CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-1-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.

(iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculators is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If for a square matrix A, $A^2 - 3A + I = O$ and $A^{-1} = xA + yI$, then the value of $x + y$ is: [1 mark]

(A) -2

(B) 2

(C) 3

(D) -3

Correct Answer: (B) 2

Solution:

Concept:

- For an invertible matrix A, $AA^{-1} = I$.
- Multiply a matrix polynomial identity by A^{-1} to express the inverse using A and I.

Step 1: Express the inverse The given identity can be rearranged as $A(3I - A) = I$, so A is invertible. Multiplying $A^2 - 3A + I = O$ by A^{-1} gives

$$A - 3I + A^{-1} = O.$$

Thus $A^{-1} = -A + 3I$.

Step 2: Identify the coefficients Comparing this expression with the stated form $xA + yI$, the coefficients are $x = -1$ and $y = 3$. Therefore their sum is obtained from $-1 + 3$. Options (A) and (D) give negative sums, while (C) uses only the coefficient of I and omits the coefficient of A.

Step 3: Conclusion $x + y = 2$. Hence, the correct option is (B).

Quick Tip to Solve: Rearrange the identity as $A(3I - A) = I$ to read off the inverse immediately.

2. If $|A| = 2$, where A is a 2×2 matrix, then $|4A^{-1}|$ equals: [1 mark]

(A) 4

(B) 2

(C) 8

(D) $\frac{1}{32}$

Correct Answer: (C) 8

Solution:

Concept:

- For an $n \times n$ matrix, $|kA| = k^n|A|$.
- For an invertible matrix, $|A^{-1}| = 1/|A|$.

Quick Tip to Solve: Use the matrix order as the exponent on the scalar before taking the reciprocal determinant.

Step 1: Evaluate the inverse determinant Since $|A| = 2 \neq 0$, the inverse exists and

$$|A^{-1}| = \frac{1}{2}.$$

The inverse changes the determinant to its reciprocal; it does not change the order of the matrix.

Step 2: Apply the scalar factor The matrix has order 2, so multiplying all its entries by 4 multiplies its determinant by 4^2. Hence

$$|4A^{-1}| = 4^2|A^{-1}| = 16 \times \frac{1}{2} = 8.$$

Option (A) uses only one factor of 4, (B) leaves the original determinant unchanged, and (D) incorrectly divides by the scalar factor.

Step 3: Conclusion $|4A^{-1}| = 8$. Hence, the correct option is (C).

Quick Tip to Solve: Use the matrix order as the exponent on the scalar before taking the reciprocal determinant.

3. Let A be a 3×3 matrix such that $|\operatorname{adj} A| = 64$. Then $|A|$ is equal to: [1 mark]

(A) 8 only

(B) -8 only

(C) 64

(D) 8 or -8

Correct Answer: (D) 8 or -8

Solution:

Concept:

- For an $n \times n$ matrix, $|\operatorname{adj} A| = |A|^{n-1}$.
- An equation $t^2 = a^2$ admits both $t = a$ and $t = -a$.

Step 1: Use the adjoint identity The order of A is 3. Therefore

$$|\operatorname{adj} A| = |A|^{3-1} = |A|^2.$$

Using the given value gives $|A|^2 = 64$. The exponent is 2, not 3, because the adjoint determinant involves one less than the order.

Step 2: Retain both possible signs Taking the square root gives $|A| = \pm 8$. The vertical bars here denote the determinant of a matrix, which can be negative; they do not denote its absolute value. Both signs have square 64. Options (A) and (B) each omit a valid sign, while (C) confuses the determinant with the given adjoint determinant.

Step 3: Conclusion $|A| = 8$ or -8. Hence, the correct option is (D).

Quick Tip to Solve: Determinant bars do not imply positivity: retain both roots when its square is given.

4. If $A = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$ and $2A + B$ is a null matrix, then B is equal to: [1 mark]

(A) $\begin{pmatrix} 6 & 8 \\ 10 & 4 \end{pmatrix}$

(B) $\begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}$

(C) $\begin{pmatrix} 5 & 8 \\ 10 & 3 \end{pmatrix}$

(D) $\begin{pmatrix} -5 & -8 \\ -10 & -3 \end{pmatrix}$

Correct Answer: (B) $\begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}$

Solution:

Concept:

- The null matrix O has every entry equal to zero.
- Matrix addition and scalar multiplication operate entry by entry.

Step 1: Isolate the matrix The statement that $2A + B$ is a null matrix means $2A + B = O$. Subtracting $2A$ from both sides gives $B = -2A$, so every entry of A must be doubled and its sign reversed.

Step 2: Compute and check the entries Consequently,

$$B = -2 \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}.$$

Adding this matrix to $2A$ gives zero in all four positions. Option (A) has the wrong signs; options (C) and (D) have incorrect diagonal entries, so they cannot make the sum null.

Step 3: Conclusion $B = \begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}$. Hence, the correct option is (B).

Quick Tip to Solve: For $kA + B = O$, form $B = -kA$ and check all entries, including the diagonal.

5. If $\frac{d}{dx}(f(x)) = \log x$, then $f(x)$ equals: [1 mark]

(A) $-\frac{1}{x} + C$

(B) $x(\log x - 1) + C$

(C) $x(\log x + x) + C$

(D) $\frac{1}{x} + C$

Correct Answer: (B) $x(\log x - 1) + C$

Solution:

Concept:

- An antiderivative of $\log x$ is found by integration by parts on $x > 0$.
- $\int u\,dv = uv - \int v\,du$, with an arbitrary integration constant C.

Step 1: Integrate by parts Choose $u = \log x$ and $dv = dx$. Then $du = dx/x$ and $v = x$, giving

$$f(x) = x\log x - \int 1\,dx = x\log x - x + C.$$

Factor out x to compare this with the printed options.

Step 2: Verify by differentiation The derivative of $x(\log x - 1) + C$ is $\log x + 1 - 1 = \log x$. The derivatives of (A) and (D) are respectively $1/x^2$ and $-1/x^2$, while (C) differentiates to $\log x + 1 + 2x$. These do not match the given derivative.

Step 3: Conclusion $f(x) = x(\log x - 1) + C$, for $x > 0$. Hence, the correct option is (B).

Quick Tip to Solve: Check an antiderivative option by differentiating; the product rule cancels the extra 1.

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6. $\int_0^{\pi/6} \sec^2\left(x - \frac{\pi}{6}\right) dx$ is equal to: [1 mark]

(A) $\frac{1}{\sqrt{3}}$

(B) $-\frac{1}{\sqrt{3}}$

(C) $\sqrt{3}$

(D) $-\sqrt{3}$

Correct Answer: (A) $\frac{1}{\sqrt{3}}$

Solution:

Concept:

- $\frac{d}{dx}\tan(x - a) = \sec^2(x - a)$ for a constant a.
- A definite integral equals the upper endpoint value of an antiderivative minus its lower endpoint value.

Step 1: Find the antiderivative The printed shift is $\pi/6$. Its derivative with respect to x is 1, so an

Quick Tip to Solve: Compute the shifted argument at each endpoint first, then subtract the tangent values.

antiderivative of the integrand is $\tan(x - \pi/6)$. The argument ranges from $-\pi/6$ to 0; no pole occurs in this interval.

Step 2: Substitute the endpoints Thus

$$\int_0^{\pi/6} \sec^2(x - \pi/6)\, dx = \tan 0 - \tan(-\pi/6) = 0 - \left(-\frac{1}{\sqrt{3}}\right).$$

The integral is positive because the integrand is positive, ruling out (B) and (D). Option (C) uses the tangent value at $\pi/3$, instead of the actual lower argument $-\pi/6$.

Step 3: Conclusion The integral is $\frac{1}{\sqrt{3}}$. Hence, the correct option is (A).

Quick Tip to Solve: Compute the shifted argument at each endpoint first, then subtract the tangent values.

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7. The sum of the order and the degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = \sin y$ is: [1 mark]

(A) 5

(B) 2

(C) 3

(D) 4

Correct Answer: (C) 3

Solution:

Concept:

- The order is the order of the highest derivative present in the equation.
- The degree is the power of the highest-order derivative when the equation is polynomial in its derivatives.

Step 1: Identify the order The equation contains dy/dx and d^2y/dx^2. The highest derivative is the second derivative, so the order is 2. The power 3 on the first derivative does not raise the order of the equation.

Step 2: Identify the degree The equation is already polynomial in the derivatives. The highest-order derivative d^2y/dx^2 occurs to the first power, so the degree is 1. The term $\sin y$ involves the dependent variable and does not prevent defining this degree. Therefore add 2 and 1; option (A) incorrectly uses the cube on the first derivative, (B) reports only the order, and (D) is neither the required sum nor a valid degree calculation.

Step 3: Conclusion The sum of the order and degree is $2 + 1 = 3$. Hence, the correct option is (C).

Quick Tip to Solve: Find the highest-order derivative first; its power, not the largest power elsewhere, gives degree.

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8. The value of p for which the vectors $2\hat{i} + p\hat{j} + \hat{k}$ and $-4\hat{i} - 6\hat{j} + 26\hat{k}$ are perpendicular to each other, is: [1 mark]

(A) 3

(B) -3

(C) $-\frac{17}{3}$

(D) $\frac{17}{3}$

Correct Answer: (A) 3

Solution:

Concept:

- For nonzero vectors, perpendicularity is equivalent to a zero scalar product: $\vec{u} \cdot \vec{v} = 0$.
- The scalar product equals the sum of products of corresponding Cartesian components.

Step 1: Form the perpendicularity equation The two given vectors have components $(2, p, 1)$ and $(-4, -6, 26)$. Hence their scalar product is

$$2(-4) + p(-6) + 1(26) = 18 - 6p.$$

Both vectors are nonzero, so the zero-dot-product condition applies directly.

Step 2: Solve and check the alternatives Perpendicularity requires $18 - 6p = 0$, giving $p = 3$. Substituting this value yields $-8 - 18 + 26 = 0$. The values $-3, -17/3, 17/3$ produce scalar products $36, 52, -16$, respectively; none is zero.

Step 3: Conclusion Therefore $\boxed{p = 3}$, and the correct option is (A).

Quick Tip to Solve: For perpendicular vectors, multiply matching components and set their sum to zero; do not equate the components.

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9. The value of $(\hat{i} \times \hat{j}) \cdot \hat{j} + (\hat{j} \times \hat{i}) \cdot \hat{k}$ is: [1 mark]

(A) 2

(B) 0

(C) 1

(D) -1

Correct Answer: (D) -1

Solution:

Concept:

- The right-handed unit vectors satisfy $\hat{i} \times \hat{j} = \hat{k}$ and $\hat{j} \times \hat{i} = -\hat{k}$.
- Orthogonal unit vectors have zero dot product, whereas $\hat{k} \cdot \hat{k} = 1$.

Quick Tip to Solve: Write each cross product first, preserving its order, and only then take the indicated dot product.

Step 1: Evaluate the first scalar product Using the cyclic cross-product rule,

$$(\hat{i} \times \hat{j}) \cdot \hat{j} = \hat{k} \cdot \hat{j} = 0.$$

The result is zero because the two unit vectors are perpendicular, not because either vector has zero magnitude.

Step 2: Evaluate the second scalar product and add Reversing the order of a cross product changes its sign. Thus

$$(\hat{j} \times \hat{i}) \cdot \hat{k} = (-\hat{k}) \cdot \hat{k} = -1.$$

Adding the terms gives $0 - 1 = -1$. Option 1 misses the reversed-product sign, option 0 ignores the second term, and option 2 incorrectly treats both terms as positive unit scalar products.

Step 3: Conclusion The required value is $\boxed{-1}$; the correct option is (D).

Quick Tip to Solve: Write each cross product first, preserving its order, and only then take the indicated dot product.

