

# CBSE Class 12 Mathematics Question Paper 2022 with Solution PDF-Set 65-5-3

## Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 2 hours

Maximum Marks: 40

### General Instructions

- (i) This question paper contains **three Sections-A, B and C**.
- (ii) Each section is compulsory.
- (iii) **Section-A** has 6 short-answer type-I questions of 2 marks each.
- (iv) **Section-B** has 4 short-answer type-II questions of 3 marks each.
- (v) **Section-C** has 4 long-answer type questions of 4 marks each.
- (vi) There is an internal choice in some questions.
- (vii) Question 14 is a case study based question with two subparts of 2 marks each.

## Section A (12 marks)

Question numbers 1 to 6 carry 2 marks each.

1. The Cartesian equation of a line  $AB$  is:  $\frac{2x-1}{12} = \frac{y+2}{2} = \frac{z-3}{3}$ . Find the direction cosines of a line parallel to line  $AB$ . [2 marks]

### Solution:

#### Concept:

- A line in symmetric form  $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$  has direction ratios  $a, b, c$ . A parallel line has proportional direction ratios.
- The direction cosines corresponding to direction ratios  $a, b, c$  are  $l = \frac{a}{\sqrt{a^2+b^2+c^2}}$ ,  $m = \frac{b}{\sqrt{a^2+b^2+c^2}}$  and  $n = \frac{c}{\sqrt{a^2+b^2+c^2}}$ ; they satisfy  $l^2 + m^2 + n^2 = 1$ .

**Step 1: Put the line into standard symmetric form** The coefficient of  $x$  in the first numerator must be removed before reading the direction ratios. Since  $2x - 1 = 2(x - \frac{1}{2})$ ,

$$\frac{2x-1}{12} = \frac{x-\frac{1}{2}}{6}.$$

Hence the given line is

$$\frac{x-\frac{1}{2}}{6} = \frac{y-(-2)}{2} = \frac{z-3}{3}.$$

Its direction ratios are therefore 6, 2, 3, not 12, 2, 3. Any parallel line can use the same direction ratios.

**Step 2: Normalize the direction ratios** The length of the direction vector (6, 2, 3) is

$$\sqrt{6^2 + 2^2 + 3^2} = \sqrt{36 + 4 + 9} = 7.$$

Dividing each component by 7 gives  $l = 6/7$ ,  $m = 2/7$  and  $n = 3/7$ . As a check, their squared sum is  $(36 + 4 + 9)/49 = 1$ . Reversing the direction of the same line changes all three signs together.

**Step 3: Conclusion** The direction cosines are  $\left(\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$ . The opposite orientation has  $\left(-\frac{6}{7}, -\frac{2}{7}, -\frac{3}{7}\right)$ .

**Quick Tip to Solve:** First make every numerator's coordinate coefficient equal to one: here the first direction ratio is  $12/2 = 6$ .

2. Find:  $\int \frac{dx}{\sqrt{4x - x^2}}.$

[2 marks]

**Solution:**

**Concept:**

- Complete the square to express a quadratic under a square root as  $a^2 - u^2$ . The standard antiderivative is  $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}(u/a) + C$ , for  $a > 0$  and  $|u| < a$ .
- In an indefinite integral,  $C$  denotes an arbitrary real constant. Here  $\sin^{-1}$  denotes the principal inverse sine function, not the reciprocal of sine.

**Step 1: Complete the square and substitute** Rewrite the expression under the square root:

$$4x - x^2 = 4 - (x^2 - 4x + 4) = 4 - (x - 2)^2.$$

Put  $u = x - 2$ , so that  $du = dx$ . The required integral becomes

$$I = \int \frac{du}{\sqrt{2^2 - u^2}}.$$

For a real integrand with a nonzero denominator,  $4x - x^2 > 0$ ; hence  $0 < x < 4$ , equivalently  $-2 < u < 2$ .

**Step 2: Apply the inverse sine formula and check** Using the standard formula with  $a = 2$ , we obtain  $I = \sin^{-1}(u/2) + C$ . Substituting back gives  $I = \sin^{-1}((x - 2)/2) + C$ . Its derivative is

$$\frac{d}{dx} \sin^{-1}\left(\frac{x - 2}{2}\right) = \frac{1/2}{\sqrt{1 - (x - 2)^2/4}} = \frac{1}{\sqrt{4 - (x - 2)^2}} = \frac{1}{\sqrt{4x - x^2}},$$

which reproduces the given integrand throughout  $0 < x < 4$ .

**Step 3: Conclusion**  $\int \frac{dx}{\sqrt{4x - x^2}} = \sin^{-1}\left(\frac{x - 2}{2}\right) + C$ , valid for  $0 < x < 4$ .

**Quick Tip to Solve:** Rewrite  $4x - x^2$  as  $4 - (x - 2)^2$ ; the shift  $u = x - 2$  immediately exposes the inverse sine formula.

3(a). Events  $A$  and  $B$  are such that  $P(A) = \frac{1}{2}$ ,  $P(B) = \frac{7}{12}$  and  $P(\overline{A} \cup \overline{B}) = \frac{1}{4}$ . Find whether the events  $A$  and  $B$  are independent or not.

[2 marks]

**Solution:**

**Concept:**

- Events  $A$  and  $B$  are independent precisely when  $P(A \cap B) = P(A)P(B)$ . The intersection probability cannot exceed either individual probability.
- De Morgan's law gives  $\overline{A} \cup \overline{B} = \overline{A \cap B}$ . Thus  $P(A \cap B) = 1 - P(\overline{A} \cup \overline{B})$ ; the bars in the question apply separately to  $A$  and  $B$ .

**Quick Tip to Solve:** Separate complement bars give  $\overline{A} \cup \overline{B} = \overline{A \cap B}$ ; check the resulting intersection against  $P(A)$  before judging independence.

