CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-1-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) This question paper is divided into **five** Sections – **A, B, C, D and E**.

(iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculators is **not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. $\int \frac{1+\tan x}{1-\tan x}\,dx$ **is equal to:** [1 mark]

(A) $\sec^2\left(\frac{\pi}{4}+x\right)+C$

(B) $\sec^2\left(\frac{\pi}{4}-x\right)+C$

(C) $\log\left|\sec\left(\frac{\pi}{4}+x\right)\right|+C$

(D) $\log\left|\sec\left(\frac{\pi}{4}-x\right)\right|+C$

Correct Answer: (C) $\log\left|\sec\left(\frac{\pi}{4}+x\right)\right|+C$

Solution:

Concept:

- The tangent addition formula gives $\tan(a+x)=\dfrac{\tan a+\tan x}{1-\tan a\tan x}$.
- For a variable u, $\int \tan u\,du=\log|\sec u|+C$, on an interval where the integrand is defined.

Step 1: Recognise the shifted tangent Using $\tan(\pi/4)=1$, rewrite the integrand as

$$\frac{1+\tan x}{1-\tan x}=\tan\left(\frac{\pi}{4}+x\right).$$

The shift has derivative one, so there is no extra chain-rule factor.

Step 2: Integrate and check the choices Set $u=\pi/4+x$, so $du=dx$. Then the antiderivative is $\log|\sec u|+C$. Differentiating either squared-secant choice gives an extra factor $2\sec^2 u$, so (A) and (B) fail. Choice (D) differentiates to $-\tan(\pi/4-x)$, which is not the given integrand.

Step 3: Conclusion

$$\boxed{\log\left|\sec\left(\frac{\pi}{4}+x\right)\right|+C}$$

Hence, the correct option is (C).

Quick Tip to Solve: When a tangent fraction contains $1+\tan x$ above $1-\tan x$, try the addition angle $\pi/4+x$.

2. $\int_0^{\pi/6} \sec^2\left(x - \frac{\pi}{6}\right) dx$ **is equal to:** [1 mark]

(A) $\frac{1}{\sqrt{3}}$

(B) $-\frac{1}{\sqrt{3}}$

(C) $\sqrt{3}$

(D) $-\sqrt{3}$

Correct Answer: (A) $\frac{1}{\sqrt{3}}$

Solution:

Concept:

- $\frac{d}{du}\tan u = \sec^2 u$. A constant shift of the variable leaves $du = dx$.
- A definite integral equals the antiderivative at the upper limit minus its value at the lower limit.

Step 1: Find the antiderivative For $u = x - \pi/6$, we have $du = dx$. Thus an antiderivative of the integrand is $\tan(x - \pi/6)$. The shifted angle ranges from $-\pi/6$ to 0, so there is no singularity in this interval.

Step 2: Evaluate in the correct order

$$\int_0^{\pi/6} \sec^2\left(x - \frac{\pi}{6}\right) dx = \tan 0 - \tan\left(-\frac{\pi}{6}\right) = 0 - \left(-\frac{1}{\sqrt{3}}\right).$$

The integrand is positive, eliminating (B) and (D). The tangent of $\pi/6$ is $1/\sqrt{3}$, not $\sqrt{3}$, eliminating (C).

Step 3: Conclusion

$$\boxed{\frac{1}{\sqrt{3}}}$$

Hence, the correct option is (A).

Quick Tip to Solve: Write both shifted endpoint angles first; subtracting the negative lower tangent makes this integral positive.

3. The sum of the order and the degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = \sin y$ is: [1 mark]

(A) 5

(B) 2

(C) 3

(D) 4

Correct Answer: (C) 3

Solution:

Concept:

- The order is the order of the highest derivative present.
- The degree is the power of that highest-order derivative when the equation is polynomial in its derivatives.

Step 1: Identify the highest derivative The derivatives appearing are dy/dx and d^2y/dx^2. The second derivative is the highest, so the order is 2. The term $\sin y$ contains no derivative and does not affect the order.

Step 2: Determine the degree and compare The equation is already polynomial in its derivatives. The highest-order derivative d^2y/dx^2 occurs to the first power, so the degree is 1. The cube belongs to the first derivative and must not be used as the degree. Therefore the required sum is $2 + 1$; choices (A), (B) and (D) do not equal this sum.

Step 3: Conclusion

$$\boxed{\text{Order} + \text{degree} = 2 + 1 = 3}$$

Hence, the correct option is (C).

Quick Tip to Solve: Locate the highest derivative first, then inspect only its exponent; ignore the cube of the lower derivative.

4. The value of p for which the vectors $2\hat{i} + p\hat{j} + \hat{k}$ and $-4\hat{i} - 6\hat{j} + 26\hat{k}$ are perpendicular to each other, is: [1 mark]

(A) 3

(B) -3

(C) $-\frac{17}{3}$

(D) $\frac{17}{3}$

Correct Answer: (A) 3

Solution:

Concept:

- Nonzero vectors $\vec{u}$ and $\vec{v}$ are perpendicular precisely when $\vec{u} \cdot \vec{v} = 0$.

Quick Tip to Solve: Multiply matching components with their signs intact, then set their sum to zero for perpendicular vectors.

- The dot product is the sum of products of corresponding Cartesian components.

Step 1: Form the dot product For the given vectors,

$$(2\hat{i} + p\hat{j} + \hat{k}) \cdot (-4\hat{i} - 6\hat{j} + 26\hat{k}) = 2(-4) + p(-6) + 1(26) = 18 - 6p.$$

The last component contributes 26, since the first vector has coefficient 1 along $\hat{k}$.

Step 2: Use perpendicularity and test Setting $18 - 6p = 0$ gives $6p = 18$. Substituting $p = 3$ makes the dot product zero. For the other options $-3, -17/3, 17/3$, the dot products are respectively $36, 52, -16$, all nonzero.

Step 3: Conclusion

$$\boxed{p = 3}$$

Hence, the correct option is (A).

Quick Tip to Solve: Multiply matching components with their signs intact, then set their sum to zero for perpendicular vectors.

5. If the vector $\hat{i} - b\hat{j} + \hat{k}$ is equally inclined to the coordinate axes, then the value of b is: [1 mark]

(A) -1

(B) 1

(C) $-\sqrt{3}$

(D) $-\dfrac{1}{\sqrt{3}}$

Correct Answer: (A) -1

Solution:

Concept:

- For a vector with components (a, b, c), its direction cosines are its components divided by its magnitude.
- Equal inclination to the positive coordinate axes means equal direction cosines, including their signs.

Step 1: Write the direction cosines The vector has components $(1, -b, 1)$ and magnitude $\sqrt{2 + b^2}$. Hence its direction cosines are

$$l = \frac{1}{\sqrt{2 + b^2}}, \quad m = \frac{-b}{\sqrt{2 + b^2}}, \quad n = \frac{1}{\sqrt{2 + b^2}}.$$

Step 2: Equate the inclinations Equal angles imply $l = m = n$, so the common positive denominator gives $1 = -b = 1$. Choice (B), $b = 1$, makes the middle direction cosine negative and hence gives a different angle. Choices (C) and (D) make the middle component respectively $\sqrt{3}$ and $1/\sqrt{3}$, neither equal to 1.

Quick Tip to Solve: Equal inclination requires equal signed components here; equate $-b$ to 1, not their absolute values.

Step 3: Conclusion

$$\boxed{b = -1}$$

Hence, the correct option is (A).

Quick Tip to Solve: Equal inclination requires equal signed components here; equate $-b$ to 1, not their absolute values.

6. If $\vec{a} + \vec{b} = \hat{i}$ and $\vec{a} = 2\hat{i} - 2\hat{j} + 2\hat{k}$, then $|\vec{b}|$ equals: [1 mark]

(A) $\sqrt{14}$

(B) 3

(C) $\sqrt{12}$

(D) $\sqrt{17}$

Correct Answer: (B) 3

Solution:

Concept:

- From $\vec{a} + \vec{b} = \vec{c}$, isolate $\vec{b} = \vec{c} - \vec{a}$ component by component.
- The magnitude of $x\hat{i} + y\hat{j} + z\hat{k}$ is $\sqrt{x^2 + y^2 + z^2}$.

Step 1: Find the components of the unknown vector

$$\vec{b} = \hat{i} - (2\hat{i} - 2\hat{j} + 2\hat{k}) = -\hat{i} + 2\hat{j} - 2\hat{k}.$$

Subtraction changes the signs of every component of $\vec{a}$, including its negative $\hat{j}$ component.

Step 2: Calculate and check the magnitude

$$|\vec{b}| = \sqrt{(-1)^2 + 2^2 + (-2)^2} = \sqrt{1 + 4 + 4} = \sqrt{9}.$$

Its squared magnitude is 9, whereas the squared magnitudes offered in (A), (C) and (D) are $14, 12, 17$. Therefore those three choices cannot be correct.

Step 3: Conclusion

$$\boxed{|\vec{b}| = 3}$$

Hence, the correct option is (B).

Quick Tip to Solve: Isolate $\vec{b}$ before taking magnitudes; the magnitude of a difference is computed from its actual components.

7. Direction cosines of a line perpendicular to both x-axis and z-axis are: [1 mark]

(A) $1, 0, 1$

(B) $1, 1, 1$

(C) $0, 0, 1$

(D) $0, 1, 0$

Correct Answer: (D) $0, 1, 0$

Solution:

Concept:

- Direction cosines (l, m, n) obey $l^2 + m^2 + n^2 = 1$.
- Perpendicularity to an axis makes the direction cosine associated with that axis zero.

Step 1: Use the two perpendicularity conditions Perpendicularity to the x-axis gives $l = \cos 90^\circ = 0$. Perpendicularity to the z-axis similarly gives $n = 0$. Such a line is parallel to the y-axis.

Step 2: Apply normalization and inspect each option The identity now yields $m^2 = 1$, so either orientation gives $m = \pm 1$. The listed triple $(0, 1, 0)$ is therefore valid. Options (A) and (B) have a nonzero x-component and also fail normalization. Option (C) has $n = 1$, so it is parallel rather than perpendicular to the z-axis.

Step 3: Conclusion

$$\boxed{(l, m, n) = (0, 1, 0)}$$

Hence, the correct option is (D).

Quick Tip to Solve: A line perpendicular to both the x- and z-axes runs along the remaining axis, so only its middle direction cosine is nonzero.

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8. If $P\left(\frac{A}{B}\right) = 0.3$, $P(A) = 0.4$ and $P(B) = 0.8$, then $P\left(\frac{B}{A}\right)$ is equal to: [1 mark]

(A) 0.6

(B) 0.3

(C) 0.06

(D) 0.4

Correct Answer: (A) 0.6

Solution:

Concept:

- Conditional probability is $P(A \mid B) = P(A \cap B)/P(B)$, when $P(B) > 0$.
- The reverse conditional probability is $P(B \mid A) = P(A \cap B)/P(A)$, when $P(A) > 0$.

Step 1: Recover the common intersection probability The printed notation $P(A/B)$ denotes $P(A \mid B)$. Thus

$$P(A \cap B) = P(A \mid B)P(B) = 0.3 \times 0.8 = 0.24.$$

Quick Tip to Solve: First find $P(A \cap B)$ by multiplying by the conditioning probability; then divide by the new conditioning probability.

Both conditioning events have positive probability, so these formulas apply.

Step 2: Reverse the conditioning

$$P(B \mid A) = \frac{0.24}{0.4} = 0.6.$$

Choice (B) incorrectly preserves the original conditional probability; conditional probabilities need not be symmetric. Choice (C) has a decimal-place error, and choice (D) simply repeats $P(A)$, rather than using the intersection.

Step 3: Conclusion

$$\boxed{P(B/A) = 0.6}$$

Hence, the correct option is (A).

Quick Tip to Solve: First find $P(A \cap B)$ by multiplying by the conditioning probability; then divide by the new conditioning probability.

9. For what value of k may the function $f(x) = \begin{cases} k(3x^2 - 5x), & x \leq 0, \\ \cos x, & x > 0 \end{cases}$ become continuous? [1 mark]

(A) 0

(B) 1

(C) $-\frac{1}{2}$

(D) No value

Correct Answer: (D) No value

Solution:

Concept:

- Continuity at a joining point requires the left limit, right limit and function value to agree.
- Polynomials and cosine are continuous on their respective intervals; only the junction $x = 0$ needs checking.