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10. If $\vec{a} + \vec{b} = \hat{i}$ and $\vec{a} = 2\hat{i} - 2\hat{j} + 2\hat{k}$, then $|\vec{b}|$ equals: [1 mark]

(A) $\sqrt{14}$

(B) 3

(C) $\sqrt{12}$

(D) $\sqrt{17}$

Correct Answer: (B) 3

Solution:

Concept:

- From $\vec{a} + \vec{b} = \vec{c}$, obtain $\vec{b} = \vec{c} - \vec{a}$ by componentwise subtraction.
- For $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, its magnitude is $\sqrt{b_1^2 + b_2^2 + b_3^2}$.

Step 1: Find the unknown vector The vector on the right is $\hat{i} = (1, 0, 0)$. Therefore

$$\vec{b} = \hat{i} - (2\hat{i} - 2\hat{j} + 2\hat{k}) = -\hat{i} + 2\hat{j} - 2\hat{k}.$$

Subtraction reverses the signs of every component of $\vec{a}$, including its negative second component.

Step 2: Calculate and compare magnitudes Using the three components,

$$|\vec{b}| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = \sqrt{1 + 4 + 4} = \sqrt{9}.$$

The squared magnitude is 9, so neither $\sqrt{14}$, $\sqrt{12}$, nor $\sqrt{17}$ agrees with the component calculation. Vector magnitudes cannot be subtracted as ordinary signed components.

Step 3: Conclusion $\boxed{|\vec{b}| = 3}$, so the correct option is (B).

Quick Tip to Solve: Subtract the vectors before taking a magnitude; check the signs of all three resulting components.

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11. Direction cosines of the line $\frac{x-1}{2}=\frac{1-y}{3}=\frac{2z-1}{12}$ are: [1 mark]

(A) $\frac{2}{7},\frac{3}{7},\frac{6}{7}$

(B) $\frac{2}{\sqrt{157}},-\frac{3}{\sqrt{157}},\frac{12}{\sqrt{157}}$

(C) $\frac{2}{7},-\frac{3}{7},-\frac{6}{7}$

(D) $\frac{2}{7},-\frac{3}{7},\frac{6}{7}$

Correct Answer: (D) $\frac{2}{7},-\frac{3}{7},\frac{6}{7}$

Solution:

Concept:

- A line with direction ratios (a,b,c) has direction cosines $(a,b,c)/\sqrt{a^2+b^2+c^2}$.
- Rewrite numerators as coordinate minus constant before reading their denominators as direction ratios.

Step 1: Express the line in a common parameter Let each given ratio equal t. Then $x=1+2t$, $y=1-3t$, and $z=\frac{1}{2}+6t$. The last equation follows from $2z-1=12t$, so the direction ratios are $(2,-3,6)$, not $(2,-3,12)$.

Step 2: Normalize and check the options The normalization factor is

$$\sqrt{2^2+(-3)^2+6^2}=\sqrt{49}=7.$$

Thus the direction cosines in this orientation are $(2/7,-3/7,6/7)$; their squared sum is one. Option A has the wrong sign of the second component, B reads 12 without dividing the z numerator by 2, and C has the wrong sign of the third component.

Step 3: Conclusion The direction cosines are $\boxed{\left(\frac{2}{7},-\frac{3}{7},\frac{6}{7}\right)}$; the correct option is (D).

Quick Tip to Solve: Introduce a common parameter when a numerator contains a coefficient such as 2z; it reveals the true direction ratios.

12. If $P\left(\frac{A}{B}\right) = 0.3$, $P(A) = 0.4$ and $P(B) = 0.8$, then $P\left(\frac{B}{A}\right)$ is equal to: [1 mark]

(A) 0.6

(B) 0.3

(C) 0.06

(D) 0.4

Correct Answer: (A) 0.6

Solution:

Concept:

- Conditional probability is $P(A \mid B) = P(A \cap B)/P(B)$, provided $P(B) > 0$.
- The same intersection appears in both conditional probabilities: $P(B \mid A) = P(A \cap B)/P(A)$.

Step 1: Find the intersection probability The printed notation $P(A/B)$ means $P(A \mid B)$. From the given values,

$$P(A \cap B) = P(A \mid B)P(B) = 0.3 \times 0.8 = 0.24.$$

This probability does not exceed either marginal probability, as required.

Step 2: Reverse the conditioning event Now condition on A, whose probability is 0.4:

$$P(B \mid A) = \frac{0.24}{0.4} = 0.6.$$

The value 0.3 is the given conditional probability in the opposite direction. The alternatives 0.06 and 0.4 do not follow from dividing the intersection probability by $P(A)$.

Step 3: Conclusion $\boxed{P(B/A) = 0.6}$, and the correct option is (A).

Quick Tip to Solve: Find the intersection first, then divide by the probability of the event after the conditioning bar.

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13. The value of k for which $f(x) = \begin{cases} 3x + 5, & x \geq 2 \\ kx^2, & x < 2 \end{cases}$ is a continuous function, is: [1 mark]

(A) $-\frac{11}{4}$

(B) $\frac{4}{11}$

(C) 11

(D) $\frac{11}{4}$

Correct Answer: (D) $\frac{11}{4}$

Solution:

Concept:

- Continuity at $x = a$ requires $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$.

Quick Tip to Solve: At a piecewise joining point, evaluate the branch containing equality and match the other branch limit to that value.

- Each polynomial branch is continuous within its own interval; only the joining point needs checking.

Step 1: Evaluate the value and one-sided limits At the joining point $x = 2$, the branch $3x + 5$ applies because its interval includes equality. Hence

$$f(2) = 3(2) + 5 = 11, \qquad \lim_{x \to 2^+} f(x) = 11.$$

Approaching from the left uses kx^2, giving $\lim_{x \to 2^-} f(x) = 4k$.

Step 2: Match the two branches Continuity therefore requires $4k = 11$, or $k = 11/4$. This also makes the left-hand limit equal to the actual function value. The options $-11/4, 4/11, 11$ give left limits $-11, 16/11, 44$, respectively, which disagree with 11. No other point can cause a discontinuity because both branches are polynomials.

Step 3: Conclusion The required value is $\boxed{k = \dfrac{11}{4}}$; the correct option is (D).

Quick Tip to Solve: At a piecewise joining point, evaluate the branch containing equality and match the other branch limit to that value.

14. If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $(3I + 4A)(3I - 4A) = x^2 I$, then the value(s) of x is/are: [1 mark]

(A) $\pm\sqrt{7}$

(B) 0

(C) ± 5

(D) 25

Correct Answer: (C) ± 5

Solution:

Concept:

- The identity matrix I commutes with any square matrix A, so mixed terms cancel in $(3I + 4A)(3I - 4A)$.
- Matrix powers must be computed from the actual entries; the printed matrix here satisfies $A^2 = -I$.

Step 1: Compute the square of the given matrix Using the printed lower-left entry -1,

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

The negative sign is essential to the product that follows.

Step 2: Expand the product and solve for x Since $IA = AI = A$,

$$(3I + 4A)(3I - 4A) = 9I - 12A + 12A - 16A^2 = 25I.$$

Comparing with $x^2 I$ gives $x^2 = 25$. Both signs of the square root solve this equation. The alternatives

Quick Tip to Solve: Compute A squared before using a difference-of-squares identity, and take both roots when solving x squared equals a positive number.

$\pm\sqrt{7}$ and 0 square to 7 and 0; 25 squares to 625, so they do not satisfy the matrix identity.

Step 3: Conclusion $\boxed{x = \pm 5}$, and the correct option is (C).

Quick Tip to Solve: Compute A squared before using a difference-of-squares identity, and take both roots when solving x squared equals a positive number.

15. The general solution of the differential equation $x\,dy - (1 + x^2)\,dx = dx$ is: [1 mark]

(A) $y = 2x + \dfrac{x^3}{3} + C$

(B) $y = 2\log x + \dfrac{x^3}{3} + C$

(C) $y = \dfrac{x^2}{2} + C$

(D) $y = 2\log x + \dfrac{x^2}{2} + C$

Correct Answer: (D) $y = 2\log x + \dfrac{x^2}{2} + C$

Solution:

Concept:

- For a separable differential equation, express dy/dx as a function of x, then integrate with respect to x.
- Use $\int x\,dx = x^2/2$ and $\int dx/x = \log|x|$, on an interval excluding zero.

Step 1: Isolate the derivative Transferring the differential terms gives $x\,dy = (2 + x^2)\,dx$. Thus, for $x \neq 0$,

$$\frac{dy}{dx} = \frac{2}{x} + x.$$

The coefficient of the logarithmic term must therefore be 2.

Step 2: Integrate and compare the choices Integration gives

$$y = 2\log|x| + \frac{x^2}{2} + C.$$

On $x > 0$, this is the printed logarithmic expression. Option (A) differentiates to $2 + x^2$, option (B) to $2/x + x^2$, and option (C) to x; none matches $2/x + x$. Option (D) does.

Step 3: Conclusion The general solution is $y = 2\log|x| + x^2/2 + C$, represented by option (D) on $x > 0$.

Quick Tip to Solve: Divide by x before integrating; the constant 2 then produces a logarithm, not a linear term.

16. **If $f(x) = a(x - \cos x)$ is strictly decreasing in $\mathbb{R}$, then a belongs to:** [1 mark]

(A) $\{0\}$

(B) $(0, \infty)$

(C) $(-\infty, 0)$

(D) $(-\infty, \infty)$

Correct Answer: (C) $(-\infty, 0)$

Solution:

Concept:

- A strictly decreasing function satisfies $f(v) < f(u)$ whenever $u < v$.
- Here $f\prime(x) = a(1 + \sin x)$, and $1 + \sin x \geq 0$.

Step 1: Determine the sign of the parameter For $a > 0$, the derivative is positive except at isolated points, so the function increases. For $a = 0$, the function is constant and cannot be strictly decreasing. Thus only negative values of a can work.

Step 2: Verify strict decrease despite isolated zeros Let $g(x) = x - \cos x$. For any $u < v$,

$$g(v) - g(u) = \int_u^v (1 + \sin t)\, dt > 0.$$

The integrand is continuous, nonnegative, and is not identically zero on any interval; its zeros are isolated. Multiplication by $a < 0$ makes this difference negative. Hence every negative parameter gives strict decrease. Options (A), (B) and (D) include zero or positive values and fail.

Step 3: Conclusion $a \in (-\infty, 0)$. Hence, the correct option is (C).

Quick Tip to Solve: Isolated zeros of the derivative do not destroy strict monotonicity; check its sign on intervals.

17. **The corner points of the feasible region in the graphical representation of a linear programming problem are $(2, 72)$, $(15, 20)$ and $(40, 15)$. If $z = 18x + 9y$ be the objective function, then:** [1 mark]

(A) z is maximum at $(2, 72)$, minimum at $(15, 20)$

(B) z is maximum at $(15, 20)$, minimum at $(40, 15)$

(C) z is maximum at $(40, 15)$, minimum at $(15, 20)$

(D) z is maximum at $(40, 15)$, minimum at $(2, 72)$

Correct Answer: (C) z is maximum at $(40, 15)$, minimum at $(15, 20)$

Solution:

Concept:

- The corner-point method evaluates a linear objective at every vertex of the feasible region.
- Compare the values of $z = 18x + 9y$ to identify both extrema.

Quick Tip to Solve: Compute 9(2x+y) at each vertex to compare the objective with less arithmetic.

Step 1: Evaluate all three vertices At the stated corner points,

$$z(2, 72) = 18(2) + 9(72) = 684,$$
$$z(15, 20) = 18(15) + 9(20) = 450,$$
$$z(40, 15) = 18(40) + 9(15) = 855.$$

Both coordinates contribute to the objective, so the largest coordinate alone is not decisive.

Step 2: Compare the objective values The ordering is $450 < 684 < 855$. Thus $(15, 20)$ supplies the minimum and $(40, 15)$ the maximum. Option (A) wrongly selects $(2, 72)$ for the maximum, option (B) reverses the true extreme vertices, and option (D) wrongly selects $(2, 72)$ for the minimum.

Step 3: Conclusion The maximum is 855 at $(40, 15)$, and the minimum is 450 at $(15, 20)$. Correct option: (C).

Quick Tip to Solve: Compute 9(2x+y) at each vertex to compare the objective with less arithmetic.

