CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-1-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) This question paper is divided into **five** Sections – **A, B, C, D and E**.

(iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19 and 20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculators is **not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $\vec{a}+\vec{b}=\hat{i}$ and $\vec{a}=2\hat{i}-2\hat{j}+2\hat{k}$, then $|\vec{b}|$ equals: [1 mark]

(A) $\sqrt{14}$

(B) 3

(C) $\sqrt{12}$

(D) $\sqrt{17}$

Correct Answer: (B) 3

Solution:

Concept:

- Vector subtraction is performed component by component in the orthonormal basis $\hat{i},\hat{j},\hat{k}$.
- For $\vec{v}=v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$, its magnitude is $|\vec{v}|=\sqrt{v_1^2+v_2^2+v_3^2}$.

Step 1: Isolate the required vector From $\vec{a}+\vec{b}=\hat{i}$, subtract $\vec{a}$ on both sides. Thus

$$\vec{b}=\hat{i}-(2\hat{i}-2\hat{j}+2\hat{k})=-\hat{i}+2\hat{j}-2\hat{k}.$$

The minus sign changes the sign of every component of $\vec{a}$.

Step 2: Calculate the magnitude and compare options The squared magnitude is

$$|\vec{b}|^2=(-1)^2+2^2+(-2)^2=1+4+4=9.$$

A magnitude is non-negative, so its value is the positive square root of 9. Options (A), (C) and (D) have squared magnitudes 14, 12 and 17 respectively; none equals 9.

Step 3: Conclusion $|\vec{b}|=3$. Hence, the correct option is (B).

Quick Tip to Solve: First subtract the vectors componentwise; then square the components, keeping the magnitude non-negative.

2. The direction ratios of a line parallel to z-axis are: [1 mark]

(A) $\langle 1,\ 1,\ 0\rangle$

(B) $\langle 1,\ 1,\ 1\rangle$

(C) $\langle 0,\ 0,\ 0\rangle$

(D) $\langle 0,\ 0,\ 1\rangle$

Correct Answer: (D) $\langle 0,\ 0,\ 1\rangle$

Solution:

Concept:

- Direction ratios are the components (a,b,c) of a non-zero vector parallel to a line.

Quick Tip to Solve: For a line parallel to a coordinate axis, only the direction ratio corresponding to that axis can be non-zero.

- A line parallel to the z-axis has no change in x or y, so its direction ratios satisfy $a = b = 0$ and $c \neq 0$.

Step 1: Identify the direction of the axis The unit vector along the positive z-axis is $\hat{k} = 0\hat{i} + 0\hat{j} + 1\hat{k}$. Thus a direction vector for a parallel line is $(0, 0, 1)$; any non-zero scalar multiple also gives valid direction ratios.

Step 2: Check all the proposed triples Option (A), $(1, 1, 0)$, lies in a direction parallel to the xy-plane. Option (B), $(1, 1, 1)$, changes all three coordinates. Option (C), $(0, 0, 0)$, is the zero vector and has no direction. Only option (D) has both horizontal components zero and a non-zero vertical component.

Step 3: Conclusion The required direction ratios are $\langle 0, 0, 1 \rangle$. Hence, the correct option is (D).

Quick Tip to Solve: For a line parallel to a coordinate axis, only the direction ratio corresponding to that axis can be non-zero.

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3. Ashima can hit a target 2 out of 3 times. She tried to hit the target twice. The probability that she missed the target exactly once is: [1 mark]

(A) $\frac{2}{3}$

(B) $\frac{1}{3}$

(C) $\frac{4}{9}$

(D) $\frac{1}{9}$

Correct Answer: (C) $\frac{4}{9}$

Solution:

Concept:

- The probability of a hit in one trial is $p = 2/3$, so the probability of a miss is $q = 1 - p = 1/3$.
- For independent trials, multiply probabilities within a sequence and add probabilities of mutually exclusive sequences.

Step 1: List the outcomes with exactly one miss In two attempts, exactly one miss can occur as a hit followed by a miss, or a miss followed by a hit. These two ordered outcomes are mutually exclusive. Assuming independent attempts, each has probability $(2/3)(1/3) = 2/9$.

Step 2: Add the probabilities and rule out distractors

$$P(\text{exactly one miss}) = \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}.$$

The values $2/3$ and $1/3$ describe a hit and a miss in one attempt, respectively. The value $1/9 = (1/3)^2$ describes two misses, not exactly one.

Step 3: Conclusion The required probability is $\frac{4}{9}$. Hence, the correct option is (C).

Quick Tip to Solve: Exactly one miss in two attempts has two possible positions; remember the factor 2 in $2pq$.

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4. The function $f(x) = |x| - x$ is: [1 mark]

(A) continuous but not differentiable at $x = 0$.

(B) continuous and differentiable at $x = 0$.

(C) neither continuous nor differentiable at $x = 0$.

(D) differentiable but not continuous at $x = 0$.

Correct Answer: (A) continuous but not differentiable at $x = 0$.

Solution:

Concept:

- A function is continuous at zero if its left and right limits equal its value $f(0)$.
- Differentiability at zero requires the left and right derivatives to be equal. Use the piecewise definition of $|x|$.

Step 1: Check continuity with the piecewise expression Since $|x| = -x$ for $x < 0$ and $|x| = x$ for $x \geq 0$,

$$f(x) = \begin{cases} -2x, & x < 0, \\ 0, & x \geq 0. \end{cases}$$

Both one-sided limits at zero are 0, and $f(0) = 0$. Therefore the function is continuous at zero.

Step 2: Compare the one-sided derivatives The left derivative at zero is

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = -2,$$

whereas the right derivative is

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = 0.$$

Their inequality excludes differentiability and therefore option (B). Continuity excludes option (C). Option (D) also fails: differentiability always implies continuity.

Step 3: Conclusion The function is continuous but not differentiable at $x = 0$. Hence, the correct option is (A).

Quick Tip to Solve: Expand an absolute-value expression on each side of zero; compare both the values and the slopes separately.

5. If $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $(3I + 4A)(3I - 4A) = x^2 I$, then the value(s) of x is/are: [1 mark]

(A) $\pm\sqrt{7}$

(B) 0

(C) ± 5

(D) 25

Correct Answer: (C) ± 5

Solution:

Concept:

- The identity matrix I commutes with every square matrix, so the cross terms cancel in $(3I + 4A)(3I - 4A)$.
- Compute A^2 before simplifying the matrix polynomial; $x^2 = 25$ permits both real square roots.

Step 1: Square the given matrix

$$A^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I.$$

The negative sign in A^2 is essential.

Step 2: Expand the product and determine the scalar

$$\begin{aligned} (3I + 4A)(3I - 4A) &= 9I - 12A + 12A - 16A^2 \\ &= 9I + 16I = 25I. \end{aligned}$$

Comparing with $x^2 I$ gives $x^2 = 25$. Option (A) would give $7I$, option (B) gives the zero matrix, and option (D), $x = 25$, would give $625I$; all are inconsistent with the product.

Step 3: Conclusion $x = \pm 5$. Hence, the correct option is (C).

Quick Tip to Solve: Check whether $A^2 = -I$ before using the difference of squares, and distinguish x from x^2.

6. The general solution of the differential equation $x\,dy - (1+x^2)\,dx = dx$ is: [1 mark]

(A) $y = 2x + \dfrac{x^3}{3} + C$

(B) $y = 2\log x + \dfrac{x^3}{3} + C$

(C) $y = \dfrac{x^2}{2} + C$

(D) $y = 2\log x + \dfrac{x^2}{2} + C$

Correct Answer: (D) $y = 2\log x + \dfrac{x^2}{2} + C$

Solution:

Concept:

- Rearrange the differential equation into dy/dx form before integrating; division by x requires $x \neq 0$.
- The relevant antiderivatives are $\int dx/x = \log|x| + C$ and $\int x\,dx = x^2/2 + C$.

Step 1: Isolate the derivative Adding $(1+x^2)\,dx$ to both sides gives

$$x\,dy = (2+x^2)\,dx, \qquad \frac{dy}{dx} = \frac{2}{x} + x \quad (x \neq 0).$$

The right-hand dx contributes the second constant term; it must not be dropped.

Step 2: Integrate and test the printed options Integration gives

$$y = 2\log|x| + \frac{x^2}{2} + C.$$

On $x > 0$, this is $2\log x + x^2/2 + C$, as printed in option (D). Differentiating (A), (B) and (C) gives $2 + x^2$, $2/x + x^2$ and x, respectively; none equals the required derivative. Differentiating (D) gives exactly $2/x + x$. The absolute-value form also applies on any interval with $x < 0$.

Step 3: Conclusion The general solution is $y = 2\log|x| + \dfrac{x^2}{2} + C$ for $x \neq 0$; for the printed positive-domain form, the correct option is (D).

Quick Tip to Solve: Bring every dx term to the same side before division, then differentiate the proposed answer as a quick check.

7. If $f(x) = a(x - \cos x)$ is strictly decreasing in $\mathbb{R}$, then a belongs to: [1 mark]

(A) $\{0\}$

(B) $(0, \infty)$

(C) $(-\infty, 0)$

(D) $(-\infty, \infty)$

Correct Answer: (C) $(-\infty, 0)$

Solution:

Concept:

- For $g(x) = x - \cos x$, the derivative is $g'(x) = 1 + \sin x \geq 0$ for all real x.
- The derivative vanishes only at isolated points, so g is strictly increasing; multiplying it by a negative constant reverses strict inequalities.

Step 1: Establish strict increase of the inner function For $u < v$,

$$g(v) - g(u) = \int_u^v (1 + \sin t)\, dt > 0.$$

The integrand is continuous and non-negative, and cannot be identically zero on any interval of positive length: its zeros $t = -\pi/2 + 2k\pi$, $k \in \mathbb{Z}$, are isolated. Thus g is strictly increasing even though its derivative is zero at some points.

Step 2: Determine the allowable sign of the constant If $a < 0$, then $g(u) < g(v)$ implies $ag(u) > ag(v)$, so $f = ag$ is strictly decreasing. For $a > 0$, it is strictly increasing, excluding (B). For $a = 0$, it is constant, excluding (A). The whole real line in (D) wrongly includes both zero and positive values.

Step 3: Conclusion $a \in (-\infty, 0)$. Hence, the correct option is (C).

Quick Tip to Solve: Isolated zeros of a derivative do not destroy strict monotonicity; check its sign throughout each interval.

8. The corner points of the feasible region in the graphical representation of a linear programming problem are $(2, 72)$, $(15, 20)$ and $(40, 15)$. If $z = 18x + 9y$ be the objective function, then: [1 mark]

(A) z is maximum at $(2, 72)$, minimum at $(15, 20)$

(B) z is maximum at $(15, 20)$, minimum at $(40, 15)$

(C) z is maximum at $(40, 15)$, minimum at $(15, 20)$

(D) z is maximum at $(40, 15)$, minimum at $(2, 72)$

Correct Answer: (C) z is maximum at $(40, 15)$, minimum at $(15, 20)$

Solution:

Concept:

- The corner-point method evaluates a linear objective function at the vertices of the feasible region.
- Here the objective is $z = 18x + 9y$, where x, y are the coordinates of each corner point.

Step 1: Evaluate the objective at all three corners At $(2, 72)$, $z = 18(2) + 9(72) = 684$. At $(15, 20)$, $z = 18(15) + 9(20) = 450$. At $(40, 15)$, $z = 18(40) + 9(15) = 855$. These values must be compared; the

Quick Tip to Solve: Factor out 9 and compare $2x + y$ at the three corners: $76, 50, 95$; the ordering is unchanged.

largest coordinate by itself does not decide an optimum.

Step 2: Compare the values and check the options The ordering is $450 < 684 < 855$, so the maximum belongs to $(40, 15)$ and the minimum to $(15, 20)$. Option (A) chooses the wrong maximum, (B) reverses the actual extreme points, and (D) chooses the wrong minimum.

Step 3: Conclusion $z_{\max} = 855$ at $(40, 15)$, and $z_{\min} = 450$ at $(15, 20)$. Hence, the correct option is (C).

Quick Tip to Solve: Factor out 9 and compare $2x + y$ at the three corners: $76, 50, 95$; the ordering is unchanged.

9. The number of corner points of the feasible region determined by the constraints $x - y \geq 0,\ 2y \leq x + 2,\ x \geq 0,\ y \geq 0$ is: [1 mark]

(A) 2

(B) 3

(C) 4

(D) 5

Correct Answer: (A) 2

Solution:

Concept:

- The feasible region is the intersection of the half-planes specified by all the inequalities.
- A corner point is a feasible intersection where the boundary changes direction; an unbounded region can still have finitely many corners.

Step 1: Find the feasible boundary intersections The constraints give $0 \leq y \leq x$ and $y \leq (x + 2)/2$, with $x \geq 0$. On the horizontal axis, $y = 0$ gives the corner $(0, 0)$. The oblique lines intersect where $x = (x + 2)/2$, hence at $(2, 2)$. Both points satisfy every constraint.

