.

Step 2: Compute the intersection and compare with the product Rearranging the addition rule and using a common denominator gives

$$\begin{aligned} P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &= \frac{1}{2} + \frac{7}{12} - \frac{3}{4} \\ &= \frac{6}{12} + \frac{7}{12} - \frac{9}{12} = \frac{1}{3}. \end{aligned}$$

On the other hand,

$$P(A)P(B) = \frac{1}{2} \times \frac{7}{12} = \frac{7}{24}.$$

The intersection probability is $\frac{1}{3} = \frac{8}{24}$, which differs from $\frac{7}{24}$; hence the equality required for independence does not hold.

Step 3: Conclusion The events A and B are not independent because $P(A \cap B) = \frac{1}{3} \neq \frac{7}{24} = P(A)P(B)$.

Quick Tip to Solve: First convert the complement of a union into the union probability, then use the addition rule to test independence.

4(b). A box B_1 contains 1 white ball and 3 red balls. Another box B_2 contains 2 white balls and 3 red balls. If one ball is drawn at random from each of the boxes B_1 and B_2 , then find the probability that the two balls drawn are of the same colour. [2 marks]

Solution:

Concept:

- For a random draw, the probability of a colour is the number of balls of that colour divided by the total number of balls.
- Draws from separate boxes are independent, so their probabilities multiply. Mutually exclusive colour outcomes have probabilities that add.

Step 1: Find the probability that both balls are white The first box contains $1 + 3 = 4$ balls and the second contains $2 + 3 = 5$ balls. Drawing from one box does not change the contents of the other box. Thus, if W_1, W_2 denote white draws from the respective boxes,

$$P(W_1 \cap W_2) = P(W_1)P(W_2) = \frac{1}{4} \times \frac{2}{5} = \frac{2}{20}.$$

Step 2: Find the probability that both balls are red and combine Let R_1, R_2 denote red draws from the respective boxes. By independence,

$$P(R_1 \cap R_2) = \frac{3}{4} \times \frac{3}{5} = \frac{9}{20}.$$

The two balls have the same colour precisely when both are white or both are red. These possibilities are mutually exclusive: a pair cannot simultaneously consist of two white balls and two red balls. Consequently,

$$P(\text{same colour}) = \frac{2}{20} + \frac{9}{20} = \frac{11}{20}.$$

Mixed-colour pairs are excluded from the required event.

Step 3: Conclusion The probability that the two balls drawn have the same colour is $\boxed{\frac{11}{20}}$.

Quick Tip to Solve: List the two matching pairs, white-white and red-red, multiply within each pair, then add the results.

5. Let X be a random variable which assumes values x_1, x_2, x_3, x_4 such that $2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4)$. Find the probability distribution of X . [2 marks]

Solution:

Concept:

- A probability distribution assigns non-negative probabilities to all possible values of X , with total probability equal to 1.
- Introduce a common value for the given equal multiples of probabilities, then determine it by

Quick Tip to Solve: Convert equal multiples into the ratio of reciprocals: the probability ratio is $15 : 10 : 30 : 6$, whose sum is 61.

normalization.

Step 1: Express all four probabilities using one constant Let the common value in the given relation be k . Then

$$2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4) = k.$$

Dividing each equality by its coefficient gives

$$P(X = x_1) = \frac{k}{2}, \quad P(X = x_2) = \frac{k}{3}, \quad P(X = x_3) = k, \quad P(X = x_4) = \frac{k}{5}.$$

Thus the probabilities are inversely proportional to the corresponding coefficients, rather than directly proportional to them.

Step 2: Normalize and obtain the distribution Since the four listed values comprise the possible values of X , their probabilities must sum to one:

$$\begin{aligned} \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} &= 1, \\ k \left(\frac{15 + 10 + 30 + 6}{30} \right) &= 1, \\ \frac{61k}{30} &= 1, \quad k = \frac{30}{61}. \end{aligned}$$

Substitution gives the probabilities $\frac{15}{61}, \frac{10}{61}, \frac{30}{61}, \frac{6}{61}$, respectively. They are all non-negative and their numerators add to 61, so they constitute a valid distribution. They also satisfy the original equality, each multiple being $\frac{30}{61}$.

Step 3: Conclusion The probability distribution is

X	x_1	x_2	x_3	x_4
$P(X)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

Quick Tip to Solve: Convert equal multiples into the ratio of reciprocals: the probability ratio is 15 : 10 : 30 : 6, whose sum is 61.