**Step 1: Use the printed complement condition** Applying De Morgan's law to the given probability,

$$P(A \cap B) = 1 - P(\overline{A} \cup \overline{B}) = 1 - \frac{1}{4} = \frac{3}{4}.$$

This condition concerns the complement of an intersection. It must not be replaced by the complement of the union  $A \cup B$ , because that would be a different event.

**Step 2: Compare with the independence criterion and probability bounds** The product of the individual probabilities is

$$P(A)P(B) = \frac{1}{2} \cdot \frac{7}{12} = \frac{7}{24},$$

whereas the printed condition gives  $P(A \cap B) = 3/4 = 18/24$ . These are unequal, so the independence criterion is not satisfied. There is also a stronger inconsistency: since  $A \cap B \subseteq A$ , we must have  $P(A \cap B) \leq P(A) = 1/2$ . The derived value  $3/4$  violates this bound. Equivalently,  $\overline{A} \subseteq \overline{A} \cup \overline{B}$  would require the latter probability to be at least  $P(\overline{A}) = 1/2$ , not  $1/4$ .

**Step 3: Conclusion** The independence equality fails:  $\frac{3}{4} \neq \frac{7}{24}$ . Moreover, the printed probabilities are inconsistent, so no pair of events can satisfy all three conditions.

**Quick Tip to Solve:** Separate complement bars give  $\overline{A} \cup \overline{B} = \overline{A \cap B}$ ; check the resulting intersection against  $P(A)$  before judging independence.

OR

**3(b).** A box  $B_1$  contains 1 white ball and 3 red balls. Another box  $B_2$  contains 2 white balls and 3 red balls. If one ball is drawn at random from each of the boxes  $B_1$  and  $B_2$ , then find the probability that the two balls drawn are of the same colour. [2 marks]

**Solution:**

**Concept:**

- For equally likely balls, the probability of a colour is its number of balls divided by the total number of balls in that box.
- The draws from the two separate boxes are independent, so joint probabilities multiply. The events of drawing two white balls and two red balls are mutually exclusive, so their probabilities add.

**Step 1: Find the colour probabilities for each box** Box  $B_1$  contains  $1 + 3 = 4$  balls, while box  $B_2$  contains  $2 + 3 = 5$  balls. Writing  $W_i$  and  $R_i$  for white and red draws from box  $B_i$ ,

$$P(W_1) = \frac{1}{4}, \quad P(R_1) = \frac{3}{4}, \quad P(W_2) = \frac{2}{5}, \quad P(R_2) = \frac{3}{5}.$$

**Step 2: Combine the two ways to get the same colour** The balls have the same colour precisely when

**Quick Tip to Solve:** List both matching colours before adding: white-white and red-red each need a product of the probabilities from the two boxes.

both are white or both are red. Since these outcomes cannot occur together, independence gives

$$\begin{aligned} P(\text{same colour}) &= P(W_1)P(W_2) + P(R_1)P(R_2) \\ &= \frac{1}{4} \cdot \frac{2}{5} + \frac{3}{4} \cdot \frac{3}{5} = \frac{2+9}{20}. \end{aligned}$$

**Step 3: Conclusion** The required probability is  $\boxed{\frac{11}{20}}$ .

**Quick Tip to Solve:** List both matching colours before adding: white-white and red-red each need a product of the probabilities from the two boxes.

4. Find the general solution of the following differential equation:  $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$ . [2 marks]

**Solution:**

**Concept:**

- A separable differential equation can be solved by collecting the factors involving  $y$  with  $dy$  and those involving  $x$  with  $dx$ , then integrating.
- Use  $e^{x-y} = e^x e^{-y}$  and  $\int e^y dy = e^y$ . A single arbitrary constant  $C$  accounts for the constants of integration on both sides.

**Step 1: Separate the variables** Factor the right-hand side of the given equation:

$$\frac{dy}{dx} = e^{-y}(e^x + x^2).$$

As  $e^y$  is positive for every real  $y$ , multiplication by it introduces no loss of solutions. Hence

$$e^y dy = (e^x + x^2) dx.$$

**Step 2: Integrate both sides and check** Integration gives

$$\int e^y dy = \int (e^x + x^2) dx, \quad e^y = e^x + \frac{x^3}{3} + C.$$

Differentiating this relation yields  $e^y y' = e^x + x^2$ ; dividing by  $e^y$  reproduces the given differential equation. For a real-valued explicit solution, the expression inside the logarithm must be positive.

**Step 3: Conclusion** The general solution is  $\boxed{e^y = e^x + \frac{x^3}{3} + C}$ , equivalently  $y = \ln(e^x + x^3/3 + C)$  on any interval where  $e^x + x^3/3 + C > 0$ .

**Quick Tip to Solve:** Factor out  $e^{-y}$  first, then multiply by  $e^y$ ; this immediately makes the equation separable.

5. If  $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ ,  $\vec{a} \cdot \vec{b} = 1$  and  $\vec{a} \times \vec{b} = \hat{j} - \hat{k}$ , then find  $|\vec{b}|$ .

[2 marks]

**Solution:**

**Concept:**

- For two vectors  $\vec{a}$  and  $\vec{b}$ , the identity  $|\vec{a} \times \vec{b}|^2 + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$  follows from  $\sin^2 \theta + \cos^2 \theta = 1$ , where  $\theta$  is their angle.
- The square of a vector's magnitude is the sum of the squares of its components along the mutually perpendicular unit vectors  $\hat{i}, \hat{j}, \hat{k}$ .

