CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-2-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This Question Paper contains **38** questions. All questions are compulsory.

(ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.

(iii) In **Section A** – Question Nos. **1 to 18** are Multiple Choice Questions (MCQs) and Question Nos. **19 & 20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B** – Question Nos. **21 to 25** are Very Short Answer (VSA) type questions of **2** marks each.

(v) In **Section C** – Question Nos. **26 to 31** are Short Answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D** – Question Nos. **32 to 35** are Long Answer (LA) type questions carrying **5** marks each.

(vii) In **Section E** – Question Nos. **36 to 38** are source based/case based/passage based/integrated units of assessment questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section **B**, **3** questions in Section **C**, **2** questions in Section **D** and **2** questions in Section **E**.

(ix) Use of calculators is **NOT** allowed.

Section A (20 marks)

(Multiple Choice Questions)

Each question carries 1 mark.

1. If $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then A^{2023} is equal to [1 mark]

(A) $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(B) $\begin{pmatrix} 0 & 2023 \\ 0 & 0 \end{pmatrix}$

(C) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

(D) $\begin{pmatrix} 2023 & 0 \\ 0 & 2023 \end{pmatrix}$

Correct Answer: (C) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Solution:

Concept:

- A matrix is nilpotent if some positive integer power of it is the zero matrix.
- Matrix powers use repeated matrix multiplication, not multiplication of each entry by the exponent.

Step 1: Calculate the square For the printed matrix, direct multiplication gives

$$A^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Each entry is zero because every row-column product vanishes.

Step 2: Use the higher-power property For every integer $n \geq 2$, $A^n = A^2 A^{n-2}$ is zero. Since $2023 \geq 2$, this applies here. Option (A) is A itself; option (B) incorrectly scales its nonzero entry; option (D) is a nonzero diagonal matrix. None of these is a higher power of this nilpotent matrix.

Step 3: Conclusion $A^{2023} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence, the correct option is (C).

Quick Tip to Solve: Compute the square first: once a matrix power is zero, every higher power is also zero.

2. If $\begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix} = P + Q$, where P is a symmetric and Q is a skew symmetric matrix, then Q is equal to [1 mark]

(A) $\begin{pmatrix} 2 & 5/2 \\ 5/2 & 4 \end{pmatrix}$

(B) $\begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}$

(C) $\begin{pmatrix} 0 & 5/2 \\ -5/2 & 0 \end{pmatrix}$

(D) $\begin{pmatrix} 2 & -5/2 \\ 5/2 & 4 \end{pmatrix}$

Correct Answer: (B) $\begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}$

Solution:

Concept:

- A symmetric matrix satisfies $P^T = P$, while a skew-symmetric matrix satisfies $Q^T = -Q$.
- For any square matrix $M = P + Q$, its skew-symmetric part is $Q = (M - M^T)/2$.

Step 1: Find the skew-symmetric part Transposing reverses the off-diagonal entries, so

$$Q = \frac{1}{2}\left[\begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix} - \begin{pmatrix} 2 & 5 \\ 0 & 4 \end{pmatrix}\right] = \begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}.$$

Step 2: Check the options The diagonal of a real skew-symmetric matrix must be zero, ruling out (A) and (D). Option (C) has zero diagonal but the opposite off-diagonal signs: subtracting it from the given matrix does not give a symmetric matrix. For the computed matrix, the remaining part has both off-diagonal entries $5/2$, as required.

Step 3: Conclusion $Q = \begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}$. Hence, the correct option is (B).

Quick Tip to Solve: Use half of the matrix minus its transpose; keep the subtraction order to avoid reversing both signs.

3. If $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & a & 1 \end{pmatrix}$ is non-singular matrix and $a \in A$, then the set A is [1 mark]

(A) $\mathbb{R}$

(B) $\{0\}$

(C) $\{4\}$

(D) $\mathbb{R} - \{4\}$

Correct Answer: (D) $\mathbb{R} - \{4\}$

Solution:

Concept:

- A square matrix is non-singular exactly when its determinant is nonzero.
- Expansion along the first row uses alternating signs $+, -, +$.