6. If $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{a} \cdot \vec{b} = 1$ and $\vec{a} \times \vec{b} = \hat{j} - \hat{k}$, then find $|\vec{b}|$.

[2 marks]

Solution:

Concept:

- For vectors $\vec{a}$ and $\vec{b}$, $(\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2 = |\vec{a}|^2 |\vec{b}|^2$.
- The squared magnitude of $x\hat{i} + y\hat{j} + z\hat{k}$ is $x^2 + y^2 + z^2$, since the unit vectors are mutually perpendicular.

Step 1: Calculate the known squared magnitudes From $\vec{a} = \hat{i} + \hat{j} + \hat{k}$,

$$|\vec{a}|^2 = 1^2 + 1^2 + 1^2 = 3.$$

Quick Tip to Solve: Add the squared dot product and squared cross-product magnitude, then divide by the known vector's squared magnitude.

The given cross product is $\hat{j} - \hat{k}$, so

$$|\vec{a} \times \vec{b}|^2 = 0^2 + 1^2 + (-1)^2 = 2.$$

Also, $(\vec{a} \cdot \vec{b})^2 = 1^2 = 1$. The negative component of the cross product becomes positive on squaring; the squared components are added, never subtracted.

Step 2: Apply the dot-cross magnitude identity Substitution in the identity gives

$$1 + 2 = 3|\vec{b}|^2, \quad |\vec{b}|^2 = 1.$$

Since a magnitude is nonnegative, we select the positive square root. The identity accounts for the components of $\vec{b}$ parallel and perpendicular to $\vec{a}$, and therefore allows its magnitude to be found directly without solving three component equations.

Step 3: Conclusion Hence, $|\vec{b}| = 1$.

Quick Tip to Solve: Add the squared dot product and squared cross-product magnitude, then divide by the known vector's squared magnitude.

Section B (12 marks)

Question numbers 7 to 10 carry 3 marks each.

7(a). Find: $\int e^x \sin 2x \, dx$.

[3 marks]

Solution:

Concept:

- Integration by parts states $\int u \, dv = uv - \int v \, du$, where u and v are differentiable functions.
- Differentiation gives $\frac{d}{dx} \sin 2x = 2 \cos 2x$ and $\frac{d}{dx} \cos 2x = -2 \sin 2x$. Applying integration by parts twice returns the original integral.

Step 1: Apply integration by parts to the sine integral Let $I = \int e^x \sin 2x \, dx$. Choose $u = \sin 2x$ and $dv = e^x \, dx$. Then $du = 2 \cos 2x \, dx$ and $v = e^x$, giving

$$I = e^x \sin 2x - 2 \int e^x \cos 2x \, dx.$$

Denote the remaining integral by $J = \int e^x \cos 2x \, dx$. Thus $I = e^x \sin 2x - 2J$. The factor 2 is required by the chain rule for the angle $2x$.

Step 2: Integrate the cosine integral by parts For J , take $u = \cos 2x$ and $dv = e^x \, dx$. Hence $du = -2 \sin 2x \, dx$ and $v = e^x$, so

$$J = e^x \cos 2x + 2 \int e^x \sin 2x \, dx = e^x \cos 2x + 2I.$$

Quick Tip to Solve: For an exponential times a sine or cosine, integrate by parts twice and collect the returning original integral on the left.

Substituting this relation into the expression for I ,

$$\begin{aligned} I &= e^x \sin 2x - 2(e^x \cos 2x + 2I), \\ 5I &= e^x(\sin 2x - 2 \cos 2x). \end{aligned}$$

These relations are understood up to additive constants, which are collected into a single arbitrary constant at the end.

Step 3: Verify the antiderivative Differentiating the resulting expression checks both the coefficient and the sign:

$$\begin{aligned} \frac{d}{dx} \left[\frac{e^x}{5} (\sin 2x - 2 \cos 2x) \right] &= \frac{e^x}{5} [\sin 2x - 2 \cos 2x + 2 \cos 2x + 4 \sin 2x] \\ &= e^x \sin 2x. \end{aligned}$$

Step 4: Conclusion Therefore, $\int e^x \sin 2x \, dx = \frac{e^x}{5} (\sin 2x - 2 \cos 2x) + C$, where C is an arbitrary real constant.

Quick Tip to Solve: For an exponential times a sine or cosine, integrate by parts twice and collect the returning original integral on the left.