**Step 1: Evaluate the known magnitudes** The given vector is  $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ , so

$$|\vec{a}|^2 = 1^2 + 1^2 + 1^2 = 3.$$

Also,  $\vec{a} \times \vec{b} = \hat{j} - \hat{k}$  has components  $(0, 1, -1)$ . Therefore

$$|\vec{a} \times \vec{b}|^2 = 0^2 + 1^2 + (-1)^2 = 2, \quad (\vec{a} \cdot \vec{b})^2 = 1.$$

**Step 2: Apply the dot-cross identity** Substitute the known values into the identity:

$$\begin{aligned} |\vec{a}|^2 |\vec{b}|^2 &= (\vec{a} \cdot \vec{b})^2 + |\vec{a} \times \vec{b}|^2 \\ 3|\vec{b}|^2 &= 1 + 2 = 3, \end{aligned}$$

so  $|\vec{b}|^2 = 1$ . Since a magnitude is non-negative, the positive square root is required.

**Step 3: Conclusion** The required magnitude is  $|\vec{b}| = 1$ .

**Quick Tip to Solve:** When both dot and cross products are supplied, use their squared-magnitude identity to find the unknown magnitude without solving for components.

6. Let  $X$  be a random variable which assumes values  $x_1, x_2, x_3, x_4$  such that  $2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4)$ . Find the probability distribution of  $X$ .

[2 marks]

**Solution:**

**Concept:**

- A probability distribution lists each possible value of a random variable with its probability. Every probability is non-negative and their sum is 1.
- If several multiples of probabilities are equal, introduce their common value first. This converts the given relation into a single normalisation equation.

**Step 1: Express the probabilities using a common value** Write  $p_i = P(X = x_i)$ , for  $i = 1, 2, 3, 4$ . Let the common value in the given relation be  $k$ . Then

$$2p_1 = 3p_2 = p_3 = 5p_4 = k, \quad p_1 = \frac{k}{2}, \quad p_2 = \frac{k}{3}, \quad p_3 = k, \quad p_4 = \frac{k}{5}.$$

**Quick Tip to Solve:** For equal multiples of probabilities, take the reciprocal ratio: here  $1/2 : 1/3 : 1 : 1/5 = 15 : 10 : 30 : 6$ , and divide each entry by 61.

Thus the probabilities are inversely proportional to the respective multipliers 2, 3, 1, 5, rather than directly proportional to them.

**Step 2: Normalise and check the probabilities** Since these are all the possible values of  $X$ ,

$$\begin{aligned}\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} &= 1, \\ k \left( \frac{15 + 10 + 30 + 6}{30} \right) &= 1, \\ \frac{61k}{30} &= 1, \quad k = \frac{30}{61}.\end{aligned}$$

Substitution gives  $p_1 = 15/61$ ,  $p_2 = 10/61$ ,  $p_3 = 30/61$  and  $p_4 = 6/61$ . All four are positive and sum to  $(15 + 10 + 30 + 6)/61 = 1$ . Also,  $2p_1 = 3p_2 = p_3 = 5p_4 = 30/61$ , confirming the given relation.

**Step 3: Conclusion** The required probability distribution is

| $X$    | $x_1$           | $x_2$           | $x_3$           | $x_4$          |
|--------|-----------------|-----------------|-----------------|----------------|
| $P(X)$ | $\frac{15}{61}$ | $\frac{10}{61}$ | $\frac{30}{61}$ | $\frac{6}{61}$ |

**Quick Tip to Solve:** For equal multiples of probabilities, take the reciprocal ratio: here  $1/2 : 1/3 : 1 : 1/5 = 15 : 10 : 30 : 6$ , and divide each entry by 61.

## Section B (12 marks)

Question numbers 7 to 10 carry 3 marks each.

**7(a).** If  $\vec{a}$  and  $\vec{b}$  are two vectors such that  $|\vec{a} + \vec{b}| = |\vec{b}|$ , then prove that  $(\vec{a} + 2\vec{b})$  is perpendicular to  $\vec{a}$ . [3 marks]

**Solution:**

**Concept:**

- For any vector  $\vec{u}$ ,  $|\vec{u}|^2 = \vec{u} \cdot \vec{u}$ . Squaring an equality of magnitudes therefore gives an equality of scalar products.
- The dot product is distributive and symmetric:  $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$  and  $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ .
- Two non-zero vectors are perpendicular when their dot product is zero, since  $\vec{u} \cdot \vec{v} = |\vec{u}| |\vec{v}| \cos \phi$ , where  $\phi$  is their angle.

**Step 1: Square the given magnitude equality** The hypothesis is  $|\vec{a} + \vec{b}| = |\vec{b}|$ . Since magnitudes are non-negative, we can square both sides to obtain

$$(\vec{a} + \vec{b}) \cdot (\vec{a} + \vec{b}) = \vec{b} \cdot \vec{b}.$$

This replaces the length condition by an algebraic condition involving the same two vectors, without introducing their angle as a separate unknown.

**Quick Tip to Solve:** Square a given equality of vector magnitudes first; after cancelling the common squared length, factor the remaining expression as the dot product you need to prove is zero.

**Step 2: Expand and simplify the scalar product** Using distributivity and symmetry gives

$$\vec{a} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = \vec{b} \cdot \vec{b},$$
$$|\vec{a}|^2 + 2\vec{a} \cdot \vec{b} = 0.$$

The two mixed terms are equal, which is why their coefficient is 2. Cancelling  $|\vec{b}|^2$  on both sides leaves precisely the expression needed for the required perpendicularity.

**Step 3: Evaluate the dot product of the required vectors** Now calculate

$$(\vec{a} + 2\vec{b}) \cdot \vec{a} = \vec{a} \cdot \vec{a} + 2\vec{b} \cdot \vec{a} = |\vec{a}|^2 + 2\vec{a} \cdot \vec{b} = 0.$$

Thus the required vectors have zero scalar product. In the non-zero case their angle has zero cosine and is therefore  $90^\circ$ . The scalar-product identity remains valid in any degenerate zero-vector case as well.