Step 1: Evaluate the determinant Let M be the printed matrix. Expanding along its first row gives

$$|M| = (3 - a) - 2(2 - 3) + (2a - 9) = a - 4.$$

The middle minor is -1, so its contribution is $+2$, not -2.

Step 2: Apply non-singularity and compare sets The condition is $a - 4 \neq 0$, that is, $a \neq 4$. Every other real value is permitted. Option (A) wrongly includes the singular value; (B) includes only one allowed value and omits all the others; (C) contains precisely the excluded value.

Step 3: Conclusion $A = \mathbb{R} \setminus \{4\}$. Hence, the correct option is (D).

Quick Tip to Solve: Compute the determinant as a function of the parameter, then exclude its zero rather than selecting that zero.

4. If $|A| = |kA|$, where A is a square matrix of order 2, then sum of all possible values of k is [1 mark]

(A) 1

(B) -1

(C) 2

(D) 0

Correct Answer: (D) 0

Solution:

Concept:

- For a square matrix of order n, $|kA| = k^n|A|$, since each of its n rows is multiplied by k.
- The printed equation is $(k^2 - 1)|A| = 0$ for an order-two matrix.

Step 1: Find the finite values intended by the options When $|A| \neq 0$, division by $|A|$ gives $k^2 = 1$. Thus both $k = 1$ and $k = -1$ satisfy the equation. Their sum is $1 + (-1) = 0$; (A) and (B) each retain just one value, and (C) would result from adding absolute values.

Quick Tip to Solve: Before cancelling a determinant, check whether it can be zero; for a nonzero order-two determinant solve k squared equals one.

Step 2: Check the omitted singular case The question does not explicitly state that A is non-singular. If $|A| = 0$, then $|kA| = k^2 \cdot 0 = 0$ for every real k, and the sum of all possible values is not defined. The finite answer represented by the printed options therefore uses the non-singular interpretation; the determinant identity alone does not justify division when $|A| = 0$.

Step 3: Conclusion Under the non-singular interpretation implicit in the finite options, the sum is 0, giving option (D). If A is singular, all real k satisfy the equation and no such finite sum exists.

Quick Tip to Solve: Before cancelling a determinant, check whether it can be zero; for a nonzero order-two determinant solve k squared equals one.

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5. If $\frac{d}{dx}[f(x)] = ax + b$ and $f(0) = 0$, then $f(x)$ is equal to [1 mark]

(A) $a + b$

(B) $\frac{ax^2}{2} + bx$

(C) $\frac{ax^2}{2} + bx + c$

(D) b

Correct Answer: (B) $\frac{ax^2}{2} + bx$

Solution:

Concept:

- An antiderivative of $ax + b$ is $ax^2/2 + bx + C$, where a, b are constants.
- An initial value determines the otherwise arbitrary integration constant.

Step 1: Integrate the derivative Integrating each term separately gives

$$f(x) = \int (ax + b)\, dx = \frac{ax^2}{2} + bx + C.$$

Differentiating the quadratic term gives ax, and differentiating the linear term gives b.

Step 2: Use the initial value and check options At $x = 0$, the two variable terms vanish, so $f(0) = C = 0$. Thus no free constant remains. The constant functions in (A) and (D) have zero derivative in general, while (C) leaves an arbitrary constant despite the supplied initial value; it reduces to (B) only after setting that constant to zero.

Step 3: Conclusion $f(x) = \frac{ax^2}{2} + bx$. Hence, the correct option is (B).

Quick Tip to Solve: Integrate first, then substitute x equals zero: every positive-power term disappears and reveals the constant.

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6. Degree of the differential equation $\sin x + \cos\left(\frac{dy}{dx}\right) = y^2$ is [1 mark]

(A) 2

(B) 1

(C) not defined

(D) 0

Correct Answer: (C) not defined

Solution:

Concept:

- The order is the order of the highest derivative present in a differential equation.
- The degree is the power of the highest-order derivative only when the equation is polynomial in its derivatives.