Step 1: Evaluate the left side For $x \leq 0$, $f(x) = k(3x^2 - 5x)$. Consequently, for every finite real k,

$$\lim_{x \to 0^-} f(x) = k(0 - 0) = 0, \qquad f(0) = 0.$$

Multiplying by a constant cannot change the zero limit of this polynomial.

Step 2: Compare the right side For $x > 0$, $f(x) = \cos x$, so

$$\lim_{x \to 0^+} f(x) = \cos 0 = 1.$$

The one-sided limits are unequal, independently of k. Thus neither $k = 0$, $k = 1$ nor $k = -\frac{1}{2}$ restores continuity at the origin.

Step 3: Conclusion No value of k makes the function continuous. Hence, the correct option is (D).

Quick Tip to Solve: At a piecewise junction, compute both one-sided limits before trying to solve for the parameter.

10. If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $(3I + 4A)(3I - 4A) = x^2 I$, then the value(s) of x is/are: [1 mark]

(A) $\pm\sqrt{7}$

(B) 0

(C) ± 5

(D) 25

Correct Answer: (C) ± 5

Solution:

Concept:

- For the identity matrix I, $IA = AI = A$, so the cross terms in $(aI + bA)(aI - bA)$ cancel.
- Compute A^2 first to simplify a matrix polynomial without multiplying four separate matrices.

Step 1: Square the matrix Using the given entries,

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

The negative sign matters: this matrix does not square to the identity.

Step 2: Use the product identity Expanding with commuting identity factors gives

$$(3I + 4A)(3I - 4A) = 9I - 16A^2 = 9I + 16I = 25I.$$

Hence $x^2 = 25$. Both square roots must be retained. Option (A) would give $7I$, option (B) would give the zero matrix, and option (D) would give $625I$.

Step 3: Conclusion $x = \pm 5$. Hence, the correct option is (C).

Quick Tip to Solve: When a matrix squares to minus the identity, the difference-of-squares expression becomes a sum.

11. The general solution of the differential equation $x\,dy - (1+x^2)\,dx = dx$ is: [1 mark]

(A) $y = 2x + \dfrac{x^3}{3} + C$

(B) $y = 2\log x + \dfrac{x^3}{3} + C$

(C) $y = \dfrac{x^2}{2} + C$

(D) $y = 2\log x + \dfrac{x^2}{2} + C$

Correct Answer: (D) $y = 2\log x + \dfrac{x^2}{2} + C$

Solution:

Concept:

- Rearrange an equation in differentials to isolate dy/dx, then integrate term by term.
- On $x > 0$, $\int dx/x = \log x$; on an interval with $x < 0$, use $\log|x|$.

Step 1: Isolate the derivative The printed equation is $x\,dy - (1+x^2)\,dx = dx$. Moving the second term to the right gives

$$x\,dy = (2+x^2)\,dx, \qquad \frac{dy}{dx} = \frac{2}{x} + x \quad (x \neq 0).$$

The extra dx contributes a second constant term before division by x.

Step 2: Integrate and check the choices Integration yields

$$y = 2\log|x| + \frac{x^2}{2} + C.$$

On $x > 0$, this is the form in (D). Its derivative is $2/x + x$, as required. The derivatives of (A), (B) and (C) are respectively $2 + x^2$, $2/x + x^2$ and x; each misses a required term or power. The constant C is arbitrary on each interval avoiding zero.

Step 3: Conclusion $y = 2\log x + \dfrac{x^2}{2} + C$ on $x > 0$. Hence, the correct option is (D).

Quick Tip to Solve: After dividing by x, reduce each power before integrating; differentiating the options checks the result.

12. If $f(x) = a(x - \cos x)$ is strictly decreasing in $\mathbb{R}$, then a belongs to: [1 mark]

(A) $\{0\}$

(B) $(0, \infty)$

(C) $(-\infty, 0)$

(D) $(-\infty, \infty)$

Correct Answer: (C) $(-\infty, 0)$

Solution:

Concept:

- A differentiable function with negative derivative except at isolated points can still be strictly decreasing.
- For real x, $-1 \leq \sin x \leq 1$, so $1 + \sin x \geq 0$.

Step 1: Determine the sign of the derivative Differentiate the given expression:

$$f\prime(x) = a(1 + \sin x).$$

If $a > 0$, the derivative is nonnegative and positive away from isolated zeros, so the function increases. If $a = 0$, the function is constant and therefore is not strictly decreasing.

Step 2: Verify strict decrease when a is negative For $a < 0$, $f\prime(x) \leq 0$. To justify strictness, for any $u < v$,

$$f(v) - f(u) = a \int_u^v (1 + \sin t)\, dt < 0.$$

The integrand is nonnegative and vanishes only at isolated points, so its integral over every interval of positive length is positive. Thus isolated zeros of the derivative do not obstruct strict decrease. The set of all real numbers in (D) includes the excluded zero and positive values.

Step 3: Conclusion $a \in (-\infty, 0)$. Hence, the correct option is (C).

Quick Tip to Solve: Do not exclude negative a just because the derivative vanishes at isolated points; check strict change on an interval.

13. The corner points of the feasible region in the graphical representation of a linear programming problem are $(2, 72)$, $(15, 20)$ and $(40, 15)$. If $z = 18x + 9y$ be the objective function, then: [1 mark]

(A) z is maximum at $(2, 72)$, minimum at $(15, 20)$

(B) z is maximum at $(15, 20)$, minimum at $(40, 15)$

(C) z is maximum at $(40, 15)$, minimum at $(15, 20)$

(D) z is maximum at $(40, 15)$, minimum at $(2, 72)$

Correct Answer: (C) z is maximum at $(40, 15)$, minimum at $(15, 20)$

Solution:

Concept:

- A linear objective function attains its finite extremum at a corner point of the feasible region.
- Compare the objective values themselves, rather than comparing only one coordinate of the vertices.

Step 1: Evaluate at the given vertices Substitute each ordered pair into $z = 18x + 9y$:

$$z(2, 72) = 18(2) + 9(72) = 36 + 648 = 684,$$
$$z(15, 20) = 18(15) + 9(20) = 270 + 180 = 450,$$
$$z(40, 15) = 18(40) + 9(15) = 720 + 135 = 855.$$

Both terms are needed because an increase in one coordinate can offset a decrease in the other.

Step 2: Compare and test the statements The three values satisfy $450 < 684 < 855$. Therefore the vertex $(40, 15)$ gives the greatest value and $(15, 20)$ gives the least value. Option (A) incorrectly places the maximum at the first vertex; (B) reverses the relevant extreme vertices; (D) incorrectly places the minimum at $(2, 72)$.

Step 3: Conclusion Maximum $z = 855$ at $(40, 15)$; minimum $z = 450$ at $(15, 20)$. Hence, the correct option is (C).

Quick Tip to Solve: For a linear objective, make a short vertex-value list and rank the numerical values before choosing the statement.

14. The number of corner points of the feasible region determined by the constraints $x - y \geq 0,\ 2y \leq x + 2,\ x \geq 0,\ y \geq 0$ is: [1 mark]

(A) 2

(B) 3

(C) 4

(D) 5

Correct Answer: (A) 2

Solution:

Concept:

- A corner point is a feasible intersection of boundary lines where the direction of the region boundary changes.
- Here the inequalities reduce to $y \leq x$, $y \leq (x+2)/2$, $x \geq 0$ and $y \geq 0$.

Step 1: Find feasible intersections The two sloping boundaries meet where $y = x$ and $2y = x + 2$, giving $(x, y) = (2, 2)$. The coordinate axes meet at $(0, 0)$, which satisfies both inequalities. The intersection of $x = 0$ with $2y = x + 2$ is $(0, 1)$, but it violates $y \leq x$. The intersection with $y = 0$ is $(-2, 0)$, outside the first quadrant.

Step 2: Describe the boundary and count The feasible boundary follows the nonnegative x-axis, then the segment of $y = x$ from $(0, 0)$ to $(2, 2)$, and the ray $y = (x+2)/2$ for $x \geq 2$. The region is unbounded to the right, which does not add a vertex at infinity. Only $(0, 0)$ and $(2, 2)$ are distinct corner points, so choices of three, four or five count inadmissible intersections or nonexistent endpoints.

Step 3: Conclusion There are 2 corner points. Hence, the correct option is (A).

Quick Tip to Solve: Check every boundary intersection against all inequalities; an unbounded ray has no extra endpoint at infinity.

15. If for a square matrix A, $A^2 - 3A + I = O$ and $A^{-1} = xA + yI$, then the value of $x + y$ is: [1 mark]

(A) -2

(B) 2

(C) 3

(D) -3

Correct Answer: (B) 2

Solution:

Concept:

- Multiply a matrix polynomial identity by A^{-1} to express the inverse in terms of A and I.
- The identity $A^2 - 3A + I = O$ also gives $A(3I - A) = I$, confirming invertibility.

Step 1: Express the inverse Multiplying the given equation by A^{-1}, and using $AA^{-1} = I$, gives

$$A - 3I + A^{-1} = O.$$

Quick Tip to Solve: Multiply the polynomial identity by the inverse and isolate it; read the coefficient of A with its sign.

Hence

$$A^{-1} = 3I - A.$$

This expression may also be checked by multiplication: $A(3I - A) = 3A - A^2 = I$, using the original identity.

Step 2: Identify the requested coefficients Comparing this expression with $A^{-1} = xA + yI$, the coefficients supplied by the identity are $x = -1$ and $y = 3$. Their sum is $-1 + 3 = 2$. The value -2 comes from reversing the subtraction; 3 omits the coefficient of A, and -3 is neither the constant coefficient nor their sum. Keep the signs when rearranging the matrix equation.

Step 3: Conclusion $x + y = 2$. Hence, the correct option is (B).

Quick Tip to Solve: Multiply the polynomial identity by the inverse and isolate it; read the coefficient of A with its sign.

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16. If $\left|\frac{A^{-1}}{2}\right| = \frac{1}{k\,|A|}$, where A is a 3×3 matrix, then the value of k is: [1 mark]

(A) $\frac{1}{8}$

(B) 8

(C) 2

(D) $\frac{1}{2}$

Correct Answer: (B) 8

Solution:

Concept:

- For an invertible matrix, $|A^{-1}| = 1/|A|$.
- For an $n \times n$ matrix B, $|cB| = c^n|B|$, since each of its n rows receives the factor c.

Step 1: Apply the scalar rule in order three The matrix inside the determinant is $\frac{1}{2}A^{-1}$. Since it has three rows,

$$\left|\frac{A^{-1}}{2}\right| = \left(\frac{1}{2}\right)^3 |A^{-1}| = \frac{1}{8}|A^{-1}|.$$

Dividing the matrix by two divides every row by two, so the determinant is divided by eight, rather than by two.

Step 2: Use the inverse determinant and compare As A^{-1} exists, $|A| \neq 0$. Therefore

$$\left|\frac{A^{-1}}{2}\right| = \frac{1}{8|A|} = \frac{1}{k|A|}.$$

Comparing the nonzero denominators gives $k = 8$. Option (A) confuses the scalar determinant factor with the denominator coefficient; (C) ignores the matrix order, and (D) applies only the scalar itself.

Step 3: Conclusion $k = 8$. Hence, the correct option is (B).

Quick Tip to Solve: Raise a scalar multiplier to the matrix order before using the determinant formula for the inverse.

17. Let A be a 3×3 matrix such that $|\operatorname{adj} A| = 64$. Then $|A|$ is equal to: [1 mark]

(A) 8 only

(B) -8 only

(C) 64

(D) 8 or -8

Correct Answer: (D) 8 or -8

Solution:

Concept:

- For a square matrix of order n, $|\operatorname{adj} A| = |A|^{n-1}$.
- A positive square has two real square roots, with opposite signs.