Step 2: Count the corners without closing the unbounded region For $0 \leq x \leq 2$, the upper boundary is $y = x$; for $x \geq 2$, it is $y = (x + 2)/2$. The lower boundary is $y = 0$. Thus there are exactly two corners, $(0, 0)$ and $(2, 2)$. The intercept $(-2, 0)$ of the second line is excluded by $x \geq 0$. No extra vertex arises at infinity; options (B), (C), and (D) overcount.

Step 3: Conclusion The feasible region has 2 corner points. Hence, the correct option is (A).

Quick Tip to Solve: Check each intersection against every inequality, and do not add a closing edge to an unbounded feasible region.

10. If for a square matrix A, $A^2 - 3A + I = O$ and $A^{-1} = xA + yI$, then the value of $x + y$ is: [1 mark]

(A) -2

(B) 2

(C) 3

(D) -3

Correct Answer: (B) 2

Solution:

Concept:

- I denotes the identity matrix and O the zero matrix, of the same order as A.
- An identity of the form $A(3I - A) = I$ identifies $3I - A$ as the inverse of A.

Step 1: Rearrange the polynomial identity From $A^2 - 3A + I = O$, obtain $3A - A^2 = I$. Factoring gives $A(3I - A) = I$. Thus the given matrix has an inverse and $A^{-1} = 3I - A$. Equivalently, multiplying the original identity by A^{-1} yields $A - 3I + A^{-1} = O$.

Step 2: Identify the coefficients in the inverse expression Writing this as $A^{-1} = (-1)A + 3I$ gives $x = -1$ and $y = 3$ in the required expression. Therefore $x + y = -1 + 3 = 2$. Among the printed options, only (B) has this value; (A), (C), and (D) do not agree with the coefficients obtained from the identity.

Step 3: Conclusion $x + y = 2$. Hence, the correct option is (B).

Quick Tip to Solve: Rearrange a matrix polynomial to isolate I, then factor out A to read off its inverse directly.

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11. If $\begin{pmatrix} x & 2 \\ 3 & x-1 \end{pmatrix}$ is a singular matrix, then the product of all possible values of x is: [1 mark]

(A) 6

(B) -6

(C) 0

(D) -7

Correct Answer: (B) -6

Solution:

Concept:

- A square matrix is singular if and only if its determinant is zero.
- For $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is $ad - bc$.

Step 1: Apply the singularity condition The determinant of the given matrix is $x(x-1) - 2(3)$. Setting it equal to zero gives $x^2 - x - 6 = 0$. The off-diagonal product is subtracted, so the constant term is -6, not $+6$.

Quick Tip to Solve: After forming the monic determinant equation, use the constant term as the product of its roots.

Step 2: Find and multiply the possible values Factor the quadratic: $x^2 - x - 6 = (x-3)(x+2) = 0$. Thus the two possible values are $x = 3$ and $x = -2$, whose product is $3(-2) = -6$. Option (A) loses the negative sign, (C) incorrectly includes zero as a root, and (D) is not the root product.

Step 3: Conclusion The product of all possible values of x is -6. Hence, the correct option is (B).

Quick Tip to Solve: After forming the monic determinant equation, use the constant term as the product of its roots.

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12. Let A be a 3×3 matrix such that $|\operatorname{adj} A| = 64$. Then $|A|$ is equal to: [1 mark]

(A) 8 only

(B) -8 only

(C) 64

(D) 8 or -8

Correct Answer: (D) 8 or -8

Solution:

Concept:

- For a square matrix of order n, $|\operatorname{adj} A| = |A|^{n-1}$.
- For order 3, this relation becomes $|\operatorname{adj} A| = |A|^2$; squaring removes the sign.

Step 1: Use the determinant identity for order three The matrix has order 3, and the given determinant of its adjoint is 64. Therefore $|A|^{3-1} = 64$, or $|A|^2 = 64$. Here the vertical bars denote a determinant, not the absolute value of a number.

Step 2: Retain both signs of the determinant Solving the quadratic equation gives $|A| = 8$ or $|A| = -8$, because each value has square 64. Nothing in the question restricts the determinant to be positive. Options (A) and (B) each omit a valid sign, while (C) confuses the determinant of the adjoint with that of the matrix.

Step 3: Conclusion $|A| = 8$ or -8. Hence, the correct option is (D).

Quick Tip to Solve: For a 3×3 matrix, take both square roots of the adjoint determinant unless another condition fixes the sign.

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13. **If $A = \begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix}$ and $2A + B$ is a null matrix, then B is equal to:** [1 mark]

(A) $\begin{pmatrix} 6 & 8 \\ 10 & 4 \end{pmatrix}$

(B) $\begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}$

(C) $\begin{pmatrix} 5 & 8 \\ 10 & 3 \end{pmatrix}$

(D) $\begin{pmatrix} -5 & -8 \\ -10 & -3 \end{pmatrix}$

Correct Answer: (B) $\begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}$

Solution:

Concept:

- A null matrix has zero in every entry, and it is the additive identity for matrices of the same order.
- Scalar multiplication multiplies every matrix entry by that scalar.

Step 1: Isolate the unknown matrix The given condition is $2A + B = O$. Subtracting $2A$ from both sides gives $B = -2A$. This operation must be applied to the whole matrix, so every entry of B is negative twice the corresponding entry of A.

Step 2: Multiply the entries and check the choices Thus

$$B = -2\begin{pmatrix} 3 & 4 \\ 5 & 2 \end{pmatrix} = \begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}.$$

Adding this to $2A$ cancels each entry. Option (A) gives $+2A$, and options (C) and (D) have incorrect diagonal entries. Only option (B) makes the sum a null matrix.

Step 3: Conclusion $B = \begin{pmatrix} -6 & -8 \\ -10 & -4 \end{pmatrix}$. Hence, the correct option is (B).

Quick Tip to Solve: From $kA + B = O$, write $B = -kA$ immediately and scale every entry, including the diagonal entries.

14. The primitive of $\dfrac{2}{1+\cos 2x}$ is: [1 mark]

(A) $\sec^2 x$

(B) $2\sec^2 x \tan x$

(C) $\tan x$

(D) $-\cot x$

Correct Answer: (C) $\tan x$

Solution:

Concept:

- A primitive is an antiderivative: differentiating it gives the original integrand.
- The identity $1+\cos 2x = 2\cos^2 x$ and $\dfrac{d}{dx}\tan x = \sec^2 x$ apply wherever $\cos x \neq 0$.

Step 1: Simplify the integrand before integrating Using the double-angle identity gives

$$\frac{2}{1+\cos 2x} = \frac{2}{2\cos^2 x} = \sec^2 x.$$

Therefore its primitive is $\int \sec^2 x\, dx = \tan x + C$, on each interval where the given integrand is defined.

Step 2: Check by differentiation and eliminate the distractors Differentiating $\tan x$ gives $\sec^2 x$, exactly the simplified integrand. Option (A) differentiates to $2\sec^2 x \tan x$; option (B) differentiates to $4\sec^2 x \tan^2 x + 2\sec^4 x$, and option (D) differentiates to $\csc^2 x$. These are not the required function.

Step 3: Conclusion The primitive is $\tan x + C$; the listed primitive is $\tan x$. Hence, the correct option is (C).

Quick Tip to Solve: Use the double-angle identity first; then differentiate the proposed primitive to confirm the integrand.

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15. $\displaystyle\int_0^{\pi/6} \sec^2\left(x - \frac{\pi}{6}\right) dx$ is equal to: [1 mark]

(A) $\dfrac{1}{\sqrt{3}}$

(B) $-\dfrac{1}{\sqrt{3}}$

(C) $\sqrt{3}$

(D) $-\sqrt{3}$

Correct Answer: (A) $\dfrac{1}{\sqrt{3}}$

Solution:

Concept:

- The fundamental theorem gives $\int_a^b f(x)\, dx = F(b) - F(a)$, where F is an antiderivative of f.

Quick Tip to Solve: First shift both limits by $-\pi/6$; subtracting the negative lower-end tangent makes the integral positive.

- The derivative of $\tan u$ is $\sec^2 u$; here $u = x - \pi/6$ has derivative 1.

Step 1: Find an antiderivative Since the inner expression has derivative one, no extra numerical factor is needed. Thus an antiderivative of $\sec^2(x - \pi/6)$ is $\tan(x - \pi/6)$. The argument lies between $-\pi/6$ and 0, where tangent is continuous.

Step 2: Apply the limits and compare the options

$$\begin{aligned} I &= \left[\tan\left(x - \frac{\pi}{6}\right)\right]_0^{\pi/6} \\ &= \tan 0 - \tan\left(-\frac{\pi}{6}\right) \\ &= 0 - \left(-\frac{1}{\sqrt{3}}\right) = \frac{1}{\sqrt{3}}. \end{aligned}$$

The integrand is positive, so the negative values in (B) and (D) cannot occur. Option (C) uses the tangent value at $\pi/3$, not at the stated endpoint.

Step 3: Conclusion The integral equals $\boxed{1/\sqrt{3}}$. Hence, the correct option is (A).

Quick Tip to Solve: First shift both limits by $-\pi/6$; subtracting the negative lower-end tangent makes the integral positive.

16. The sum of the order and the degree of the differential equation $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = \sin y$ is: [1 mark]

(A) 5

(B) 2

(C) 3

(D) 4

Correct Answer: (C) 3

Solution:

Concept:

- The order of a differential equation is the order of its highest derivative.
- The degree is the power of the highest-order derivative when the equation is polynomial in its derivatives.

Step 1: Identify the highest derivative The derivatives present are dy/dx and d^2y/dx^2. The second derivative is the highest-order one, so the order is 2. The cubic power of the first derivative does not raise the order to three.

Step 2: Find the degree and test the alternatives Write the equation as

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 - \sin y = 0.$$

It is polynomial in its derivatives, and the highest-order derivative occurs to the first power. Therefore

Quick Tip to Solve: Circle the highest-order derivative first; only its exponent determines the degree, not the largest exponent anywhere.

the degree is 1, even though the first derivative is cubed. The requested sum is $2 + 1 = 3$. Option (A) incorrectly adds order two to the exponent three; (B) gives only the order; (D) is not the sum of order and degree.

Step 3: Conclusion The order is 2, the degree is 1, and their sum is $\boxed{3}$. Hence, the correct option is (C).

Quick Tip to Solve: Circle the highest-order derivative first; only its exponent determines the degree, not the largest exponent anywhere.

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17. The value of p for which the vectors $2\hat{i} + p\hat{j} + \hat{k}$ and $-4\hat{i} - 6\hat{j} + 26\hat{k}$ are perpendicular to each other, is: [1 mark]

(A) 3

(B) -3

(C) $-\frac{17}{3}$

(D) $\frac{17}{3}$

Correct Answer: (A) 3

Solution:

Concept:

- Two nonzero vectors are perpendicular if and only if their scalar product is zero.
- For Cartesian components, $(a_1, a_2, a_3) \cdot (b_1, b_2, b_3) = a_1b_1 + a_2b_2 + a_3b_3$.

Step 1: Use the perpendicularity condition Let $\vec{a} = 2\hat{i} + p\hat{j} + \hat{k}$ and $\vec{b} = -4\hat{i} - 6\hat{j} + 26\hat{k}$. Both are nonzero regardless of p, so perpendicularity is equivalent to $\vec{a} \cdot \vec{b} = 0$. Computing component by component gives

$$2(-4) + p(-6) + 1(26) = 0.$$

Step 2: Solve and check each proposed value The equation becomes $-8 - 6p + 26 = 18 - 6p = 0$, giving $p = 3$. For the options $3, -3, -17/3, 17/3$, the corresponding scalar products $18 - 6p$ are respectively $0, 36, 52, -16$. Only the first value makes the vectors perpendicular.

Step 3: Conclusion The required value is $\boxed{p = 3}$. Hence, the correct option is (A).

Quick Tip to Solve: Keep the signs of corresponding components when taking the dot product; the constant terms here combine to $26 - 8 = 18$.

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18. For what value of λ, the projection of vector $\hat{i}+\lambda\hat{j}$ on vector $\hat{i}-\hat{j}$ is $\sqrt{2}$? [1 mark]

(A) -1

(B) 1

(C) 0

(D) 3

Correct Answer: (A) -1

Solution:

Concept:

- The scalar projection of $\vec{a}$ on a nonzero vector $\vec{b}$ is $\vec{a}\cdot\vec{b}/|\vec{b}|$.
- The direction of the vector onto which projection is taken determines the sign of this scalar projection.

Step 1: Compute the scalar projection Here $\vec{a}=\hat{i}+\lambda\hat{j}$ and $\vec{b}=\hat{i}-\hat{j}$. Their scalar product is $1-\lambda$, while $|\vec{b}|=\sqrt{1^2+(-1)^2}=\sqrt{2}$. Thus the projection on the given vector is $(1-\lambda)/\sqrt{2}$.