Step 1: Inspect how the derivative occurs The only derivative is dy/dx, so the equation has order one. However, it appears inside $\cos(dy/dx)$, rather than as a polynomial power. Moving terms gives $\cos(dy/dx) = y^2 - \sin x$, which still has a trigonometric function of the derivative.

Step 2: Apply the definition of degree This equation is not polynomial in its derivatives, so the degree cannot be assigned. The exponent 2 belongs to y, not its derivative, ruling out (A). Option (B) confuses order with degree. Option (D) is also incorrect: an undefined degree is not the number zero.

Step 3: Conclusion The degree is not defined. Hence, the correct option is (C).

Quick Tip to Solve: Check whether the derivative lies inside a trigonometric function before looking for its exponent.

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7. The integrating factor of the differential equation $(1-y^2)\frac{dx}{dy} + yx = ay$, $(-1 < y < 1)$ is [1 mark]

(A) $\frac{1}{y^2-1}$

(B) $\frac{1}{\sqrt{y^2-1}}$

(C) $\frac{1}{1-y^2}$

(D) $\frac{1}{\sqrt{1-y^2}}$

Correct Answer: (D) $\frac{1}{\sqrt{1-y^2}}$

Solution:

Concept:

- For $dx/dy + P(y)x = Q(y)$, the integrating factor is $e^{\int P(y)\,dy}$.
- Here y is the independent variable; the interval $-1 < y < 1$ ensures $1 - y^2 > 0$.

Quick Tip to Solve: Divide by the coefficient of dx/dy first, then substitute one minus y squared in the exponent integral.

Step 1: Put the equation in linear form Dividing by the nonzero coefficient $1 - y^2$ yields

$$\frac{dx}{dy} + \frac{y}{1-y^2}x = \frac{ay}{1-y^2}.$$

Thus the coefficient used in the integrating factor is $P(y) = y/(1-y^2)$, not the right-hand side.

Step 2: Integrate the coefficient and compare options With $u = 1 - y^2$, $du = -2y\,dy$. Therefore

$$\int \frac{y}{1-y^2}\,dy = -\frac{1}{2}\log(1-y^2), \qquad e^{\int P(y)dy} = (1-y^2)^{-1/2}.$$

Options (A) and (C) use the wrong power; (B) has a negative radicand on the given interval, whereas the obtained factor is real and positive.

Step 3: Conclusion The integrating factor is $\frac{1}{\sqrt{1-y^2}}$. Hence, the correct option is (D).

Quick Tip to Solve: Divide by the coefficient of dx/dy first, then substitute one minus y squared in the exponent integral.

8. Unit vector along $\overrightarrow{PQ}$, where coordinates of P and Q respectively are $(2, 1, -1)$ and $(4, 4, -7)$, is [1 mark]

(A) $2\hat{i} + 3\hat{j} - 6\hat{k}$

(B) $-2\hat{i} - 3\hat{j} + 6\hat{k}$

(C) $\frac{-2\hat{i}}{7} - \frac{3\hat{j}}{7} + \frac{6\hat{k}}{7}$

(D) $\frac{2\hat{i}}{7} + \frac{3\hat{j}}{7} - \frac{6\hat{k}}{7}$

Correct Answer: (D) $\frac{2\hat{i}}{7} + \frac{3\hat{j}}{7} - \frac{6\hat{k}}{7}$

Solution:

Concept:

- The vector from P to Q is obtained by subtracting the coordinates of P from those of Q.
- A unit vector along a nonzero vector $\mathbf{v}$ is $\mathbf{v}/|\mathbf{v}|$.

Step 1: Find the directed vector and its magnitude Using destination minus starting point,

$$\overrightarrow{PQ} = (4-2)\hat{i} + (4-1)\hat{j} + (-7+1)\hat{k} = 2\hat{i} + 3\hat{j} - 6\hat{k}.$$

Its magnitude is $\sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{49} = 7$.