OR

7(b). Find: $\int \frac{2x}{(x^2 + 1)(x^2 + 2)} \, dx.$

[3 marks]

Solution:

Concept:

- Substitute $t = x^2$, so that $dt = 2x \, dx$. This converts the repeated quadratic expressions into linear factors.
- Partial fractions give $\frac{1}{(t+1)(t+2)} = \frac{1}{t+1} - \frac{1}{t+2}$.
- The logarithmic integration rule is $\int \frac{dt}{t+a} = \ln |t+a| + C$, where C is an arbitrary constant.

Step 1: Make a suitable substitution Let I denote the required integral. Both denominator factors contain x^2 , while the numerator $2x$ is the derivative of x^2 . Thus put $t = x^2$, giving $dt = 2x \, dx$. We obtain

$$I = \int \frac{dt}{(t+1)(t+2)}.$$

The substituted integrand is a rational function with two distinct linear factors in the denominator.

Step 2: Resolve into partial fractions and integrate Set

$$\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}.$$

Quick Tip to Solve: If the numerator is $2x$ and the denominator contains only x^2 , substitute $t = x^2$ before using partial fractions.

Clearing the denominator gives $1 = A(t+2) + B(t+1)$. Substituting $t = -1$ yields $A = 1$, while $t = -2$ yields $B = -1$. Hence,

$$\begin{aligned} I &= \int \left(\frac{1}{t+1} - \frac{1}{t+2} \right) dt \\ &= \ln|t+1| - \ln|t+2| + C. \end{aligned}$$

Step 3: Substitute back and check the derivative On replacing t by x^2 , both logarithm arguments are positive for every real x , so their absolute-value signs may be omitted. Differentiation verifies the result:

$$\frac{d}{dx} [\ln(x^2+1) - \ln(x^2+2)] = \frac{2x}{x^2+1} - \frac{2x}{x^2+2} = \frac{2x}{(x^2+1)(x^2+2)}.$$

Thus the antiderivative has the correct derivative throughout the real domain of the integrand.

Step 4: Conclusion The required integral is $\ln \left(\frac{x^2+1}{x^2+2} \right) + C$, where C is an arbitrary constant.

Quick Tip to Solve: If the numerator is $2x$ and the denominator contains only x^2 , substitute $t = x^2$ before using partial fractions.

8. Find the equation of the plane passing through the line of intersection of the planes $\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 10$ and $\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$ and passing through the point $(-2, 3, 1)$. [3 marks]

Solution:

Concept:

- If two planes have equations $P_1 = 0$ and $P_2 = 0$, the family $P_1 + \lambda P_2 = 0$ contains their common line, because both expressions vanish at each point of that line.
- A plane passes through a given point precisely when that point's coordinates satisfy its equation. This condition determines the parameter λ .
- For $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, the vector equation $\vec{r} \cdot (a\hat{i} + b\hat{j} + c\hat{k}) = d$ is equivalent to $ax + by + cz = d$.

Step 1: Express the given planes in Cartesian form Write $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Expanding the two dot products gives

$$P_1 = x + y + z - 10 = 0, \quad P_2 = 2x + 3y - z + 4 = 0.$$

Their normal vectors $(1, 1, 1)$ and $(2, 3, -1)$ are not proportional, so the planes intersect in a line. Every point on this line satisfies both equations simultaneously.

Step 2: Apply the prescribed point to the family of planes A plane containing the intersection line can be written as

$$(x + y + z - 10) + \lambda(2x + 3y - z + 4) = 0.$$

To make it pass through $(-2, 3, 1)$, substitute these coordinates. The two bracketed expressions have values

$$-2 + 3 + 1 - 10 = -8, \quad 2(-2) + 3(3) - 1 + 4 = 8.$$

Thus $-8 + 8\lambda = 0$, giving $\lambda = 1$. This determines the required member of the family.

Quick Tip to Solve: Evaluate each given plane's left side at the prescribed point before expanding the plane family; here their values -8 and 8 immediately give $\lambda = 1$.

Step 3: Simplify and verify the resulting equation With $\lambda = 1$, adding the given plane expressions produces

$$(x + y + z - 10) + (2x + 3y - z + 4) = 3x + 4y - 6.$$

The coefficient of z cancels. At the prescribed point, $3(-2) + 4(3) - 6 = -6 + 12 - 6 = 0$, so the point lies on this plane. Furthermore, on the intersection line each of P_1 and P_2 is zero; their sum is therefore zero there as well. Both required conditions are satisfied.