Step 2: Use the specified projection and compare options

$$\frac{1-\lambda}{\sqrt{2}}=\sqrt{2} \quad\Longrightarrow\quad 1-\lambda=2 \quad\Longrightarrow\quad \lambda=-1.$$

For $\lambda=-1,1,0,3$, the scalar projections are respectively $\sqrt{2},0,1/\sqrt{2},-\sqrt{2}$. The final value has the opposite direction and sign; taking an absolute value would incorrectly admit option (D).

Step 3: Conclusion The required value is $\boxed{\lambda=-1}$. Hence, the correct option is (A).

Quick Tip to Solve: Divide the dot product by the magnitude of the target vector and retain its sign when matching the stated scalar projection.

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19. Assertion (A): The range of the function $f(x)=2\sin^{-1}x+\dfrac{3\pi}{2}$, where $x\in[-1,1]$, is $\left[\dfrac{\pi}{2},\dfrac{5\pi}{2}\right]$.

Reason (R): The range of the principal value branch of $\sin^{-1}(x)$ is $[0,\pi]$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (C) Assertion (A) is true, but Reason (R) is false.

Solution:

Concept:

- The principal range of $\sin^{-1}x$, for $x\in[-1,1]$, is $[-\pi/2,\pi/2]$.
- Multiplying an interval by a positive constant preserves its endpoint order; adding a constant shifts both endpoints.

Quick Tip to Solve: Recall inverse sine uses $[-\pi/2,\pi/2]$; $[0,\pi]$ is the principal range of inverse cosine.

Step 1: Judge Assertion (A) Start with $-\pi/2 \le \sin^{-1} x \le \pi/2$. Multiplication by 2, followed by addition of $3\pi/2$, gives

$$\frac{\pi}{2} \le 2\sin^{-1} x + \frac{3\pi}{2} \le \frac{5\pi}{2}.$$

Both endpoints are attained at $x = -1$ and $x = 1$, and continuity fills the interval. Assertion (A) is true.

Step 2: Judge Reason (R) The stated interval $[0, \pi]$ is not the principal range of inverse sine. For example, $\sin^{-1}(-1) = -\pi/2$, which is outside that stated interval. The correct principal range is $[-\pi/2, \pi/2]$. Therefore Reason (R) is false.

Step 3: Check whether the reason explains the assertion A false reason cannot explain the true assertion. Options (A) and (B) require both statements to be true, while (D) requires a false assertion and a true reason. Only option (C) matches the truth values obtained.

Step 4: Conclusion The assertion is true and the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Recall inverse sine uses $[-\pi/2, \pi/2]$; $[0, \pi]$ is the principal range of inverse cosine.

20. Assertion (A): The equation of the line passing through the points $(1, 2, 3)$ and $(3, -1, 3)$ is $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-3}{0}$.

Reason (R): The equation of the line passing through the points (x_1, y_1, z_1), (x_2, y_2, z_2) is given by $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true, but Reason (R) is false.

(D) Assertion (A) is false, but Reason (R) is true.

Correct Answer: (D) Assertion (A) is false, but Reason (R) is true.

Solution:

Concept:

- A line through P and Q has direction vector $\overrightarrow{PQ} = Q - P$.
- The two-point symmetric equation uses these coordinate differences as direction ratios; a zero direction ratio means the corresponding coordinate stays constant.

Step 1: Judge Assertion (A) For $P = (1, 2, 3)$ and $Q = (3, -1, 3)$, $Q - P = (2, -3, 0)$. Thus a correct parametrisation is $x = 3 + 2t,\ y = -1 - 3t,\ z = 3$. The asserted direction ratios $(2, 3, 0)$ have the wrong sign for the second component. Indeed, at P, $(x-3)/2 = -1$ but $(y+1)/3 = 1$, so P does not satisfy the asserted equality. Assertion (A) is false.

Step 2: Judge Reason (R) The displayed formula is the standard two-point equation: its direction ratios are $x_2 - x_1, y_2 - y_1, z_2 - z_1$. A zero denominator in this notation expresses a constant coordinate,

Quick Tip to Solve: Compute every direction ratio as second coordinate minus first coordinate; when a ratio is zero, state that coordinate as constant.

rather than ordinary division by zero. In this problem the final component means $z = 3$. With that standard interpretation, Reason (R) is true.

Step 3: Check the explanation and the answer codes Applying the reason correctly gives $(x-1)/2 = (y-2)/(-3)$, together with $z = 3$, not the assertion. Thus the assertion is false and the reason is true. Options (A), (B) and (C) all require a true assertion, so each is excluded.

Step 4: Conclusion The assertion is false and the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Compute every direction ratio as second coordinate minus first coordinate; when a ratio is zero, state that coordinate as constant.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21. If the product of two positive numbers is 9, find the numbers so that the sum of their squares is minimum. [2 marks]

Solution:

Concept:

- For a function of one variable, an interior minimum must satisfy $S'(x) = 0$; $S''(x) > 0$ establishes a local minimum.
- A product constraint lets one positive variable be expressed in terms of the other, reducing the optimisation to one variable.

Step 1: Express the sum in one variable Let the two positive numbers be x and y. Since $xy = 9$, we have $y = 9/x$, with $x > 0$. Hence their sum of squares is

$$S(x) = x^2 + y^2 = x^2 + \frac{81}{x^2}.$$

The positivity condition determines the domain and will rule out negative roots.

Step 2: Find the stationary point Differentiating gives

$$S'(x) = 2x - \frac{162}{x^3}.$$

At a stationary point,

$$2x - \frac{162}{x^3} = 0 \quad \Longrightarrow \quad 2x^4 = 162 \quad \Longrightarrow \quad x^4 = 81.$$

As $x > 0$, the admissible root is $x = 3$. The constraint then gives $y = 9/3 = 3$.

Step 3: Verify the minimum The second derivative is

$$S''(x) = 2 + \frac{486}{x^4} > 0 \qquad (x > 0).$$

Thus S is strictly convex throughout its domain, so its sole stationary point is the global minimum. As

Quick Tip to Solve: With a fixed positive product, write the sum of squares as $(x-y)^2 + 2xy$; equality immediately identifies equal numbers.

a direct check,

$$x^2 + y^2 = (x - y)^2 + 2xy = (x - y)^2 + 18 \geq 18,$$

with equality precisely when $x = y$. The candidate $(3, 3)$ meets both the equality condition and the product constraint.

Step 4: Conclusion The two positive numbers are $\boxed{3 \text{ and } 3}$; the minimum sum of their squares is 18.

Quick Tip to Solve: With a fixed positive product, write the sum of squares as $(x - y)^2 + 2xy$; equality immediately identifies equal numbers.

22(a). A function $f : A \to B$ defined as $f(x) = 2x$ is both one-one and onto. If $A = \{1, 2, 3, 4\}$, then find the set B. [2 marks]

Solution:

Concept:

- For a function $f : A \to B$, the range is $f(A) = \{f(x) : x \in A\}$, the set of values actually obtained from the domain.
- A function is onto when every element of its codomain B has a preimage in A; thus $B = f(A)$. It is one-one when distinct domain elements have distinct images.

Step 1: Calculate the images of the domain elements Use the given rule $f(x) = 2x$ for each element of $A = \{1, 2, 3, 4\}$:

$$f(1) = 2, \quad f(2) = 4,$$
$$f(3) = 6, \quad f(4) = 8.$$

Hence the set of values attained by the function is $f(A) = \{2, 4, 6, 8\}$. These four images are distinct, consistently with the stated one-one property.

Step 2: Use the onto condition to identify the codomain Because f maps into B, all four calculated images must belong to B. Conversely, because f is onto, every element of B must be one of these images. No additional element can occur in B, as it would have no preimage among $1, 2, 3, 4$. The two inclusions $f(A) \subseteq B$ and $B \subseteq f(A)$ therefore give equality.

Step 3: Conclusion $\boxed{B = \{2, 4, 6, 8\}}$.

Quick Tip to Solve: When a function is onto, calculate its complete range and use that range as the codomain.

OR

22(b). Evaluate:

$\sin^{-1}\left(\sin\frac{3\pi}{4}\right)+\cos^{-1}\left(\cos\frac{3\pi}{4}\right)+\tan^{-1}(1)$ [2 marks]

Solution:

Concept:

- The principal-value ranges are $[-\pi/2,\pi/2]$ for $\sin^{-1}$, $[0,\pi]$ for $\cos^{-1}$, and $(-\pi/2,\pi/2)$ for $\tan^{-1}$.
- An inverse trigonometric expression returns the angle in its principal range with the required trigonometric value; it does not always return the original angle.

Step 1: Evaluate the sine and cosine terms using their ranges The angle $3\pi/4$ lies outside the principal range of $\sin^{-1}$. Since $\sin(3\pi/4)=\sin(\pi/4)=1/\sqrt{2}$, and $\pi/4$ is in that principal range,

$$\sin^{-1}\left(\sin\frac{3\pi}{4}\right)=\frac{\pi}{4}.$$

In contrast, $3\pi/4$ is in $[0,\pi]$, the principal range of $\cos^{-1}$. Therefore

$$\cos^{-1}\left(\cos\frac{3\pi}{4}\right)=\frac{3\pi}{4}.$$

Step 2: Evaluate the tangent term and combine As $\tan(\pi/4)=1$ and $\pi/4\in(-\pi/2,\pi/2)$, we have $\tan^{-1}(1)=\pi/4$. Substituting the three principal values into the given expression gives

$$\frac{\pi}{4}+\frac{3\pi}{4}+\frac{\pi}{4}=\frac{(1+3+1)\pi}{4}.$$

All terms are in radians, so they can be added directly.

Step 3: Conclusion The required value is $\boxed{\frac{5\pi}{4}}$.

Quick Tip to Solve: Before cancelling an inverse function with its trigonometric function, check the principal-value range.

23. Find all the vectors of magnitude $3\sqrt{3}$ which are collinear to vector $\hat{i}+\hat{j}+\hat{k}$. [2 marks]

Solution:

Concept:

- A vector collinear with a nonzero vector $\vec{u}$ has the form $\vec{v}=t\vec{u}$, where t is a real scalar.
- The magnitude of $t\vec{u}$ is $|t|\,|\vec{u}|$. Positive and negative values of t give the same and opposite directions, respectively.

Step 1: Express the required vector as a scalar multiple Let $\vec{u}=\hat{i}+\hat{j}+\hat{k}$. Its components are $(1,1,1)$,

Quick Tip to Solve: For all vectors of a fixed magnitude collinear with a given vector, include both signs of the unit-vector multiple.

so

$$|\vec{u}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}.$$

Every vector collinear with $\vec{u}$ can be written as $\vec{v} = t(\hat{i} + \hat{j} + \hat{k})$. This representation includes vectors pointing in either direction along the same line; taking only a positive multiple would omit a possible vector.

Step 2: Impose the magnitude condition The required magnitude is $3\sqrt{3}$, hence

$$|\vec{v}| = |t|\sqrt{3} = 3\sqrt{3} \quad \Longrightarrow \quad |t| = 3.$$

Since t is real, the two possibilities are $t = 3$ and $t = -3$. Substituting these into the scalar-multiple expression gives components $(3, 3, 3)$ and $(-3, -3, -3)$. Each has magnitude $\sqrt{9+9+9} = 3\sqrt{3}$, and each is collinear with the given vector.

Step 3: Conclusion All the required vectors are $\boxed{3\hat{i} + 3\hat{j} + 3\hat{k}}$ and $\boxed{-3\hat{i} - 3\hat{j} - 3\hat{k}}$.

Quick Tip to Solve: For all vectors of a fixed magnitude collinear with a given vector, include both signs of the unit-vector multiple.

24(a). Position vectors of the points A, B and C as shown in the figure below are $\vec{a}$, $\vec{b}$ and $\vec{c}$ respectively. If $\vec{AC} = \frac{5}{4}\vec{AB}$, express $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$. [2 marks]



Solution:

Concept:

- If points A, B, C have position vectors $\vec{a}, \vec{b}, \vec{c}$, then $\overrightarrow{AB} = \vec{b} - \vec{a}$ and $\overrightarrow{AC} = \vec{c} - \vec{a}$.
- Vector displacement is always the position vector of the endpoint minus that of the starting point. The given scalar relation can therefore be solved directly for $\vec{c}$.

Step 1: Translate the displacement relation into position vectors The figure places A, B, C on the same line, with C beyond B in the direction from A to B. This agrees with the positive factor $5/4 > 1$ in the given relation. Replacing the two displacement vectors by differences of position vectors yields

$$\vec{c} - \vec{a} = \frac{5}{4}(\vec{b} - \vec{a}).$$

Both displacements start at A, so the subtracted position vector is $\vec{a}$ in each expression.