Step 2: Normalize and check direction Dividing all three components by the positive magnitude preserves the direction and makes the length one. Option (A) has the correct direction but length seven. Option (B) has length seven and the reverse direction. Option (C) is a unit vector but points from Q to P, so it is opposite to the requested direction.

Quick Tip to Solve: Use Q minus P for direction, then divide by seven; reversing the subtraction gives the opposite unit vector.

Step 3: Conclusion The required unit vector is $\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}$. Hence, the correct option is (D).

Quick Tip to Solve: Use Q minus P for direction, then divide by seven; reversing the subtraction gives the opposite unit vector.

9. Position vector of the mid-point of line segment AB is $3\hat{i} + 2\hat{j} - 3\hat{k}$. If position vector of the point A is $2\hat{i} + 3\hat{j} - 4\hat{k}$, then position vector of the point B is [1 mark]

(A) $\frac{5\hat{i}}{2} + \frac{5\hat{j}}{2} - \frac{7\hat{k}}{2}$

(B) $4\hat{i} + \hat{j} - 2\hat{k}$

(C) $5\hat{i} + 5\hat{j} - 7\hat{k}$

(D) $\frac{\hat{i}}{2} - \frac{\hat{j}}{2} + \frac{\hat{k}}{2}$

Correct Answer: (B) $4\hat{i} + \hat{j} - 2\hat{k}$

Solution:

Concept:

- The midpoint position vector is $\vec{m} = (\vec{a} + \vec{b})/2$, where $\vec{a}, \vec{b}$ are the endpoint position vectors.
- Apply vector addition and subtraction component by component.

Step 1: Rearrange the midpoint formula From $2\vec{m} = \vec{a} + \vec{b}$, the unknown endpoint satisfies $\vec{b} = 2\vec{m} - \vec{a}$. Thus doubling the midpoint precedes subtracting the known endpoint.

Step 2: Substitute and check the options

$$\vec{b} = 2(3\hat{i} + 2\hat{j} - 3\hat{k}) - (2\hat{i} + 3\hat{j} - 4\hat{k}) = 4\hat{i} + \hat{j} - 2\hat{k}.$$

Its average with $\vec{a}$ gives the stated midpoint. Option (A) averages the midpoint with A, (C) adds their vectors, and (D) halves their difference; none is the required endpoint.

Step 3: Conclusion The position vector of B is $4\hat{i} + \hat{j} - 2\hat{k}$. Hence, the correct option is (B).

Quick Tip to Solve: Recover an endpoint by doubling the midpoint and subtracting the other endpoint.

10. Projection of vector $2\hat{i}+3\hat{j}$ on the vector $3\hat{i}-2\hat{j}$ is [1 mark]

(A) 0

(B) 12

(C) $\frac{12}{\sqrt{13}}$

(D) $\frac{-12}{\sqrt{13}}$

Correct Answer: (A) 0

Solution:

Concept:

- The scalar projection of $\vec{a}$ on nonzero $\vec{b}$ is $\vec{a}\cdot\vec{b}/|\vec{b}|$.
- A zero dot product means the vectors are perpendicular and the projection is zero.

Step 1: Calculate the dot product Take $\vec{a}=2\hat{i}+3\hat{j}$ and $\vec{b}=3\hat{i}-2\hat{j}$. Orthogonality of the unit vectors gives $\vec{a}\cdot\vec{b}=2(3)+3(-2)=6-6=0$.

Step 2: Evaluate the projection and compare The length of the direction vector is $|\vec{b}|=\sqrt{3^2+(-2)^2}=\sqrt{13}$, which is nonzero. Hence the projection is $0/\sqrt{13}=0$. The value 12 arises by ignoring the negative sign in the second component; both nonzero fractions also fail because the dot product is zero.

Step 3: Conclusion The projection is 0. Hence, the correct option is (A).

Quick Tip to Solve: Check the dot product first: perpendicular vectors have zero projection.