**Step 4: Conclusion** Hence  $(\vec{a} + 2\vec{b}) \cdot \vec{a} = 0$ , proving that  $\vec{a} + 2\vec{b}$  is perpendicular to  $\vec{a}$  by the scalar-product criterion.

**Quick Tip to Solve:** Square a given equality of vector magnitudes first; after cancelling the common squared length, factor the remaining expression as the dot product you need to prove is zero.

OR

**7(b). If  $\vec{a}$  and  $\vec{b}$  are unit vectors and  $\theta$  is the angle between them, then prove that  $\sin \frac{\theta}{2} = \frac{1}{2} |\vec{a} - \vec{b}|$ .** [3 marks]

**Solution:**

**Concept:**

- A unit vector has magnitude 1. For vectors with angle  $\theta$ , the scalar product is  $\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos \theta$ ; for unit vectors it is  $\cos \theta$ .
- The squared magnitude of a difference is  $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b}$ .
- The half-angle identity is  $1 - \cos \theta = 2 \sin^2(\theta/2)$ . The angle between two non-zero vectors lies in  $[0, \pi]$ , so  $\sin(\theta/2) \geq 0$ .

**Step 1: Expand the squared magnitude of the difference** By the definition and distributive property of the scalar product,

$$|\vec{a} - \vec{b}|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})$$
$$= \vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{b} - 2\vec{a} \cdot \vec{b}.$$

The negative mixed terms combine because the scalar product is symmetric. We use the difference here because the desired formula contains  $|\vec{a} - \vec{b}|$ .

**Step 2: Use the unit-vector condition and the half-angle identity** Since  $|\vec{a}| = |\vec{b}| = 1$  and their angle is

**Quick Tip to Solve:** For unit vectors, expand the squared sum or difference first; a difference produces  $1 - \cos \theta$ , which converts directly to a sine half-angle identity.

$\theta$ , we have  $\vec{a} \cdot \vec{b} = \cos \theta$ . Therefore

$$\begin{aligned} |\vec{a} - \vec{b}|^2 &= 1 + 1 - 2 \cos \theta \\ &= 2(1 - \cos \theta) \\ &= 4 \sin^2 \frac{\theta}{2}. \end{aligned}$$

The unit-vector hypothesis is essential: it makes both squared magnitudes equal to 1 and removes the magnitude factors from the mixed scalar product.

**Step 3: Take the non-negative square root** Both  $|\vec{a} - \vec{b}|$  and  $\sin(\theta/2)$  are non-negative because  $0 \leq \theta \leq \pi$ . Thus taking square roots gives  $|\vec{a} - \vec{b}| = 2 \sin(\theta/2)$ , without a sign ambiguity. This also covers  $\theta = 0$  and  $\theta = \pi$ , for which the difference has magnitude 0 and 2, respectively.

**Step 4: Conclusion** Hence

$$\sin \frac{\theta}{2} = \frac{1}{2} |\vec{a} - \vec{b}|.$$

**Quick Tip to Solve:** For unit vectors, expand the squared sum or difference first; a difference produces  $1 - \cos \theta$ , which converts directly to a sine half-angle identity.

**8(a). Find:**  $\int e^x \sin 2x \, dx$ .

[3 marks]

**Solution:**

**Concept:**

- Integration by parts states that  $\int u \, dv = uv - \int v \, du$ . Choose the trigonometric factor as  $u$  and the exponential factor as  $dv$ .
- Applying integration by parts twice brings back the original integral. Treating it as an unknown  $I$  then gives an algebraic equation for its antiderivative.

**Step 1: Apply integration by parts for the first time** Put  $I = \int e^x \sin 2x \, dx$ . Take  $u = \sin 2x$  and  $dv = e^x \, dx$ , so that  $du = 2 \cos 2x \, dx$  and  $v = e^x$ . Therefore,

$$I = e^x \sin 2x - 2 \int e^x \cos 2x \, dx.$$

Denote the remaining integral by  $J = \int e^x \cos 2x \, dx$ . Thus  $I = e^x \sin 2x - 2J$ .

**Step 2: Apply integration by parts to the remaining integral** For  $J$ , take  $u = \cos 2x$  and  $dv = e^x \, dx$ . Here  $du = -2 \sin 2x \, dx$  and  $v = e^x$ . The negative derivative changes the sign of the new integral:

$$J = e^x \cos 2x + 2 \int e^x \sin 2x \, dx = e^x \cos 2x + 2I.$$

**Quick Tip to Solve:** After the second integration by parts, move the repeated integral to the left; do not keep integrating indefinitely.

Substitute this expression into the first relation and collect the terms containing  $I$ :

$$I = e^x \sin 2x - 2(e^x \cos 2x + 2I),$$

$$5I = e^x(\sin 2x - 2 \cos 2x).$$

These relations are taken for compatible antiderivatives; the arbitrary constant is inserted in the final expression.

**Step 3: Check the antiderivative** Differentiating the expression obtained, using the product rule, gives

$$\frac{d}{dx} \left[ \frac{e^x}{5} (\sin 2x - 2 \cos 2x) \right] = \frac{e^x}{5} [(\sin 2x - 2 \cos 2x) + (2 \cos 2x + 4 \sin 2x)]$$

$$= e^x \sin 2x.$$

This reproduces the given integrand and checks both the coefficient and the minus sign.

**Step 4: Conclusion** Hence,  $\int e^x \sin 2x \, dx = \frac{e^x}{5} (\sin 2x - 2 \cos 2x) + C$ , where  $C$  is an arbitrary real constant.

**Quick Tip to Solve:** After the second integration by parts, move the repeated integral to the left; do not keep integrating indefinitely.

OR

**8(b). Find:**  $\int \frac{2x}{(x^2 + 1)(x^2 + 2)} \, dx.$

[3 marks]

**Solution:**

**Concept:**

- A substitution is useful when the numerator contains the derivative of a repeated expression in the denominator. Here  $d(x^2 + 1) = 2x \, dx$ .
- Partial fractions give  $\frac{1}{t(t+1)} = \frac{1}{t} - \frac{1}{t+1}$ . Also,  $\int \frac{1}{t} \, dt = \ln |t| + C$ , where  $C$  is an arbitrary constant.