Quick Tip to Solve: Write each directed segment as endpoint minus starting point before rearranging a relation between position vectors.

Step 2: Isolate the position vector of C Expand the right-hand side and then add $\vec{a}$ to both sides:

$$\begin{aligned}\vec{c}-\vec{a} &= \frac{5}{4}\vec{b}-\frac{5}{4}\vec{a},\\ \vec{c} &= \frac{5}{4}\vec{b}+\left(1-\frac{5}{4}\right)\vec{a}\\ &= \frac{5}{4}\vec{b}-\frac{1}{4}\vec{a}.\end{aligned}$$

The coefficients sum to one, as expected for a position vector on the line through A and B. The negative coefficient of $\vec{a}$ reflects that C is beyond B.

Step 3: Conclusion $\boxed{\vec{c}=\dfrac{5\vec{b}-\vec{a}}{4}}$.

Quick Tip to Solve: Write each directed segment as endpoint minus starting point before rearranging a relation between position vectors.

OR

24(b). Check whether the lines given by equations $x = 2\lambda + 2,\ y = 7\lambda + 1,\ z = -3\lambda - 3$ and $x = -\mu - 2,\ y = 2\mu + 8,\ z = 4\mu + 5$ are perpendicular to each other or not. [2 marks]

Solution:

Concept:

- For a parametric line $x = x_0 + lt,\ y = y_0 + mt,\ z = z_0 + nt$, the parameter coefficients (l, m, n) are direction ratios; the constants specify a point on the line.
- Two line directions are perpendicular when the scalar product of their nonzero direction vectors is zero: $\vec{d_1} \cdot \vec{d_2} = 0$.

Step 1: Read the direction ratios from the equations The first line has equations $x = 2\lambda+2$, $y = 7\lambda+1$, $z = -3\lambda - 3$. Its direction vector is therefore $\vec{d_1} = 2\hat{i} + 7\hat{j} - 3\hat{k}$. For the second line, $x = -\mu - 2$, $y = 2\mu + 8$, $z = 4\mu + 5$, the direction vector is $\vec{d_2} = -\hat{i} + 2\hat{j} + 4\hat{k}$. Only the coefficients of the parameters describe direction; the constant terms do not enter the angle calculation.

Step 2: Check the perpendicularity condition Compute the dot product using corresponding components:

$$\vec{d_1} \cdot \vec{d_2} = 2(-1) + 7(2) + (-3)(4) = -2 + 14 - 12 = 0.$$

Both direction vectors are nonzero. Consequently, in the formula

$$\cos\theta = \frac{\vec{d_1} \cdot \vec{d_2}}{|\vec{d_1}|\,|\vec{d_2}|},$$

the numerator is zero and the denominator is positive. Thus the angle between the given line directions is 90°.

Step 3: Conclusion Yes, the given lines have perpendicular directions, since their direction vectors

Quick Tip to Solve: To test perpendicularity of parametric lines, dot the parameter coefficients and ignore the constant terms.

have zero dot product.

Quick Tip to Solve: To test perpendicularity of parametric lines, dot the parameter coefficients and ignore the constant terms.

25. If $x = \sqrt{a^{\tan^{-1} t}}$, $y = \sqrt{a^{\cot^{-1} t}}$, then show that $x\,\dfrac{dy}{dx} + y = 0$. [2 marks]

Solution:

Concept:

- For a real parameter t, the principal inverse trigonometric values satisfy $\tan^{-1} t + \cot^{-1} t = \pi/2$.
- For a positive exponential base a, $\sqrt{a^u} = a^{u/2}$. A constant product $xy = K$ gives $x\,dy/dx + y = 0$ by the product rule, wherever y is a differentiable function of x.

Step 1: Establish a constant product Multiplying the two given expressions and using the inverse trigonometric identity, we obtain

$$xy = a^{\frac{1}{2}\tan^{-1} t}\, a^{\frac{1}{2}\cot^{-1} t} = a^{\frac{1}{2}(\tan^{-1} t + \cot^{-1} t)} = a^{\pi/4}.$$

The right-hand side depends only on the fixed base a, so the product is independent of the parameter t.

Step 2: Differentiate the identity with respect to x For the usual parametric interpretation, take $a > 0$ and $a \neq 1$. Then

$$\frac{dx}{dt} = \frac{x \log a}{2(1 + t^2)} \neq 0,$$

so differentiation with respect to x is valid. Differentiating $xy = a^{\pi/4}$, the product rule gives

$$x\frac{dy}{dx} + y\frac{dx}{dx} = 0.$$

The first term accounts for the variation in y, while the second accounts for the variation in x. If $a = 1$, both coordinates are constant and the parametric derivative is undefined.

Step 3: Conclusion Thus, wherever the given parametrization defines dy/dx, $\boxed{x\dfrac{dy}{dx} + y = 0}$, as required.

Quick Tip to Solve: Multiply the two coordinates first: complementary inverse trigonometric exponents make their product constant.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26(a). Evaluate: $\int_0^{\pi/2} e^x \sin x \, dx$. [3 marks]

Solution:

Concept:

- Integration by parts gives $\int u\,dv = uv - \int v\,du$. For definite integrals, evaluate the product at both limits.
- Applying integration by parts twice to an exponential times a sine produces the original integral again; its value can then be found algebraically.

Step 1: Apply integration by parts to the sine factor Let

$$I = \int_0^{\pi/2} e^x \sin x \, dx.$$

Choose $u = \sin x$, $dv = e^x\,dx$, so $du = \cos x\,dx$ and $v = e^x$. Thus

$$I = [e^x \sin x]_0^{\pi/2} - \int_0^{\pi/2} e^x \cos x \, dx = e^{\pi/2} - J,$$

where $J = \int_0^{\pi/2} e^x \cos x\,dx$. The lower endpoint of the product is zero because $\sin 0 = 0$.

Step 2: Apply integration by parts to the cosine integral For J, take $u = \cos x$ and $dv = e^x\,dx$. Since $du = -\sin x\,dx$, the subtraction in the integration-by-parts formula turns the remaining integral into a positive term:

$$J = [e^x \cos x]_0^{\pi/2} + \int_0^{\pi/2} e^x \sin x \, dx = -1 + I.$$

Here the endpoint contribution is $e^{\pi/2}\cos(\pi/2) - e^0 \cos 0 = 0 - 1 = -1$.

Step 3: Solve the resulting equation and check the primitive Substituting the expression for J gives

$$I = e^{\pi/2} - (-1 + I), \qquad 2I = e^{\pi/2} + 1.$$

As a check, an antiderivative is $\frac{1}{2}e^x(\sin x - \cos x)$; its derivative is

$$\tfrac{1}{2}e^x(\sin x - \cos x + \cos x + \sin x) = e^x \sin x.$$

Its upper value is $e^{\pi/2}/2$, and its lower value is $-1/2$, confirming the endpoint subtraction.

Step 4: Conclusion Therefore, $\boxed{\int_0^{\pi/2} e^x \sin x \, dx = \frac{e^{\pi/2} + 1}{2}}$.

Quick Tip to Solve: After two integrations by parts, move the repeated integral to the left; keep the negative lower-end contribution.

OR

26(b). Find: $\int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$. [3 marks]

Solution:

Concept:

- The tangent subtraction identity is $\tan U - \tan V = \sin(U-V)/(\cos U \cos V)$, wherever the denominators are nonzero.
- Since $\frac{d}{dx}\log|\cos(x-c)| = -\tan(x-c)$ for a constant c, tangent integrals give logarithms of cosine factors.

Step 1: Rewrite the reciprocal product as a difference of tangents Put $U = x-b$ and $V = x-a$. Their difference is $a-b$, which is independent of x. Hence

$$\tan(x-b) - \tan(x-a) = \frac{\sin(a-b)}{\cos(x-b)\cos(x-a)}.$$

For $\sin(a-b) \neq 0$, the given integrand therefore equals

$$\frac{\tan(x-b) - \tan(x-a)}{\sin(a-b)}.$$

This identity converts a product in the denominator into two standard integrals.

Step 2: Integrate the two tangent terms Taking the constant factor outside and integrating term by term gives

$$\begin{aligned}\int \frac{dx}{\cos(x-a)\cos(x-b)} &= \frac{-\log|\cos(x-b)| + \log|\cos(x-a)|}{\sin(a-b)} + C \\ &= \frac{1}{\sin(a-b)}\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right| + C.\end{aligned}$$

Absolute values ensure a real logarithm on every interval where both cosine factors are nonzero. Differentiating the logarithmic expression returns the tangent difference divided by $\sin(a-b)$, and hence the original integrand.

Step 3: Account for coincident shifts modulo pi If $\sin(a-b) = 0$, write $b = a + k\pi$, where k is an integer. Then

$$\cos(x-b) = (-1)^k\cos(x-a), \qquad \frac{1}{\cos(x-a)\cos(x-b)} = (-1)^k\sec^2(x-a).$$

This special case integrates directly to $(-1)^k\tan(x-a)+C$. In particular, when $a = b$, it is $\tan(x-a)+C$.

Step 4: Conclusion For $\sin(a-b) \neq 0$, the required integral is $\boxed{\frac{1}{\sin(a-b)}\log\left|\frac{\cos(x-a)}{\cos(x-b)}\right| + C}$. If $b = a + k\pi$, it is $\boxed{(-1)^k\tan(x-a) + C}$.

Quick Tip to Solve: Use tan(x-b) minus tan(x-a) so that the numerator is sin(a-b); this keeps the logarithmic sign consistent.

27. Evaluate: $\int_0^{\pi/2} [\log(\sin x) - \log(2\cos x)]\,dx$**.** [3 marks]

Solution:

Concept:

- For positive arguments, $\log(uv) = \log u + \log v$. Thus $\log(2\cos x) = \log 2 + \log(\cos x)$ for $0 < x < \pi/2$.
- The definite-integral reflection property $\int_0^L f(x)\,dx = \int_0^L f(L-x)\,dx$ follows from the substitution $u = L - x$.
- Endpoint logarithmic singularities are integrable: $\int_0^{\varepsilon} \log x\,dx = \varepsilon\log\varepsilon - \varepsilon$. Therefore the sine and cosine logarithm integrals can be treated as convergent improper integrals.

Step 1: Separate the constant logarithm Denote the given integral by I. On the open integration interval both sine and cosine are positive, so the logarithm rule gives

$$\begin{aligned} I &= \int_0^{\pi/2} \log(\sin x)\,dx - \int_0^{\pi/2} \log(\cos x)\,dx - \int_0^{\pi/2} \log 2\,dx \\ &= A - B - \frac{\pi}{2}\log 2. \end{aligned}$$

The terms A and B denote the sine and cosine logarithm integrals, respectively. Near their singular endpoints they behave like $\log x$ or $\log(\pi/2 - x)$, so each converges.

Step 2: Show that the two variable logarithm integrals are equal In A, substitute $u = \pi/2 - x$. Then $dx = -du$, and the limits $0, \pi/2$ become $\pi/2, 0$. Also $\sin x = \cos u$. Consequently,

$$A = \int_{\pi/2}^{0} \log(\cos u)(-du) = \int_0^{\pi/2} \log(\cos u)\,du = B.$$

This substitution can first be applied with the singular endpoints excluded and then extended by limits. Equality of these convergent improper integrals justifies cancelling them.

Step 3: Evaluate the constant contribution Using $A - B = 0$, the only remaining contribution is

$$-\int_0^{\pi/2} \log 2\,dx = -[x\log 2]_0^{\pi/2} = -\frac{\pi}{2}\log 2.$$

The factor 2 inside the second logarithm is essential: it contributes the nonzero constant term even though the sine and cosine logarithm integrals are equal.

Step 4: Conclusion Hence, $\boxed{\int_0^{\pi/2} [\log(\sin x) - \log(2\cos x)]\,dx = -\frac{\pi}{2}\log 2}$.

Quick Tip to Solve: Reflect x to pi/2 minus x to cancel the sine and cosine logarithms, after separating the factor 2 inside the logarithm.

28. Find: $\displaystyle\int \frac{x^2}{(x^2+4)(x^2+9)}\,dx$. [3 marks]

Solution:

Concept:

- For distinct quadratic factors, express the rational function as a sum of simpler fractions. Here constants in the numerators suffice because both the numerator and denominator are even functions of x.
- The standard integral is $\displaystyle\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C$, where $a>0$ and C is an arbitrary constant.

Step 1: Resolve the integrand into partial fractions Let

$$\frac{x^2}{(x^2+4)(x^2+9)} = \frac{A}{x^2+4}+\frac{B}{x^2+9}.$$

Multiplying by the common denominator gives

$$x^2 = A(x^2+9)+B(x^2+4) = (A+B)x^2+9A+4B.$$

Comparing the coefficient of x^2 and the constant term, we get $A+B=1$ and $9A+4B=0$. Subtracting four times the first equation from the second yields $5A=-4$, so $A=-4/5$ and $B=9/5$.