11. Equation of a line passing through point $(1,1,1)$ and parallel to z-axis is [1 mark]

(A) $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$

(B) $\frac{x-1}{1}=\frac{y-1}{1}=\frac{z-1}{1}$

(C) $\frac{x}{0}=\frac{y}{0}=\frac{z-1}{1}$

(D) $\frac{x-1}{0}=\frac{y-1}{0}=\frac{z-1}{1}$

Correct Answer: (D) $\frac{x-1}{0}=\frac{y-1}{0}=\frac{z-1}{1}$

Solution:

Concept:

- A line through (x_0,y_0,z_0) with direction ratios (a,b,c) has parametric equations $x=x_0+at,\ y=y_0+bt,\ z=z_0+ct$.
- Parallelism to the z-axis gives direction ratios $(0,0,1)$.

Step 1: Use the given point and direction The line through $(1,1,1)$ parallel to the z-axis is $x=1,\ y=1,\ z=1+t$, where t is real. Only the z-coordinate varies; the other two remain fixed.

Step 2: Match the printed symmetric notation This is conventionally written $(x-1)/0=(y-1)/0=(z-1)/1$. Here zero denominators encode fixed coordinates $x=1, y=1$; they are not ordinary

Quick Tip to Solve: For an axis-parallel line, keep the other two coordinates fixed at the given point.

numerical divisions. Options (A) and (B) have direction ratios $(1, 1, 1)$, while (C) fixes $x = y = 0$, so it misses the given point.

Step 3: Conclusion The required line is $x = 1,\ y = 1,\ z = 1 + t$, represented by option (D).

Quick Tip to Solve: For an axis-parallel line, keep the other two coordinates fixed at the given point.

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12. If the sum of numbers obtained on throwing a pair of dice is 9, then the probability that number obtained on one of the dice is 4, is [1 mark]

(A) $\frac{1}{9}$

(B) $\frac{4}{9}$

(C) $\frac{1}{18}$

(D) $\frac{1}{2}$

Correct Answer: (D) $\frac{1}{2}$

Solution:

Concept:

- Conditional probability is $P(A \mid B) = P(A \cap B)/P(B)$ for $P(B) > 0$.
- For two fair dice, ordered outcomes are equally likely; retain only outcomes satisfying the given sum.

Step 1: Restrict the sample space Let B be the event that the sum is 9. The possible ordered pairs are $(3, 6), (4, 5), (5, 4), (6, 3)$. These four outcomes remain equally likely after conditioning on B.

Step 2: Count the favourable outcomes Let A mean that at least one die shows 4. Within B, only $(4, 5)$ and $(5, 4)$ satisfy A. Therefore $P(A \mid B) = (2/36)/(4/36) = 2/4 = 1/2$. The fractions in (A), (B), and (C) use an incorrect count or fail to condition on the known sum; $1/18 = 2/36$ is the unconditioned intersection probability.

Step 3: Conclusion The required probability is $\frac{1}{2}$. Hence, the correct option is (D).

Quick Tip to Solve: When a dice sum is given, count favourable pairs only among pairs with that sum.

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13. Anti-derivative of $\frac{\tan x - 1}{\tan x + 1}$ with respect to x is [1 mark]

(A) $\sec^2\left(\frac{\pi}{4} - x\right) + c$

(B) $-\sec^2\left(\frac{\pi}{4} - x\right) + c$

(C) $\log\left|\sec\left(\frac{\pi}{4} - x\right)\right| + c$

(D) $-\log\left|\sec\left(\frac{\pi}{4} - x\right)\right| + c$

Correct Answer: (C) $\log\left|\sec\left(\frac{\pi}{4} - x\right)\right| + c$

Solution:

Concept:

- The tangent subtraction identity is $\tan(x - \pi/4) = (\tan x - 1)/(1 + \tan x)$.
- An antiderivative of $\tan u$ is $\log|\sec u| + C$, with the chain rule applied to u.