**Step 1: Substitute for the quadratic expression** Let  $I = \int \frac{2x}{(x^2 + 1)(x^2 + 2)} \, dx$ . Set  $t = x^2 + 1$ , giving  $dt = 2x \, dx$  and  $x^2 + 2 = t + 1$ . The integral becomes

$$I = \int \frac{1}{t(t+1)} \, dt.$$

The numerator is absorbed completely into the differential, so no additional factor of two is needed.

**Step 2: Resolve the rational function into partial fractions** Write  $\frac{1}{t(t+1)} = \frac{A}{t} + \frac{B}{t+1}$ . Multiplying by  $t(t+1)$  yields

$$1 = A(t+1) + Bt = (A+B)t + A.$$

Comparing the constant terms gives  $A = 1$ , while comparing the coefficients of  $t$  gives  $A + B = 0$ ,

**Quick Tip to Solve:** Substitute  $t = x^2 + 1$  first: the two quadratic factors become consecutive linear factors, making the partial fractions immediate.

hence  $B = -1$ . Therefore,

$$I = \int \left( \frac{1}{t} - \frac{1}{t+1} \right) dt = \ln |t| - \ln |t+1| + C.$$

For real  $x$ , both  $t = x^2 + 1$  and  $t + 1 = x^2 + 2$  are strictly positive, so their absolute-value signs can be omitted after substitution.

**Step 3: Substitute back and verify** Returning to  $x$ , the antiderivative is  $\ln(x^2 + 1) - \ln(x^2 + 2) + C$ . Its derivative is

$$\begin{aligned} \frac{2x}{x^2 + 1} - \frac{2x}{x^2 + 2} &= \frac{2x[(x^2 + 2) - (x^2 + 1)]}{(x^2 + 1)(x^2 + 2)} \\ &= \frac{2x}{(x^2 + 1)(x^2 + 2)}. \end{aligned}$$

This checks the order of the logarithms and reproduces the integrand for every real  $x$ .

**Step 4: Conclusion** Hence,  $\int \frac{2x}{(x^2 + 1)(x^2 + 2)} dx = \ln \left( \frac{x^2 + 1}{x^2 + 2} \right) + C$ , where  $C$  is an arbitrary real constant.

**Quick Tip to Solve:** Substitute  $t = x^2 + 1$  first: the two quadratic factors become consecutive linear factors, making the partial fractions immediate.

**9. Evaluate:**  $\int_1^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4-x}} dx.$

[3 marks]

**Solution:**

**Concept:**

- The reflection property of definite integrals is  $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$ . It follows from the substitution  $t = a + b - x$ , with the transformed limits reversed.
- If  $f(x) + f(a + b - x) = 1$  throughout the interval, adding the original and reflected integrals gives  $2I = b - a$ . Here the endpoints sum to 4, which exchanges the two square roots.

**Step 1: Check the integrand on the given interval** Let

$$I = \int_1^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4-x}} dx.$$

For  $1 \leq x \leq 3$ , both  $x$  and  $4 - x$  lie between 1 and 3. Thus both square roots are real and positive, and the denominator cannot vanish. The integrand is continuous on the entire interval, so the reflection property applies.

**Step 2: Reflect the variable about the midpoint** Use  $t = 4 - x$ , so  $dx = -dt$ . When  $x = 1$ ,  $t = 3$ , and when  $x = 3$ ,  $t = 1$ . Therefore,

$$\begin{aligned} I &= \int_3^1 \frac{\sqrt{4-t}}{\sqrt{4-t} + \sqrt{t}} (-dt) \\ &= \int_1^3 \frac{\sqrt{4-t}}{\sqrt{4-t} + \sqrt{t}} dt. \end{aligned}$$

**Quick Tip to Solve:** When two terms in a denominator exchange under  $x \mapsto a + b - x$ , add the reflected integrals before attempting an antiderivative.

Since the integration variable is a dummy variable, rename  $t$  as  $x$ :

$$I = \int_1^3 \frac{\sqrt{4-x}}{\sqrt{x} + \sqrt{4-x}} dx.$$

The limits are unchanged after reversing them, while the numerator becomes the other square root.

**Step 3: Add the two equal representations** Adding the original integral and its reflected form, using their common denominator, gives

$$\begin{aligned} 2I &= \int_1^3 \left( \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4-x}} + \frac{\sqrt{4-x}}{\sqrt{x} + \sqrt{4-x}} \right) dx \\ &= \int_1^3 1 dx = [x]_1^3 = 3 - 1 = 2. \end{aligned}$$

The simplification holds at every point in the interval because the denominator is positive. Dividing the equality by two yields the required value without finding an indefinite integral.

**Step 4: Conclusion** Hence,  $\int_1^3 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4-x}} dx = 1$ .

**Quick Tip to Solve:** When two terms in a denominator exchange under  $x \mapsto a + b - x$ , add the reflected integrals before attempting an antiderivative.

**10. Find the distance of the point  $(2, 3, 4)$  measured along the line  $\frac{x-4}{3} = \frac{y+5}{6} = \frac{z+1}{2}$  from the plane  $3x + 2y + 2z + 5 = 0$ .** [3 marks]

**Solution:**

**Concept:**

- A line in symmetric form  $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$  has direction ratios  $(a, b, c)$ . A displacement parallel to it can therefore be written as  $t(a, b, c)$ .
- To measure a point's distance from a plane along a specified direction, find where the line through the point parallel to that direction meets the plane. The required distance is the length of this displacement, rather than the perpendicular distance to the plane.