Step 2: Integrate the two standard terms Thus the required integral becomes

$$\begin{aligned} I &= -\frac{4}{5}\int \frac{dx}{x^2+2^2}+\frac{9}{5}\int \frac{dx}{x^2+3^2} \\ &= -\frac{4}{5}\cdot\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)+\frac{9}{5}\cdot\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+C. \end{aligned}$$

As a check, differentiating the two inverse-tangent terms gives $-4/[5(x^2+4)]+9/[5(x^2+9)]$. On taking their common denominator, the numerator is $(-4(x^2+9)+9(x^2+4))/5 = x^2$, exactly the given integrand. Both denominator factors are positive for every real x.

Step 3: Conclusion The required integral is $\boxed{-\frac{2}{5}\tan^{-1}\left(\frac{x}{2}\right)+\frac{3}{5}\tan^{-1}\left(\frac{x}{3}\right)+C}$.

Quick Tip to Solve: Compare the constant term as well as the coefficient of x squared before integrating; the partial-fraction coefficient of the first term is negative.

29(a). Find the general solution of the differential equation: $\displaystyle\frac{dy}{dx}-\frac{2y}{x}=\sin\frac{1}{x}$. [3 marks]

Solution:

Concept:

- A linear differential equation $y\prime+P(x)y=Q(x)$ has integrating factor IF $=e^{\int P(x)\,dx}$. Multiplication by this factor turns its left-hand side into the derivative of y IF.
- Here $x\neq 0$. Solve on an interval entirely on one side of zero, and retain an arbitrary constant for

Quick Tip to Solve: After multiplying by the integrating factor, use t = 1/x and keep the minus sign in dt; this converts the integral to a cosine.

the general solution.

Step 1: Identify the integrating factor Comparing the given equation

$$\frac{dy}{dx} - \frac{2}{x}y = \sin\left(\frac{1}{x}\right)$$

with the linear form gives $P(x) = -2/x$ and $Q(x) = \sin(1/x)$. Hence

$$\text{IF} = e^{-2\int dx/x} = e^{-2\ln|x|} = x^{-2}.$$

Multiplying every term by x^{-2} gives

$$\frac{1}{x^2}\frac{dy}{dx} - \frac{2y}{x^3} = \frac{1}{x^2}\sin\left(\frac{1}{x}\right).$$

By the product rule, the left-hand side is $\dfrac{d}{dx}(y/x^2)$.

Step 2: Integrate using a reciprocal substitution Integration therefore gives

$$\frac{y}{x^2} = \int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\,dx + C.$$

Set $t = 1/x$. Then $dt = -dx/x^2$, so the integral is $-\int \sin t\,dt = \cos t$. Consequently, the integrated equation is $y/x^2 = \cos(1/x) + C$. For verification, differentiate the resulting expression to obtain

$$\frac{dy}{dx} = 2x\left[\cos\left(\frac{1}{x}\right) + C\right] + \sin\left(\frac{1}{x}\right).$$

Subtracting $2y/x$ cancels the first term and leaves precisely $\sin(1/x)$. The constant is unrestricted because no initial value is specified.

Step 3: Conclusion The general solution is $\boxed{y = x^2\left[\cos\left(\frac{1}{x}\right) + C\right]}$, on any interval with $x \neq 0$.

Quick Tip to Solve: After multiplying by the integrating factor, use t = 1/x and keep the minus sign in dt; this converts the integral to a cosine.

OR

29(b). Find the particular solution of the differential equation $\dfrac{dy}{dx} = \sin(x+y) + \sin(x-y)$, given that $y = 0$ when $x = \dfrac{\pi}{4}$. [3 marks]

Solution:

Concept:

- The identity $\sin(x+y) + \sin(x-y) = 2\sin x\cos y$ converts the given equation into a separable differential equation.

Quick Tip to Solve: Apply the sum-to-product identity first, then substitute the initial point into the integrated equation; ln(1) makes the constant immediate.

- The standard antiderivative is $\int \sec y\, dy = \ln|\sec y + \tan y| + C$. The initial condition determines the constant of integration.

Step 1: Use the identity and separate the variables By the sum-to-product identity,

$$\frac{dy}{dx} = 2\sin x \cos y.$$

At the initial point $y = 0$, we have $\cos y = 1$, so division by $\cos y$ is valid on a neighbourhood of that point. Separating variables gives

$$\sec y\, dy = 2\sin x\, dx.$$

Integrating both sides, we obtain

$$\ln|\sec y + \tan y| = -2\cos x + C.$$

The constant solutions with $\cos y = 0$ do not satisfy the given initial condition and hence are not relevant to this particular solution.

Step 2: Apply the specified initial condition and verify Substitute $x = \pi/4$ and $y = 0$ into the integrated relation. Since $\sec 0 + \tan 0 = 1$, we get

$$0 = -2\cos\frac{\pi}{4} + C = -\sqrt{2} + C,$$

so $C = \sqrt{2}$. To verify the differential equation, note that

$$\frac{d}{dy}\ln|\sec y + \tan y| = \frac{\sec y \tan y + \sec^2 y}{\sec y + \tan y} = \sec y.$$

Differentiating the relation with this constant gives $\sec y\, dy/dx = 2\sin x$, or $dy/dx = 2\sin x \cos y$, which equals the original right-hand side. At the initial point, both sides of the integrated relation are zero. The solution branch through $y = 0$ lies in $-\pi/2 < y < \pi/2$, where the logarithm is well defined.

Step 3: Conclusion The particular solution through the given point is $\boxed{\ln|\sec y + \tan y| = \sqrt{2} - 2\cos x}$, taking the branch through $y = 0$.

Quick Tip to Solve: Apply the sum-to-product identity first, then substitute the initial point into the integrated equation; $\ln(1)$ makes the constant immediate.

30. Solve the following linear programming problem graphically: Maximize $z = 600x + 400y$ subject to the constraints

$x + 2y \leq 12,$

$2x + y \leq 12,$

$x + 1.25y \geq 5,$

$x,\ y \geq 0.$ [3 marks]

Solution:

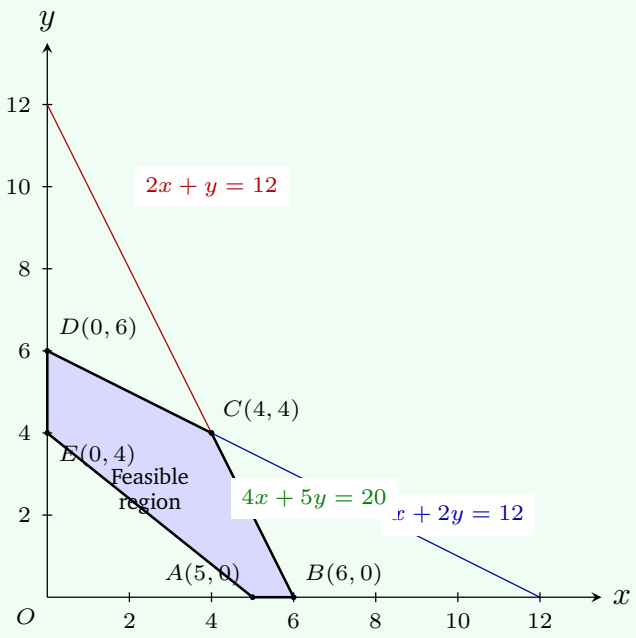
Concept:

- The feasible region is the common part of the half-planes satisfying all constraints, restricted to the first quadrant by $x, y \geq 0$.
- For a bounded feasible polygon, a linear objective $z = 600x + 400y$ attains its maximum at a corner point. Evaluate the objective at every feasible vertex.

Step 1: Draw the boundary lines and choose the required half-planes The boundary $x + 2y = 12$ has intercepts $(12, 0)$ and $(0, 6)$, while $2x + y = 12$ has intercepts $(6, 0)$ and $(0, 12)$. Both inequalities select the side containing the origin. The third boundary is

$$x + 1.25y = 5 \quad \Longleftrightarrow \quad 4x + 5y = 20,$$

with intercepts $(5, 0)$ and $(0, 4)$. Its inequality selects the side away from the origin. Shade only the intersection of these three half-planes in the first quadrant; it is the polygon shown below.



Feasible region ABCDE and its boundary lines

Step 2: Determine all corners of the feasible region On the x-axis, the constraints give $5 \leq x \leq 6$, so two vertices are $A(5, 0)$ and $B(6, 0)$. On the y-axis, they give $4 \leq y \leq 6$, so the other axis vertices are

Quick Tip to Solve: Use the intercepts to shade each half-plane first; the lower constraint removes the origin and creates two additional vertices.

$D(0,6)$ and $E(0,4)$. For the intersection of the upper boundaries,

$$x+2y=12, \quad 2x+y=12,$$
$$x-y=0, \qquad x=y=4.$$

Thus $C=(4,4)$; it satisfies the third inequality since $4+1.25(4)=9\geq 5$. The two upper boundary lines do not cut the segment joining A to E, so these five points are all the feasible vertices.

Step 3: Evaluate the objective at the feasible vertices The feasible region is bounded and nonempty. Apply the corner-point method:

Vertex	$z=600x+400y$
$A(5,0)$	3000
$B(6,0)$	3600
$C(4,4)$	4000
$D(0,6)$	2400
$E(0,4)$	1600

The value at C exceeds the values at all four other corners.

Step 4: Conclusion The maximum value is $\boxed{z_{\max}=4000}$, attained at $\boxed{x=4,\ y=4}$.

Quick Tip to Solve: Use the intercepts to shade each half-plane first; the lower constraint removes the origin and creates two additional vertices.

31(a). The probability distribution of a random variable X is given below:

X	1	2	3
$P(X)$	$\frac{k}{2}$	$\frac{k}{3}$	$\frac{k}{6}$

(i) Find the value of k. (ii) Find $P(1\leq X<3)$. (iii) Find $E(X)$, the mean of X. [3 marks]

Solution:

Concept:

- Probabilities in a discrete distribution are nonnegative and sum to one: $\sum_x P(X=x)=1$.
- The probability of an interval is the sum over the values lying in it. The expectation is $E(X)=\sum_x xP(X=x)$, a probability-weighted mean.

Step 1: (i) Normalize the distribution to find k The random variable has only the three listed values $1,2,3$. Their probabilities exhaust the sample space, so

$$\frac{k}{2}+\frac{k}{3}+\frac{k}{6}=1,$$
$$k\left(\frac{3+2+1}{6}\right)=1,$$
$$k=1.$$

Quick Tip to Solve: For a strict upper bound, exclude the endpoint before summing; for the mean, multiply every value by its own probability.

Thus the three probabilities are $1/2, 1/3, 1/6$. They are all nonnegative, each is at most one, and their sum equals one, confirming that the resulting distribution is valid.

Step 2: (ii) Sum the probabilities in the specified interval The event $1 \le X < 3$ includes $X = 1$ and $X = 2$, but excludes $X = 3$ because the upper bound is strict. These two events are mutually exclusive. Therefore,

$$P(1 \le X < 3) = P(X = 1) + P(X = 2) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}.$$

Equivalently, its complement in the given support is just $X = 3$, giving $1 - P(X = 3) = 1 - 1/6 = 5/6$.

Step 3: (iii) Calculate the expectation Multiply each possible value by its corresponding probability and add:

$$\begin{aligned} E(X) &= 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{3}\right) + 3\left(\frac{1}{6}\right) \\ &= \frac{1}{2} + \frac{2}{3} + \frac{1}{2} = \frac{5}{3}. \end{aligned}$$

The mean lies between the smallest and largest possible values, 1 and 3, as required. The values are not equally likely, so taking their ordinary arithmetic mean would not give the expectation.

Step 4: Conclusion $\boxed{k = 1, \quad P(1 \le X < 3) = \frac{5}{6}, \quad E(X) = \frac{5}{3}}$.

Quick Tip to Solve: For a strict upper bound, exclude the endpoint before summing; for the mean, multiply every value by its own probability.

OR

31(b). A **and** B **are independent events such that** $P(A \cap \bar{B}) = \frac{1}{4}$ **and** $P(\bar{A} \cap B) = \frac{1}{6}$**. Find** $P(A)$ **and** $P(B)$**.** [3 marks]

Solution:

Concept:

- For independent events A, B, write $p = P(A)$, $q = P(B)$. Then $P(A \cap B) = pq$, and complements also factor: $P(A \cap \bar{B}) = p(1 - q)$.
- Both probabilities must lie in $[0, 1]$. Solve the simultaneous equations and retain every pair satisfying the original equations.

Step 1: Translate the given probabilities using independence Set $P(A) = p$ and $P(B) = q$. Independence gives

$$p(1 - q) = \frac{1}{4},$$
$$(1 - p)q = \frac{1}{6}.$$

Quick Tip to Solve: Subtract the two complement-intersection equations to cancel the product pq; then check both roots against the original conditions.