Step 1: Apply the adjoint determinant identity The order of the given matrix is 3. Thus $|\operatorname{adj} A| = |A|^{3-1} = |A|^2$. Substituting the given value gives

$$|A|^2 = 64.$$

Step 2: Check all the printed choices Taking both square roots gives $|A| = 8$ or $|A| = -8$. Options (A) and (B) each exclude a possible sign without justification. Option (C) confuses the determinant of the adjoint with the determinant of the matrix: $64^2 \neq 64$. Both signs satisfy the given condition, so option (D) includes precisely the possible values.

Step 3: Conclusion $\boxed{|A| = 8 \text{ or } -8}$. Hence, the correct option is (D).

Quick Tip to Solve: For a matrix of order three, square the determinant to get the adjoint determinant; retain both square-root signs.

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18. If $A = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$ and $2A + B$ is a null matrix, then B is equal to: [1 mark]

(A) $\begin{pmatrix} 6 & 8 \\ 10 & 4 \end{pmatrix}$

(B) $\begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}$

(C) $\begin{pmatrix} 5 & 8 \\ 10 & 3 \end{pmatrix}$

(D) $\begin{pmatrix} -5 & -8 \\ -10 & -3 \end{pmatrix}$

Correct Answer: (B) $\begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}$

Solution:

Concept:

- The null matrix has every entry equal to zero.
- Scalar multiplication multiplies every entry; from $2A+B=O$, additive cancellation gives $B=-2A$.

Step 1: Isolate the required matrix The equation is an equality of matrices of the same order. Subtracting $2A$ from both sides yields

$$B = O - 2A = -2\begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}.$$

Step 2: Multiply and compare the choices Entrywise multiplication gives

$$B = \begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}.$$

Adding this to $2A$ cancels all four entries. Option (A) has the wrong sign. Options (C) and (D) also have incorrect diagonal entries: doubling 3 and 2 gives 6 and 4, not 5 and 3.

Step 3: Conclusion $\boxed{B = \begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}}$. Hence, the correct option is (B).

Quick Tip to Solve: Use B = -2A and check the two diagonal entries first to eliminate the choices with 5 and 3.

19. Assertion (A): The equation of the line passing through the points $(1,2,3)$ and $(3,-1,3)$ is $\frac{x-3}{2}=\frac{y+1}{3}=\frac{z-3}{0}$.

Reason (R): The equation of the line passing through the points (x_1,y_1,z_1) and (x_2,y_2,z_2) is $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- A line through two distinct points has direction vector equal to their coordinate difference.
- In symmetric notation a zero direction component means the corresponding coordinate remains constant; it does not mean ordinary division by zero.

Quick Tip to Solve: Compute both point differences with the same subtraction order; one reversed component changes the line.

Step 1: Test the Assertion The direction from $(1, 2, 3)$ to $(3, -1, 3)$ is $(2, -3, 0)$. Through the second point, the correct line is

$$\frac{x-3}{2} = \frac{y+1}{-3}, \qquad z = 3.$$

The printed assertion uses $+3$ in the second denominator, reversing just the y-component. At $(1, 2, 3)$, the first ratio is -1 and the second is 1, so the assertion is false.

Step 2: Test the Reason The reason states the usual two-point equation, using direction ratios $x_2 - x_1, y_2 - y_1, z_2 - z_1$. This is true in standard symmetric-line notation. If a difference is zero, its coordinate is fixed at the corresponding point coordinate; here $z = 3$.

Step 3: Decide whether the Reason explains the Assertion The true two-point formula produces the negative y-direction ratio, and therefore contradicts the asserted line. Options (A), (B) and (C) all require a true assertion and are excluded. The appropriate truth combination is false Assertion and true Reason.

Step 4: Conclusion The Assertion is false and the Reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Compute both point differences with the same subtraction order; one reversed component changes the line.

20. Assertion (A): The number of onto functions from a set P containing 5 elements to a set Q containing 2 elements is 30.

Reason (R): Number of onto functions from a set containing m elements to a set containing n elements is n^m. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (C) Assertion (A) is true and Reason (R) is false.

Solution:

Concept:

- A function is onto when every element of its codomain has a preimage.
- From an m-element domain to an n-element codomain there are n^m total functions, which need not all be onto.

Step 1: Test the Assertion by counting For each of the five elements of P, there are two independent choices in Q, so the total is $2^5 = 32$. A function fails to be onto precisely when all five inputs map to the same element of Q. There are two such constant functions. Thus the onto count is

$$32 - 2 = 30,$$

Quick Tip to Solve: With exactly two codomain elements, subtract the two constant functions from the total function count.

and the Assertion is true.

Step 2: Test the Reason The expression n^m counts all functions, including those that omit codomain elements. In this very example it gives 32 rather than the onto count 30. Consequently the Reason is false.

Step 3: Check the explanatory relation and choices The Reason cannot explain the true Assertion because it uses the total-function count without excluding the two constant functions. Options (A) and (B) require a true Reason, and option (D) requires a false Assertion; each is excluded.

Step 4: Conclusion The Assertion is true and the Reason is false. Hence, the correct option is (C).

Quick Tip to Solve: With exactly two codomain elements, subtract the two constant functions from the total function count.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21(a). Position vectors of the points A, B and C as shown in the figure below are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. If $\vec{AC} = \frac{5}{4}\vec{AB}$, express $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$. [2 marks]



Solution:

Concept:

- If points A, B, C have position vectors $\vec{a}, \vec{b}, \vec{c}$, then $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = \vec{c} - \vec{a}$.
- A positive scalar multiple preserves the direction of a vector; a factor greater than one places the endpoint farther along the same ray.

Step 1: Express the directed segments as position-vector differences The given figure places B between A and C. Using a common origin for the three position vectors, substitute the differences into the stated vector relation:

$$\vec{c} - \vec{a} = \frac{5}{4}(\vec{b} - \vec{a}).$$

The subtraction order follows the arrow from the starting point A to each endpoint, so no sign change is needed.

Quick Tip to Solve: Write each segment as endpoint position vector minus starting-point position vector before rearranging.

Step 2: Isolate the position vector of C Expand the right-hand side and add $\vec{a}$ to both sides:

$$\begin{aligned}\vec{c}-\vec{a} &= \frac{5}{4}\vec{b}-\frac{5}{4}\vec{a},\\ \vec{c} &= \frac{5}{4}\vec{b}+\left(1-\frac{5}{4}\right)\vec{a}\\ &= \frac{5}{4}\vec{b}-\frac{1}{4}\vec{a}.\end{aligned}$$

As a check, $\vec{c}-\vec{b}=\frac{1}{4}(\vec{b}-\vec{a})$; thus C lies beyond B and $BC=AB/4$, agreeing with the figure and the factor $5/4$.

Step 3: Conclusion $\boxed{\vec{c}=\dfrac{5\vec{b}-\vec{a}}{4}}$.

Quick Tip to Solve: Write each segment as endpoint position vector minus starting-point position vector before rearranging.

OR

21(b). Check whether the lines given by equations $x=2\lambda+2,\ y=7\lambda+1,\ z=-3\lambda-3$ and $x=-\mu-2,\ y=2\mu+8,\ z=4\mu+5$ are perpendicular to each other or not. [2 marks]

Solution:

Concept:

- The coefficients of the parameter in a line equation give its direction ratios.
- Two line directions are perpendicular when their dot product is zero: $\vec{u}\cdot\vec{v}=u_1v_1+u_2v_2+u_3v_3$.

Step 1: Identify the direction vectors The first line can be written as

$$(x,y,z)=(2,1,-3)+\lambda(2,7,-3),$$

so its direction vector is $\vec{u}=(2,7,-3)$. Similarly the second line is

$$(x,y,z)=(-2,8,5)+\mu(-1,2,4),$$

giving $\vec{v}=(-1,2,4)$. The constant terms specify points on the lines and do not determine their directions.

Step 2: Apply the perpendicularity test Compute the scalar product using corresponding components:

$$\vec{u}\cdot\vec{v}=2(-1)+7(2)+(-3)(4)=-2+14-12=0.$$

Both direction vectors are nonzero, with lengths $\sqrt{62}$ and $\sqrt{21}$. Hence $\cos\theta=(\vec{u}\cdot\vec{v})/(\sqrt{62}\sqrt{21})=0$, so the angle between the given line directions is 90°.

Step 3: Conclusion The direction vectors have zero dot product. Therefore the given lines are perpendicular to each other (their directions make 90°).

Quick Tip to Solve: Use only the parameter coefficients as direction ratios; the constant coordinates do not enter the dot product.

22. If $y = (x + \sqrt{x^2 - 1})^2$, then show that $(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 4y^2$. [2 marks]

Solution:

Concept:

- The chain rule gives $\frac{d}{dx}(u^2) = 2u\,u'$, where $u = x + \sqrt{x^2 - 1}$.
- For $|x| > 1$, $\frac{d}{dx}\sqrt{x^2 - 1} = \frac{x}{\sqrt{x^2 - 1}}$. Differentiation is performed where the radical is nonzero.

Step 1: Differentiate the given expression Put $u = x + \sqrt{x^2 - 1}$, so that $y = u^2$. Applying the chain rule directly,

$$\frac{dy}{dx} = 2\left(x + \sqrt{x^2 - 1}\right)\left(1 + \frac{x}{\sqrt{x^2 - 1}}\right).$$

The bracket from differentiating u contains both the derivative of x and that of the radical; neither contribution may be omitted.

Step 2: Express the derivative in terms of y and square Combining the terms in the second bracket gives

$$1 + \frac{x}{\sqrt{x^2 - 1}} = \frac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}}.$$

Consequently,

$$\frac{dy}{dx} = \frac{2\left(x + \sqrt{x^2 - 1}\right)^2}{\sqrt{x^2 - 1}} = \frac{2y}{\sqrt{x^2 - 1}}.$$

Squaring this equation and multiplying by $x^2 - 1$ eliminates the denominator and establishes the required relation wherever the derivative exists.

Step 3: Conclusion Hence $\boxed{(x^2 - 1)\left(\frac{dy}{dx}\right)^2 = 4y^2}$, for $|x| > 1$. The endpoint values $x = \pm 1$ are excluded from the differentiation.

Quick Tip to Solve: Combine the derivative bracket over the radical: its numerator is exactly the original inner expression.

23. Find the sub-intervals in which $f(x) = \log(2 + x) - \frac{x}{2 + x}$, $x > -2$ is increasing or decreasing. [2 marks]

Solution:

Concept:

- A differentiable function is increasing on an interval where $f'(x) > 0$ and decreasing where $f'(x) < 0$.
- Use $\frac{d}{dx}\log(2 + x) = \frac{1}{2 + x}$, with the natural logarithm convention, and the quotient rule for $x/(2 + x)$.

Quick Tip to Solve: After simplifying the derivative, ignore its positive square denominator and test only the sign of x.

Step 1: Calculate the derivative The stated domain is $x > -2$, so the logarithm is defined and $2 + x \neq 0$. By the quotient rule,

$$\frac{d}{dx}\left(\frac{x}{2+x}\right) = \frac{(2+x) - x}{(2+x)^2} = \frac{2}{(2+x)^2}.$$

Thus

$$f'(x) = \frac{1}{2+x} - \frac{2}{(2+x)^2} = \frac{x}{(2+x)^2}.$$

The denominator is strictly positive throughout the domain.

Step 2: Test the sign on each sub-interval The only zero of the derivative in the domain is $x = 0$. For $-2 < x < 0$, its numerator x is negative, so $f'(x) < 0$. For $x > 0$, its numerator is positive, so $f'(x) > 0$. At $x = 0$, the derivative is zero; this point separates the decreasing and increasing intervals. No interval with $x \leq -2$ can be considered because it lies outside the given domain.

Step 3: Conclusion The function is $\boxed{\text{decreasing on } (-2, 0)}$ and $\boxed{\text{increasing on } (0, \infty)}$.

Quick Tip to Solve: After simplifying the derivative, ignore its positive square denominator and test only the sign of x.

24(a). A function $f : A \to B$ defined as $f(x) = 2x$ is both one-one and onto. If $A = \{1, 2, 3, 4\}$, then find the set B. [2 marks]

Solution:

Concept:

- The image of A under f is $f(A) = \{f(x) : x \in A\}$; every image value must belong to the codomain B.
- An onto function satisfies $f(A) = B$. A one-one function assigns different outputs to distinct inputs.