Step 4: Conclusion The required plane is $\boxed{3x + 4y = 6}$, or equivalently $\boxed{\vec{r} \cdot (3\hat{i} + 4\hat{j}) = 6}$.

Quick Tip to Solve: Evaluate each given plane's left side at the prescribed point before expanding the plane family; here their values -8 and 8 immediately give $\lambda = 1$.

9(a). Let $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{i} - \hat{j}$ and $\vec{c} = \hat{i} + \hat{j} + \hat{k}$. If $\hat{n}$ is a unit vector such that $\vec{a} \cdot \hat{n} = 0$ and $\vec{b} \cdot \hat{n} = 0$, then find $|\vec{c} \cdot \hat{n}|$. [3 marks]

Solution:

Concept:

- In the orthonormal basis $\hat{i}, \hat{j}, \hat{k}$, a dot product is the sum of products of corresponding components. A zero dot product expresses perpendicularity.
- A unit vector $\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$ satisfies $x^2 + y^2 + z^2 = 1$. Both signs of a perpendicular unit vector must be considered before taking the requested absolute value.

Step 1: Translate the two perpendicularity conditions Write $\hat{n} = x\hat{i} + y\hat{j} + z\hat{k}$, with real components. Using the given vectors, we obtain

$$\vec{a} \cdot \hat{n} = x + y = 0,$$

$$\vec{b} \cdot \hat{n} = x - y = 0.$$

Adding the equations yields $2x = 0$, so $x = 0$. Substitution into either equation gives $y = 0$. Thus the direction perpendicular to both given vectors has only a $\hat{k}$ -component. These two independent restrictions determine the normal direction, but do not yet determine its sign.

Step 2: Apply the unit-length restriction and find the dot product Since $\hat{n}$ is a unit vector,

$$x^2 + y^2 + z^2 = 1 \implies z^2 = 1.$$

Hence $z = 1$ or $z = -1$, giving $\hat{n} = \pm\hat{k}$. With $\vec{c} = \hat{i} + \hat{j} + \hat{k}$, the component formula gives

$$\vec{c} \cdot \hat{n} = x + y + z = z = \pm 1.$$

Both choices satisfy each original perpendicularity condition and have unit length. Their dot products with $\vec{c}$ have opposite signs; taking the absolute value removes this orientation dependence.

Step 3: Conclusion Therefore, $\boxed{|\vec{c} \cdot \hat{n}| = 1}$.

Quick Tip to Solve: Add and subtract the zero-dot-product equations, then apply the unit-length condition to the remaining component.

9(b). If $\vec{a}$ and $\vec{b}$ are unit vectors inclined at an angle 30° to each other, then find the area of the parallelogram with $(\vec{a} + 3\vec{b})$ and $(3\vec{a} + \vec{b})$ as adjacent sides. [3 marks]

Solution:

Concept:

- The area of a parallelogram with adjacent side vectors $\vec{u}$ and $\vec{v}$ is $A = |\vec{u} \times \vec{v}|$. The magnitude is used because area is non-negative.
- For two vectors making angle θ , $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \theta$. Cross products distribute over addition, with $\vec{a} \times \vec{a} = \vec{0}$ and $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$.

Step 1: Expand the cross product of the adjacent sides Take $\vec{u} = \vec{a} + 3\vec{b}$ and $\vec{v} = 3\vec{a} + \vec{b}$. The area formula uses these entire side vectors, rather than the original unit vectors alone. By distributivity,

$$\begin{aligned}\vec{u} \times \vec{v} &= 3(\vec{a} \times \vec{a}) + \vec{a} \times \vec{b} + 9(\vec{b} \times \vec{a}) + 3(\vec{b} \times \vec{b}) \\ &= \vec{a} \times \vec{b} - 9\vec{a} \times \vec{b} \\ &= -8\vec{a} \times \vec{b}.\end{aligned}$$

The self-products vanish because a vector is parallel to itself. Reversing the order of the remaining mixed product changes its sign, which accounts for the subtraction.

Step 2: Substitute the unit lengths and included angle Since $\vec{a}$ and $\vec{b}$ are unit vectors and their included angle is 30° ,

$$|\vec{a} \times \vec{b}| = 1 \cdot 1 \cdot \sin 30^\circ = \frac{1}{2}.$$

Taking the magnitude of the expanded cross product,

$$A = |-8\vec{a} \times \vec{b}| = 8|\vec{a} \times \vec{b}| = 8\left(\frac{1}{2}\right).$$

The negative sign specifies the normal's orientation; it cannot make the geometric area negative. No lengths for the combined sides need to be calculated separately.