Subtract the second equation from the first. The product terms cancel:

$$p - pq - q + pq = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}.$$

Hence $p - q = 1/12$, so $p = q + 1/12$. In particular, the probability of A exceeds that of B; this is a useful sign check before substitution.

Step 2: Solve the quadratic and retain both candidate pairs Substitute $p = q + 1/12$ into $(1-p)q = 1/6$:

$$\left(\frac{11}{12} - q\right) q = \frac{1}{6},$$
$$12q^2 - 11q + 2 = 0,$$
$$(3q - 2)(4q - 1) = 0.$$

Thus $q = 2/3$ or $q = 1/4$. Using $p = q + 1/12$, the respective values of p are $3/4$ and $1/3$. Every value lies in $[0, 1]$, so neither pair is rejected on probability bounds.

Step 3: Verify both pairs in the original equations For $(p, q) = (3/4, 2/3)$,

$$p(1 - q) = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}, \qquad (1 - p)q = \frac{1}{4} \cdot \frac{2}{3} = \frac{1}{6}.$$

For $(p, q) = (1/3, 1/4)$,

$$p(1 - q) = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}, \qquad (1 - p)q = \frac{2}{3} \cdot \frac{1}{4} = \frac{1}{6}.$$

Both pairs satisfy the given conditions exactly. The information supplied therefore admits two possible pairs, and a complete solution includes both.

Step 4: Conclusion $\boxed{(P(A), P(B)) = \left(\frac{3}{4}, \frac{2}{3}\right) \text{ or } \left(\frac{1}{3}, \frac{1}{4}\right)}$.

Quick Tip to Solve: Subtract the two complement-intersection equations to cancel the product pq; then check both roots against the original conditions.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32(a). **If** $A = \begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}$ **and** $B^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix}$**, find** $(AB)^{-1}$**.** [5 marks]

Solution:

Concept:

- For invertible square matrices, $(AB)^{-1} = B^{-1}A^{-1}$. The order reverses because $(AB)(B^{-1}A^{-1}) = AIA^{-1} = I$.

Quick Tip to Solve: Since B inverse is already supplied, find only A inverse and left-multiply it by B inverse; reverse the factor order before multiplying.

- For a square matrix A, $A^{-1} = \dfrac{\operatorname{adj} A}{|A|}$, provided $|A| \neq 0$. Here $\operatorname{adj} A$ is the transpose of the cofactor matrix.
- The cofactor of entry a_{ij} is $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant obtained by deleting row i and column j.

Step 1: Check that the inverse of A exists Expand the determinant along the first row, retaining the alternating signs:

$$\begin{aligned} |A| &= 1(3 \cdot 1 - 0 \cdot (-2)) - 2((-1) \cdot 1 - 0 \cdot 0) \\ &\quad + (-2)((-1)(-2) - 3 \cdot 0) \\ &= 3 + 2 - 4 = 1. \end{aligned}$$

Since this determinant is nonzero, A^{-1} exists. Also, the given notation B^{-1} means that B is invertible, so the inverse-of-a-product identity applies.

Step 2: Find the cofactor matrix and adjugate The first-row cofactors are $C_{11} = 3$, $C_{12} = -(-1) = 1$ and $C_{13} = 2$. For the second row, $C_{21} = -(2-4) = 2$, $C_{22} = 1$ and $C_{23} = -(-2) = 2$. For the third row, $C_{31} = 6$, $C_{32} = -(0-2) = 2$ and $C_{33} = 3 + 2 = 5$. Thus

$$C = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{pmatrix}, \qquad \operatorname{adj} A = C^T = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

As $|A| = 1$, division by the determinant leaves this adjugate unchanged: $A^{-1} = \operatorname{adj} A$.

Step 3: Multiply in the reversed order Using $(AB)^{-1} = B^{-1}A^{-1}$, compute each entry by multiplying a row of the first matrix by a column of the second:

$$B^{-1}A^{-1} = \begin{pmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}.$$

The first row is $(9-1+2,\ 6-1+2,\ 18-2+5)$; the second is $(-45+6-10,\ -30+6-10,\ -90+12-25)$; and the third is $(15-2+4,\ 10-2+4,\ 30-4+10)$. This explicitly preserves the correct order of the factors; interchanging them is not a valid matrix identity.

Step 4: Conclusion Therefore, $\boxed{(AB)^{-1} = \begin{pmatrix} 10 & 7 & 21 \\ -49 & -34 & -103 \\ 17 & 12 & 36 \end{pmatrix}}$.

Quick Tip to Solve: Since B inverse is already supplied, find only A inverse and left-multiply it by B inverse; reverse the factor order before multiplying.

OR

32(b). Solve the following system of equations by matrix method:

$x + 2y + 3z = 6$

$2x - y + z = 2$

$3x + 2y - 2z = 3$ [5 marks]

Solution:

Concept:

- A system of three linear equations is written as $MX = F$, where M contains the coefficients, X is the column of unknowns and F is the column of constants.
- If $|M| \neq 0$, the unique solution is $X = M^{-1}F = \dfrac{1}{|M|}(\text{adj}\, M)F$. The adjugate is the transpose of the cofactor matrix.

Step 1: Express the equations in matrix form Take the coefficients of x, y, z in the same order in every equation. In particular, the second equation has coefficient -1 for y, while the third has coefficient -2 for z. Then the given system becomes

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}.$$

Denote these three matrices by M, X, F, respectively, so that $MX = F$. We will find the inverse of M before multiplying by the constants column.

Step 2: Evaluate the determinant Expanding along the first row gives

$$\begin{aligned} |M| &= 1\{(-1)(-2) - 1 \cdot 2\} - 2\{2(-2) - 1 \cdot 3\} \\ &\quad + 3\{2 \cdot 2 - (-1) \cdot 3\} \\ &= 1(0) - 2(-7) + 3(7) = 35. \end{aligned}$$

Thus $|M| \neq 0$. The inverse exists and the equations have a unique solution; there is no need to consider a singular-system case.

Step 3: Calculate the inverse using cofactors Including the cofactor signs, the first row of cofactors is $(0, 7, 7)$. The second row is $(10, -11, 4)$, obtained from $-(2(-2) - 3 \cdot 2) = 10$, $1(-2) - 3 \cdot 3 = -11$, and $-(1 \cdot 2 - 2 \cdot 3) = 4$. The third row is $(5, 5, -5)$. Therefore

$$C = \begin{pmatrix} 0 & 7 & 7 \\ 10 & -11 & 4 \\ 5 & 5 & -5 \end{pmatrix}, \quad \text{adj}\, M = \begin{pmatrix} 0 & 10 & 5 \\ 7 & -11 & 5 \\ 7 & 4 & -5 \end{pmatrix}.$$

Consequently $M^{-1} = \dfrac{1}{35}\,\text{adj}\, M$. Transposing the cofactor matrix is essential before using it as the adjugate.

Quick Tip to Solve: Keep the columns ordered as x, y, z throughout, and check the final values in all three original equations to catch sign errors.

Step 4: Multiply by the constants and check Premultiplication of $MX = F$ by M^{-1} gives

$$X = \frac{1}{35}\begin{pmatrix} 0 & 10 & 5 \\ 7 & -11 & 5 \\ 7 & 4 & -5 \end{pmatrix}\begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix}$$
$$= \frac{1}{35}\begin{pmatrix} 0+20+15 \\ 42-22+15 \\ 42+8-15 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

As a check in the original equations, these values give $1+2+3=6$, $2-1+1=2$, and $3+2-2=3$, agreeing with all three constants.

Step 5: Conclusion Hence the unique solution by the matrix method is $\boxed{x=1,\ y=1,\ z=1}$.

Quick Tip to Solve: Keep the columns ordered as x, y, z throughout, and check the final values in all three original equations to catch sign errors.

33(a). Find the vector and the Cartesian equations of a line passing through the point $(1, 2, -4)$ and parallel to the line joining the points $A(3, 3, -5)$ and $B(1, 0, -11)$. Hence, find the distance between the two lines. [5 marks]

Solution:

Concept:

- A line through a point with position vector $\vec{p}$ and nonzero direction vector $\vec{d}$ has vector equation $\vec{r} = \vec{p} + \lambda\vec{d}$, where $\lambda \in \mathbb{R}$.
- If $\vec{d} = a\hat{i} + b\hat{j} + c\hat{k}$, its Cartesian equation is $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$, provided the three direction components are nonzero.
- For two parallel lines with common direction $\vec{d}$, the perpendicular distance is $D = \frac{|(\vec{a}-\vec{p})\times\vec{d}|}{|\vec{d}|}$, where $\vec{a}$ and $\vec{p}$ are position vectors of points on the respective lines.

Step 1: Obtain the common direction Let $P = (1, 2, -4)$ be the given point. The line joining $A = (3, 3, -5)$ and $B = (1, 0, -11)$ has direction

$$\overrightarrow{AB} = (1-3)\hat{i} + (0-3)\hat{j} + (-11+5)\hat{k} = -2\hat{i} - 3\hat{j} - 6\hat{k}.$$

Multiplying a direction vector by a nonzero constant does not change the line's direction. We therefore use $\vec{d} = 2\hat{i} + 3\hat{j} + 6\hat{k}$ for both lines. Its magnitude is $|\vec{d}| = \sqrt{4+9+36} = 7$.

Step 2: Form the required vector and Cartesian equations The required line passes through P and has direction $\vec{d}$. Substituting these in the line equation gives

$$\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k}), \qquad \lambda \in \mathbb{R}.$$

Quick Tip to Solve: For parallel lines, take any connecting vector and divide the magnitude of its cross product with the common direction by the direction's magnitude.

Thus $x = 1 + 2\lambda$, $y = 2 + 3\lambda$ and $z = -4 + 6\lambda$. Eliminating λ gives the Cartesian equation

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}.$$

At $\lambda = 0$, this returns the specified point, while its direction is a scalar multiple of $\overrightarrow{AB}$, verifying both requirements.

Step 3: Calculate the perpendicular separation of the lines The original line may be written as $\vec{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + t\vec{d}$. A connecting vector from the required line to this line is $\overrightarrow{PA} = 2\hat{i} + \hat{j} - \hat{k}$. Only the component of this connecting vector perpendicular to $\vec{d}$ measures the distance between the lines; its component along $\vec{d}$ does not contribute. Hence

$$\begin{aligned}
\overrightarrow{PA} \times \vec{d} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix} \\
&= 9\hat{i} - 14\hat{j} + 4\hat{k}, \\
|\overrightarrow{PA} \times \vec{d}| &= \sqrt{81 + 196 + 16} = \sqrt{293}, \\
D &= \frac{\sqrt{293}}{7}.
\end{aligned}$$

As a check, the squared perpendicular component is $|\overrightarrow{PA}|^2 - (\overrightarrow{PA} \cdot \vec{d})^2/|\vec{d}|^2 = 6 - 1/49 = 293/49$, agreeing with the cross-product calculation.

Step 4: Conclusion The required vector equation is $\boxed{\vec{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})}$, and its Cartesian equation is $\boxed{\frac{x-1}{2} = \frac{y-2}{3} = \frac{z+4}{6}}$. The distance between the two parallel lines is $\boxed{\frac{\sqrt{293}}{7} \text{ units}}$.

Quick Tip to Solve: For parallel lines, take any connecting vector and divide the magnitude of its cross product with the common direction by the direction's magnitude.

OR

33(b). Find the equations of the line passing through the points $A(1, 2, 3)$ and $B(3, 5, 9)$. Hence, find the coordinates of the points on this line which are at a distance of 14 units from point B. [5 marks]

Solution:

Concept:

- The direction vector of the line through two distinct points A and B is $\overrightarrow{AB} = \vec{b} - \vec{a}$, where $\vec{a}$ and $\vec{b}$ are their position vectors.
- The vector equation of a line through $A(x_0, y_0, z_0)$ with direction (a, b, c) is $\vec{r} = \vec{a} + t(a\hat{i} + b\hat{j} + c\hat{k})$. Eliminating t gives its Cartesian equation.
- If a point on a line is written relative to B as $\vec{r} = \vec{b} + s\vec{d}$, its distance from B is $|s|\,|\vec{d}|$. The absolute value allows points on both sides of B.

Quick Tip to Solve: Anchor the parameter at B when distance is measured from B; use both signs of the parameter to obtain the two points on opposite sides.

Step 1: Find the direction vector of the line For $A = (1, 2, 3)$ and $B = (3, 5, 9)$, subtract corresponding coordinates:

$$\overrightarrow{AB} = (3-1)\hat{i} + (5-2)\hat{j} + (9-3)\hat{k} = 2\hat{i} + 3\hat{j} + 6\hat{k}.$$

Write $\vec{d} = 2\hat{i} + 3\hat{j} + 6\hat{k}$. Its length is

$$|\vec{d}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7.$$

Thus one unit change of the parameter in the direction $\vec{d}$ represents a displacement of length 7, not length 1.