Step 1: Find the image of every element of A The domain consists of precisely four elements: $A = \{1, 2, 3, 4\}$. Substituting each into the given rule $f(x) = 2x$ gives

$$f(1) = 2, \quad f(2) = 4,$$
$$f(3) = 6, \quad f(4) = 8.$$

Therefore the complete image set is $f(A) = \{2, 4, 6, 8\}$. These four outputs are distinct, in agreement with the stated one-one property.

Step 2: Use the onto condition to determine the codomain Since $f : A \to B$ is a function, all four calculated outputs must be in B. Since it is also onto, every element of B must occur as the image of some element of A. Thus B cannot contain an additional element: no fifth input is available to produce it. Equally, none of the four displayed output values can be omitted. Together these two inclusions force $B = f(A)$; the one-one property is already verified by the distinct images.

Step 3: Conclusion Hence $\boxed{B = \{2, 4, 6, 8\}}$.

Quick Tip to Solve: For an onto map on a finite domain, list every image; the codomain is exactly that image set.

OR

24(b). Evaluate:

$$\sin^{-1}\left(\sin\frac{3\pi}{4}\right)+\cos^{-1}\left(\cos\frac{3\pi}{4}\right)+\tan^{-1}(1)$$ [2 marks]

Solution:

Concept:

- The principal-value range of $\sin^{-1}$ is $[-\pi/2,\pi/2]$, while that of $\cos^{-1}$ is $[0,\pi]$.
- The principal-value range of $\tan^{-1}$ is $(-\pi/2,\pi/2)$. An inverse trigonometric function returns the angle in its own range.

Step 1: Evaluate the sine inverse using its principal range The angle $3\pi/4$ lies outside the principal range of $\sin^{-1}$, so the inverse sine cannot simply cancel the sine. Using the supplementary-angle identity,

$$\sin\frac{3\pi}{4}=\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}.$$

Because $\pi/4$ belongs to $[-\pi/2,\pi/2]$, we obtain $\sin^{-1}(\sin(3\pi/4))=\pi/4$.

Step 2: Evaluate the remaining principal values and add Unlike the sine inverse case, $3\pi/4$ belongs to the principal range $[0,\pi]$ of $\cos^{-1}$. Therefore

$$\cos^{-1}\left(\cos\frac{3\pi}{4}\right)=\frac{3\pi}{4}.$$

Also, $\tan(\pi/4)=1$ and $\pi/4$ is in the principal range of $\tan^{-1}$, giving $\tan^{-1}(1)=\pi/4$. Adding the three values now gives

$$\frac{\pi}{4}+\frac{3\pi}{4}+\frac{\pi}{4}=\frac{5\pi}{4}.$$

Each term must be evaluated separately with its correct range.

Step 3: Conclusion The required value is $\boxed{\frac{5\pi}{4}}$.

Quick Tip to Solve: Check each inverse function range first: sine reflects the angle to pi/4, while cosine retains 3pi/4.

25. For two non-zero vectors $\vec{a}$ and $\vec{b}$, if $|\vec{a}-\vec{b}|=|\vec{a}+\vec{b}|$, then find the angle between $\vec{a}$ and $\vec{b}$. [2 marks]

Solution:

Concept:

- The squared magnitude satisfies $|\vec{v}|^2=\vec{v}\cdot\vec{v}$, so expand squared vector lengths using the distributive law.
- For nonzero vectors, $\vec{a}\cdot\vec{b}=|\vec{a}|\,|\vec{b}|\cos\theta$, where $0\le\theta\le\pi$ is their angle.

Step 1: Square the given equality and expand Both magnitudes are nonnegative, so squaring preserves

Quick Tip to Solve: Equal lengths of a vector sum and difference cancel the squared terms, forcing the dot product to zero.

the equality:

$$|\vec{a} - \vec{b}|^2 = |\vec{a} + \vec{b}|^2.$$

Using the dot product on each side yields

$$|\vec{a}|^2 + |\vec{b}|^2 - 2\vec{a} \cdot \vec{b} = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a} \cdot \vec{b}.$$

The cross term changes sign between the difference and the sum; the individual squared lengths remain unchanged.

Step 2: Deduce the angle from the dot product Cancelling the equal squared-length terms gives $-2\vec{a} \cdot \vec{b} = 2\vec{a} \cdot \vec{b}$, hence $4\vec{a} \cdot \vec{b} = 0$. Thus $\vec{a} \cdot \vec{b} = 0$. Since both vectors are nonzero, $|\vec{a}|\,|\vec{b}| > 0$, and we may divide the dot-product formula to obtain

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}|\,|\vec{b}|} = 0.$$

Within the angle range $[0, \pi]$, this identifies the perpendicular angle uniquely.

Step 3: Conclusion The vectors are perpendicular; the angle between them is $\boxed{\theta = \frac{\pi}{2} = 90^\circ}$.

Quick Tip to Solve: Equal lengths of a vector sum and difference cancel the squared terms, forcing the dot product to zero.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26(a). Find the general solution of the differential equation: $\dfrac{dx}{dy} = \dfrac{e^{x/y}\left(\dfrac{x}{y} - 1\right)}{1 + e^{x/y}}$. [3 marks]

Solution:

Concept:

- A homogeneous equation has its right-hand side expressed in terms of the ratio x/y. Treat y as the independent variable and put $v = x/y$, so $x = vy$ and $dx/dy = v + y\,dv/dy$.
- The logarithmic derivative of $v + e^v$ is $(1 + e^v)/(v + e^v)$. Integrating a separable equation introduces an arbitrary constant. The original equation requires $y \neq 0$.

Step 1: Make the homogeneous substitution Substitute $x = vy$ into the given equation. Then

$$v + y\frac{dv}{dy} = \frac{e^v(v-1)}{1+e^v}, \qquad y\frac{dv}{dy} = -\frac{v+e^v}{1+e^v}.$$

The cancellation in the numerator follows from $e^v(v-1) - v(1+e^v) = -e^v - v$.

Quick Tip to Solve: Use $x = vy$ because the derivative is dx/dy; the numerator after separation is exactly the derivative of $v + e^v$.

Step 2: Separate and integrate For $v + e^v \neq 0$, separation gives

$$\frac{1 + e^v}{v + e^v}\, dv = -\frac{dy}{y}.$$

Since the numerator is the derivative of the denominator,

$$\ln|v + e^v| = -\ln|y| + C_1.$$

Hence $y(v + e^v) = C$, where the arbitrary real constant absorbs the possible signs. Replacing v by x/y expresses the family in the original variables.

Step 3: Check the constant-ratio solution and the family Division by $v + e^v$ temporarily excludes a constant ratio satisfying $v + e^v = 0$. This ratio gives $x = vy$ and is included by $C = 0$ in the same family. As an independent check, differentiate $x + ye^{x/y} = C$ with respect to y:

$$(1 + e^{x/y})\frac{dx}{dy} + e^{x/y}\left(1 - \frac{x}{y}\right) = 0.$$

Rearrangement recovers the printed differential equation, so no extra family has been introduced.

Step 4: Conclusion The general solution is

$$\boxed{x + ye^{x/y} = C}, \qquad y \neq 0,$$

where C is an arbitrary real constant.

Quick Tip to Solve: Use $x = vy$ because the derivative is dx/dy; the numerator after separation is exactly the derivative of $v + e^v$.

OR

26(b). Find the particular solution of the differential equation $\frac{dy}{dx} + \cot x \cdot y = \cos^2 x$, given that when $x = \frac{\pi}{2}$, $y = 0$. [3 marks]

Solution:

Concept:

- For the linear differential equation $y' + P(x)y = Q(x)$, an integrating factor is IF $= e^{\int P(x)\,dx}$. Multiplying by it turns the left side into a product derivative.
- Here $P(x) = \cot x$ and $Q(x) = \cos^2 x$. Work on the interval $0 < x < \pi$, containing the given initial point, where $\sin x \neq 0$.

Step 1: Find and apply the integrating factor We have $\int \cot x\, dx = \ln|\sin x|$, so on the interval containing $\pi/2$ take IF $= \sin x$. Multiplying the equation by $\sin x$ gives

$$\sin x \frac{dy}{dx} + y\cos x = \sin x \cos^2 x.$$

Quick Tip to Solve: Multiply by sin x immediately: it produces the derivative of y sin x, and the initial condition makes the integration constant zero.

By the product rule this becomes $\dfrac{d}{dx}(y\sin x) = \sin x\cos^2 x$.

Step 2: Integrate and use the initial condition Put $u = \cos x$, so $du = -\sin x\,dx$. Thus

$$y\sin x = \int \sin x\cos^2 x\,dx = -\frac{\cos^3 x}{3} + C.$$

At $x = \pi/2$, $y = 0$, while $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$. Substitution therefore gives $C = 0$, fixing the member of the general family uniquely.

Step 3: Verify the selected solution The selected relation is $y\sin x = -\cos^3 x/3$. Its derivative is $y'\sin x + y\cos x = \sin x\cos^2 x$. Dividing by $\sin x \neq 0$ gives exactly $y' + y\cot x = \cos^2 x$. The relation also yields $y = 0$ at $x = \pi/2$, verifying both the differential equation and the prescribed value.

Step 4: Conclusion The particular solution is

$$\boxed{y = -\frac{\cos^3 x}{3\sin x}}, \qquad 0 < x < \pi.$$

Quick Tip to Solve: Multiply by sin x immediately: it produces the derivative of y sin x, and the initial condition makes the integration constant zero.

27. Solve the following linear programming problem graphically: Maximize $P = 100x + 5y$ subject to the constraints

$x + y \leq 300,$

$3x + y \leq 600,$

$y \leq x + 200,$

$x,\ y \geq 0.$ [3 marks]

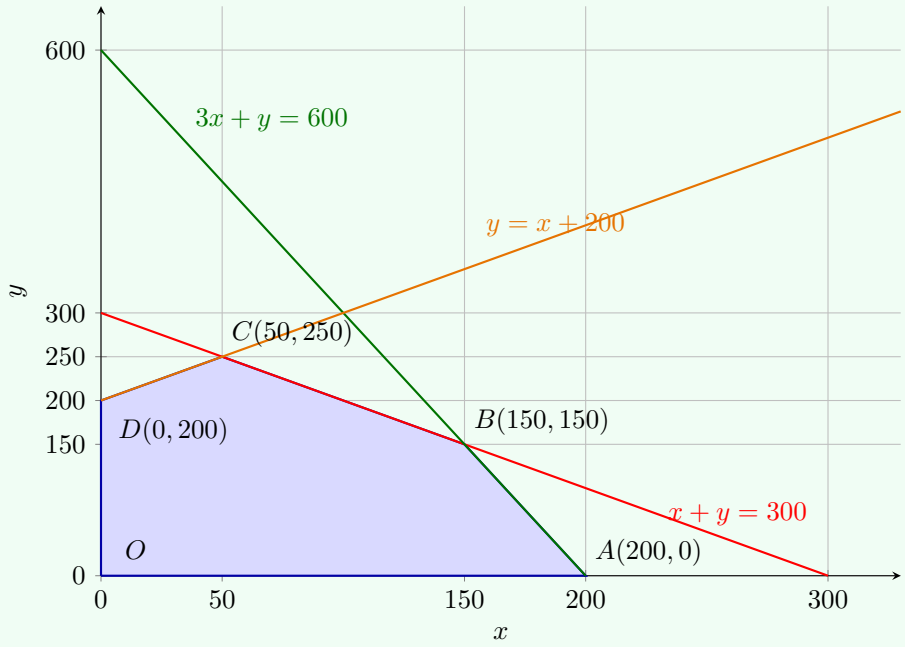
Solution:

Concept:

- The feasible region is the intersection of all constraint half-planes in the first quadrant, since $x, y \geq 0$. Every boundary line is included because the inequalities are non-strict.
- A linear objective $P = 100x + 5y$ attains its maximum at a vertex of a nonempty bounded feasible polygon. Compare its value at every vertex.

Step 1: Draw the boundaries and select the feasible side Draw $x + y = 300$ through $(300, 0), (0, 300)$, and $3x + y = 600$ through $(200, 0), (0, 600)$. Draw $y = x + 200$ through $(0, 200), (100, 300)$. The origin satisfies all three inequalities, so take the side containing it in each case and restrict to the first quadrant. The shaded common region in the graph has all its boundary segments included.