18. The number of corner points of the feasible region determined by the constraints $x - y \geq 0,\ 2y \leq x + 2,\ x \geq 0,\ y \geq 0$ is: [1 mark]

(A) 2

(B) 3

(C) 4

(D) 5

Correct Answer: (A) 2

Solution:

Concept:

- A corner point is a feasible intersection of boundary lines where the boundary changes direction.
- The constraints become $0 \leq y \leq x$ and $y \leq (x + 2)/2$, with $x \geq 0$.

Step 1: Find candidate boundary intersections The axes meet at $(0, 0)$, which satisfies every inequality. The two sloping lines intersect when $y = x$ and $2y = x + 2$; hence $x = y = 2$. On the y-axis, $y \leq x = 0$ forces $y = 0$, so no other feasible vertex occurs there.

Step 2: Discard infeasible intersections and count The line $2y = x + 2$ meets the x-axis at $(-2, 0)$, outside $x \geq 0$, and the y-axis at $(0, 1)$, violating $y \leq x$. The lower boundary is the nonnegative x-axis; the upper boundary follows $y = x$ to $(2, 2)$ and then $y = (x + 2)/2$. The region is unbounded, but infinity is not a vertex. There are precisely two corner points; options (B), (C) and (D) overcount infeasible intersections.

Step 3: Conclusion The corner points are $(0, 0)$ and $(2, 2)$. Their number is 2, so option (A) is correct.

Quick Tip to Solve: Check every line intersection against all inequalities; an unbounded ray adds no corner at infinity.

19. Assertion (A): The range of the function $f(x) = 2\sin^{-1} x + \frac{3\pi}{2}$, where $x \in [-1, 1]$, is $\left[\frac{\pi}{2}, \frac{5\pi}{2}\right]$.

Reason (R): The range of the principal value branch of $\sin^{-1}(x)$ is $[0, \pi]$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (C) Assertion (A) is true and Reason (R) is false.

Solution:

Concept:

- The principal range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$.
- Multiplying an interval by a positive number preserves its endpoint order; adding a constant shifts both endpoints.

Step 1: Judge the Assertion For $x \in [-1, 1]$,

$$-\pi \le 2\sin^{-1} x \le \pi.$$

Adding $3\pi/2$ gives

$$\frac{\pi}{2} \le f(x) \le \frac{5\pi}{2}.$$

Both endpoints occur, at $x = -1$ and $x = 1$, respectively, and continuity supplies every intermediate value. The Assertion is true.

Step 2: Judge the Reason The claimed range $[0, \pi]$ is not the principal range of inverse sine. For example, $\sin^{-1}(-1) = -\pi/2$, a negative value. The stated interval is instead the principal range of inverse cosine. Therefore the Reason is false.

Step 3: Select the appropriate logical code A false Reason cannot explain the true Assertion. Options (A) and (B) require both statements to be true, while option (D) requires the Assertion to be false. Only option (C) has the required truth values.

Step 4: Conclusion The Assertion is true and the Reason is false. Correct option: (C).

Quick Tip to Solve: Write the principal interval first, then transform both endpoints in the same order as the function.

20. Assertion (A): Equation of a line passing through the points $(1, 2, 3)$ and $(3, -1, 3)$ is $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-3}{0}$.

Reason (R): Equation of a line passing through points (x_1, y_1, z_1) and (x_2, y_2, z_2) is given by $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (D) Assertion (A) is false and Reason (R) is true.

Solution:

Concept:

- The line through two distinct points has direction vector equal to the difference of their position vectors.
- A zero z-direction component means z is constant; the customary zero-denominator symmetric notation is read as this condition.

Step 1: Judge the Assertion The direction from $(1, 2, 3)$ to $(3, -1, 3)$ is $(2, -3, 0)$. However, the asserted ratios give $x = 3 + 2t$, $y = -1 + 3t$, $z = 3$. To reach $x = 1$, one needs $t = -1$, which gives $y = -4$, not 2. Thus the asserted line misses the first point, and the Assertion is false.

Step 2: Judge the Reason The standard two-point formula uses the differences $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ as direction ratios, so the Reason is true. For these points it yields

$$\frac{x-1}{2} = \frac{y-2}{-3}, \qquad z = 3.$$

Equivalently, using the second point as base, the correct equation is $(x-3)/2 = (y+1)/(-3)$, together with $z = 3$.

Step 3: Assess the explanation and choose the code The correct general formula exposes the wrong sign in the asserted y-direction ratio. It therefore cannot establish that Assertion. Options (A), (B) and (C) all require the Assertion to be true and are excluded.

Step 4: Conclusion The Assertion is false and the Reason is true. Hence the correct option is (D).

Quick Tip to Solve: When checking a symmetric line equation, substitute both points and retain the signs of coordinate differences.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21(a). A function $f : A \to B$ defined as $f(x) = 2x$ is both one-one and onto. If $A = \{1, 2, 3, 4\}$, then find the set B. [2 marks]

Solution:

Concept:

- A function is onto when its image equals its codomain: $f(A) = B$.
- A function is one-one when distinct elements of its domain have distinct images.

Step 1: Calculate every image in the domain The given domain contains only the four elements $1, 2, 3, 4$. Apply the stated rule to each:

$$f(1) = 2(1) = 2,$$
$$f(2) = 2(2) = 4,$$
$$f(3) = 2(3) = 6,$$
$$f(4) = 2(4) = 8.$$

Consequently the complete image set is $f(A) = \{2, 4, 6, 8\}$. No other output is possible because there are no other domain elements.

Step 2: Use surjectivity to fix the codomain Every calculated image must belong to B, since $f : A \to B$ is a function. Conversely, the onto condition requires each member of B to be the image of some member of A. Therefore B contains exactly the four calculated values: adding any other value would violate onto, while omitting one would contradict the specified function.

Step 3: Verify the one-one condition If $f(x_1) = f(x_2)$, then $2x_1 = 2x_2$, so division by 2 gives $x_1 = x_2$. Thus the four outputs are distinct, consistent with the stipulated one-one property. The proposed codomain satisfies both requirements.

Step 4: Conclusion The required set is $\boxed{B = \{2, 4, 6, 8\}}$.

Quick Tip to Solve: For an onto function on a finite domain, list every image and take exactly that set as the codomain.

OR

21(b). Evaluate: $\sin^{-1}\left(\sin\frac{3\pi}{4}\right) + \cos^{-1}\left(\cos\frac{3\pi}{4}\right) + \tan^{-1}(1)$. [2 marks]

Solution:

Concept:

- Inverse trigonometric functions return principal values: $\sin^{-1} t \in [-\pi/2, \pi/2]$, $\cos^{-1} t \in [0, \pi]$, and $\tan^{-1} t \in (-\pi/2, \pi/2)$.

Quick Tip to Solve: Check each inverse functions principal range before cancelling it with the corresponding trigonometric function.

- Angles outside a principal interval must first be replaced by an angle in that interval having the same trigonometric value.

Step 1: Evaluate the sine inverse The angle $3\pi/4$ lies outside the principal range of sine inverse. Using $\sin(\pi - \theta) = \sin\theta$, we obtain

$$\sin\frac{3\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

Since $\pi/4 \in [-\pi/2, \pi/2]$, $\sin^{-1}(\sin(3\pi/4)) = \pi/4$.

Step 2: Evaluate the remaining terms and add The angle $3\pi/4$ already belongs to $[0, \pi]$, so $\cos^{-1}(\cos(3\pi/4)) = 3\pi/4$. Also, $\tan(\pi/4) = 1$, with $\pi/4$ in the principal range of tangent inverse. Hence $\tan^{-1}(1) = \pi/4$. Adding the three principal values gives

$$\frac{\pi}{4} + \frac{3\pi}{4} + \frac{\pi}{4} = \frac{5\pi}{4}.$$

Step 3: Conclusion The required value is $\boxed{5\pi/4}$.

Quick Tip to Solve: Check each inverse functions principal range before cancelling it with the corresponding trigonometric function.

22. Find all the vectors of magnitude $3\sqrt{3}$ which are collinear to vector $\hat{i} + \hat{j} + \hat{k}$. [2 marks]

Solution:

Concept:

- Vectors collinear with a nonzero vector $\vec{a}$ have the form $\lambda\vec{a}$, where λ is a real scalar.
- The magnitude satisfies $|\lambda\vec{a}| = |\lambda|\,|\vec{a}|$; a specified positive magnitude therefore allows both parallel and opposite directions.

Step 1: Write the general collinear vector Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$. Its components are $(1, 1, 1)$, so

$$|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

Any vector collinear with $\vec{a}$ can be written as $\vec{v} = \lambda(\hat{i} + \hat{j} + \hat{k})$. This includes vectors pointing along $\vec{a}$ and vectors pointing in the reverse direction.

Step 2: Impose the required magnitude Using the given magnitude,

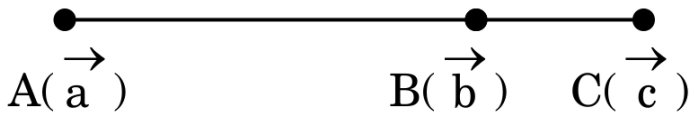
$$|\vec{v}| = |\lambda|\sqrt{3} = 3\sqrt{3}.$$

Dividing by the nonzero number $\sqrt{3}$ gives $|\lambda| = 3$, hence $\lambda = 3$ or $\lambda = -3$. Both values are necessary: magnitude is nonnegative and cannot determine the sign of the scalar. Each resulting vector has squared magnitude $9 + 9 + 9 = 27$, hence magnitude $3\sqrt{3}$.

Step 3: Conclusion The required vectors are $\boxed{3\hat{i} + 3\hat{j} + 3\hat{k}}$ and $\boxed{-3\hat{i} - 3\hat{j} - 3\hat{k}}$.

Quick Tip to Solve: For a fixed nonzero magnitude, include both signs of the unit vector along the given direction.

23(a). **Position vectors of the points A, B and C as shown in the figure below are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. If $\vec{AC} = \frac{5}{4}\vec{AB}$, express $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$.** [2 marks]



Solution:

Concept:

- If points A and B have position vectors $\vec{a}$ and $\vec{b}$, then $\overrightarrow{AB} = \vec{b} - \vec{a}$.
- A directed displacement is obtained by subtracting the starting point's position vector from the endpoint's position vector. This applies even when the endpoint lies beyond a segment.

Step 1: Convert the displacement relation into position vectors Using a common origin for the three position vectors, we have

$$\overrightarrow{AC} = \vec{c} - \vec{a}, \qquad \overrightarrow{AB} = \vec{b} - \vec{a}.$$

Substitute these expressions into the given vector relation:

$$\vec{c} - \vec{a} = \frac{5}{4}(\vec{b} - \vec{a}).$$

The factor $5/4 > 1$ means that C lies beyond B in the direction from A to B, as in the printed figure.

Step 2: Isolate the position vector of C Expand the right-hand side and add $\vec{a}$ to both sides:

$$\begin{aligned}\vec{c} &= \vec{a} + \frac{5}{4}\vec{b} - \frac{5}{4}\vec{a} \\ &= \frac{5}{4}\vec{b} - \frac{1}{4}\vec{a}.\end{aligned}$$

As a check, subtracting $\vec{b}$ yields $\overrightarrow{BC} = \frac{1}{4}(\vec{b} - \vec{a}) = \frac{1}{4}\overrightarrow{AB}$, consistent with the displayed order A, B, C.

Step 3: Conclusion Thus $\boxed{\vec{c} = (5\vec{b} - \vec{a})/4}$.

Quick Tip to Solve: Convert each directed segment to endpoint minus starting point before rearranging the equation.

OR

23(b). **Check whether the lines given by equations $x = 2\lambda + 2$, $y = 7\lambda + 1$, $z = -3\lambda - 3$ and $x = -\mu - 2$, $y = 2\mu + 8$, $z = 4\mu + 5$ are perpendicular to each other or not.** [2 marks]

Solution:

Concept:

- For a parametric line $x = x_0 + a\lambda$, $y = y_0 + b\lambda$, $z = z_0 + c\lambda$, the parameter coefficients (a, b, c) are direction ratios.