Step 2: Write both equations of the line Using point A and the direction vector found above, the vector equation is

$$\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + t(2\hat{i} + 3\hat{j} + 6\hat{k}), \qquad t \in \mathbb{R}.$$

Its coordinate form is $x = 1 + 2t$, $y = 2 + 3t$, $z = 3 + 6t$. Eliminating the common parameter gives

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}.$$

Setting $t = 0$ gives A, and setting $t = 1$ gives B; hence this equation indeed represents the line through the two prescribed points.

Step 3: Use the distance condition with B as the reference point To measure distance from B directly, express a general point Q of the same line as

$$Q = (3 + 2s,\ 5 + 3s,\ 9 + 6s), \qquad s \in \mathbb{R}.$$

Therefore $\overrightarrow{BQ} = (2s, 3s, 6s)$, and

$$\begin{aligned} BQ &= \sqrt{(2s)^2 + (3s)^2 + (6s)^2} \\ &= 7|s|. \end{aligned}$$

The prescribed distance $BQ = 14$ gives $|s| = 2$. There are two possibilities, $s = 2$ and $s = -2$, corresponding to travel in opposite directions along the line.

Step 4: Substitute and verify both points For $s = 2$, the coordinates are $(3+4, 5+6, 9+12) = (7, 11, 21)$. For $s = -2$, they are $(3 - 4, 5 - 6, 9 - 12) = (-1, -1, -3)$. Their displacement vectors from B are respectively $(4, 6, 12)$ and $(-4, -6, -12)$. Each has length $\sqrt{16 + 36 + 144} = 14$, as required. In the equation based at A, their parameter values are $t = 3$ and $t = -1$, respectively, verifying that both also lie on the original line.

Step 5: Conclusion The vector equation is $\boxed{\vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + t(2\hat{i} + 3\hat{j} + 6\hat{k})}$, and the Cartesian equation is $\boxed{\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{6}}$. The two points at distance 14 units from B are $\boxed{(7, 11, 21) \text{ and } (-1, -1, -3)}$.

Quick Tip to Solve: Anchor the parameter at B when distance is measured from B; use both signs of the parameter to obtain the two points on opposite sides.

34. Find the area of the region bounded by the lines $y = 4x + 5$, $x + y = 5$ and $4y = x + 5$, using integration. [5 marks]

Solution:

Concept:

- The area between an upper curve $y = f(x)$ and a lower curve $y = g(x)$, from $x = a$ to $x = b$, is $\int_a^b [f(x) - g(x)]\,dx$, provided $f(x) \geq g(x)$ on that interval.
- The intersections of the three boundary lines determine the vertices and the limits of integration. If the upper boundary changes at a vertex, split the integral at its x-coordinate.

Step 1: Express the lines and find their intersections Write the three lines as

$$y = 4x + 5, \qquad y = 5 - x, \qquad y = \frac{x+5}{4}.$$

The first two meet where $4x + 5 = 5 - x$, giving $x = 0$ and $y = 5$. Thus one vertex is $A(0, 5)$. The first and third meet where

$$4x + 5 = \frac{x+5}{4} \implies 16x + 20 = x + 5 \implies x = -1, \quad y = 1.$$

Thus another vertex is $B(-1, 1)$. Finally, the second and third meet where

$$5 - x = \frac{x+5}{4} \implies 20 - 4x = x + 5 \implies x = 3, \quad y = 2.$$

The third vertex is $C(3, 2)$. The bounded triangle extends from $x = -1$ to $x = 3$.

Step 2: Identify the upper and lower boundaries The line BC, namely $y = (x + 5)/4$, is the lower boundary over the whole region. For $-1 \leq x \leq 0$, the upper boundary is BA, namely $y = 4x + 5$. For $0 \leq x \leq 3$, the upper boundary is AC, namely $y = 5 - x$. Hence the upper boundary changes at $x = 0$, and the required area is the sum of two integrals.

Step 3: Evaluate the two integrals Let S denote the required area. Then

$$\begin{aligned} S &= \int_{-1}^{0} \left(4x + 5 - \frac{x+5}{4}\right) dx + \int_{0}^{3} \left(5 - x - \frac{x+5}{4}\right) dx \\ &= \frac{15}{4} \int_{-1}^{0} (x + 1)\,dx + \frac{5}{4} \int_{0}^{3} (3 - x)\,dx \\ &= \frac{15}{4} \left[\frac{x^2}{2} + x\right]_{-1}^{0} + \frac{5}{4} \left[3x - \frac{x^2}{2}\right]_{0}^{3} \\ &= \frac{15}{4} \left(\frac{1}{2}\right) + \frac{5}{4} \left(\frac{9}{2}\right) \\ &= \frac{15}{8} + \frac{45}{8}. \end{aligned}$$

Step 4: Conclusion The required area is $\boxed{\frac{15}{2} \text{ square units}}$.

Quick Tip to Solve: First find the three vertices; then split at x=0, where the upper boundary switches from y=4x+5 to y=5-x.

35. A relation R is defined on a set of real numbers $\mathbb{R}$ as $R = \{(x, y) : x \cdot y$ is an irrational number$\}$. Check whether R is reflexive, symmetric and transitive or not. [5 marks]

Solution:

Concept:

- A relation R on $\mathbb{R}$ is reflexive if $(x, x) \in R$ for every real number x, and symmetric if $(x, y) \in R$ implies $(y, x) \in R$.
- Transitivity means that $(x, y) \in R$ and $(y, z) \in R$ together imply $(x, z) \in R$, for all $x, y, z \in \mathbb{R}$. One counterexample disproves a universal property.
- Here $(x, y) \in R$ precisely when the real product xy is irrational, that is, when it cannot be written as a ratio of two integers with a nonzero denominator.

Step 1: Test reflexivity using a diagonal pair For reflexivity, every real number x would have to satisfy $(x, x) \in R$. By the definition of this relation, this requires x^2 to be irrational for every real x. Choose $x = 1$. Then

$$x \cdot x = 1 \cdot 1 = 1,$$

which is rational. Hence $(1, 1) \notin R$, although $1 \in \mathbb{R}$. Therefore the condition for reflexivity fails. It is unnecessary to test all real numbers after finding this counterexample. Even taking an irrational number would not guarantee an irrational square: for example, $(\sqrt{2})^2 = 2$.

Step 2: Prove symmetry for arbitrary related elements Let $x, y \in \mathbb{R}$ be arbitrary, and suppose $(x, y) \in R$. The defining condition says that xy is irrational. Multiplication of real numbers is commutative, so

$$yx = xy.$$

Thus yx is the very same irrational number, which gives $(y, x) \in R$. This implication applies to every related pair, so R is symmetric. Pairs whose product is rational do not affect this proof, since the implication starts only with a pair that belongs to R.

Step 3: Disprove transitivity with a three-element example Choose

$$x = \sqrt{2}, \qquad y = 1, \qquad z = \sqrt{2}.$$

These are all real numbers; the definition of transitivity does not require them to be distinct. Check both premises separately:

$$xy = \sqrt{2} \cdot 1 = \sqrt{2}, \qquad yz = 1 \cdot \sqrt{2} = \sqrt{2}.$$

Since $\sqrt{2}$ is irrational, both $(\sqrt{2}, 1) \in R$ and $(1, \sqrt{2}) \in R$. However,

$$xz = \sqrt{2} \cdot \sqrt{2} = 2,$$

which is rational. Consequently $(\sqrt{2}, \sqrt{2}) \notin R$. The two premises of transitivity are true while the required consequent is false, so R is not transitive.

Step 4: Conclusion The relation R is $\boxed{\text{not reflexive, symmetric, and not transitive}}$.

Quick Tip to Solve: Use x=1 to test reflexivity, commute xy for symmetry, and use the chain sqrt(2), 1, sqrt(2) to disprove transitivity.

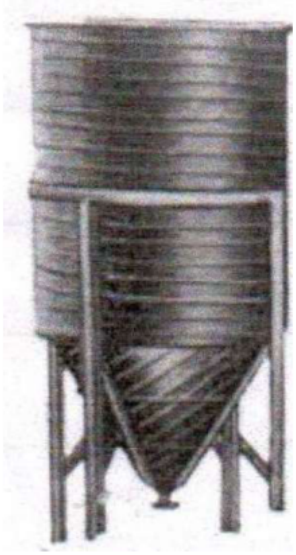
Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. A tank, as shown in the figure below, formed using a combination of a cylinder and a cone, offers better drainage as compared to a flat bottomed tank.

A tap is connected to such a tank whose conical part is full of water. Water is dripping out from a tap at the bottom at the uniform rate of 2 cm^3/s. The semi-vertical angle of the conical tank is $45°$.

On the basis of given information, answer the following questions. [4 marks]



36(i). Find the volume of water in the tank in terms of its radius r. [1 mark]

Solution:

Concept:

- For a right circular cone, $V = \frac{1}{3}\pi r^2 h$, where r is the water-surface radius and h is its vertical depth.
- The semi-vertical angle is measured between the axis and the slant side; hence $\tan 45° = r/h$.

Step 1: Relate the radius and depth Take a vertical section through the axis of the conical water volume. The radius is opposite the semi-vertical angle and the depth is adjacent to it. Therefore

$$\frac{r}{h} = \tan 45° = 1, \qquad h = r.$$

This relation holds for the water cone as its level falls, because its side follows the fixed conical tank.

Step 2: Express the volume using one variable The water occupies the conical part, so use the cone-volume formula rather than the cylinder-volume formula. Substituting the geometrical relation gives

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (r).$$

If the radius is measured in centimetres, this expression gives volume in cubic centimetres.

Step 3: Conclusion The volume of water is $\boxed{V = \frac{\pi r^3}{3}\ \text{cm}^3}$, with r in centimetres.

Quick Tip to Solve: Use the semi-vertical angle to eliminate h before applying the cone-volume formula.

36(ii). Find rate of change of radius at an instant when $r = 2\sqrt{2}$ cm. [1 mark]

Solution:

Concept:

- For this cone, $\tan 45^\circ = r/h = 1$, so $h = r$ and $V = \pi r^3/3$.
- A rate of outflow reduces the stored volume: $dV/dt = -2\,\text{cm}^3/\text{s}$. The chain rule relates this to dr/dt.

Step 1: Differentiate the volume relation Both radius and volume depend on time t. Differentiate $V = \pi r^3/3$ with respect to time:

$$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt}.$$

The outflow has magnitude $2\,\text{cm}^3/\text{s}$, but the derivative of the water volume is negative since water leaves the tank.

Step 2: Substitute the given radius At the specified instant, $r = 2\sqrt{2}\,\text{cm}$, so $r^2 = 8\,\text{cm}^2$. Hence

$$-2 = 8\pi \frac{dr}{dt}, \qquad \frac{dr}{dt} = -\frac{2}{8\pi}\,\text{cm/s}.$$

The negative sign is consistent with the water surface shrinking as the conical part drains.

Step 3: Conclusion The rate of change of radius is $\boxed{\frac{dr}{dt} = -\frac{1}{4\pi}\,\text{cm/s}}$.

Quick Tip to Solve: Put a minus sign on the volume rate before differentiating: outflow means the stored volume decreases.

36(iii)(a). Find the rate at which the wet surface of the conical tank is decreasing at an instant when radius $r = 2\sqrt{2}$ cm. [2 marks]

Solution:

Concept:

- The semi-vertical angle gives $h = r$. Thus the slant height of the water cone is $l = \sqrt{r^2 + h^2} = \sqrt{2}r$.
- The wet surface is the curved tank wall in contact with water: $S = \pi r l$. The free water surface is not part of this wet wall.
- Volume decreases at $dV/dt = -2\,\text{cm}^3/\text{s}$; use the chain rule to obtain the wall-area rate.

Step 1: Write the wet surface area in terms of radius The wetted portion of the conical wall extends from the vertex to the current water level. Its slant height is $l = \sqrt{r^2 + h^2}$. Since $h = r$, we have

$$l = \sqrt{2}r, \qquad S = \pi r l = \sqrt{2}\pi r^2.$$

Only the curved surface is included; adding a circular base would count the free water surface rather than the tank wall.

Quick Tip to Solve: The wet surface means the curved conical wall: substitute $l = \sqrt{2}r$ in $S = \pi r l$.

Step 2: Find the instantaneous radius rate Using $V = \pi r^3/3$, differentiate with respect to time:

$$\frac{dV}{dt} = \pi r^2 \frac{dr}{dt}.$$

At $r = 2\sqrt{2}$ cm, the outflow gives

$$-2 = 8\pi \frac{dr}{dt}, \qquad \frac{dr}{dt} = -\frac{1}{4\pi} \text{ cm/s}.$$

Step 3: Differentiate the wet surface area Apply the chain rule to $S = \sqrt{2}\pi r^2$:

$$\frac{dS}{dt} = 2\sqrt{2}\pi r \frac{dr}{dt}.$$

Substitution at the specified instant yields

$$\frac{dS}{dt} = 2\sqrt{2}\pi(2\sqrt{2})\left(-\frac{1}{4\pi}\right) = -2 \text{ cm}^2/\text{s}.$$

This signed derivative is negative, as expected for a shrinking wetted wall.