Quick Tip to Solve: Check every pairwise intersection against the third constraint before calling it a vertex; (100,300) is outside the feasible polygon.



Feasible polygon OABCD in the first quadrant; each boundary is included.

Step 2: Find all feasible vertices On the axes the limiting vertices are $O = (0, 0)$, $A = (200, 0)$, and $D = (0, 200)$. The intersection of $x + y = 300$ and $3x + y = 600$ gives $2x = 300$, hence $B = (150, 150)$. Intersecting $x + y = 300$ with $y = x + 200$ gives $C = (50, 250)$. The intersection of $3x + y = 600$ with $y = x + 200$ is $(100, 300)$; it violates $x + y \leq 300$ and is excluded. Thus the feasible polygon is $OABCD$.

Step 3: Evaluate the objective at the vertices Compute $P = 100x + 5y$ at each feasible vertex:

Vertex	x	y	P
O	0	0	0
A	200	0	20000
B	150	150	15750
C	50	250	6250
D	0	200	1000

The polygon is bounded and nonempty. Comparing all five values determines the greatest objective value; no omitted intersection is feasible.

Step 4: Conclusion The maximum value is

$$\boxed{P_{\max} = 20000}$$

attained at $\boxed{x = 200,\ y = 0}$.

Quick Tip to Solve: Check every pairwise intersection against the third constraint before calling it a vertex; (100,300) is outside the feasible polygon.

28(a). The probability distribution of a random variable X is given below:

X	1	2	3
$P(X)$	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k}{6}$

(i) Find the value of k. (ii) Find $P(1 \leq X < 3)$. (iii) Find $E(X)$, the mean of X. [3 marks]

Solution:

Concept:

- A probability distribution assigns non-negative probabilities to every possible value of the random variable, and their total must equal one.
- For a discrete random variable, the mean is the weighted sum $E(X) = \sum_x xP(X = x)$; each value is weighted by its own probability.

Step 1: Find the normalising constant The values $1, 2, 3$ are all the possible values of X, so their probabilities exhaust the sample space. Therefore,

$$\frac{k}{2} + \frac{k}{3} + \frac{k}{6} = 1.$$

Using the common denominator 6, this becomes $k(3 + 2 + 1)/6 = 1$, which determines $k = 1$. Substitution gives probabilities $1/2, 1/3, 1/6$, each non-negative and together summing to one.

Step 2: Identify the values in the stated event The inequality $1 \leq X < 3$ includes $X = 1$ and $X = 2$, but excludes $X = 3$. Since these two outcomes are mutually exclusive, their probabilities add:

$$P(1 \leq X < 3) = P(X = 1) + P(X = 2) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

The strict upper inequality is essential: including the last value would give the total probability instead.

Step 3: Calculate the expectation Multiply each possible value by the corresponding probability and add all three contributions:

$$\begin{aligned} E(X) &= 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{6}\right) \\ &= \frac{1}{2} + \frac{2}{3} + \frac{1}{2} = \frac{5}{3}. \end{aligned}$$

This is a probability-weighted average, rather than the unweighted average of the three displayed values. It lies between the smallest and largest possible values, as a mean must.

Step 4: Conclusion Thus (i) $\boxed{k = 1}$, (ii) $\boxed{P(1 \leq X < 3) = 5/6}$, and (iii) $\boxed{E(X) = 5/3}$.

Quick Tip to Solve: Normalise the probability row first, and check whether each endpoint of the requested interval is included before adding probabilities.

OR

28(b). A **and** B **are independent events such that** $P(A \cap \bar{B}) = \frac{1}{4}$ **and** $P(\bar{A} \cap B) = \frac{1}{6}$**. Find** $P(A)$ **and** $P(B)$**.** [3 marks]

Solution:

Concept:

- Independence gives $P(A \cap B) = P(A)P(B)$, and also $P(A \cap \bar{B}) = P(A)[1 - P(B)]$.
- Write $p = P(A)$ and $q = P(B)$, with $0 \le p, q \le 1$. All admissible roots of the probability equations must be retained.

Step 1: Translate the two given probabilities Since A and B are independent, the probability involving either complementary event factors as well. For example, subtracting $P(A \cap B) = pq$ from $P(A) = p$ gives $P(A \cap \bar{B}) = p - pq$. The two conditions therefore become

$$p(1 - q) = \frac{1}{4}, \qquad (1 - p)q = \frac{1}{6}.$$

Subtracting the second equation from the first cancels the common product pq, giving $p - q = 1/12$, or $p = q + 1/12$.

Step 2: Solve the resulting quadratic Insert $p = q + 1/12$ into $q(1 - p) = 1/6$. Then

$$\left(\frac{11}{12} - q\right) q = \frac{1}{6},$$
$$12q^2 - 11q + 2 = 0,$$
$$(3q - 2)(4q - 1) = 0.$$

Thus the two possible values are $q = 2/3$ and $q = 1/4$. Using $p = q + 1/12$, these give $p = 3/4$ and $p = 1/3$, respectively. Both pairs are within the probability interval, so neither can be discarded.

Step 3: Verify both probability pairs For $(p, q) = (3/4, 2/3)$, the products are $(3/4)(1/3) = 1/4$ and $(1/4)(2/3) = 1/6$. For $(p, q) = (1/3, 1/4)$, they are $(1/3)(3/4) = 1/4$ and $(2/3)(1/4) = 1/6$. Each pair satisfies both original conditions; the information supplied does not select only one of them.

Step 4: Conclusion $\boxed{(P(A), P(B)) = (3/4, 2/3) \text{ or } (1/3, 1/4)}$. Both pairs are valid.

Quick Tip to Solve: Subtract the two independence equations to eliminate the product, and retain every root that passes both original probability conditions.

29(a). Evaluate: $\int_0^{\pi/2} e^x \sin x \, dx$**.** [3 marks]

Solution:

Concept:

- Integration by parts gives $\int u \, dv = uv - \int v \, du$. Apply it twice to an exponential multiplied by a sine.

Quick Tip to Solve: After two integrations by parts the original integral returns; move it to the left and evaluate both endpoints, including the negative value at zero.

- A definite integral is evaluated by substituting the upper and lower limits into one antiderivative and subtracting the lower value.

Step 1: Apply integration by parts once First obtain an antiderivative, writing $I = \int e^x \sin x \, dx$. Choose $u = \sin x$ and $dv = e^x \, dx$, so $du = \cos x \, dx$ and $v = e^x$. Integration by parts gives

$$I = e^x \sin x - \int e^x \cos x \, dx.$$

Denote the remaining integral by J. The exponential stays unchanged under integration, while differentiating sine produces cosine, which allows a second application to return to the original integral.

Step 2: Apply integration by parts again and isolate the integral For $J = \int e^x \cos x \, dx$, choose $u = \cos x$ and $dv = e^x \, dx$. Since $du = -\sin x \, dx$,

$$J = e^x \cos x + I.$$

Substitution into the first equation gives $I = e^x \sin x - e^x \cos x - I$, up to an additive constant. Hence an antiderivative is

$$F(x) = \frac{e^x}{2}(\sin x - \cos x).$$

Indeed, differentiating this expression gives $F'(x) = e^x \sin x$, confirming the signs and factor $1/2$.

Step 3: Substitute both limits The integrand is continuous throughout $[0, \pi/2]$, so the fundamental theorem of calculus applies:

$$\begin{aligned} \int_0^{\pi/2} e^x \sin x \, dx &= F(\pi/2) - F(0) \\ &= \frac{e^{\pi/2}}{2}(1 - 0) - \frac{1}{2}(0 - 1) \\ &= \frac{e^{\pi/2} + 1}{2}. \end{aligned}$$

In particular, the lower-limit value is negative; subtracting it contributes a positive $1/2$.

Step 4: Conclusion $\boxed{\int_0^{\pi/2} e^x \sin x \, dx = \frac{e^{\pi/2} + 1}{2}}$.

Quick Tip to Solve: After two integrations by parts the original integral returns; move it to the left and evaluate both endpoints, including the negative value at zero.

OR

29(b). Find: $\int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$. [3 marks]

Solution:

Concept:

- The tangent subtraction identity is $\tan u - \tan v = \sin(u-v)/(\cos u \cos v)$, wherever both cosines are nonzero.
- Use $\int \tan(x-c)\,dx = -\ln|\cos(x-c)| + C$, with c constant. Antiderivatives are considered on intervals where the integrand is defined.

Step 1: Express the integrand as a difference of tangents Set $u = x-a$ and $v = x-b$. Their difference is $u - v = b - a$, so

$$\tan(x-a) - \tan(x-b) = \frac{\sin(b-a)}{\cos(x-a)\cos(x-b)}.$$

For $\sin(b-a) \neq 0$, divide this identity by the constant $\sin(b-a)$. This converts the product of shifted cosines in the denominator into two standard trigonometric integrals.

Step 2: Integrate and check the sign Integrating the two tangents separately gives

$$\begin{aligned}\int \frac{dx}{\cos(x-a)\cos(x-b)} &= \frac{-\ln|\cos(x-a)| + \ln|\cos(x-b)|}{\sin(b-a)} + C \\ &= \frac{1}{\sin(b-a)} \ln\left|\frac{\cos(x-b)}{\cos(x-a)}\right| + C.\end{aligned}$$

The derivative of the logarithm in the last line is $\tan(x-a) - \tan(x-b)$. The subtraction identity therefore recovers precisely the original integrand after division by $\sin(b-a)$. Absolute values make the logarithm real on each interval avoiding the zeros of either cosine.

Step 3: Include the exceptional constant choices If $b - a = n\pi$, where n is an integer, the preceding division is unavailable. Instead,

$$\cos(x-b) = (-1)^n \cos(x-a),$$

so the integrand is $(-1)^n \sec^2(x-a)$. Its integral is $(-1)^n \tan(x-a) + C$. In particular, when $a = b$, the answer reduces to $\tan(x-a) + C$. This covers the parameter values excluded from the logarithmic formula.

Step 4: Conclusion For $\sin(b-a) \neq 0$, the answer is $\boxed{\frac{1}{\sin(b-a)} \ln\left|\frac{\cos(x-b)}{\cos(x-a)}\right| + C}$. If $b - a = n\pi$, it is $\boxed{(-1)^n \tan(x-a) + C}$.

Quick Tip to Solve: Use the tangent subtraction identity with its order preserved; if the constant sine vanishes, simplify the product directly to a squared cosine.

30. Evaluate: $\int_0^{\pi/2} |\sin x - \cos x|\, dx$. [3 marks]

Solution:

Concept:

- For a real function $f(x)$, its absolute value equals $f(x)$ where it is nonnegative and $-f(x)$ where it is negative. Split the integral at each sign change before integrating.
- On $[0, \pi/2]$, sine and cosine are nonnegative. They are equal at $\pi/4$; cosine is larger before this point and sine is larger after it.

Step 1: Locate the sign change Let $I = \int_0^{\pi/2} |\sin x - \cos x|\, dx$. The equation $\sin x = \cos x$ gives $\tan x = 1$, with the unique solution $x = \pi/4$ in this interval. Therefore

$$I = \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx.$$

The integrand is zero at the splitting point, so including that endpoint in either piece does not affect the integral. The signs in these two expressions ensure both integrands are nonnegative.

Step 2: Evaluate the two pieces The respective antiderivatives are $\sin x + \cos x$ and $-\cos x - \sin x$. Substituting the limits gives

$$\begin{aligned} I &= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2} \\ &= (\sqrt{2} - 1) + (-1 + \sqrt{2}). \end{aligned}$$

Each contribution is positive. This also checks that removing the modulus over the entire interval without splitting would be wrong: the positive and negative signed contributions would cancel, whereas the absolute value adds their magnitudes.

Step 3: Conclusion The required value is $\boxed{2(\sqrt{2} - 1)}$.

Quick Tip to Solve: Split at $\pi/4$, where sine equals cosine; the two pieces are equal by reflection about that point.