Quick Tip to Solve: Read direction ratios from parameter coefficients, keeping the minus signs on -1 and -3.

- The angle between two lines is the angle between their direction vectors. Their directions are perpendicular precisely when their scalar product is zero.

Step 1: Identify the direction vectors For the first line, the coefficients of λ in the three coordinates give

$$\vec{d_1} = 2\hat{i} + 7\hat{j} - 3\hat{k}.$$

For the second line, the coefficients of μ give

$$\vec{d_2} = -\hat{i} + 2\hat{j} + 4\hat{k}.$$

The constant terms specify points on the lines; they do not contribute to the direction ratios or the angle calculation.

Step 2: Calculate the scalar product Multiply corresponding components and add:

$$\vec{d_1} \cdot \vec{d_2} = 2(-1) + 7(2) + (-3)(4) = -2 + 14 - 12 = 0.$$

Both direction vectors are nonzero, with magnitudes $\sqrt{62}$ and $\sqrt{21}$. Therefore the angle θ between the lines satisfies

$$\cos\theta = \frac{|\vec{d_1} \cdot \vec{d_2}|}{|\vec{d_1}||\vec{d_2}|} = 0.$$

This gives a right angle between their directions.

Step 3: Conclusion Yes. The lines have perpendicular directions; the angle between them is $\boxed{90^\circ}$.

Quick Tip to Solve: Read direction ratios from parameter coefficients, keeping the minus signs on -1 and -3.

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24. If $y = \left(x + \sqrt{x^2 - 1}\right)^2$, then show that $(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 4y^2$. [2 marks]

Solution:

Concept:

- The chain rule gives $\frac{d}{dx}[u(x)]^2 = 2u(x)u\prime(x)$.
- For $x^2 - 1 > 0$, $\frac{d}{dx}\sqrt{x^2 - 1} = \frac{x}{\sqrt{x^2 - 1}}$. The required derivative identity is considered where the function is differentiable.

Step 1: Differentiate using the chain rule Put $s = \sqrt{x^2 - 1}$, so $y = (x + s)^2$. On $|x| > 1$, $s > 0$ and $s\prime = x/s$. Therefore

$$\begin{aligned}\frac{dy}{dx} &= 2(x + s)\left(1 + \frac{x}{s}\right) \\ &= 2(x + s)\frac{s + x}{s} \\ &= \frac{2(x + s)^2}{s} = \frac{2y}{s}.\end{aligned}$$

This simplification uses the same factor $x + s$ twice; no cancellation of $x + s$ is required.

Quick Tip to Solve: After differentiating, rewrite 1+x/s as (s+x)/s to recover the original squared expression y.

Step 2: Square and use the radical relation Squaring the derivative expression gives

$$\left(\frac{dy}{dx}\right)^2 = \frac{4y^2}{s^2} = \frac{4y^2}{x^2-1}.$$

Multiplication by x^2-1, which is positive on either differentiable real interval $x > 1$ or $x < -1$, yields the stated relation. The real function is defined for $|x| \geq 1$, but its derivative is not finite at the endpoints $x = \pm 1$, so those endpoints are excluded from a derivative identity.

Step 3: Conclusion Hence $\boxed{(x^2-1)\left(\frac{dy}{dx}\right)^2 = 4y^2}$, wherever the real derivative exists.

Quick Tip to Solve: After differentiating, rewrite 1+x/s as (s+x)/s to recover the original squared expression y.

25. Show that the function $f(x) = \dfrac{16\sin x}{4+\cos x} - x$ is strictly decreasing in $\left(\dfrac{\pi}{2}, \pi\right)$. [2 marks]

Solution:

Concept:

- A differentiable function is strictly decreasing on an interval if its derivative is negative at every point of that interval.
- Use the quotient rule and the identity $\sin^2 x + \cos^2 x = 1$; for $\pi/2 < x < \pi$, we have $-1 < \cos x < 0$.

Step 1: Differentiate the printed function The denominator $4+\cos x$ is positive throughout the given interval, so the function is differentiable there. By the quotient rule,

$$\begin{aligned} f\prime(x) &= 16\frac{\cos x(4+\cos x) + \sin^2 x}{(4+\cos x)^2} - 1 \\ &= \frac{16(4\cos x + 1) - (4+\cos x)^2}{(4+\cos x)^2} \\ &= \frac{56\cos x - \cos^2 x}{(4+\cos x)^2} = \frac{\cos x(56-\cos x)}{(4+\cos x)^2}. \end{aligned}$$

Step 2: Determine the sign on the interval For every $x \in (\pi/2, \pi)$, the factor $\cos x$ is negative, whereas $56-\cos x$ and $(4+\cos x)^2$ are positive. Hence $f\prime(x) < 0$ at every point. In particular, if $a < b$ lie in this interval, the mean value theorem gives $f(b) - f(a) = f\prime(c)(b-a) < 0$ for some $c \in (a, b)$.

Step 3: Conclusion Thus $f(x) = \dfrac{16\sin x}{4+\cos x} - x$ is strictly decreasing on $(\pi/2, \pi)$.

Quick Tip to Solve: Combine the derivative into one fraction and factor its numerator before checking the sign of cosine.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26. Evaluate: $\displaystyle\int_0^{\pi/2} \left[\log(\sin x) - \log(2\cos x)\right] dx$**.** [3 marks]

Solution:

Concept:

- The reflection property is $\int_0^a g(x)\,dx = \int_0^a g(a-x)\,dx$, whenever these integrals exist.
- For positive arguments, $\log(2\cos x) = \log 2 + \log(\cos x)$. Here $\log$ denotes the natural logarithm.

Step 1: Separate the constant logarithm Let the required integral be I. For $0 < x < \pi/2$, both sine and cosine are positive. Therefore

$$I = \int_0^{\pi/2} \log(\sin x)\,dx - \int_0^{\pi/2} \log(\cos x)\,dx - \int_0^{\pi/2} \log 2\,dx.$$

The endpoint singularities are understood as improper limits. They converge: near zero, $\log(\sin x)$ behaves as $\log x$, whose integral is finite, and near $\pi/2$ the corresponding cosine logarithm has the same behaviour.

Step 2: Apply reflection to the sine integral Put $u = \pi/2 - x$. The limits change from $0, \pi/2$ to $\pi/2, 0$, and $dx = -du$. Thus

$$\begin{aligned}\int_0^{\pi/2} \log(\sin x)\,dx &= \int_{\pi/2}^{0} \log(\cos u)(-du) \\ &= \int_0^{\pi/2} \log(\cos u)\,du.\end{aligned}$$

Consequently the first two integrals cancel exactly. Their individual values need not be calculated.

Step 3: Evaluate the remaining constant integral Only the term from $\log 2$ remains, so

$$I = -\log 2\,[x]_0^{\pi/2} = -\frac{\pi}{2}\log 2.$$

The minus sign follows from subtracting the whole expression $\log(2\cos x)$, including its constant part.

Step 4: Conclusion The required value is $\boxed{-\frac{\pi}{2}\log 2}$.

Quick Tip to Solve: Reflect about $\pi/4$ to cancel the sine and cosine logarithms; only the constant logarithm survives.

27. Find: $\displaystyle\int \frac{1}{\sqrt{x}\,(\sqrt{x}+1)\,(\sqrt{x}+2)}\,dx$**.** [3 marks]

Solution:

Concept:

- For a radical expression in $\sqrt{x}$, use $t = \sqrt{x}$, giving $x = t^2$ and $dx = 2t\,dt$.

Quick Tip to Solve: Use $t = \sqrt{x}$ first: the differential cancels the extra square-root factor in the denominator.

- The identity $\dfrac{1}{(t+1)(t+2)} = \dfrac{1}{t+1} - \dfrac{1}{t+2}$ reduces the integral to logarithmic terms.

Step 1: Remove the square root by substitution On the real domain of the integrand, $x > 0$; hence $t = \sqrt{x} > 0$. Write $x = t^2$, so $dx = 2t\,dt$. Then

$$I = \int \frac{dx}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+2)} = \int \frac{2t\,dt}{t(t+1)(t+2)} = 2\int \frac{dt}{(t+1)(t+2)}.$$

The factor t cancels because $t > 0$.

Step 2: Find and integrate the partial fractions Write $1/((t+1)(t+2)) = A/(t+1) + B/(t+2)$. Multiplying by the denominator gives $1 = A(t+2) + B(t+1)$. Setting $t = -1$ gives $A = 1$, and setting $t = -2$ gives $B = -1$. Therefore

$$I = 2\int \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt = 2\log|t+1| - 2\log|t+2| + C.$$

Step 3: Return to the original variable and check Since $t = \sqrt{x} > 0$, both logarithm arguments are positive. Substitution yields $2\log((\sqrt{x}+1)/(\sqrt{x}+2)) + C$. Differentiating this expression gives

$$\frac{1}{\sqrt{x}}\left(\frac{1}{\sqrt{x}+1} - \frac{1}{\sqrt{x}+2}\right) = \frac{1}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+2)},$$

which verifies the antiderivative.

Step 4: Conclusion $\boxed{2\log\left(\dfrac{\sqrt{x}+1}{\sqrt{x}+2}\right) + C}$, for $x > 0$.

Quick Tip to Solve: Use $t = \sqrt{x}$ first: the differential cancels the extra square-root factor in the denominator.

28(a). Find the particular solution of the differential equation $\dfrac{dy}{dx} + \sec^2 x \cdot y = \tan x \cdot \sec^2 x$, given that $y(0) = 0$. [3 marks]

Solution:

Concept:

- A linear differential equation $y' + P(x)y = Q(x)$ has integrating factor $e^{\int P(x)\,dx}$.
- Here $P(x) = \sec^2 x$, so the integrating factor is $e^{\tan x}$; use the initial condition to determine the integration constant.

Step 1: Obtain the integrating factor The given equation is already in linear form:

$$\frac{dy}{dx} + \sec^2 x\, y = \tan x \sec^2 x.$$

Since $\int \sec^2 x\,dx = \tan x$, the integrating factor is IF $= e^{\tan x}$. We work on the interval $(-\pi/2, \pi/2)$ containing $x = 0$, where the coefficients are continuous and tangent is defined.

Quick Tip to Solve: After using the integrating factor, substitute $t = \tan x$ to reduce the integral to $\int te^t\,dt$.

Step 2: Integrate after multiplication Multiplying through by $e^{\tan x}$ makes the left-hand side the derivative of a product:

$$\frac{d}{dx}(ye^{\tan x}) = \tan x \sec^2 x\, e^{\tan x}.$$

Put $t = \tan x$, so $dt = \sec^2 x\, dx$. Integration by parts gives

$$\begin{aligned} ye^{\tan x} &= \int te^t\, dt \\ &= te^t - e^t + C \\ &= e^{\tan x}(\tan x - 1) + C. \end{aligned}$$

Therefore $y = \tan x - 1 + Ce^{-\tan x}$.

Step 3: Apply the condition and verify At $x = 0$, $\tan 0 = 0$ and $y(0) = 0$, hence $0 = -1 + C$, so $C = 1$. For this value, $y\prime = \sec^2 x - \sec^2 x e^{-\tan x}$. Adding $\sec^2 x\, y$ cancels the constant and exponential terms and leaves $\tan x \sec^2 x$, precisely the required right-hand side.

Step 4: Conclusion The particular solution is $\boxed{y = \tan x - 1 + e^{-\tan x}}$, on the interval containing zero where $\tan x$ is defined.

Quick Tip to Solve: After using the integrating factor, substitute $t = \tan x$ to reduce the integral to $\int te^t\, dt$.

OR

28(b). Solve the differential equation given by $x\, dy - y\, dx - \sqrt{x^2 + y^2}\, dx = 0$. [3 marks]

Solution:

Concept:

- A homogeneous equation has the form $dy/dx = F(y/x)$. Substitution $y = vx$ gives $dy/dx = v + x\, dv/dx$.
- Use $\int dv/\sqrt{1 + v^2} = \log(v + \sqrt{1 + v^2})$, with the non-negative square root.