Step 4: Conclusion The wet surface is decreasing at $\boxed{2 \text{ cm}^2/\text{s}}$; equivalently, $dS/dt = -2 \text{ cm}^2/\text{s}$.

Quick Tip to Solve: The wet surface means the curved conical wall: substitute $l = \sqrt{2}r$ in $S = \pi r l$.

OR

36(iii)(b). Find the rate of change of height h at an instant when slant height is 4 cm. [2 marks]

Solution:

Concept:

- The semi-vertical angle is $45°$, so $r = h$. The slant height satisfies $l^2 = r^2 + h^2 = 2h^2$.
- The water volume is $V = \pi h^3/3$. Outflow gives the signed volume rate $dV/dt = -2 \text{ cm}^3/\text{s}$.

Step 1: Determine the depth from the slant height In an axial right triangle, the two perpendicular sides are the water radius and depth. Since $\tan 45° = r/h = 1$, they are equal. At the specified instant $l = 4$ cm, so

$$l^2 = r^2 + h^2 = 2h^2, \qquad 16 = 2h^2.$$

Taking the positive root because height is a length gives $h = 2\sqrt{2}$ cm. The value 4 cm is the slant height, not the vertical depth.

Step 2: Relate the volume rate to the depth rate Eliminate the radius from the cone-volume formula using $r = h$:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3.$$

Quick Tip to Solve: Convert the given slant height to vertical height using $l = \sqrt{2}h$ before substituting in the rate equation.

Differentiating with respect to time gives

$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt}.$$

The tank loses water, so the stored volume has derivative $-2\,\text{cm}^3/\text{s}$.

Step 3: Evaluate at the specified instant Substitute $h^2 = 8$ into the related-rate equation:

$$-2 = 8\pi \frac{dh}{dt}, \qquad \frac{dh}{dt} = -\frac{2}{8\pi}\,\text{cm/s}.$$

The negative sign indicates that the level falls. It also agrees with $h = r$, which makes the depth and radius change at identical rates.

Step 4: Conclusion The rate of change of height is $\boxed{\frac{dh}{dt} = -\frac{1}{4\pi}\,\text{cm/s}}$.

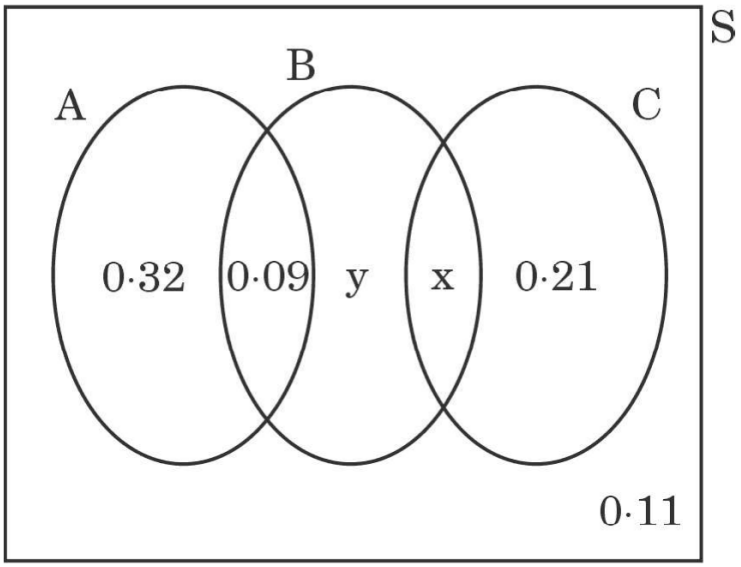
Quick Tip to Solve: Convert the given slant height to vertical height using $l = \sqrt{2}h$ before substituting in the rate equation.

37. There are different types of Yoga which involve the usage of different poses of Yoga Asanas, Meditation and Pranayam as shown in the figure below:

The Venn diagram below represents the probabilities of three different types of Yoga, A, B and C performed by the people of a society. Further, it is given that probability of a member performing type C Yoga is 0.44.

On the basis of the above information, answer the following questions: [4 marks]





37(i). Find the value of x. [1 mark]

Solution:

Concept:

- Each number in the Venn diagram is the probability of its disjoint region, rather than of an entire circle.
- The probability of an event equals the sum of the probabilities of all disjoint regions inside its circle.

Step 1: Identify the regions belonging to C The circle for type C contains the region shared with B, labelled x, and the region for type C alone, labelled 0.21. There is no region common to A and C in the printed diagram. Thus

$$P(C) = x + 0.21.$$

Step 2: Use the supplied probability The question supplies $P(C) = 0.44$. Therefore

$$x + 0.21 = 0.44, \qquad x = 0.44 - 0.21.$$

The label 0.21 does not represent all of C: it excludes the overlap with B. The other regions outside the C circle need not be included in this calculation.

Step 3: Conclusion The required value is $\boxed{x = 0.23}$.

Quick Tip to Solve: For $P(C)$, add only the labels inside circle C, including its overlap with B.

37(ii). Find the value of y. [1 mark]

Solution:

Concept:

- The disjoint regions of the sample space, including the region outside all three circles, have total probability 1.
- The C circle gives $x + 0.21 = P(C) = 0.44$, so $x = 0.23$.

Step 1: Determine the overlap value needed The region $B \cap C$ has probability x, while type C alone has probability 0.21. Hence

$$x + 0.21 = 0.44, \qquad x = 0.23.$$

This overlap is a single disjoint region in the diagram and must be counted once in the sample-space total.

Step 2: Add all disjoint regions The printed diagram gives 0.32 for A alone, 0.09 for $A \cap B$, y for B alone, 0.23 for $B \cap C$, 0.21 for C alone, and 0.11 outside the circles. Therefore

$$0.32 + 0.09 + y + 0.23 + 0.21 + 0.11 = 1.$$

The known probabilities sum to 0.96, leaving $y = 1 - 0.96$.

Step 3: Conclusion The required value is $\boxed{y = 0.04}$.

Quick Tip to Solve: Include the outside probability 0.11 when making the total equal to 1.

37(iii)(a). Find $P\left(\frac{C}{B}\right)$. [2 marks]

Solution:

Concept:

- The notation $P(C/B)$ denotes conditional probability: $P(C \mid B) = P(C \cap B)/P(B)$, provided $P(B) > 0$.
- The denominator includes every region in circle B; the numerator includes only the overlap of B and C.

Step 1: Determine the two unknown region probabilities From the two regions in circle C,

$$x + 0.21 = 0.44, \qquad x = 0.23.$$

The six disjoint regions of the whole sample space have total probability 1: $0.32 + 0.09 + y + 0.23 + 0.21 + 0.11 = 1$. Hence $y = 0.04$. The region outside the circles must be included in this total.

Step 2: Read the intersection and conditioning event The common region of B and C is exactly the region labelled x, so $P(B \cap C) = 0.23$. Circle B includes its overlap with A, its own exclusive region, and its overlap with C. Consequently

$$P(B) = 0.09 + 0.04 + 0.23 = 0.36.$$

Quick Tip to Solve: Read the event after the conditioning bar as the whole denominator circle; here it is B.

The overlap with A remains in the denominator because those people still perform type B Yoga.

Step 3: Apply conditional probability Knowing that the member performs type B restricts the sample space to circle B. The favourable portion of this restricted event is its overlap with C. Therefore

$$P(C \mid B) = \frac{P(C \cap B)}{P(B)} = \frac{0.23}{0.36} = \frac{23}{36}.$$

Using 0.44 as the denominator would instead condition on C, which is a different event.

Step 4: Conclusion The required conditional probability is $\boxed{P(C/B) = \frac{23}{36}}$.

Quick Tip to Solve: Read the event after the conditioning bar as the whole denominator circle; here it is B.

OR

37(iii)(b). Find the probability that a randomly selected person of the society does Yoga of type A or B but not C. [2 marks]

Solution:

Concept:

- The event $(A \cup B) \cap C'$ means performing at least one of types A and B, while performing no type C; here C' is the complement of C.
- The probabilities of disjoint regions in a Venn diagram are added. All regions of the sample space together have probability 1.

Step 1: Determine the two unknown regions The type C region consists of the overlap $B \cap C$, labelled x, and the region for C alone, labelled 0.21. Thus

$$x + 0.21 = 0.44 \quad \Rightarrow \quad x = 0.23.$$

Adding all mutually exclusive regions, including the outside region, gives

$$0.32 + 0.09 + y + 0.23 + 0.21 + 0.11 = 1,$$

so $y = 1 - 0.96 = 0.04$.

Step 2: Identify and add the eligible regions The required event includes the A-only region, the overlap $A \cap B$, and the B-only region. These have probabilities 0.32, 0.09 and 0.04, respectively. The overlap with C must be excluded, even though it belongs to B. Since the three included regions do not overlap with one another,

$$P\big((A \cup B) \cap C'\big) = 0.32 + 0.09 + 0.04 = 0.45.$$

Step 3: Conclusion The probability that a randomly selected person performs type A or B but not type C is $\boxed{0.45}$.

Quick Tip to Solve: Include the A-B overlap once, but exclude the entire B-C overlap when the question says not C.

38. The equation of the path traced by a roller-coaster is given by the polynomial $f(x) = a(x+9)(x+1)(x-3)$. If the roller-coaster crosses y-axis at a point $(0,-1)$, answer the following. [4 marks]

38(i). Find the value of a. [2 marks]

Solution:

Concept:

- A graph crosses the y-axis where $x = 0$. Thus a stated y-intercept $(0, k)$ gives the condition $f(0) = k$.
- In $f(x) = a(x+9)(x+1)(x-3)$, the constant a scales the product. Substitute the known point into this expression to form a linear equation for a.

Step 1: Translate the given intercept into an equation The roller-coaster crosses the y-axis at $(0,-1)$. Every point on the path has coordinates $(x, f(x))$; hence the given point requires $f(0) = -1$. Substitute $x = 0$ into the factored polynomial:

$$f(0) = a(0+9)(0+1)(0-3) = a \cdot 9 \cdot 1 \cdot (-3) = -27a.$$

The negative sign comes from the third factor; the other two factors are positive.

Step 2: Solve for the coefficient and check the intercept Equate the calculated value of $f(0)$ to the given ordinate:

$$-27a = -1 \quad \Rightarrow \quad a = \frac{-1}{-27} = \frac{1}{27}.$$

As a check, using this value yields $f(0) = \frac{1}{27} \cdot 9 \cdot 1 \cdot (-3) = -1$, agreeing with the stated point. The coefficient is positive because both sides of the equation $-27a = -1$ are negative.

Step 3: Conclusion The required coefficient is $\boxed{a = \frac{1}{27}}$.

Quick Tip to Solve: For a given y-intercept, substitute x=0 directly into the factored polynomial before expanding it.

38(ii). Find $f''(x)$ at $x = 1$. [2 marks]

Solution:

Concept:

- A y-intercept of $(0,-1)$ imposes $f(0) = -1$, determining the constant a before evaluating the derivative.
- For a polynomial, differentiate each term using $\frac{d}{dx}(x^n) = nx^{n-1}$. The second derivative $f''(x)$ is obtained by differentiating $f'(x)$ once more.

Step 1: Determine the coefficient and expand the polynomial From the given intercept,

$$-1 = f(0) = a \cdot 9 \cdot 1 \cdot (-3) = -27a, \qquad a = \frac{1}{27}.$$

Quick Tip to Solve: Expand the three linear factors first, differentiate twice, and substitute x=1 only after differentiating.

First multiply $(x+9)(x+1) = x^2 + 10x + 9$. Therefore

$$\begin{aligned} f(x) &= \frac{1}{27}(x^2 + 10x + 9)(x - 3) \\ &= \frac{1}{27}(x^3 + 7x^2 - 21x - 27). \end{aligned}$$

Keeping a constant is essential: it is the fixed coefficient of the given path, not a function of x.

Step 2: Differentiate twice and evaluate at the requested point Differentiate the expanded expression term by term:

$$f'(x) = \frac{1}{27}(3x^2 + 14x - 21), \qquad f''(x) = \frac{1}{27}(6x + 14).$$

The constant term disappears in the first differentiation, and the linear term in $f'(x)$ contributes the constant 14 in the second differentiation. Now substitute $x = 1$ into the second derivative:

$$f''(1) = \frac{6(1) + 14}{27} = \frac{20}{27}.$$

Step 3: Conclusion The required second derivative at $x = 1$ is $\boxed{f''(1) = \frac{20}{27}}$.

Quick Tip to Solve: Expand the three linear factors first, differentiate twice, and substitute x=1 only after differentiating.