Step 3: Conclusion The required area is 4 square units.

Quick Tip to Solve: For sides $p\vec{a} + q\vec{b}$ and $r\vec{a} + s\vec{b}$, use $|ps - qr| |\vec{a} \times \vec{b}|$ directly after expanding once.

10. Evaluate: $\int_0^{\pi/2} \frac{1}{1 + (\tan x)^{2/3}} dx$.

[3 marks]

Solution:

Concept:

- Reflection property: $\int_0^a f(x) dx = \int_0^a f(a-x) dx$. It follows by substituting $u = a - x$, reversing the limits and renaming the variable.

Quick Tip to Solve: Replace x by $\pi/2 - x$: the tangent power becomes its reciprocal, so the original and reflected fractions add to 1.

- For $0 < x < \pi/2$, $\tan(\pi/2 - x) = \cot x = 1/\tan x$. The positive quantity $(\tan x)^{2/3}$ therefore becomes its reciprocal under reflection.
- Complementary integrands satisfy $f(x) + f(a - x) = 1$. Adding their equal integrals then gives $2I = a$, where I is the required integral.

Step 1: Reflect the integration variable Write

$$I = \int_0^{\pi/2} \frac{1}{1 + (\tan x)^{2/3}} dx.$$

Substitute $u = \pi/2 - x$, so $dx = -du$. The limits $x = 0$ and $x = \pi/2$ become $u = \pi/2$ and $u = 0$, respectively. Reversing the limits and renaming the integration variable yields

$$I = \int_0^{\pi/2} \frac{1}{1 + (\cot x)^{2/3}} dx.$$

Thus we have a second expression for exactly the same integral, without needing an antiderivative.

Step 2: Express both fractions using tangent On the open interval, $\tan x > 0$, so $(\cot x)^{2/3} = 1/(\tan x)^{2/3}$. Multiplying numerator and denominator of the reflected integrand by $(\tan x)^{2/3}$ gives

$$I = \int_0^{\pi/2} \frac{(\tan x)^{2/3}}{1 + (\tan x)^{2/3}} dx.$$

The original integrand has limiting values 1 at 0 and 0 at $\pi/2$. Its finite endpoint limits ensure that the undefined tangent at the upper endpoint introduces no divergence.

Step 3: Add the two equal integrals Add the original expression and the reflected expression:

$$\begin{aligned} 2I &= \int_0^{\pi/2} \left[\frac{1}{1 + (\tan x)^{2/3}} + \frac{(\tan x)^{2/3}}{1 + (\tan x)^{2/3}} \right] dx \\ &= \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2} = \frac{\pi}{2}. \end{aligned}$$

The common denominator makes the numerator equal to the denominator at every interior point. Each expression on the left represents I , so their sum is $2I$.

Step 4: Conclusion Dividing by 2, the required integral is $I = \frac{\pi}{4}$.

Quick Tip to Solve: Replace x by $\pi/2 - x$: the tangent power becomes its reciprocal, so the original and reflected fractions add to 1.

Section C (16 marks)

Question numbers 11 to 14 carry 4 marks each.

11(a). Solve the following differential equation: $(y - \sin^2 x) dx + \tan x dy = 0$.

[4 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $\frac{dy}{dx} + P(x)y = Q(x)$, where P and Q are functions of x .
- Its integrating factor is $\text{I.F.} = e^{\int P(x) dx}$. Multiplication by this factor turns the left-hand side into the derivative of $y \text{ I.F.}$.

Step 1: Put the equation in linear form Dividing the given equation by dx and rearranging gives

$$\tan x \frac{dy}{dx} + y = \sin^2 x.$$

On an interval where $\tan x$ is defined and nonzero, divide by $\tan x$:

$$\frac{dy}{dx} + y \cot x = \sin x \cos x.$$

Thus $P(x) = \cot x$ and $Q(x) = \sin x \cos x$.

Step 2: Find the integrating factor Since $\int \cot x dx = \ln |\sin x|$, the integrating factor is $e^{\int \cot x dx} = |\sin x|$. A constant multiple of an integrating factor is also valid, so on a fixed-sign interval we use $\sin x$. Multiplying throughout gives

$$\sin x \frac{dy}{dx} + y \cos x = \sin^2 x \cos x.$$

By the product rule, the left-hand side is $\frac{d}{dx}(y \sin x)$.