Step 1: Separate the homogeneous equation For $x \neq 0$, rearrangement gives $dy/dx = (y + \sqrt{x^2 + y^2})/x$. First work on an interval with $x > 0$. Put $y = vx$, so $\sqrt{x^2 + y^2} = x\sqrt{1 + v^2}$. Hence

$$v + x\frac{dv}{dx} = v + \sqrt{1 + v^2}, \qquad \frac{dv}{\sqrt{1 + v^2}} = \frac{dx}{x}.$$

Step 2: Integrate and simplify Integrating gives $\log(v + \sqrt{1 + v^2}) = \log x + c$. Exponentiating and putting $K = e^c > 0$, we get $v + \sqrt{1 + v^2} = Kx$. Substituting back yields

$$y + \sqrt{x^2 + y^2} = Kx^2.$$

Multiply by $\sqrt{x^2 + y^2} - y$. The product on the left is x^2, so cancellation of x^2 gives $\sqrt{x^2 + y^2} - y = 1/K$. Set $a = 1/K > 0$. Squaring $\sqrt{x^2 + y^2} = y + a$ gives $x^2 = 2ay + a^2$.

Quick Tip to Solve: Rationalise the integrated square-root expression to obtain a constant first integral, then check its sign before squaring.

Step 3: Check the square-root branch and domain For this family,

$$y = \frac{x^2 - a^2}{2a}, \qquad \frac{dy}{dx} = \frac{x}{a},$$
$$\sqrt{x^2 + y^2} = \frac{x^2 + a^2}{2a}.$$

Therefore $x(dy/dx) - y - \sqrt{x^2 + y^2} = 0$ identically. The positive constant ensures the square root has the correct sign. This direct check extends the family to negative x and to $x = 0$. Conversely, on any interval with $x \neq 0$, differentiate $r - y$, where $r = \sqrt{x^2 + y^2}$:

$$\frac{d(r - y)}{dx} = \frac{x + (y - r)y'}{r} = 0,$$

since $y' = (y + r)/x$ and $(y - r)(y + r) = -x^2$. Thus the same constant first integral covers both signs of x; no negative square-root branch introduced by squaring is accepted.

Step 4: Conclusion The general solution is

$$\boxed{\sqrt{x^2 + y^2} - y = a, \quad a > 0}.$$

Equivalently, $y = (x^2 - a^2)/(2a)$, with $a > 0$.

Quick Tip to Solve: Rationalise the integrated square-root expression to obtain a constant first integral, then check its sign before squaring.

29. Solve graphically the following linear programming problem: Maximise $z = 6x + 3y$, subject to the constraints

$4x + y \geq 80,$

$3x + 2y \leq 150,$

$x + 5y \geq 115,$

$x \geq 0, \ y \geq 0.$ [3 marks]

Solution:

Concept:

- Plot each constraint boundary and choose the half-plane satisfying its inequality. Their common intersection in the first quadrant is the feasible region.
- A linear objective function on a non-empty bounded polygon attains its maximum at a vertex. Here $z = 6x + 3y$.

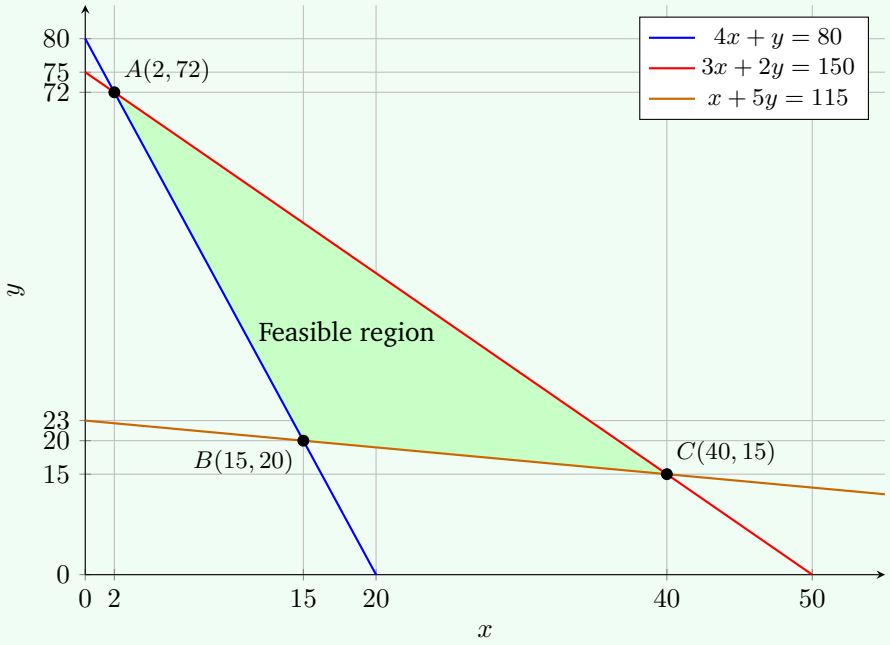
Step 1: Draw the constraint lines and choose the half-planes The boundary $4x + y = 80$ passes through $(20, 0)$ and $(0, 80)$; retain its upper side $y \geq 80 - 4x$. The line $3x + 2y = 150$ passes through $(50, 0)$ and $(0, 75)$; retain its lower side $y \leq 75 - 3x/2$. Finally $x + 5y = 115$ passes through $(0, 23)$ and $(40, 15)$; retain its upper side $y \geq 23 - x/5$. Also retain only $x \geq 0, y \geq 0$. The origin fails the two lower bounds and satisfies the upper bound, which checks these choices.

Quick Tip to Solve: Shade the common region before comparing values; intersections outside the common half-planes must never enter the corner-point table.

Step 2: Find the vertices of the common feasible region Solve each pair of boundary equations:

$$\begin{aligned} 4x+y=80,\quad 3x+2y=150 &\implies A=(2,72),\\ 4x+y=80,\quad x+5y=115 &\implies B=(15,20),\\ 3x+2y=150,\quad x+5y=115 &\implies C=(40,15). \end{aligned}$$

Each point satisfies the remaining inequality: at A, $x+5y=362\geq 115$; at B, $3x+2y=85\leq 150$; and at C, $4x+y=175\geq 80$. No feasible point lies on either coordinate axis: $x=0$ would require $y\geq 80$ and $y\leq 75$, while $y=0$ would require $x\geq 115$ and $x\leq 50$. Thus the shaded triangle ABC is the entire bounded feasible region.



The shaded triangle ABC is the feasible region.

Step 3: Compare the objective values at the vertices Evaluate $z=6x+3y$ at all three corners:

Vertex	z
$A=(2,72)$	$6(2)+3(72)=228$
$B=(15,20)$	$6(15)+3(20)=150$
$C=(40,15)$	$6(40)+3(15)=285$

Since 285 exceeds both other corner values, the vertex theorem selects C. This point also meets all non-negativity conditions and lies on both $3x+2y=150$ and $x+5y=115$.

Step 4: Conclusion The maximum value is $\boxed{z_{\max}=285}$, attained at $\boxed{(x,y)=(40,15)}$.

Quick Tip to Solve: Shade the common region before comparing values; intersections outside the common half-planes must never enter the corner-point table.

30(a). **The probability distribution of a random variable X is given below:**
(i) Find the value of k. (ii) Find $P(1 \le X < 3)$. (iii) Find $E(X)$, the mean of X. [3 marks]

X	1	2	3
P(X)	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k}{6}$

Solution:

Concept:

- A probability distribution satisfies $\sum P(X = x) = 1$, with each probability non-negative.
- For a discrete random variable, add the probabilities of exactly those support values satisfying the given inequalities.
- The mean is the probability-weighted sum $E(X) = \sum xP(X = x)$.

Step 1: Find the normalising constant The support of X is $\{1, 2, 3\}$, so these three mutually exclusive outcomes exhaust all possibilities. Their probabilities must add to one:

$$\frac{k}{2} + \frac{k}{3} + \frac{k}{6} = 1 \quad \Longrightarrow \quad \frac{3k + 2k + k}{6} = 1.$$

Thus $k = 1$. The probabilities become $1/2, 1/3, 1/6$; all are non-negative and their sum is one, so this value produces a valid distribution.

Step 2: Apply the event's endpoints carefully The printed event is $1 \le X < 3$. The lower endpoint is included and the upper endpoint is excluded. Thus $X = 1$ and $X = 2$ satisfy the event, while $X = 3$ does not. Since the two included outcomes cannot occur together,

$$P(1 \le X < 3) = P(X = 1) + P(X = 2) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Equivalently this is $1 - P(X = 3) = 1 - 1/6$, confirming the same probability.

Step 3: Calculate the mean Multiply each possible value by its corresponding probability and add:

$$E(X) = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{6}\right)$$
$$= \frac{1}{2} + \frac{2}{3} + \frac{1}{2} = \frac{5}{3}.$$

This is a weighted mean, so simply averaging $1, 2, 3$ would be incorrect because the probabilities are unequal. The result lies between the smallest and largest support values, as any valid expectation must.

Step 4: Conclusion (i) $\boxed{k = 1}$; (ii) $\boxed{P(1 \le X < 3) = 5/6}$; (iii) $\boxed{E(X) = 5/3}$.

Quick Tip to Solve: List the support values admitted by each endpoint before adding probabilities; here the equality includes X=1 and the strict upper bound excludes X=3.

OR

30(b). A **and** B **are independent events such that** $P(A \cap \bar{B}) = \frac{1}{4}$ **and** $P(\bar{A} \cap B) = \frac{1}{6}$**. Find** $P(A)$ **and** $P(B)$**.** [3 marks]

Solution:

Concept:

- Independent events satisfy $P(A \cap B) = P(A)P(B)$; the complement of either event is also independent of the other event.
- Put $p = P(A)$ and $q = P(B)$. Then $P(\bar{A}) = 1 - p$, $P(\bar{B}) = 1 - q$, and all probabilities must belong to $[0, 1]$.

Step 1: Translate the given conditions Independence gives

$$p(1 - q) = \frac{1}{4}, \qquad (1 - p)q = \frac{1}{6}.$$

Expanding gives $p - pq = 1/4$ and $q - pq = 1/6$. Subtracting eliminates the common product:

$$p - q = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}, \qquad p = q + \frac{1}{12}.$$

The order matters because the two given events have different complements.

Step 2: Solve the quadratic Substituting in the first equation,

$$\begin{aligned} (q + \tfrac{1}{12})(1 - q) &= \tfrac{1}{4}, \\ 12q^2 - 11q + 2 &= 0, \\ (3q - 2)(4q - 1) &= 0. \end{aligned}$$

Therefore $q = 2/3$ or $q = 1/4$. Using $p = q + 1/12$, the corresponding values of p are $3/4$ and $1/3$. Both pairs lie in the permissible probability range, so neither root can be discarded.

Step 3: Check both pairs For $(p, q) = (3/4, 2/3)$,

$$p(1 - q) = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}, \qquad (1 - p)q = \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6}.$$

For $(p, q) = (1/3, 1/4)$,

$$p(1 - q) = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}, \qquad (1 - p)q = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}.$$

Both satisfy the original conditions exactly, establishing that the information allows two distinct pairs.

Step 4: Conclusion $\boxed{(P(A), P(B)) = (\frac{3}{4}, \frac{2}{3}) \text{ or } (\frac{1}{3}, \frac{1}{4})}$.

Quick Tip to Solve: Subtract the mixed-event equations first to cancel the common product; solve the resulting quadratic and retain every admissible probability pair.

31(a). Evaluate: $\int_0^{\pi/2} e^x \sin x \, dx$. [3 marks]

Solution:

Concept:

- Integration by parts: $\int u\,dv = uv - \int v\,du$. Choose the trigonometric factor as u and integrate e^x.
- Applying integration by parts twice to $e^x \sin x$ brings back the original integral, allowing an algebraic solution. Definite integrals require the upper value minus the lower value.

Step 1: Apply integration by parts once Let $I = \int e^x \sin x \, dx$. Take $u = \sin x$, $dv = e^x dx$; then $du = \cos x \, dx$ and $v = e^x$. Thus

$$I = e^x \sin x - \int e^x \cos x \, dx.$$

Denote the remaining integral by J. Its evaluation requires another application of the same rule, because neither exponential nor trigonometric factor disappears after one differentiation.