31. Find: $\int \frac{1}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+2)}\, dx$. [3 marks]

Solution:

Concept:

- Use $t = \sqrt{x}$ to convert the square-root expression into a rational function. For the real integrand, $x > 0$, so $t > 0$, and $dx = 2t\, dt$.
- Partial fractions separate a product of distinct linear factors: $1/[(t+1)(t+2)] = 1/(t+1) - 1/(t+2)$. The antiderivative of $1/u$ is $\ln|u|$.

Step 1: Remove the square root by substitution Let $t = \sqrt{x}$, so $x = t^2$ and $dx = 2t\, dt$. The factor t

Quick Tip to Solve: With $t = \sqrt{x}$, retain the factor 2 from $dx = 2t\, dt$; the partial fractions then differ by a minus sign.

from the differential cancels the factor $\sqrt{x} = t$ in the denominator:

$$\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)(\sqrt{x}+2)} = 2\int \frac{dt}{(t+1)(t+2)}.$$

This cancellation is valid on the domain $x > 0$, where $t \neq 0$. The original expression is undefined at zero.

Step 2: Decompose and integrate Writing $1/[(t+1)(t+2)] = A/(t+1) + B/(t+2)$, multiplication by the denominator gives $1 = A(t+2) + B(t+1)$. Comparing coefficients gives $A + B = 0$ and $2A + B = 1$, hence $A = 1$, $B = -1$. Thus

$$\begin{aligned} 2\int \frac{dt}{(t+1)(t+2)} &= 2\int \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt \\ &= 2\ln|t+1| - 2\ln|t+2| + C. \end{aligned}$$

Substitute $t = \sqrt{x}$. Both logarithm arguments are positive on the real domain. Differentiating the logarithmic difference gives $(1/\sqrt{x})[1/(\sqrt{x}+1) - 1/(\sqrt{x}+2)]$, exactly the original integrand.

Step 3: Conclusion The required antiderivative is $\boxed{2\ln\left|\frac{\sqrt{x}+1}{\sqrt{x}+2}\right| + C}$, for $x > 0$.

Quick Tip to Solve: With $t = \sqrt{x}$, retain the factor 2 from $dx = 2t\,dt$; the partial fractions then differ by a minus sign.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32(a). Find the vector and the Cartesian equations of a line passing through the point $(1, 2, -4)$ and parallel to the line joining the points $A(3, 3, -5)$ and $B(1, 0, -11)$. Hence, find the distance between the two lines. [5 marks]

Solution:

Concept:

- A line through position vector $\vec{p}$ with nonzero direction vector $\vec{d}$ has vector equation $\vec{r} = \vec{p} + \lambda\vec{d}$. Its Cartesian equations follow by equating the three expressions for λ.
- For two parallel lines with common direction $\vec{d}$, their perpendicular distance is $D = |\vec{v} \times \vec{d}|/|\vec{d}|$, where $\vec{v}$ joins a point on one line to a point on the other. This gives the perpendicular component of their separation.

Step 1: Determine the common direction Let $P = (1, 2, -4)$. The line through $A = (3, 3, -5)$ and $B = (1, 0, -11)$ has direction

$$\overrightarrow{AB} = (1-3, 0-3, -11+5) = (-2, -3, -6).$$

We may reverse the direction without changing the line, so take $\vec{d} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. Its magnitude is

Quick Tip to Solve: For parallel lines use a joining vector and the cross-product distance formula; the skew-line triple-product formula has a zero denominator here.

$\sqrt{4+9+36} = 7$. Since the required line is parallel to AB, this same nonzero direction vector applies to it.

Step 2: Write the vector and Cartesian equations Using P and $\vec{d}$, the vector equation is

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}), \qquad \lambda \in \mathbb{R}.$$

The component equations are $x = 1 + 2\lambda$, $y = 2 + 3\lambda$, and $z = -4 + 6\lambda$. Eliminating the parameter yields

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}.$$

At $\lambda = 0$ the point is P, and the coefficients of λ confirm parallelism to the given line. For reference, the original line may be written as $\vec{r} = (3, 3, -5) + \mu(2, 3, 6)$.

Step 3: Find the perpendicular distance between the lines A joining vector from P on the new line to A on the original line is $\vec{v} = (2, 1, -1)$. Its cross product with the common direction is

$$\begin{aligned}\vec{v} \times \vec{d} &= (1 \cdot 6 - (-1) \cdot 3)\hat{i} + ((-1) \cdot 2 - 2 \cdot 6)\hat{j} + (2 \cdot 3 - 1 \cdot 2)\hat{k} \\ &= 9\hat{i} - 14\hat{j} + 4\hat{k}.\end{aligned}$$

Therefore $|\vec{v} \times \vec{d}| = \sqrt{81 + 196 + 16} = \sqrt{293}$, and $D = \sqrt{293}/7$. As a check, $|\vec{v}|^2 = 6$ and $\vec{v} \cdot \vec{d} = 1$, so its perpendicular component has squared length $6 - 1/49 = 293/49$, giving the same distance.

Step 4: Conclusion The vector equation is $\boxed{\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})}$. The Cartesian equation is $\boxed{\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}}$, and the distance is $\boxed{\sqrt{293}/7 \text{ units}}$.

Quick Tip to Solve: For parallel lines use a joining vector and the cross-product distance formula; the skew-line triple-product formula has a zero denominator here.

OR

32(b). Find the equations of the line passing through the points $A(1, 2, 3)$ and $B(3, 5, 9)$. Hence, find the coordinates of the points on this line which are at a distance of 14 units from point B. [5 marks]

Solution:

Concept:

- A line through points with position vectors $\vec{a}$ and $\vec{b}$ has equation $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$, where t is real. Its direction ratios are the coordinate differences.
- A point $\vec{r} = \vec{b} + s\vec{d}$ is at distance $|s|\,|\vec{d}|$ from $\vec{b}$. A prescribed positive distance gives points in both directions along the line.

Step 1: Find the direction vector Subtract the coordinates of $A(1, 2, 3)$ from those of $B(3, 5, 9)$:

$$\overrightarrow{AB} = (3-1)\hat{i} + (5-2)\hat{j} + (9-3)\hat{k} = 2\hat{i} + 3\hat{j} + 6\hat{k}.$$

Quick Tip to Solve: Start the distance parameter at B: its absolute value times the direction-vector length 7 must equal 14. Keep both signs to obtain the two points.

The direction ratios are therefore $2, 3, 6$. Its magnitude is $\sqrt{2^2+3^2+6^2} = \sqrt{49} = 7$. Since the points are distinct, this vector is nonzero and determines the required line's direction uniquely.

Step 2: Obtain both forms of the equation Take A as the fixed point and the above vector as direction. Then

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + 6\hat{k}), \quad t \in \mathbb{R}.$$

Equating the coefficients gives $x = 1 + 2t$, $y = 2 + 3t$ and $z = 3 + 6t$. Eliminating the common parameter gives

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}.$$

Here $t = 0$ produces A, and $t = 1$ produces B, checking that both given points lie on the line. None of the direction ratios is zero, so all three fractions are valid.

Step 3: Determine points at the specified distance Measure a fresh parameter from B. A general point on the same line is $P = (3 + 2s, 5 + 3s, 9 + 6s)$, with $s \in \mathbb{R}$. The displacement from B is $(2s, 3s, 6s)$, hence

$$PB = \sqrt{(2s)^2 + (3s)^2 + (6s)^2} = \sqrt{49s^2} = 7|s|.$$

The required distance is 14, so $7|s| = 14$, giving $s = 2$ or $s = -2$. For $s = 2$, the coordinates are $(3+4, 5+6, 9+12) = (7, 11, 21)$. For $s = -2$, they are $(3-4, 5-6, 9-12) = (-1, -1, -3)$. As a check, the displacements from B are $(4, 6, 12)$ and $(-4, -6, -12)$. Both have magnitude $\sqrt{16+36+144} = 14$; they point in opposite directions along the same line. Thus both satisfy the distance condition, and no further points do.

Step 4: Conclusion The vector equation is $\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + t(2\hat{i} + 3\hat{j} + 6\hat{k})$. The Cartesian equation is $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}$. The required points are $\boxed{(7, 11, 21) \text{ and } (-1, -1, -3)}$.

Quick Tip to Solve: Start the distance parameter at B: its absolute value times the direction-vector length 7 must equal 14. Keep both signs to obtain the two points.

33. Find the area of the region $\{(x, y) : x^2 + y^2 \leq 1 \leq x + y\}$, using integration. [5 marks]

Solution:

Concept:

- The inequalities $x^2 + y^2 \leq 1$ and $x + y \geq 1$ select the part inside the unit circle and on or above the line $y = 1 - x$.
- For vertical strips between upper curve $y = u(x)$ and lower curve $y = l(x)$, the area is $\int_a^b [u(x) - l(x)]\, dx$, where a, b are the horizontal limits of the region.
- The trigonometric substitution $x = \sin\theta$ evaluates $\int \sqrt{1 - x^2}\, dx$ on $0 \leq x \leq 1$, using $\cos^2\theta = (1 + \cos 2\theta)/2$.

Step 1: Find the boundary intersections The chained inequality requires both $x^2 + y^2 \leq 1$ and $1 \leq x + y$. The bounding line is $y = 1 - x$. Substituting in the circle gives

$$x^2 + (1 - x)^2 = 1 \quad \Longrightarrow \quad 2x(x - 1) = 0.$$

Quick Tip to Solve: Read the chained inequality as two simultaneous conditions, then integrate the upper semicircle minus the line from their intersections x=0 to x=1.

Thus the boundaries meet at $(0,1)$ and $(1,0)$. Every admissible point has $-1 \le x \le 1$ and $-1 \le y \le 1$ by the circle inequality. If $x < 0$, then $x + y < 1$, because $y \le 1$; similarly $y < 0$ is impossible. Hence the required region lies in the first quadrant, between these intersections.

Step 2: Set up the area integral For $0 \le x \le 1$, the upper circular boundary is $y = \sqrt{1 - x^2}$ and the lower straight boundary is $y = 1 - x$. Both are nonnegative. Moreover,

$$1 - x^2 - (1 - x)^2 = 2x(1 - x) \ge 0,$$

so the circular boundary is indeed at least as high as the line throughout the interval. A vertical strip has height $\sqrt{1 - x^2} - (1 - x)$, and therefore

$$A = \int_0^1 \left[\sqrt{1 - x^2} - (1 - x)\right] dx.$$

This integral covers exactly the circular segment satisfying both inequalities; no second region or symmetry multiplier is needed.

Step 3: Evaluate the two integrals In the first integral put $x = \sin\theta$. The limits $x = 0, 1$ become $\theta = 0, \pi/2$, and $dx = \cos\theta\, d\theta$. Since $\cos\theta \ge 0$ on this interval, $\sqrt{1 - x^2} = \cos\theta$. Consequently,

$$\int_0^1 \sqrt{1 - x^2}\, dx = \int_0^{\pi/2} \cos^2\theta\, d\theta$$
$$= \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{4}.$$

The straight-line integral is

$$\int_0^1 (1 - x)\, dx = \left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

Subtracting these values gives $A = \pi/4 - 1/2$. The value is positive, as expected because the circular arc lies above the chord except at its endpoints.

Step 4: Conclusion The required area is $\boxed{\dfrac{\pi - 2}{4} \text{ square units}}$.

Quick Tip to Solve: Read the chained inequality as two simultaneous conditions, then integrate the upper semicircle minus the line from their intersections x=0 to x=1.

34. A relation R is defined on a set of real numbers $\mathbb{R}$ as $R = \{(x, y) : x \cdot y$ is an irrational number$\}$. Check whether R is reflexive, symmetric and transitive or not. [5 marks]

Solution:

Concept:

- Reflexivity requires $(x, x) \in R$ for every real x, so every square x^2 would have to be irrational.
- Symmetry requires $(x, y) \in R \Rightarrow (y, x) \in R$. Transitivity requires $(x, y), (y, z) \in R \Rightarrow (x, z) \in R$.
- One counterexample disproves a universal property, while a proof must apply to arbitrary elements

Quick Tip to Solve: Use 1 to test reflexivity and the chain $1, \sqrt{2}, 1$ to test transitivity; commutativity proves symmetry.

satisfying its hypotheses.