Step 3: Integrate and verify Integrating both sides with respect to x , and taking $u = \sin x$, gives

$$y \sin x = \int \sin^2 x \cos x dx = \int u^2 du = \frac{\sin^3 x}{3} + C,$$

where C is an arbitrary constant. To check the working, differentiate this relation:

$$y' \sin x + y \cos x = \sin^2 x \cos x.$$

Dividing by $\cos x$, wherever the original $\tan x$ is defined, yields $\tan x y' + y = \sin^2 x$, exactly the given differential equation.

Step 4: Conclusion The general solution is

$$y \sin x = \frac{\sin^3 x}{3} + C,$$

Quick Tip to Solve: Divide by $\tan x$ first: the coefficient becomes $\cot x$, so the integrating factor $\sin x$ immediately produces a product derivative.

or, equivalently, $y = \frac{1}{3} \sin^2 x + C \csc x$ on intervals where $\sin x \cos x \neq 0$.

Quick Tip to Solve: Divide by $\tan x$ first: the coefficient becomes $\cot x$, so the integrating factor $\sin x$ immediately produces a product derivative.

OR

11(b). Find the general solution of the differential equation: $(x^3 + y^3) dy = x^2 y dx$.

[4 marks]

Solution:

Concept:

- A differential equation is homogeneous when its derivative can be written as a function of the ratio of its variables. Here it is convenient to regard x as a function of y .
- For a homogeneous equation in dx/dy , put $x = vy$, where $v = x/y$. The product rule then gives $dx/dy = v + y dv/dy$.
- Separation of variables reduces the transformed equation to elementary integrals. For nonzero y , $\int dy/y = \ln |y| + C$, where C is an arbitrary real constant.

Step 1: Express the equation in homogeneous form The given equation is

$$(x^3 + y^3) dy = x^2 y dx.$$

On an interval where $x \neq 0$ and $y \neq 0$, write

$$\frac{dx}{dy} = \frac{x^3 + y^3}{x^2 y} = \frac{x}{y} + \frac{y^2}{x^2}.$$

The right-hand side depends only on x/y , since $y^2/x^2 = 1/(x/y)^2$. Thus the homogeneous substitution $x = vy$ is applicable. Taking x as the dependent variable keeps the subsequent separation simple.

Step 2: Substitute and separate the variables With $x = vy$, the product rule gives

$$v + y \frac{dv}{dy} = v + \frac{1}{v^2}.$$

Cancelling v from both sides and multiplying by v^2 , we obtain

$$yv^2 \frac{dv}{dy} = 1, \quad v^2 dv = \frac{dy}{y}.$$

Each side now contains only one variable, so it can be integrated directly.

Step 3: Integrate and restore the original variables Integrating gives

$$\frac{v^3}{3} = \ln |y| + C_1.$$

Quick Tip to Solve: Use dx/dy and set $x = vy$: the v terms cancel immediately, leaving $v^2 dv = dy/y$.

Multiplying by 3, absorbing $3C_1$ into an arbitrary constant C , and replacing v by x/y , we get

$$\frac{x^3}{y^3} = 3 \ln |y| + C, \quad y \neq 0.$$

To check the result without relying on the substitution, differentiate this relation implicitly:

$$\frac{3x^2}{y^3} dx - \frac{3x^3}{y^4} dy = \frac{3}{y} dy.$$

Multiplication by $y^4/3$ yields $x^2 y dx = (x^3 + y^3) dy$, exactly the given equation. The original equation also admits the constant solution $y = 0$, which was excluded by division by y .

Step 4: Conclusion The general solution for $y \neq 0$ is

$$\boxed{\frac{x^3}{y^3} - 3 \ln |y| = C}.$$

In addition, $y = 0$ is a constant solution of the original differential equation.

Quick Tip to Solve: Use dx/dy and set $x = vy$: the v terms cancel immediately, leaving $v^2 dv = dy/y$.

12. Find the area bounded by the curves $y = |x - 1|$ and $y = 1$, using integration.

[4 marks]

Solution:

Concept:

- The area between two curves is the integral of upper ordinate minus lower ordinate: $A = \int_a^b [y_{\text{upper}} - y_{\text{lower}}] dx$, where a and b are the horizontal limits of the bounded region.
- An absolute value is evaluated piecewise. Here $|x - 1| = 1 - x$ for $x \leq 1$, and $|x - 1| = x - 1$ for $x \geq 1$; hence the integral must be split at $x = 1$.