Step 2: Apply the rule again and isolate the integral For $J = \int e^x \cos x \, dx$, take $u = \cos x$, $dv = e^x dx$. Since $du = -\sin x \, dx$,

$$J = e^x \cos x + I.$$

Substitute into the previous equation:

$$I = e^x \sin x - e^x \cos x - I.$$

Hence an antiderivative is

$$I = \frac{e^x}{2}(\sin x - \cos x) + C.$$

Differentiating this expression gives $e^x \sin x$, confirming the signs and the factor $1/2$.

Step 3: Evaluate at the printed limits Use the fundamental theorem of calculus with the upper limit $\pi/2$ and lower limit zero:

$$\begin{aligned}\int_0^{\pi/2} e^x \sin x \, dx &= \left[\frac{e^x}{2}(\sin x - \cos x)\right]_0^{\pi/2} \\ &= \frac{e^{\pi/2}}{2}(1-0) - \frac{1}{2}(0-1) \\ &= \frac{e^{\pi/2}+1}{2}.\end{aligned}$$

The lower endpoint contributes a negative antiderivative value, so subtracting it adds $1/2$. The final value is positive, consistent with the integrand on this interval.

Step 4: Conclusion $\boxed{\int_0^{\pi/2} e^x \sin x \, dx = \frac{e^{\pi/2}+1}{2}}$.

Quick Tip to Solve: After two integrations by parts, move the repeated integral to the left; evaluate the lower endpoint carefully because its antiderivative value is negative.

OR

31(b). Find: $\int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$. [3 marks]

Solution:

Concept:

- The identity $\tan u - \tan v = \frac{\sin(u-v)}{\cos u \cos v}$ follows by putting the two tangents over a common denominator.
- Use $\int \tan(x-c)\,dx = -\log|\cos(x-c)| + C$. The antiderivative is considered on intervals where both denominator factors are nonzero.

Step 1: Rewrite the product using a tangent difference Let $u = x-a$ and $v = x-b$. Then $u-v = b-a$, a constant. Consequently

$$\tan(x-a) - \tan(x-b) = \frac{\sin(b-a)}{\cos(x-a)\cos(x-b)}.$$

When $\sin(b-a) \neq 0$, divide by this constant to obtain

$$\frac{1}{\cos(x-a)\cos(x-b)} = \frac{\tan(x-a) - \tan(x-b)}{\sin(b-a)}.$$

This substitution turns the product of two cosine factors into a difference of standard integrands.

Step 2: Integrate the tangent terms Integrate each term, observing that both inner arguments have derivative one:

$$\begin{aligned}\int \frac{dx}{\cos(x-a)\cos(x-b)} &= \frac{-\log|\cos(x-a)| + \log|\cos(x-b)|}{\sin(b-a)} + C \\ &= \frac{1}{\sin(b-a)} \log\left|\frac{\cos(x-b)}{\cos(x-a)}\right| + C.\end{aligned}$$

Absolute values keep the real logarithm defined regardless of the signs of the nonzero cosine factors. Differentiating the logarithmic numerator gives $\tan(x-a) - \tan(x-b)$, so the identity in the first step verifies the answer.

Step 3: Handle the exceptional parameter values If $b-a = k\pi$, where k is an integer, the sine divisor vanishes and the logarithmic form must not be used. In this case $\cos(x-b) = (-1)^k \cos(x-a)$. Hence the integrand becomes $(-1)^k \sec^2(x-a)$, whose antiderivative is $(-1)^k \tan(x-a) + C$. This includes $a = b$, for which the original integral is simply the integral of $\sec^2(x-a)$.

Step 4: Conclusion For $\sin(b-a) \neq 0$, the answer is $\boxed{\frac{1}{\sin(b-a)} \log\left|\frac{\cos(x-b)}{\cos(x-a)}\right| + C}$. If $b-a = k\pi$, it is $\boxed{(-1)^k \tan(x-a) + C}$.

Quick Tip to Solve: Use the tangent-difference identity; preserve the matching order of the sine and logarithmic ratio, and check separately when the sine divisor is zero.

Section D (20 marks)

This section comprises long answer type questions (LA) of 5 marks each.

32. A relation R is defined on a set of real numbers $\mathbb{R}$ as $R = \{(x, y) : x \cdot y$ is an irrational number$\}$. Check whether R is reflexive, symmetric and transitive or not. [5 marks]

Solution:

Concept:

- Reflexivity requires $(x, x) \in R$ for every $x \in \mathbb{R}$; here it requires every square x^2 to be irrational.
- Symmetry requires $(x, y) \in R \Rightarrow (y, x) \in R$. Transitivity requires $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$.
- One counterexample disproves a universal property, whereas a proof must apply to arbitrary related numbers.

Step 1: Check reflexivity Choose the real number $x = 1$. Its product with itself is $1 \cdot 1 = 1$, which is rational. Hence $(1, 1) \notin R$. Reflexivity requires every diagonal pair to belong to the relation, so this missing diagonal pair disproves reflexivity. Some other diagonal pairs do belong to R: for example, if $x = \sqrt[4]{2}$, then $x^2 = \sqrt{2}$ is irrational. However, the existence of such pairs cannot establish a condition that must hold for every real number. The counterexample $x = 1$ is decisive.

Step 2: Check symmetry Let $x, y \in \mathbb{R}$ be arbitrary and suppose $(x, y) \in R$. The definition gives that xy is irrational. Since multiplication of real numbers is commutative, $yx = xy$. Thus yx is the same irrational number, and $(y, x) \in R$. This argument does not depend on a particular choice of x and y, so it proves symmetry for every pair in the relation. Reversing the order of the factors leaves their product unchanged.

Step 3: Check transitivity Take $x = 1$, $y = \sqrt{2}$, and $z = 1$. All three numbers lie in the underlying set. Their relevant products are

$$xy = \sqrt{2}, \qquad yz = \sqrt{2}, \qquad xz = 1.$$

Both xy and yz are irrational. Therefore $(1, \sqrt{2}) \in R$ and $(\sqrt{2}, 1) \in R$. However, $xz = 1$ is rational, so $(1, 1) \notin R$. We have exhibited two related pairs with the required common middle element but without the required resulting pair. This violates precisely the definition of transitivity. There is no requirement that x, y, z be distinct, so choosing $x = z = 1$ is valid.

Step 4: Conclusion The relation R is not reflexive, is symmetric, and is not transitive. It therefore is not an equivalence relation on $\mathbb{R}$.

Quick Tip to Solve: For a product relation, use commutativity for symmetry and test the triple $1, \sqrt{2}, 1$ for transitivity.

33(a). **If** $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ **and** $B^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$**, find** $(AB)^{-1}$**.** [5 marks]

Solution:

Concept:

- For invertible square matrices, $(AB)^{-1} = B^{-1}A^{-1}$. The order reverses because matrix multiplication is generally not commutative.
- If $|A| \neq 0$, then $A^{-1} = \operatorname{adj} A/|A|$. The adjugate is the transpose of the cofactor matrix, whose entries are $C_{ij} = (-1)^{i+j}M_{ij}$.

Step 1: Evaluate the determinant of A Expanding along the first row gives

$$|A| = 1(3) - 2(-1) + (-2)(2) = 3 + 2 - 4 = 1.$$

Since this determinant is nonzero, A is invertible and the adjugate formula applies.

Step 2: Compute the cofactors and the inverse Computing the signed two-by-two minors gives the cofactor matrix below. For example, $C_{12} = -((-1)(1) - 0(0)) = 1$ and $C_{31} = 2(0) - (-2)(3) = 6$. Transpose this matrix to form the adjugate. Thus the cofactor matrix and its transpose are

$$C = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{pmatrix}, \qquad \operatorname{adj} A = C^T = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

Dividing by $|A| = 1$ leaves the adjugate unchanged, so $A^{-1} = \operatorname{adj} A$. As a check, multiplication of A by this matrix gives rows $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, respectively, confirming $AA^{-1} = I$.

Step 3: Multiply the inverse factors in reversed order The required inverse is $B^{-1}A^{-1}$, with the given inverse of B on the left. Row-by-column multiplication yields

$$B^{-1}A^{-1} = \begin{pmatrix} 10 & 7 & 21 \\ -49 & -34 & -103 \\ 17 & 12 & 36 \end{pmatrix}.$$

For example, the first row is obtained as $(3\cdot3-1\cdot1+1\cdot2,\ 3\cdot2-1\cdot1+1\cdot2,\ 3\cdot6-1\cdot2+1\cdot5) = (10, 7, 21)$. Likewise the second row is $(-45+6-10, -30+6-10, -90+12-25)$, and the third is $(15-2+4, 10-2+4, 30-4+10)$. The reversal of order follows directly from $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I$, and similarly the reverse product equals I.

Step 4: Conclusion Therefore,

$$\boxed{(AB)^{-1} = \begin{pmatrix} 10 & 7 & 21 \\ -49 & -34 & -103 \\ 17 & 12 & 36 \end{pmatrix}}.$$

Quick Tip to Solve: Since B^{-1} is already given, compute only A^{-1}, then multiply $B^{-1}A^{-1}$ in that order.

OR

33(b). Solve the following system of equations by matrix method:

$x + 2y + 3z = 6$

$2x - y + z = 2$

$3x + 2y - 2z = 3$ [5 marks]

Solution:

Concept:

- Write a linear system as $AX = B$, with coefficient matrix A, unknown column X, and constant column B.
- If $|A| \neq 0$, its unique solution is $X = A^{-1}B$, where $A^{-1} = \text{adj}\, A/|A|$.
- The cofactor C_{ij} is $(-1)^{i+j}$ times the minor; the adjoint is the transpose of the cofactor matrix.

Step 1: Write the matrix equation Arrange coefficients in the order x, y, z. The three equations give

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}, \qquad AX = B.$$

The second row retains the negative coefficient of y, and the third row retains the negative coefficient of z. Multiplication of each row of A by X reproduces its corresponding original equation.

Step 2: Find the determinant and adjoint Expanding the determinant along the first row gives

$$|A| = 1(2 - 2) - 2(-4 - 3) + 3(4 + 3) = 35 \neq 0.$$

Thus A is invertible. Applying the alternating cofactor signs gives

$$C = \begin{pmatrix} 0 & 7 & 7 \\ 10 & -11 & 4 \\ 5 & 5 & -5 \end{pmatrix}.$$

For example, $C_{12} = -(2(-2) - 1(3)) = 7$, $C_{21} = -(2(-2) - 3(2)) = 10$, and $C_{32} = -(1(1) - 3(2)) = 5$. The other entries follow by deleting the appropriate row and column and evaluating the remaining two-by-two determinant. Transposing gives

$$\text{adj}\, A = \begin{pmatrix} 0 & 10 & 5 \\ 7 & -11 & 5 \\ 7 & 4 & -5 \end{pmatrix}.$$

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the constant column, and check the resulting values in all three equations.

Step 3: Apply the inverse and verify Premultiplication by A^{-1} isolates the unknown column:

$$X = \frac{1}{35}\begin{pmatrix} 0 & 10 & 5 \\ 7 & -11 & 5 \\ 7 & 4 & -5 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{35}\begin{pmatrix} 20+15 \\ 42-22+15 \\ 42+8-15 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

To verify, substitute these values into each original left-hand side: $1+2+3=6$, $2-1+1=2$, and $3+2-2=3$. Each equals its stated constant, so the computed column satisfies the whole system. Since the determinant is nonzero, no second solution exists.

Step 4: Conclusion The unique solution is $\boxed{x=1,\ y=1,\ z=1}$.

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the constant column, and check the resulting values in all three equations.