**Step 1: Write the line through the given point in the specified direction** Let  $P = (2, 3, 4)$ . From the denominators of the given line, its direction ratios are  $(3, 6, 2)$ . The point  $P$  need not lie on the given line: the words "measured along the line" specify the direction of measurement. Thus the required line through  $P$  is

$$x = 2 + 3t, \quad y = 3 + 6t, \quad z = 4 + 2t.$$

Every point on this line differs from  $P$  by a vector parallel to  $(3, 6, 2)$ .

**Step 2: Find the intersection with the plane** Let  $Q$  be the intersection of this line with  $3x + 2y + 2z + 5 = 0$

**Quick Tip to Solve:** Use the direction ratios as a displacement from the given point, solve for its parameter in the plane, and multiply the absolute parameter by the direction vector's length.

0. Substituting the coordinates gives

$$\begin{aligned}3(2 + 3t) + 2(3 + 6t) + 2(4 + 2t) + 5 &= 0, \\25 + 25t &= 0, \\t &= -1.\end{aligned}$$

Hence  $Q = (-1, -3, 2)$ . As a check,  $3(-1) + 2(-3) + 2(2) + 5 = 0$ , so this point lies on the plane. The negative parameter means that the intersection is reached opposite to the chosen direction vector; the distance itself remains nonnegative.

**Step 3: Compute the length of the displacement** The displacement from  $P$  to  $Q$  is  $(-3, -6, -2)$ . Therefore,

$$PQ = \sqrt{(-3)^2 + (-6)^2 + (-2)^2} = \sqrt{9 + 36 + 4} = \sqrt{49}.$$

This uses the length along the required direction and completes the distance calculation.

**Step 4: Conclusion** The distance of  $(2, 3, 4)$  from the plane, measured along the specified line direction, is **7 units**.

**Quick Tip to Solve:** Use the direction ratios as a displacement from the given point, solve for its parameter in the plane, and multiply the absolute parameter by the direction vector's length.

### Section C (16 marks)

Question numbers 11 to 14 carry 4 marks each.

**11. Find the area bounded by the curves  $y = |x - 1|$  and  $y = 1$ , using integration.**

[4 marks]

**Solution:**

**Concept:**

- The absolute value is defined piecewise:  $|x - 1| = 1 - x$  for  $x \leq 1$ , and  $|x - 1| = x - 1$  for  $x \geq 1$ . The expression changes at  $x = 1$ .
- The area between an upper curve  $y = f(x)$  and a lower curve  $y = g(x)$ , between their intersection abscissae  $a$  and  $b$ , is  $\int_a^b [f(x) - g(x)] dx$ . Split the integral wherever a piecewise expression changes.

**Step 1: Determine the intersections and the vertical boundaries** At the intersections of the two curves,

$$|x - 1| = 1 \implies x - 1 = \pm 1.$$

Thus  $x = 0$  or  $x = 2$ , and the intersection points are  $(0, 1)$  and  $(2, 1)$ . The absolute-value curve has its vertex at  $(1, 0)$ . On the entire interval  $0 \leq x \leq 2$ , we have  $0 \leq |x - 1| \leq 1$ . Therefore the horizontal line  $y = 1$  is the upper boundary and  $y = |x - 1|$  is the lower boundary of the bounded region.

**Step 2: Split the area integral at the vertex** Write the required area  $A$  as upper ordinate minus lower

**Quick Tip to Solve:** Find the two intersections first, then split the integral at  $x=1$ ; symmetry lets you check that each half contributes the same area.

ordinate:

$$\begin{aligned} A &= \int_0^2 (1 - |x - 1|) dx \\ &= \int_0^1 [1 - (1 - x)] dx + \int_1^2 [1 - (x - 1)] dx \\ &= \int_0^1 x dx + \int_1^2 (2 - x) dx. \end{aligned}$$

The first strip height increases from zero to one, and the second decreases from one to zero. Both integrands are nonnegative on their respective intervals, as required for an area.

**Step 3: Evaluate both definite integrals** The area on the left of the vertex is

$$A_1 = \left[ \frac{x^2}{2} \right]_0^1 = \frac{1}{2}.$$

The area on the right is

$$\begin{aligned} A_2 &= \left[ 2x - \frac{x^2}{2} \right]_1^2 \\ &= (4 - 2) - \left( 2 - \frac{1}{2} \right) = \frac{1}{2}. \end{aligned}$$

The two contributions are equal, consistent with the symmetry of the absolute-value curve about  $x = 1$ . Add these contributions to obtain the area of the complete bounded region.

**Step 4: Conclusion** The required area is  $A = A_1 + A_2 = \frac{1}{2} + \frac{1}{2} = \boxed{1 \text{ square unit}}$ .

**Quick Tip to Solve:** Find the two intersections first, then split the integral at  $x=1$ ; symmetry lets you check that each half contributes the same area.

**12(a). Solve the following differential equation:**  $(y - \sin^2 x) dx + \tan x dy = 0$ .

[4 marks]

**Solution:**

**Concept:**

- A first-order linear differential equation has the form  $\frac{dy}{dx} + P(x)y = Q(x)$ . Its integrating factor is  $\text{I.F.} = e^{\int P(x) dx}$ .
- Multiplication by the integrating factor converts the left side to a product derivative:  $\frac{d}{dx}[y \text{ I.F.}] = Q(x) \text{ I.F.}$ . Integration then introduces one arbitrary constant  $C$ .

**Step 1: Express the equation in linear form** The given equation is

$$(y - \sin^2 x) dx + \tan x dy = 0.$$

**Quick Tip to Solve:** After dividing by  $\tan x$ , use  $\sin x$  as the integrating factor; then the right side integrates immediately with  $u = \sin x$ .