Step 1: Find the points of intersection At an intersection the two ordinates are equal, so

$$|x - 1| = 1 \implies x - 1 = \pm 1.$$

Thus $x = 0$ or $x = 2$, and the intersection points are $(0, 1)$ and $(2, 1)$. Between these horizontal limits, $|x - 1| \leq 1$. Consequently the line $y = 1$ forms the upper boundary and the absolute-value curve forms the lower boundary throughout the bounded region.

Step 2: Set up the area integral on the two intervals The expression for the lower boundary changes at $x = 1$, where $y = |x - 1| = 0$. On $[0, 1]$ the vertical height of the region is $1 - (1 - x) = x$; on $[1, 2]$ it is $1 - (x - 1) = 2 - x$. Both heights are non-negative on their respective intervals. Therefore, using integration,

$$\begin{aligned} A &= \int_0^2 (1 - |x - 1|) dx \\ &= \int_0^1 x dx + \int_1^2 (2 - x) dx. \end{aligned}$$

Quick Tip to Solve: For this absolute-value curve, first find the intersections at $x = 0, 2$, then split the area integral at its vertex $x = 1$; the two halves must give equal areas.

Splitting at the vertex ensures that the sign in the absolute value is handled correctly on each side.

Step 3: Evaluate both contributions The required antiderivatives are $x^2/2$ and $2x - x^2/2$. Substituting the limits separately gives

$$\begin{aligned} A &= \left[\frac{x^2}{2} \right]_0^1 + \left[2x - \frac{x^2}{2} \right]_1^2 \\ &= \frac{1}{2} + \left((4 - 2) - \left(2 - \frac{1}{2} \right) \right) \\ &= \frac{1}{2} + \frac{1}{2}. \end{aligned}$$

The equal contributions also agree with the symmetry of $|x - 1|$ about $x = 1$. Since this is an area, the two positive integrals are added; they are not subtracted from each other.

Step 4: Conclusion Hence, the area bounded by $y = |x - 1|$ and $y = 1$ is 1 square unit.

Quick Tip to Solve: For this absolute-value curve, first find the intersections at $x = 0, 2$, then split the area integral at its vertex $x = 1$; the two halves must give equal areas.

13. In a factory, machine A produces 30% of total output, machine B produces 25% and the machine C produces the remaining output. The defective items produced by machines A , B and C are 1%, 1.2%, 2% respectively. An item is picked at random from a day's output and found to be defective. Find the probability that it was produced by machine B . [4 marks]

Solution:

Concept:

- Let A , B and C denote the events that the selected item was produced by the corresponding machines, and let D denote the event that it is defective. These machine events are mutually exclusive and exhaustive.
- The total probability theorem gives $P(D) = P(A)P(D | A) + P(B)P(D | B) + P(C)P(D | C)$.
- Bayes' theorem reverses the conditioning: $P(B | D) = \frac{P(B)P(D | B)}{P(D)}$. Here $P(D | B)$ is the defect rate of machine B , whereas $P(B | D)$ is the required probability about a known defective item.

Step 1: Write the production shares and conditional defect probabilities Since machine C produces the remaining output, its share is $100\% - 30\% - 25\% = 45\%$. Thus

$$P(A) = 0.30, \quad P(B) = 0.25, \quad P(C) = 0.45.$$

Converting the given defect percentages to probabilities gives

$$P(D | A) = 0.01, \quad P(D | B) = 0.012, \quad P(D | C) = 0.02.$$

The production shares add to 1, as required for the three possible origins of the selected item.

Step 2: Calculate the overall probability of a defective item An item can be defective and originate

Quick Tip to Solve: First multiply each machine's production share by its defect rate; then divide machine B 's contribution by the sum of all three contributions.

from any one of the three machines. Multiplying each production share by its own defect rate gives the mutually exclusive joint probabilities

$$P(A \cap D) = 0.30(0.01) = 0.003,$$

$$P(B \cap D) = 0.25(0.012) = 0.003,$$

$$P(C \cap D) = 0.45(0.02) = 0.009.$$

Adding all three contributions using the total probability theorem,

$$P(D) = 0.003 + 0.003 + 0.009 = 0.015.$$

This denominator represents all defective output, including that produced by machines A and C .