Step 1: Rewrite the integrand

$$\frac{\tan x - 1}{\tan x + 1} = \tan(x - \pi/4) = -\tan(\pi/4 - x).$$

These identities apply on intervals where the original expression is defined.

Step 2: Integrate and verify the alternatives Set $u = \pi/4 - x$, so $du = -dx$. The integral becomes $\int \tan u\, du = \log|\sec u| + C$. Differentiating this gives $-\tan u$, as required. Option (D) gives the opposite sign; differentiating (A) or (B) introduces the extra factor $2\sec^2 u$, so neither matches the integrand.

Step 3: Conclusion The antiderivative is $\log|\sec(\pi/4 - x)| + C$. Hence, the correct option is (C).

Quick Tip to Solve: Use the tangent subtraction identity, then check the sign from the inner derivative.

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14. If (a, b), (c, d) and (e, f) are the vertices of $\triangle ABC$ and Δ denotes the area of $\triangle ABC$, then $\begin{vmatrix} a & c & e \\ b & d & f \\ 1 & 1 & 1 \end{vmatrix}^2$ is equal to [1 mark]

(A) $2\Delta^2$

(B) $4\Delta^2$

(C) 2Δ

(D) 4Δ

Correct Answer: (B) $4\Delta^2$

Solution:

Concept:

- The area of a triangle is half the absolute value of its coordinate determinant.
- A determinant and its transpose are equal; squaring removes the orientation sign.

Quick Tip to Solve: First write determinant magnitude as twice the area, then square the entire relation.

Step 1: Relate the printed determinant to area Write $D = \begin{vmatrix} a & c & e \\ b & d & f \\ 1 & 1 & 1 \end{vmatrix}$. Its transpose is the usual coordinate determinant with rows $(a, b, 1), (c, d, 1), (e, f, 1)$. Therefore $\Delta = |D|/2$.

Step 2: Square the relation and examine the choices From $|D| = 2\Delta$, we obtain $D^2 = |D|^2 = (2\Delta)^2 = 4\Delta^2$. Option (A) fails to square the factor 2. Options (C) and (D) leave the area unsquared, so they do not follow from the squared determinant. The formula also holds for collinear vertices, when both sides vanish.

Step 3: Conclusion The squared determinant equals $4\Delta^2$. Hence, the correct option is (B).

Quick Tip to Solve: First write determinant magnitude as twice the area, then square the entire relation.

15. The function $f(x) = x|x|$ is [1 mark]

(A) continuous and differentiable at $x = 0$.

(B) continuous but not differentiable at $x = 0$.

(C) differentiable but not continuous at $x = 0$.

(D) neither differentiable nor continuous at $x = 0$.

Correct Answer: (A) continuous and differentiable at $x = 0$.

Solution:

Concept:

- Continuity at zero requires $\lim_{x\to 0} f(x) = f(0)$.
- The derivative at zero is $\lim_{h\to 0}[f(h) - f(0)]/h$, if the two-sided limit exists.

Step 1: Check continuity at zero For $x \geq 0$, $f(x) = x^2$; for $x < 0$, $f(x) = -x^2$. Both one-sided limits at zero equal 0, and $f(0) = 0$. Hence the function is continuous there.

Step 2: Check differentiability directly For nonzero h, $[f(h) - f(0)]/h = h|h|/h = |h|$. Its limit from either side is 0, so $f'(0) = 0$. Although $|x|$ alone has a corner, multiplying by x makes this derivative exist. Thus (B) wrongly denies differentiability, (C) denies continuity, and (D) denies both verified properties.

Step 3: Conclusion The function is continuous and differentiable at $x = 0$. Hence, the correct option is (A).

Quick Tip to Solve: At zero, use the difference quotient directly; the factor x cancels and leaves |h|.