Step 1: Check reflexivity Take $x = 1 \in \mathbb{R}$. The product $1 \cdot 1 = 1$ is rational. Hence $(1, 1) \notin R$, contradicting the requirement that every real number be related to itself. Thus R is not reflexive. Some real numbers do have irrational squares, but reflexivity requires this for every member of the underlying set. For example, the fact that $(\sqrt[4]{2}, \sqrt[4]{2}) \in R$ cannot compensate for the missing pair $(1, 1)$.

Step 2: Check symmetry Let $x, y \in \mathbb{R}$ be arbitrary and suppose $(x, y) \in R$. By definition, xy is irrational. Since real multiplication is commutative, $yx = xy$, so yx is the same irrational number. Therefore $(y, x) \in R$. This proves the required implication for every pair belonging to R, so the relation is symmetric. If a pair has rational product, neither that pair nor its reverse belongs to R; such pairs do not contradict symmetry.

Step 3: Check transitivity Choose $x = 1$, $y = \sqrt{2}$, and $z = 1$. All three numbers lie in $\mathbb{R}$, and their products are

$$xy = \sqrt{2}, \qquad yz = \sqrt{2}, \qquad xz = 1.$$

The first two products are irrational, giving $(1, \sqrt{2}) \in R$ and $(\sqrt{2}, 1) \in R$. But the third product is rational, so $(1, 1) \notin R$. Thus both hypotheses of transitivity are satisfied while the necessary conclusion fails. The choice $x = z$ is permitted because the definition does not require the three elements to be distinct. This counterexample proves that R is not transitive.

Step 4: Conclusion Therefore R is $\boxed{\text{not reflexive, symmetric, and not transitive}}$.

Quick Tip to Solve: Use 1 to test reflexivity and the chain $1, \sqrt{2}, 1$ to test transitivity; commutativity proves symmetry.

35(a). **If** $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ **and** $B^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$**, find** $(AB)^{-1}$**.** [5 marks]

Solution:

Concept:

- For invertible square matrices, $(AB)^{-1} = B^{-1}A^{-1}$; the factors occur in reverse order.
- The inverse is $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$, where the adjugate is the transpose of the cofactor matrix.

Step 1: Calculate the determinant of A Expanding along the first row gives

$$\begin{aligned} |A| &= 1(3 \cdot 1 - 0 \cdot (-2)) - 2((-1) \cdot 1 - 0 \cdot 0) \\ &\quad + (-2)((-1)(-2) - 3 \cdot 0) \\ &= 3 + 2 - 4 = 1. \end{aligned}$$

Since this determinant is nonzero, A is invertible and the adjugate formula applies without division by zero.

Step 2: Find the cofactors and inverse Delete the relevant row and column for each minor and apply

Quick Tip to Solve: Find only A^{-1}, then premultiply it by the supplied B^{-1}; reversing the factors is compulsory.

the alternating cofactor signs. The nine cofactors are

$$\begin{aligned} C_{11} &= 3, & C_{12} &= 1, & C_{13} &= 2, \\ C_{21} &= -(2-4) = 2, & C_{22} &= 1, & C_{23} &= -(-2) = 2, \\ C_{31} &= 6, & C_{32} &= -(0-2) = 2, & C_{33} &= 3+2 = 5. \end{aligned}$$

Thus the cofactor matrix and its transpose give

$$C = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{pmatrix}, \qquad A^{-1} = \frac{C^T}{1} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

The transpose is essential: using the cofactor matrix directly would not produce the inverse.

Step 3: Multiply in the required order The given matrix is already B^{-1}, so there is no need to calculate B. Using the reverse-order identity,

$$(AB)^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

Row-by-column multiplication gives the successive rows as

$$\begin{aligned} &(9-1+2,\ 6-1+2,\ 18-2+5) = (10, 7, 21), \\ &(-45+6-10,\ -30+6-10,\ -90+12-25) = (-49, -34, -103), \\ &(15-2+4,\ 10-2+4,\ 30-4+10) = (17, 12, 36). \end{aligned}$$

As an exact check, multiplying this resulting matrix on the right by A returns the given B^{-1}; this agrees with $B^{-1}A^{-1}A = B^{-1}$.

Step 4: Conclusion Hence $\boxed{(AB)^{-1} = \begin{pmatrix} 10 & 7 & 21 \\ -49 & -34 & -103 \\ 17 & 12 & 36 \end{pmatrix}}$.

Quick Tip to Solve: Find only A^{-1}, then premultiply it by the supplied B^{-1}; reversing the factors is compulsory.

OR

35(b). Solve the following system of equations by matrix method:

$x + 2y + 3z = 6$

$2x - y + z = 2$

$3x + 2y - 2z = 3$ [5 marks]

Solution:

Concept:

- A system of linear equations is written as $AX = B$, where A is its coefficient matrix, X the column of unknowns and B the constant column.
- If $|A| \neq 0$, the unique solution is $X = A^{-1}B$, with $A^{-1} = \text{adj}(A)/|A|$. The adjugate is the transpose of the cofactor matrix.

Step 1: Write the matrix equation Keeping the order of unknowns as x, y, z in every row, the given equations become

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}}_{B}.$$

The signs in the second and third rows are retained from the printed equations; this is necessary before forming the inverse.

Step 2: Check invertibility Expand along the first row:

$$\begin{aligned} |A| &= 1\{(-1)(-2) - (1)(2)\} \\ &\quad - 2\{(2)(-2) - (1)(3)\} + 3\{(2)(2) - (-1)(3)\} \\ &= 0 - 2(-7) + 3(7) = 35. \end{aligned}$$

Since the determinant is nonzero, the inverse exists and the matrix equation has exactly one solution.

Step 3: Find cofactors and the inverse Deleting the corresponding row and column and applying the sign pattern gives

$$C = \begin{pmatrix} 0 & 7 & 7 \\ 10 & -11 & 4 \\ 5 & 5 & -5 \end{pmatrix}.$$

For example, $C_{21} = -[2(-2) - 3(2)] = 10$ and $C_{22} = 1(-2) - 3(3) = -11$. Transpose the entire cofactor matrix, rather than using it directly:

$$\text{adj}(A) = C^T = \begin{pmatrix} 0 & 10 & 5 \\ 7 & -11 & 5 \\ 7 & 4 & -5 \end{pmatrix}, \qquad A^{-1} = \frac{1}{35}\begin{pmatrix} 0 & 10 & 5 \\ 7 & -11 & 5 \\ 7 & 4 & -5 \end{pmatrix}.$$

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the constants column; verify the resulting values in all three original equations.

Step 4: Multiply and check the solution Premultiply $AX = B$ by A^{-1}:

$$X = \frac{1}{35}\begin{pmatrix} 0 & 10 & 5 \\ 7 & -11 & 5 \\ 7 & 4 & -5 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$$
$$= \frac{1}{35}\begin{pmatrix} 0+20+15 \\ 42-22+15 \\ 42+8-15 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Substitution verifies all three equations: $1+2+3=6$, $2-1+1=2$, and $3+2-2=3$. Thus the computed vector satisfies the original system as well as the matrix equation.

Step 5: Conclusion The unique solution is $\boxed{x=1,\ y=1,\ z=1}$.

Quick Tip to Solve: Transpose the cofactor matrix before multiplying by the constants column; verify the resulting values in all three original equations.

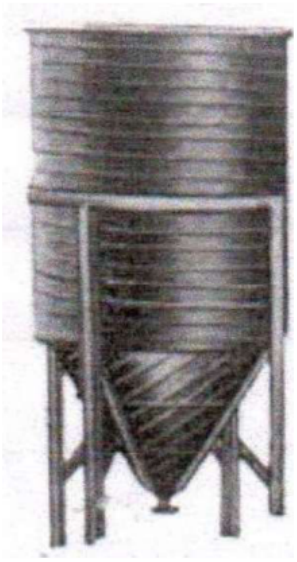
Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. A tank, as shown in the figure below, formed using a combination of a cylinder and a cone, offers better drainage as compared to a flat bottomed tank.

A tap is connected to such a tank whose conical part is full of water. Water is dripping out from a tap at the bottom at the uniform rate of 2 cm^3/s. The semi-vertical angle of the conical tank is $45°$.

On the basis of given information, answer the following questions. [4 marks]



36(i). Find the volume of water in the tank in terms of its radius r. [1 mark]

Solution:

Concept:

- For water occupying a cone of radius r and vertical height h, $V = \frac{1}{3}\pi r^2 h$.
- The semi-vertical angle is $45°$, so $\tan 45° = r/h = 1$ and hence $h = r$ throughout drainage.

Step 1: Relate radius and height The angle is measured between the cone axis and its sloping side. The right triangle in the axial section therefore gives $r/h = \tan 45° = 1$, so the water depth equals its surface radius. This relation remains valid for every smaller cone of water as the tank drains.

Step 2: Express the water volume Substitute $h = r$ into the conical volume formula:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (r).$$

Only the conical water is being considered, since the stem states that the conical part is full; the cylindrical portion does not contribute an additional water volume here.

Step 3: Conclusion $\boxed{V = \frac{\pi r^3}{3}}$, in cm^3 when r is in cm.

Quick Tip to Solve: Use the semi-vertical angle to replace the height by the radius before applying the cone volume formula.

36(ii). Find rate of change of radius at an instant when $r = 2\sqrt{2}$ cm. [1 mark]

Solution:

Concept:

- For water occupying a cone of radius r and vertical height h, $V = \frac{1}{3}\pi r^2 h$.
- The semi-vertical angle is $45°$, so $\tan 45° = r/h = 1$ and hence $h = r$ throughout drainage.
- Outflow makes the volume decrease, so $dV/dt = -2\,\text{cm}^3/\text{s}$; rates follow by differentiation with respect to time.

Step 1: Differentiate the volume relation Using $h = r$, the volume is $V = \pi r^3/3$. Both volume and radius vary with time, so the chain rule gives

$$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt}.$$

The negative volume rate represents water leaving the tank, rather than entering it.

Step 2: Evaluate at the stated radius At $r = 2\sqrt{2}\,\text{cm}$, $r^2 = 8\,\text{cm}^2$. Substitute into the differentiated relation:

$$-2 = 8\pi \frac{dr}{dt}, \qquad \frac{dr}{dt} = -\frac{2}{8\pi}.$$

The negative sign is consistent with the circular water surface becoming smaller as the water level falls.

Step 3: Conclusion $\boxed{\frac{dr}{dt} = -\frac{1}{4\pi}\ \text{cm/s}}$.

Quick Tip to Solve: Enter the outflow as a negative volume rate, and square the given radius before dividing by the area factor.

36(iii)(a). Find the rate at which the wet surface of the conical tank is decreasing at an instant when radius $r = 2\sqrt{2}$ cm. [2 marks]

Solution:

Concept:

- For water occupying a cone of radius r and vertical height h, $V = \frac{1}{3}\pi r^2 h$.
- The semi-vertical angle is $45°$, so $\tan 45° = r/h = 1$ and hence $h = r$ throughout drainage.
- The wet surface is the curved conical wall in contact with water: $S = \pi r \ell$, where $\ell = \sqrt{r^2 + h^2}$ is the slant height.

Step 1: Express wet surface area using radius Since $h = r$, the slant height of the water cone is $\ell = \sqrt{r^2 + r^2} = \sqrt{2}r$. Thus

$$S = \pi r \ell = \pi\sqrt{2} r^2.$$

The circular top surface is a free water surface; it is not part of the wet wall of the tank, so no additional πr^2 term is included.

Quick Tip to Solve: Use curved surface area only, and distinguish the negative signed derivative from the positive rate of decrease.

Step 2: Obtain the instantaneous radius rate The volume relation $V = \pi r^3/3$ gives

$$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt}.$$

Water flows out at $2\,\text{cm}^3/\text{s}$, so $dV/dt = -2$. At $r = 2\sqrt{2}$, this yields $dr/dt = -1/(4\pi)\,\text{cm/s}$.

Step 3: Differentiate the wet surface area Apply the chain rule to the area expression and evaluate at the same instant:

$$\frac{dS}{dt} = 2\pi\sqrt{2}r\frac{dr}{dt}$$
$$= 2\pi\sqrt{2}(2\sqrt{2})\left(-\frac{1}{4\pi}\right) = -2\,\text{cm}^2/\text{s}.$$

The signed derivative is negative, as expected for a decreasing wetted area.