Dividing by  $dx$ , moving the sine term to the right and then dividing by  $\tan x$ , we obtain

$$\tan x \frac{dy}{dx} + y = \sin^2 x,$$
$$\frac{dy}{dx} + y \cot x = \sin x \cos x.$$

Hence  $P(x) = \cot x$  and  $Q(x) = \sin x \cos x$ . We work on an interval on which  $\tan x$  is defined and nonzero, as required for this division.

**Step 2: Calculate the integrating factor** Since  $\int \cot x \, dx = \log |\sin x|$ , the integrating factor is

$$e^{\int \cot x \, dx} = e^{\log |\sin x|} = |\sin x|.$$

On the chosen interval,  $\sin x$  has a fixed sign. A nonzero constant multiple of an integrating factor is also an integrating factor, so we may conveniently use I.F.  $= \sin x$ .

**Step 3: Multiply and integrate the product derivative** Multiplying the linear equation by  $\sin x$  gives

$$\sin x \frac{dy}{dx} + y \cos x = \sin^2 x \cos x.$$

The left side is the derivative of  $y \sin x$ . Thus

$$\frac{d}{dx}(y \sin x) = \sin^2 x \cos x.$$

Integrate with respect to  $x$ . With  $u = \sin x$ , we have  $du = \cos x \, dx$ , so

$$y \sin x = \int \sin^2 x \cos x \, dx = \int u^2 \, du = \frac{\sin^3 x}{3} + C.$$

**Step 4: Check the integrated relation** Differentiation of the integrated relation yields

$$y' \sin x + y \cos x = \sin^2 x \cos x.$$

Dividing by  $\cos x$ , where the original equation is defined, gives  $y' \tan x + y = \sin^2 x$ . Rearranging recovers  $(y - \sin^2 x) \, dx + \tan x \, dy = 0$ , confirming the solution.

**Step 5: Conclusion** The general solution is

$$y \sin x = \frac{\sin^3 x}{3} + C,$$

or equivalently  $y = \frac{\sin^2 x}{3} + C \csc x$  on an interval where  $\sin x \neq 0$  and the original equation is defined.

**Quick Tip to Solve:** After dividing by  $\tan x$ , use  $\sin x$  as the integrating factor; then the right side integrates immediately with  $u = \sin x$ .

**12(b). Find the general solution of the differential equation:**  $(x^3 + y^3) dy = x^2 y dx$ .

[4 marks]

**Solution:**

**Concept:**

- A differential equation involving homogeneous expressions of the same degree can be reduced to a separable equation by a ratio substitution. Here both  $x^3 + y^3$  and  $x^2 y$  have degree three.
- Put  $x = vy$ , where  $v = x/y$  for  $y \neq 0$ . Then  $dx = v dy + y dv$ . After substitution, separate the terms containing  $v$  and  $y$ , and integrate.
- The logarithmic integral is  $\int dy/y = \ln |y| + C$  on an interval where  $y$  does not change sign. A solution excluded by division must be checked in the original equation.

**Step 1: Introduce the homogeneous substitution** The given equation is

$$(x^3 + y^3) dy = x^2 y dx.$$

For  $y \neq 0$ , take  $x = vy$ , so that  $dx = v dy + y dv$ . Substitution gives

$$(v^3 + 1)y^3 dy = v^2 y^3 (v dy + y dv).$$

**Step 2: Separate and integrate** Divide by  $y^3$  and cancel the common term  $v^3 dy$ :

$$\begin{aligned}(v^3 + 1) dy &= v^3 dy + v^2 y dv, \\ dy &= v^2 y dv, \\ v^2 dv &= \frac{dy}{y}.\end{aligned}$$

Integrating both sides yields

$$\frac{v^3}{3} = \ln |y| + C_1.$$

**Step 3: Restore the original variables and check the family** Substitute  $v = x/y$  and absorb the factor three into the arbitrary constant:

$$\frac{x^3}{y^3} - 3 \ln |y| = C.$$

Differentiating this relation and multiplying by  $y^4/3$  gives  $x^2 y dx - (x^3 + y^3) dy = 0$ , verifying the printed equation.

**Step 4: Check the solution excluded by division** We divided by  $y$ , so examine  $y = 0$  separately. Along this constant solution,  $dy = 0$ , and both sides of the original equation vanish. Thus  $y = 0$  is an additional solution, excluded from the logarithmic family.

**Step 5: Conclusion** The general nonzero solution is  $\boxed{\frac{x^3}{y^3} - 3 \ln |y| = C}$ , for  $y \neq 0$ , where  $C$  is an arbitrary real constant. The constant solution  $\boxed{y = 0}$  also satisfies the differential equation.

**Quick Tip to Solve:** Use  $x = vy$  here: the  $v^3 dy$  terms cancel immediately, leaving  $v^2 dv = dy/y$ . Always test  $y = 0$  after dividing by  $y$ .

**13.** There are two boxes, namely box-I and box-II. Box-I contains 3 red and 6 black balls. Box-II contains 5 red and 5 black balls. One of the two boxes, is selected at random and a ball is drawn at random. The ball drawn is found to be red. Find the probability that this red ball comes out from box-II. [4 marks]

**Solution:**

**Concept:**

- Conditional probability measures the chance of an event after some information is known. Let  $E_1$  and  $E_2$  denote selection of box-I and box-II, respectively, and let  $R$  denote drawing a red ball.
- The boxes are mutually exclusive and exhaustive possibilities. The law of total probability gives  $P(R) = P(E_1)P(R | E_1) + P(E_2)P(R | E_2)$ .
- Bayes' theorem gives the required reverse conditional probability:  $P(E_2 | R) = \frac{P(E_2)P(R | E_2)}{P(R)}$ .