34(a). Find the vector and the Cartesian equations of a line passing through the point $(1,2,-4)$ and parallel to the line joining the points $A(3,3,-5)$ and $B(1,0,-11)$. Hence, find the distance between the two lines. [5 marks]

Solution:

Concept:

- A line through a position vector $\vec{p}$, parallel to a nonzero direction vector $\vec{b}$, has vector equation $\vec{r}=\vec{p}+t\vec{b}$, where $t\in\mathbb{R}$.
- For direction ratios l,m,n, all nonzero, the Cartesian equation through (x_0,y_0,z_0) is $\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$.
- The distance between parallel lines through $\vec{p}$ and $\vec{a}$, with common direction $\vec{b}$, is $d=\frac{|(\vec{a}-\vec{p})\times\vec{b}|}{|\vec{b}|}$. This is the component of their connecting vector perpendicular to the lines.

Step 1: Determine a direction vector From the given points,

$$\overrightarrow{AB}=(1-3)\hat{i}+(0-3)\hat{j}+(-11+5)\hat{k}=-2\hat{i}-3\hat{j}-6\hat{k}.$$

A nonzero scalar multiple is equally valid as a direction vector. Choose $\vec{b}=2\hat{i}+3\hat{j}+6\hat{k}$, which has length $\sqrt{4+9+36}=7$. The required line and the line through A,B therefore share these direction ratios.

Step 2: Obtain both equations of the required line Let $P=(1,2,-4)$, so $\vec{p}=\hat{i}+2\hat{j}-4\hat{k}$. Using the point and direction form gives

$$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+t(2\hat{i}+3\hat{j}+6\hat{k}),\quad t\in\mathbb{R}.$$

Quick Tip to Solve: Reverse all three direction ratios together if convenient; for parallel-line distance, cross a connecting vector with that common direction.

Its component equations are $x = 1 + 2t$, $y = 2 + 3t$, and $z = -4 + 6t$. Eliminating t gives

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}.$$

Setting $t = 0$ recovers P, confirming that the required point lies on this line.

Step 3: Compute the perpendicular separation Use $A = (3, 3, -5)$ on the other line. The connecting vector from P to A is

$$\vec{a} - \vec{p} = 2\hat{i} + \hat{j} - \hat{k}.$$

Its cross product with the common direction vector is

$$(\vec{a} - \vec{p}) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix} = 9\hat{i} - 14\hat{j} + 4\hat{k}.$$

Thus its magnitude is $\sqrt{81 + 196 + 16} = \sqrt{293}$. Dividing this parallelogram area by the base length $|\vec{b}| = 7$ yields the perpendicular distance:

$$d = \frac{\sqrt{293}}{7}.$$

The nonzero cross product also establishes that these parallel lines are distinct. Choosing B instead of A would add a multiple of $\vec{b}$ to the connecting vector and leave the cross product unchanged.

Step 4: Conclusion The vector equation is $\boxed{\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + t(2\hat{i} + 3\hat{j} + 6\hat{k}),\ t \in \mathbb{R}}$. The Cartesian equation is $\boxed{\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}}$, and the distance between the lines is $\boxed{\frac{\sqrt{293}}{7} \text{ units}}$.

Quick Tip to Solve: Reverse all three direction ratios together if convenient; for parallel-line distance, cross a connecting vector with that common direction.

OR

34(b). Find the equations of the line passing through the points $A(1, 2, 3)$ and $B(3, 5, 9)$. Hence, find the coordinates of the points on this line which are at a distance of 14 units from point B. [5 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{a}$ and nonzero direction vector $\vec{d}$ has equation $\vec{r} = \vec{a} + t\vec{d}$, where $t \in \mathbb{R}$.
- The vector joining two distinct points gives the direction of their line. A displacement $s\vec{d}$ has length $|s|\,|\vec{d}|$; the absolute value accounts for both directions along the line.

Step 1: Obtain the direction vector The given points are $A(1, 2, 3)$ and $B(3, 5, 9)$. Subtracting

Quick Tip to Solve: The direction vector has length 7, so move from B by plus or minus twice that vector to obtain both points 14 units away.

corresponding coordinates,

$$\overrightarrow{AB} = (3-1)\hat{i} + (5-2)\hat{j} + (9-3)\hat{k} = 2\hat{i} + 3\hat{j} + 6\hat{k}.$$

Thus the direction ratios are $2, 3, 6$, and the magnitude of this direction vector is $\sqrt{2^2+3^2+6^2} = 7$. The points are distinct, so they determine a unique line.

Step 2: Write the equations of the line Taking A as the fixed point gives the vector equation

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + 6\hat{k}), \quad t \in \mathbb{R}.$$

Its component equations are $x = 1+2t$, $y = 2+3t$ and $z = 3+6t$. Eliminating the common parameter gives

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}.$$

All denominators are nonzero. For $t = 0$ the coordinates give A; for $t = 1$ they give B, verifying both specified points.

Step 3: Use the distance condition For a general point $P(1+2t, 2+3t, 3+6t)$ on the line, its displacement from B is

$$\overrightarrow{BP} = (t-1)(2\hat{i} + 3\hat{j} + 6\hat{k}).$$

Therefore $BP = 7|t-1|$. The prescribed distance gives

$$7|t-1| = 14 \quad \Longrightarrow \quad |t-1| = 2.$$

Hence $t = 3$ or $t = -1$. Both parameter values are necessary, since there is one point in each direction from B on the infinite line.

Step 4: Calculate and check the coordinates Substituting $t = 3$ gives $P_1 = (7, 11, 21)$. Substituting $t = -1$ gives $P_2 = (-1, -1, -3)$. Their displacements from B are $(4, 6, 12)$ and $(-4, -6, -12)$, respectively. Each displacement has length $\sqrt{16+36+144} = \sqrt{196} = 14$. Their midpoint is $(3, 5, 9) = B$, providing a further check on the symmetry of the results.

Step 5: Conclusion The line is $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + 6\hat{k})$, or $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$. The required points are $\boxed{(7, 11, 21) \text{ and } (-1, -1, -3)}$.

Quick Tip to Solve: The direction vector has length 7, so move from B by plus or minus twice that vector to obtain both points 14 units away.

35. Find the area of the region bounded by the curves $x^2 = y$, $y = x + 2$ and the x-axis, using integration. [5 marks]

Solution:

Concept:

- The area between an upper curve $y = f(x)$ and a lower curve $y = g(x)$ is $\int_a^b [f(x) - g(x)]\,dx$. Here the lower boundary of the required region is the x-axis, $y = 0$.
- Find boundary intersections first. If the upper boundary changes from one curve to another, split the integral at the point where that change occurs.

Step 1: Locate the intersections The parabola $y = x^2$ meets the x-axis at $O(0, 0)$, while the line $y = x + 2$ meets it at $A(-2, 0)$. For the intersections of the line and parabola,

$$x^2 = x + 2 \quad \Longrightarrow \quad (x - 2)(x + 1) = 0.$$

Thus the curves intersect at $B(-1, 1)$ and $C(2, 4)$. The enclosed region whose boundary uses all three given curves follows the line from A to B, the parabola from B to O, and the x-axis from O back to A.

Step 2: Identify the upper boundary on each interval For $-2 \le x \le -1$, the upper boundary of this region is $y = x + 2$, and the lower boundary is $y = 0$. For $-1 \le x \le 0$, its upper boundary is $y = x^2$, again with $y = 0$ below. The intersection $C(2, 4)$ belongs to the other region enclosed by just the line and the parabola; it is not on the boundary containing the x-axis segment AO.

Step 3: Integrate the part under the line The area from $x = -2$ to $x = -1$ is

$$\begin{aligned} A_1 &= \int_{-2}^{-1} (x + 2)\,dx \\ &= \left[\frac{x^2}{2} + 2x\right]_{-2}^{-1} \\ &= \left(\frac{1}{2} - 2\right) - (2 - 4) = \frac{1}{2}. \end{aligned}$$

Step 4: Integrate the part under the parabola and add The remaining part extends from $x = -1$ to $x = 0$. Its area is

$$\begin{aligned} A_2 &= \int_{-1}^{0} x^2\,dx = \left[\frac{x^3}{3}\right]_{-1}^{0} \\ &= 0 - \left(-\frac{1}{3}\right) = \frac{1}{3}. \end{aligned}$$

Adding the two positive areas gives

$$A = A_1 + A_2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Step 5: Conclusion The required area bounded by $y = x^2$, $y = x + 2$, and the x-axis is $\boxed{\frac{5}{6} \text{ square units}}$.

Quick Tip to Solve: Trace the boundary that includes the x-axis: split at x=-1, using the line on [-2,-1] and the parabola on [-1,0].

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

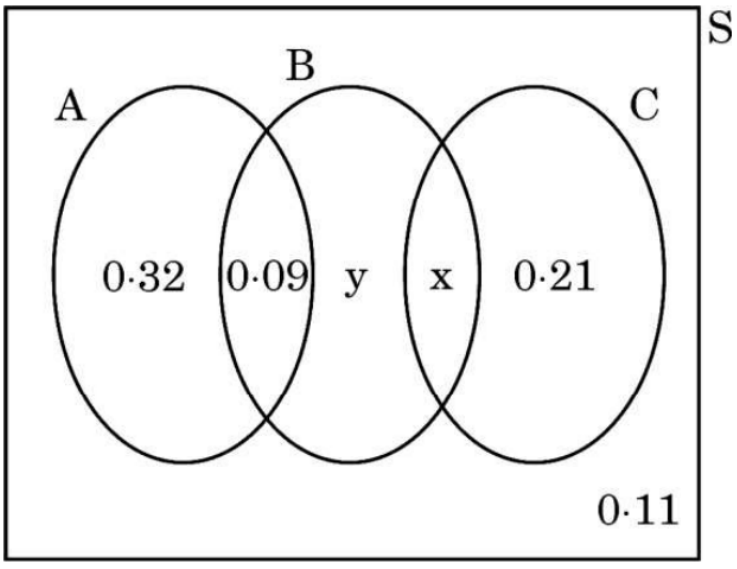
36. There are different types of Yoga which involve the usage of different poses of Yoga Asanas, Meditation and Pranayam as shown in the figure below:

The Venn diagram below represents the probabilities of three different types of Yoga, A, B and C performed by the people of a society. Further, it is given that probability of a member performing type C Yoga is 0.44.

On the basis of the above information, answer the following questions: [4 marks]



Types of Yoga



36(i). Find the value of x. [1 mark]

Solution:

Concept:

- For disjoint regions of a Venn diagram, their probabilities add to give the probability of the whole event.
- The event C contains its exclusive region and the overlap $B \cap C$, whose probability is x.

Step 1: Identify the regions of C The printed diagram shows two non-overlapping regions inside C: the region belonging only to C, of probability 0.21, and the common region of B and C, of probability x. The event A does not intersect C. Thus no other region contributes to $P(C)$.

Step 2: Use the given probability Since $P(C) = 0.44$, addition of these two region probabilities gives

$$x + 0.21 = 0.44, \qquad x = 0.44 - 0.21 = 0.23.$$

This value is positive and gives the stated total probability of C.

Step 3: Conclusion Therefore, $\boxed{x = 0.23}$.

Quick Tip to Solve: Add only the disjoint regions inside the circle for the given event.

36(ii). Find the value of y. [1 mark]

Solution:

Concept:

- The probabilities of all mutually exclusive regions of the sample space sum to 1.
- Include the region outside all three circles when using the total-probability sum.

Step 1: Determine the overlap value The two disjoint regions of C have probabilities x and 0.21. Hence $x + 0.21 = 0.44$, giving $x = 0.23$. The number y labels the region in B that lies outside both A and C.

Step 2: Sum the complete sample space The six disjoint regions have probabilities $0.32, 0.09, y, 0.23, 0.21$ and 0.11. The last probability is outside all circles and must be counted. Therefore

$$0.32 + 0.09 + y + 0.23 + 0.21 + 0.11 = 1.$$

The known probabilities sum to 0.96, so $y = 1 - 0.96 = 0.04$.

Step 3: Conclusion Therefore, $\boxed{y = 0.04}$.

Quick Tip to Solve: Total the exclusive Venn regions once each, including the outside region.

36(iii)(a). Find $P\left(\frac{C}{B}\right)$. [2 marks]

Solution:

Concept:

- Conditional probability is $P(C \mid B) = P(C \cap B)/P(B)$, provided $P(B) > 0$.
- The condition B restricts the sample space to every region inside circle B.