Step 4: Conclusion The wet surface decreases at $\boxed{2\,\text{cm}^2/\text{s}}$, with signed rate $dS/dt = -2\,\text{cm}^2/\text{s}$.

Quick Tip to Solve: Use curved surface area only, and distinguish the negative signed derivative from the positive rate of decrease.

OR

36(iii)(b). Find the rate of change of height h at an instant when slant height is 4 cm. [2 marks]

Solution:

Concept:

- For water occupying a cone of radius r and vertical height h, $V = \frac{1}{3}\pi r^2 h$.
- The semi-vertical angle is $45°$, so $\tan 45° = r/h = 1$ and hence $h = r$ throughout drainage.
- The slant height satisfies $\ell^2 = r^2 + h^2$; using $r = h$ gives $\ell = \sqrt{2}h$.

Step 1: Find the height at the specified instant The semi-vertical angle gives $r = h$. Therefore the slant height of the water cone is $\ell = \sqrt{r^2 + h^2} = \sqrt{2}h$. At the instant for which $\ell = 4\,\text{cm}$,

$$h = \frac{4}{\sqrt{2}} = 2\sqrt{2}\,\text{cm}, \qquad h^2 = 8\,\text{cm}^2.$$

The slant height is used to locate this instant; it is not a fixed value during drainage.

Step 2: Express volume in terms of height and differentiate Because $r = h$, the cone volume becomes

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3.$$

Differentiating with respect to time gives $dV/dt = \pi h^2(dh/dt)$. The tap removes water, hence $dV/dt = -2\,\text{cm}^3/\text{s}$.

Step 3: Evaluate the height rate Substitute the instantaneous value $h^2 = 8$ into the rate equation:

$$-2 = 8\pi\frac{dh}{dt}, \qquad \frac{dh}{dt} = -\frac{1}{4\pi}\,\text{cm/s}.$$

Quick Tip to Solve: Convert the slant height to vertical height using the angle before evaluating the differentiated volume equation.

The negative derivative shows that the vertical depth of water falls as the cone empties, in agreement with the physical situation.

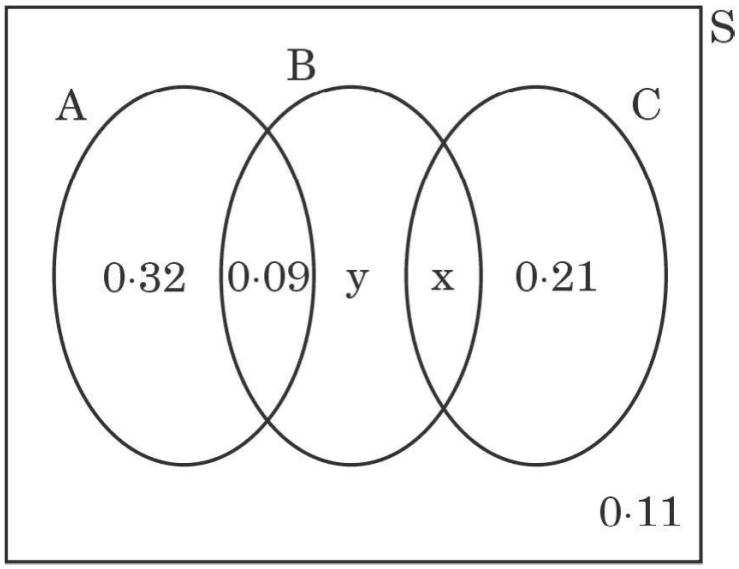
Step 4: Conclusion $\boxed{\frac{dh}{dt} = -\frac{1}{4\pi}\ \text{cm/s}}$.

Quick Tip to Solve: Convert the slant height to vertical height using the angle before evaluating the differentiated volume equation.

37. There are different types of Yoga which involve the usage of different poses of Yoga Asanas, Meditation and Pranayam as shown in the figure below:

The Venn diagram below represents the probabilities of three different types of Yoga, A, B and C performed by the people of a society. Further, it is given that probability of a member performing type C Yoga is 0.44.

On the basis of the above information, answer the following questions: [4 marks]



37(i). Find the value of x. [1 mark]

Solution:

Concept:

- The probability of an event is the sum of the probabilities of the disjoint regions inside its circle.
- The overlap $B \cap C$ has probability x, whereas the region belonging only to C has probability 0.21.

Step 1: Read the regions inside C The circle C consists of exactly two disjoint regions: its overlap with B and its exclusive region. The circle A does not intersect C, so no region of A is added. Thus

$$P(C) = x + 0.21.$$

Step 2: Use the given probability The stem gives $P(C) = 0.44$. Therefore

$$x + 0.21 = 0.44, \qquad x = 0.44 - 0.21 = 0.23.$$

Quick Tip to Solve: For x, add only the two regions within circle C; the outside value 0.11 is excluded.

This value is non-negative, as required for a probability.

Step 3: Conclusion $\boxed{x = 0.23}$.

Quick Tip to Solve: For x, add only the two regions within circle C; the outside value 0.11 is excluded.

37(ii). Find the value of y. [1 mark]

Solution:

Concept:

- The probabilities of all mutually exclusive regions in the sample space sum to 1.
- The outside region is part of the sample space even though its members practise none of A, B, C.

Step 1: Find the overlap with C From the two regions inside C,

$$x + 0.21 = 0.44 \quad \Rightarrow \quad x = 0.23.$$

This is the probability of the overlap $B \cap C$.

Step 2: Use the total probability The six disjoint regions are $0.32, 0.09, y, x, 0.21$, and the outside region 0.11. Hence

$$0.32 + 0.09 + y + 0.23 + 0.21 + 0.11 = 1.$$

The known regions sum to 0.96, leaving $y = 1 - 0.96 = 0.04$.

Step 3: Conclusion $\boxed{y = 0.04}$.

Quick Tip to Solve: Include the outside region 0.11 when summing the entire sample space to find y.

37(iii)(a). Find $P\left(\frac{C}{B}\right)$. [2 marks]

Solution:

Concept:

- For $P(B) > 0$, conditional probability is $P(C \mid B) = \dfrac{P(C \cap B)}{P(B)}$.
- When conditioning on B, the denominator includes every region inside B, including its overlaps with other events.

Step 1: Determine the missing region probabilities The circle C gives $x + 0.21 = 0.44$, so $x = 0.23$. Summing the disjoint regions of the whole sample space gives

$$y = 1 - (0.32 + 0.09 + 0.23 + 0.21 + 0.11) = 0.04.$$

Thus both unknown regions have been fixed before evaluating the conditional probability.

Step 2: Find the numerator and denominator The intersection $C \cap B$ is precisely the region labelled x, hence $P(C \cap B) = 0.23$. The circle B contains the left overlap 0.09, its exclusive region y, and the

Quick Tip to Solve: For C given B, divide the B-C overlap by the sum of all three regions inside B.

right overlap x. Therefore

$$P(B) = 0.09 + 0.04 + 0.23 = 0.36.$$

Step 3: Apply conditional probability The printed notation $P(C/B)$ means the probability of C given B. Consequently,

$$P(C \mid B) = \frac{0.23}{0.36} = \frac{23}{36}.$$

The denominator is positive and the result lies between 0 and 1, as a probability must.

Step 4: Conclusion $\boxed{P(C/B) = \frac{23}{36}}$.

Quick Tip to Solve: For C given B, divide the B-C overlap by the sum of all three regions inside B.

OR

37(iii)(b). Find the probability that a randomly selected person of the society does Yoga of type A or B but not C. [2 marks]

Solution:

Concept:

- The word "or" denotes the union $A \cup B$; "but not C" requires intersecting that union with C'.
- Add disjoint Venn regions once each, so that people who perform both A and B are not counted twice.

Step 1: Determine the exclusive B region First, $P(C) = x + 0.21 = 0.44$ gives $x = 0.23$. The total probability of the sample space is 1, including the outside region 0.11. Therefore

$$y = 1 - (0.32 + 0.09 + 0.23 + 0.21 + 0.11) = 0.04.$$

Step 2: Select the required regions The event is $(A \cup B) \cap C'$. It contains the region only in A, the overlap $A \cap B$, and the region only in B. Their probabilities are respectively $0.32, 0.09$, and $y = 0.04$. The $B \cap C$ region is excluded because those people perform C, while the outside region is excluded because it belongs to neither A nor B.

Step 3: Add the selected probabilities These three selected regions are disjoint, so the probability of their union is their sum:

$$P\big((A \cup B) \cap C'\big) = 0.32 + 0.09 + 0.04 = 0.45.$$

Including the overlap 0.09 once correctly handles the inclusive meaning of "or".

Step 4: Conclusion The required probability is $\boxed{0.45}$.

Quick Tip to Solve: Cross out every region in C first, then add the remaining regions belonging to A or B once.

38. The equation of the path traced by a roller-coaster is given by the polynomial $f(x) = a(x+9)(x+1)(x-3)$. If the roller-coaster crosses y-axis at a point $(0,-1)$, answer the following. [4 marks]

38(i). Find the value of a. [2 marks]

Solution:

Concept:

- A point (x_0, y_0) on the graph $y = f(x)$ satisfies $f(x_0) = y_0$.
- At a crossing of the y-axis, the horizontal coordinate is $x = 0$; the given point therefore supplies $f(0) = -1$.

Step 1: Translate the crossing into an equation The roller-coaster path is

$$f(x) = a(x+9)(x+1)(x-3).$$

Since its graph passes through $(0,-1)$, we substitute $x = 0$ and use the corresponding ordinate -1. This crossing is not a zero of the polynomial: its ordinate is nonzero.

Step 2: Substitute and solve for a At $x = 0$, the three factors are $9, 1$, and -3. Thus

$$f(0) = a(9)(1)(-3) = -27a.$$

Equating this to the given ordinate gives

$$-27a = -1 \quad \Rightarrow \quad a = \frac{1}{27}.$$

The negative factor -3 is essential for the sign.

Step 3: Check the value at the stated point With $a = 1/27$, substitution yields

$$f(0) = \frac{1}{27} \cdot 9 \cdot 1 \cdot (-3) = -1.$$

Hence this coefficient makes the graph pass through the specified y-axis point.

Step 4: Conclusion $\boxed{a = \frac{1}{27}}$.

Quick Tip to Solve: At a y-axis crossing set x equal to zero before expanding the cubic; the factors become constants.

38(ii). Find $f''(x)$ at $x = 1$. [2 marks]

Solution:

Concept:

- For the graph $y = f(x)$ through $(0,-1)$, the condition $f(0) = -1$ determines the constant coefficient a.
- The power rule is $\frac{d}{dx}x^n = nx^{n-1}$. Apply it twice to the expanded polynomial, keeping a constant.

Quick Tip to Solve: Expand the three linear factors once; the second derivative of the cubic is the linear expression a(6x+14).

Step 1: Determine the coefficient Using the given y-axis crossing,

$$-1 = f(0) = a(9)(1)(-3) = -27a, \qquad a = \frac{1}{27}.$$

This value must be used in the derivative because the path is specified by both its factors and the given crossing.

Step 2: Expand and differentiate twice First multiply $(x+1)(x-3) = x^2 - 2x - 3$. Then

$$(x+9)(x^2 - 2x - 3) = x^3 + 7x^2 - 21x - 27.$$

Hence

$$f(x) = a(x^3 + 7x^2 - 21x - 27),$$
$$f'(x) = a(3x^2 + 14x - 21),$$
$$f''(x) = a(6x + 14).$$

The constant term vanishes on the first differentiation, and the linear term vanishes on the second.

Step 3: Evaluate at x equal to one Substitute $x = 1$ and $a = 1/27$ into the second derivative:

$$f''(1) = \frac{1}{27}(6 \cdot 1 + 14) = \frac{20}{27}.$$

Evaluation follows differentiation, so the dependence on x is retained throughout the working.

Step 4: Conclusion $\boxed{f''(1) = \frac{20}{27}}$.

Quick Tip to Solve: Expand the three linear factors once; the second derivative of the cubic is the linear expression a(6x+14).