**Step 1: Find the prior and conditional probabilities** Since one of the two boxes is selected at random, each box has the same probability of selection:

$$P(E_1) = P(E_2) = \frac{1}{2}.$$

Box-I contains  $3 + 6 = 9$  balls, of which 3 are red. Box-II contains  $5 + 5 = 10$  balls, of which 5 are red. Therefore,

$$P(R | E_1) = \frac{3}{9} = \frac{1}{3}, \quad P(R | E_2) = \frac{5}{10} = \frac{1}{2}.$$

**Step 2: Calculate the probability of drawing a red ball** A red ball can be obtained by either selecting box-I and drawing red or selecting box-II and drawing red. These two routes cannot occur together, so their probabilities are added:

$$\begin{aligned} P(R) &= P(E_1)P(R | E_1) + P(E_2)P(R | E_2) \\ &= \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{6} + \frac{1}{4} = \frac{5}{12}. \end{aligned}$$

**Step 3: Apply Bayes' theorem to the observed red ball** The probability of both selecting box-II and drawing a red ball is

$$P(E_2 \cap R) = P(E_2)P(R | E_2) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}.$$

Restricting the outcomes to the observed event  $R$ , Bayes' theorem gives:

$$P(E_2 | R) = \frac{P(E_2 \cap R)}{P(R)} = \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{1}{4} \cdot \frac{12}{5} = \frac{3}{5}.$$

**Step 4: Conclusion** The probability that the observed red ball came from box-II is  $\boxed{\frac{3}{5}}$ .

**Quick Tip to Solve:** With equally likely boxes, divide box-II's red fraction by the sum of both boxes' red fractions:  $(1/2)/(1/3 + 1/2)$ .

#### 14. Case Study Problem:

Two motorcycles  $A$  and  $B$  are running at the speed more than the allowed speed on the roads represented by the lines  $\vec{r} = \lambda(\hat{i} + 2\hat{j} - \hat{k})$  and  $\vec{r} = (3\hat{i} + 3\hat{j}) + \mu(2\hat{i} + \hat{j} + \hat{k})$  respectively.

Based on the above information, answer the following questions:

[4 marks]

14(a). Find the shortest distance between the given lines.

[2 marks]

##### Solution:

##### Concept:

- For two non-parallel lines  $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ , their shortest distance is  $d = \frac{|(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2)|}{|\vec{b}_1 \times \vec{b}_2|}$ .
- Here  $\vec{a}_1, \vec{a}_2$  are position vectors of points on the lines, and  $\vec{b}_1, \vec{b}_2$  are their direction vectors. A zero numerator with a nonzero cross product means the lines are coplanar and intersect.

**Step 1: Identify the vectors and calculate the common normal** From the given equations,

$$\begin{aligned}\vec{a}_1 &= (0, 0, 0), & \vec{a}_2 &= (3, 3, 0), \\ \vec{b}_1 &= (1, 2, -1), & \vec{b}_2 &= (2, 1, 1).\end{aligned}$$

The cross product, perpendicular to both lines, is

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & 1 & 1 \end{vmatrix} = 3\hat{i} - 3\hat{j} - 3\hat{k}.$$

Its magnitude is  $\sqrt{9+9+9} = 3\sqrt{3} \neq 0$ , so the lines are not parallel and the formula applies.

**Step 2: Substitute into the shortest-distance formula** Since  $\vec{a}_2 - \vec{a}_1 = 3\hat{i} + 3\hat{j}$ , the scalar triple product is

$$(\vec{a}_2 - \vec{a}_1) \cdot (\vec{b}_1 \times \vec{b}_2) = 3(3) + 3(-3) + 0(-3) = 0.$$

Therefore,

$$d = \frac{|0|}{3\sqrt{3}} = 0.$$

The zero scalar triple product establishes coplanarity; together with the non-parallel directions, this shows that the lines meet, explaining the zero separation.

**Step 3: Conclusion** The shortest distance between the given lines is 0 units.

**Quick Tip to Solve:** First check the scalar triple product: if it is zero and the direction vectors are not parallel, the lines intersect and their shortest distance is zero.

14(b). Find the point at which the motorcycles may collide.

[2 marks]

**Solution:**

**Concept:**

- Two roads represented by straight lines can have a common point only when the position vectors on the two lines are equal for some values of their parameters.
- Equality of position vectors means equality of all three Cartesian coordinates. A candidate intersection obtained from two coordinates must also satisfy the remaining coordinate.

**Step 1: Write the coordinates of points on both roads** For the road of motorcycle A,

$$\vec{r} = \lambda(\hat{i} + 2\hat{j} - \hat{k}) \implies (x, y, z) = (\lambda, 2\lambda, -\lambda).$$

For the road of motorcycle B,

$$\vec{r} = 3\hat{i} + 3\hat{j} + \mu(2\hat{i} + \hat{j} + \hat{k}) \implies (x, y, z) = (3 + 2\mu, 3 + \mu, \mu).$$

At a possible collision point, the coordinates must therefore obey

$$\lambda = 3 + 2\mu, \quad 2\lambda = 3 + \mu, \quad -\lambda = \mu.$$

**Step 2: Solve the parameter equations and verify the common point** From the third equation,  $\mu = -\lambda$ . Substituting this into the first equation gives

$$\lambda = 3 - 2\lambda \implies 3\lambda = 3 \implies \lambda = 1, \quad \mu = -1.$$

The second equation is also satisfied, since  $2\lambda = 2$  and  $3 + \mu = 2$ . With these parameter values, the first road gives  $(1, 2, -1)$ , while the second gives  $(3 - 2, 3 - 1, -1) = (1, 2, -1)$ . Thus the candidate lies on both roads. This determines the spatial location where a collision is possible; a collision would require the motorcycles to reach that location simultaneously.

**Step 3: Conclusion** The motorcycles may collide at  $(1, 2, -1)$ , whose position vector is  $\hat{i} + 2\hat{j} - \hat{k}$ .

**Quick Tip to Solve:** Use the  $z$ -coordinates first: they immediately give  $\mu = -\lambda$ . Solve one more coordinate equation, then check the third.