Step 1: Find the required region probabilities From $P(C) = 0.44$, the overlap of B and C is $x = 0.44 - 0.21 = 0.23$. All regions of the sample space sum to one, including the outside region of probability 0.11. Thus

$$y = 1 - (0.32 + 0.09 + 0.23 + 0.21 + 0.11) = 0.04.$$

Step 2: Find the probability of the conditioning event Circle B contains the A-B overlap, the exclusive B region, and the B-C overlap. These regions are mutually exclusive, so

$$P(B) = 0.09 + y + x = 0.09 + 0.04 + 0.23 = 0.36.$$

Its common region with C has probability $P(C \cap B) = 0.23$.

Step 3: Apply the conditional probability formula The printed notation $P(C/B)$ denotes the probability of C given B. Therefore

$$P(C \mid B) = \frac{P(C \cap B)}{P(B)} = \frac{0.23}{0.36} = \frac{23}{36}.$$

We divide by the full probability of B, since the selected person is already known to perform type B Yoga.

Step 4: Conclusion The required probability is $\boxed{\frac{23}{36}}$.

Quick Tip to Solve: For C given B, divide the B-C overlap by the sum of all regions in B.

OR

36(iii)(b). Find the probability that a randomly selected person of the society does Yoga of type A or B but not C. [2 marks]

Solution:

Concept:

- The event described by A or B but not C is $(A \cup B) \cap C'$.
- Add mutually exclusive Venn regions once; an overlap is one region, not two.

Step 1: Determine the missing Venn entries The given probability $P(C) = 0.44$ gives $x + 0.21 = 0.44$,

Quick Tip to Solve: Exclude every region inside C before adding the regions of A or B.

so $x = 0.23$. Including all exclusive regions and the outside probability 0.11,

$$y = 1 - (0.32 + 0.09 + 0.23 + 0.21 + 0.11) = 0.04.$$

Step 2: Select the regions satisfying the event A person in A only qualifies, as does a person in both A and B, or in B only. Their respective probabilities are 0.32, 0.09 and $y = 0.04$. The region $B \cap C$ does not qualify, because the condition explicitly excludes anyone performing type C. Neither the C-only region nor the outside region belongs to the required event.

Step 3: Add the qualifying probabilities The selected three regions are disjoint. Hence

$$P\big((A \cup B) \cap C'\big) = 0.32 + 0.09 + 0.04 = 0.45.$$

The common A-B region is counted exactly once, as required by the word or.

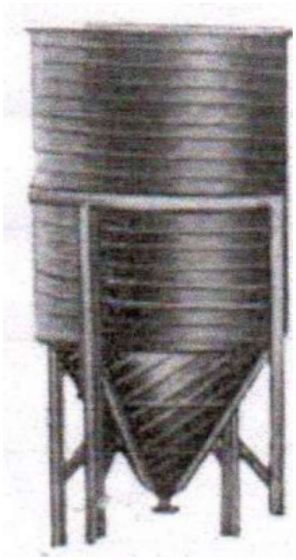
Step 4: Conclusion The required probability is $\boxed{0.45}$.

Quick Tip to Solve: Exclude every region inside C before adding the regions of A or B.

37. A tank, as shown in the figure below, formed using a combination of a cylinder and a cone, offers better drainage as compared to a flat bottomed tank.

A tap is connected to such a tank whose conical part is full of water. Water is dripping out from a tap at the bottom at the uniform rate of 2 cm^3/s. The semi-vertical angle of the conical tank is $45°$.

On the basis of given information, answer the following questions. [4 marks]



37(i). Find the volume of water in the tank in terms of its radius r. [1 mark]

Solution:

Concept:

- A cone of radius r and height h has volume $V = \frac{1}{3}\pi r^2 h$.

Quick Tip to Solve: Use the semi-vertical angle to eliminate height before writing volume in r.

- The semi-vertical angle gives $\tan 45^\circ = r/h$ for the water cone.

Step 1: Relate the radius and height The water occupies the conical portion of the tank. In a vertical cross-section, the radius r is opposite the semi-vertical angle and the height h is adjacent to it. Therefore

$$\tan 45^\circ = \frac{r}{h} = 1, \qquad h = r.$$

This relation holds for each water level because the water cone is similar to the tank cone.

Step 2: Substitute into the volume formula Using $h = r$ in the volume of the water cone,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (r) = \frac{\pi r^3}{3}.$$

If r is measured in centimetres, this volume is measured in cubic centimetres.

Step 3: Conclusion The volume is $\boxed{V = \frac{\pi r^3}{3} \text{ cm}^3}$, with r measured in centimetres.

Quick Tip to Solve: Use the semi-vertical angle to eliminate height before writing volume in r.

37(ii). Find rate of change of radius at an instant when $r = 2\sqrt{2}$ cm. [1 mark]

Solution:

Concept:

- For a semi-vertical angle of 45°, $h = r$ and $V = \pi r^3/3$.
- Water draining at 2 cm^3/s means $dV/dt = -2$ cm^3/s.

Step 1: Differentiate the volume relation The radius of the water surface varies with time. Using the chain rule on $V = \pi r^3/3$,

$$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt}.$$

The outflow decreases the volume, so the given draining rate is used with a negative sign.

Step 2: Evaluate at the specified radius At $r = 2\sqrt{2}$ cm, we have $r^2 = 8$ cm^2. Thus

$$-2 = 8\pi \frac{dr}{dt}, \qquad \frac{dr}{dt} = -\frac{1}{4\pi} \text{ cm/s}.$$

The negative sign confirms that the radius decreases as the water drains.

Step 3: Conclusion The rate of change is $\boxed{\frac{dr}{dt} = -\frac{1}{4\pi} \text{ cm/s}}$.

Quick Tip to Solve: Draining makes dV/dt negative; substitute r squared only after differentiating.

37(iii)(a). Find the rate at which the wet surface of the conical tank is decreasing at an instant when radius $r = 2\sqrt{2}$ cm. [2 marks]

Solution:

Concept:

- The wetted curved surface of the cone is $S = \pi r l$, where l is the slant height up to the water level.
- For semi-vertical angle $45°$, $h = r$ and $l = \sqrt{r^2 + h^2} = \sqrt{2}\, r$.
- The volume relation is $V = \pi r^3/3$, with $dV/dt = -2$ cm^3/s.

Step 1: Express the wet surface in terms of radius The wet surface is the tank wall in contact with water, so it is the curved surface of the water cone. It excludes the exposed horizontal water surface. Since $h = r$, its slant height is $l = \sqrt{2}r$. Therefore

$$S = \pi r l = \sqrt{2}\pi r^2.$$

Step 2: Calculate the radius rate at the instant Differentiating the volume relation with respect to time gives

$$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt}.$$

At $r = 2\sqrt{2}$ cm, $r^2 = 8$. Hence

$$-2 = 8\pi \frac{dr}{dt}, \qquad \frac{dr}{dt} = -\frac{1}{4\pi} \text{ cm/s}.$$

Step 3: Differentiate the wet surface area Using the chain rule,

$$\frac{dS}{dt} = 2\sqrt{2}\pi r \frac{dr}{dt}.$$

Substituting the instantaneous values,

$$\frac{dS}{dt} = 2\sqrt{2}\pi(2\sqrt{2})\left(-\frac{1}{4\pi}\right) = -2 \text{ cm}^2/\text{s}.$$

Thus the signed area rate is negative, and the positive rate of decrease is its magnitude.

Step 4: Conclusion The wet surface is decreasing at $\boxed{2 \text{ cm}^2/\text{s}}$.

Quick Tip to Solve: Wet surface means the curved tank wall: use pi r l, with l equal to square root 2 times r.

OR

37(iii)(b). Find the rate of change of height h at an instant when slant height is 4 cm. [2 marks]

Solution:

Concept:

- For a cone of water with radius r, height h, and slant height l, $r/h = \tan 45^\circ = 1$. Thus $r = h$ throughout drainage.
- The volume is $V = \frac{1}{3}\pi r^2 h$. Since water leaves the tank, $dV/dt = -2\,\text{cm}^3/\text{s}$.

Step 1: Express volume in terms of height The water occupies a cone similar to the conical part of the tank. Substituting $r = h$ in the volume formula gives

$$V = \frac{\pi}{3}h^3, \qquad \frac{dV}{dt} = \pi h^2 \frac{dh}{dt}.$$

This relation holds as the water level falls; both the radius and height change with time.

Step 2: Use the given slant height By Pythagoras, $l^2 = r^2 + h^2 = 2h^2$. At $l = 4\,\text{cm}$, we therefore have $h^2 = 16/2 = 8\,\text{cm}^2$. Substitute this instantaneous value after differentiation:

$$-2 = \pi(8)\frac{dh}{dt}, \qquad \frac{dh}{dt} = -\frac{1}{4\pi}\ \text{cm/s}.$$

Step 3: Conclusion The rate of change of height is $\boxed{-\frac{1}{4\pi}\ \text{cm/s}}$; the height decreases at $\frac{1}{4\pi}$ cm/s.

Quick Tip to Solve: Use the semi-vertical angle to eliminate the radius, and convert slant height to vertical height before substituting.

38. The equation of the path traced by a roller-coaster is given by the polynomial $f(x) = a(x+9)(x+1)(x-3)$. If the roller-coaster crosses y-axis at a point $(0,-1)$, answer the following. [4 marks]

38(i). Find the value of a. [2 marks]

Solution:

Concept:

- A point (x, y) on the graph $y = f(x)$ satisfies the equation $y = f(x)$.
- At the y-axis, $x = 0$. The given intercept $(0,-1)$ therefore means $f(0) = -1$, which determines the constant multiplier a.

Step 1: Apply the given intercept The path is $f(x) = a(x+9)(x+1)(x-3)$. Since it passes through $(0,-1)$, substitute $x = 0$ and $f(0) = -1$:

$$-1 = a(0+9)(0+1)(0-3) = a(9)(1)(-3).$$

The three factors have product -27, so the intercept condition is $-27a = -1$.

Quick Tip to Solve: For a factored polynomial with a given y-intercept, substitute x=0 directly; expansion is unnecessary.

Step 2: Determine and check the constant Dividing the equation by -27 gives

$$a = \frac{-1}{-27} = \frac{1}{27}.$$

To check the sign and value, substitute this constant back at $x = 0$: $f(0) = \frac{1}{27}(9)(1)(-3) = -1$, which is exactly the given vertical coordinate. The nonzero factor -27 also ensures that the constant is uniquely determined.

Step 3: Conclusion The required constant is $\boxed{a = \frac{1}{27}}$.

Quick Tip to Solve: For a factored polynomial with a given y-intercept, substitute x=0 directly; expansion is unnecessary.

38(ii). Find $f''(x)$ at $x = 1$. [2 marks]

Solution:

Concept:

- The given intercept determines the constant: $f(0) = -27a = -1$, hence $a = 1/27$.
- For a polynomial, differentiate each power using $\frac{d}{dx}x^n = nx^{n-1}$. The second derivative $f''(x)$ is obtained by differentiating $f'(x)$ once more.

Step 1: Expand the path polynomial First multiply $(x + 9)(x + 1) = x^2 + 10x + 9$. Multiplication by the remaining factor gives

$$(x^2 + 10x + 9)(x - 3) = x^3 + 7x^2 - 21x - 27.$$

With $a = 1/27$, the function is therefore $f(x) = \frac{1}{27}(x^3 + 7x^2 - 21x - 27)$.

Step 2: Differentiate twice and evaluate Treating a as the constant already determined by the intercept, differentiate term by term:

$$f'(x) = \frac{1}{27}(3x^2 + 14x - 21), \qquad f''(x) = \frac{1}{27}(6x + 14).$$

Now put $x = 1$ in the second derivative to obtain $f''(1) = \frac{6+14}{27} = \frac{20}{27}$. Substitution is done after both differentiations, so that the function's dependence on x is retained.

Step 3: Conclusion The required value is $\boxed{f''(1) = \frac{20}{27}}$.

Quick Tip to Solve: Expand this cubic once, differentiate twice, and only then substitute x=1 in the second derivative.