Step 3: Apply Bayes' theorem to the observed defective item The item is already known to be defective, so the sample space is restricted to defective items. The appropriate numerator is the joint probability that the item came from B and is defective. Therefore

$$P(B | D) = \frac{P(B \cap D)}{P(D)} = \frac{0.003}{0.015} = \frac{3}{15} = \frac{1}{5}.$$

The conditioning is essential: the unconditional production share 0.25 alone does not account for the different defect rates of the machines.

Step 4: Conclusion The probability that the defective item was produced by machine B is $\boxed{\frac{1}{5}}$, or 20%.

Quick Tip to Solve: First multiply each machine's production share by its defect rate; then divide machine B 's contribution by the sum of all three contributions.

14. Case Study Problem:

Two motorcycles A and B are running at the speed more than the allowed speed on the roads represented by the lines $\vec{r} = \lambda(\hat{i} + 2\hat{j} - \hat{k})$ and $\vec{r} = (3\hat{i} + 3\hat{j}) + \mu(2\hat{i} + \hat{j} + \hat{k})$ respectively.

Based on the above information, answer the following questions:

[4 marks]

14(a). Find the shortest distance between the given lines.

[2 marks]

Solution:

Concept:

- For lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$, the vectors $\vec{b}_1, \vec{b}_2$ give their directions and $\vec{a}_1, \vec{a}_2$ locate points on them.
- For nonparallel lines, the shortest distance is $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$. A zero distance occurs when these lines intersect.

Step 1: Identify the vectors and their common normal Here $\vec{a}_1 = (0, 0, 0)$, $\vec{a}_2 = (3, 3, 0)$, $\vec{b}_1 = (1, 2, -1)$

Quick Tip to Solve: Compute the scalar triple product first: a zero numerator with a nonzero cross product shows the lines intersect.

and $\vec{b}_2 = (2, 1, 1)$. Thus

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 3\hat{i} - 3\hat{j} - 3\hat{k}.$$

This nonzero vector confirms that the lines are not parallel; its magnitude is $3\sqrt{3}$.

Step 2: Substitute in the distance formula The displacement between the chosen points is $\vec{a}_2 - \vec{a}_1 = 3\hat{i} + 3\hat{j}$. Its scalar product with the common normal is

$$(3\hat{i} + 3\hat{j}) \cdot (3\hat{i} - 3\hat{j} - 3\hat{k}) = 9 - 9 = 0.$$

Hence the distance calculation gives $d = |0|/(3\sqrt{3})$. The displacement has no component along the common normal, consistent with the roads meeting.

Step 3: Conclusion The shortest distance between the given lines is $\boxed{0}$ units.

Quick Tip to Solve: Compute the scalar triple product first: a zero numerator with a nonzero cross product shows the lines intersect.

14(b). Find the point at which the motorcycles may collide.

[2 marks]

Solution:

Concept:

- A possible collision point must lie on both road lines. Equate corresponding Cartesian coordinates to find the common point.
- The parameters λ and μ describe positions on the two lines independently. A common position is a possible collision location when the motorcycles reach it simultaneously.

Step 1: Equate the coordinates on the roads A general point on the first road is $(\lambda, 2\lambda, -\lambda)$, while a general point on the second road is $(3 + 2\mu, 3 + \mu, \mu)$. Therefore a common point must satisfy

$$\lambda = 3 + 2\mu, \quad 2\lambda = 3 + \mu, \quad -\lambda = \mu.$$

All three equations must hold for the same pair of parameter values; satisfying just one coordinate is insufficient.

Step 2: Solve the parameters and verify the common position From the third equation, $\mu = -\lambda$. Substitute this into the first equation:

$$\lambda = 3 - 2\lambda \implies 3\lambda = 3 \implies \lambda = 1, \quad \mu = -1.$$

The second equation also holds, since $2(1) = 3 + (-1) = 2$. Substitution into the first line gives $(1, 2, -1)$, and into the second gives $(3 - 2, 3 - 1, -1) = (1, 2, -1)$. Thus this position belongs to both roads; an actual collision requires simultaneous arrival there.

Step 3: Conclusion The motorcycles may collide at $\boxed{(1, 2, -1)}$, whose position vector is $\boxed{\hat{i} + 2\hat{j} - \hat{k}}$.

Quick Tip to Solve: Use the z-coordinates first to obtain $\mu = -\lambda$, then check the remaining coordinate after solving.