16. If $\tan\left(\frac{x+y}{x-y}\right) = k$, then $\frac{dy}{dx}$ is equal to [1 mark]

(A) $\frac{-y}{x}$

(B) $\frac{y}{x}$

(C) $\sec^2\left(\frac{y}{x}\right)$

(D) $-\sec^2\left(\frac{y}{x}\right)$

Correct Answer: (B) $\frac{y}{x}$

Solution:

Concept:

- For a constant k, differentiating $\tan u = k$ yields $\sec^2 u\, u' = 0$.
- Use the quotient rule for $u = (x+y)/(x-y)$, treating y as a differentiable function of x.

Step 1: Differentiate the implicit relation Put $u = (x+y)/(x-y)$. Wherever the printed relation is defined, $x - y \neq 0$ and $\sec^2 u > 0$. Therefore $u' = 0$, and the quotient rule gives

$$(1+y')(x-y) - (x+y)(1-y') = 0.$$

Step 2: Simplify and check the options Expanding the numerator gives $-2y + 2xy' = 0$. Thus $xy' = y$, or $y' = y/x$ at points with $x \neq 0$, as used in the printed options. Option (A) reverses the sign, while (C) and (D) retain unrelated secant factors: the secant factor from differentiating tangent cancels because it is nonzero.

Step 3: Conclusion $\frac{dy}{dx} = \frac{y}{x}$ where the displayed quotient is defined. Hence, the correct option is (B).

Quick Tip to Solve: Cancel the nonzero secant-squared factor before expanding the quotient-rule numerator.

17. The objective function $Z = ax + by$ of an LPP has maximum value 42 at $(4, 6)$ and minimum value 19 at $(3, 2)$. Which of the following is true? [1 mark]

(A) $a = 9, b = 1$

(B) $a = 5, b = 2$

(C) $a = 3, b = 5$

(D) $a = 5, b = 3$

Correct Answer: (C) $a = 3, b = 5$

Solution:

Concept:

- An objective function has the form $Z = ax + by$, where a, b are fixed coefficients.
- Substitute each given point and its objective value to obtain two simultaneous linear equations.

Step 1: Use the two stated values At $(4, 6)$, the value is 42, so $4a + 6b = 42$. At $(3, 2)$, the value is 19,

Quick Tip to Solve: Check both point-value equations; one matching value does not identify the coefficients.

so $3a + 2b = 19$. Multiplying the second equation by 3 and subtracting the first gives $5a = 15$, since $57 - 42 = 15$, but $9a - 4a = 5a$, hence $a = 3$. Then $9 + 2b = 19$ gives $b = 5$.

Step 2: Check every printed option For options (A), (B), (C), (D), the pairs of objective values at $(4, 6)$ and $(3, 2)$ are respectively $(42, 29)$, $(32, 19)$, $(42, 19)$ and $(38, 21)$. Thus only option (C) satisfies both stated values; matching just one point is insufficient.

Step 3: Conclusion The coefficients are $a = 3,\ b = 5$. Hence, the correct option is (C).

Quick Tip to Solve: Check both point-value equations; one matching value does not identify the coefficients.

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18. The corner points of the feasible region of a linear programming problem are $(0, 4)$, $(8, 0)$ and $\left(\frac{20}{3}, \frac{4}{3}\right)$. If $Z = 30x + 24y$ is the objective function, then (maximum value of Z - minimum value of Z) is equal to [1 mark]

(A) 40

(B) 96

(C) 120

(D) 136

Correct Answer: 144; none of the printed options is correct.

Solution:

Concept:

- A linear objective function on a bounded polygonal feasible region attains its maximum and minimum at corner points.
- Evaluate $Z = 30x + 24y$ at every listed corner and compare the resulting values.

Step 1: Evaluate all corner values At $(0, 4)$, $Z = 30(0) + 24(4) = 96$. At $(8, 0)$, $Z = 30(8) + 24(0) = 240$. At $\left(\frac{20}{3}, \frac{4}{3}\right)$, $Z = 30\left(\frac{20}{3}\right) + 24\left(\frac{4}{3}\right) = 200 + 32 = 232$.

Step 2: Compare the extrema with the options The maximum of $96, 240, 232$ is 240, and the minimum is 96. Their difference is $240 - 96 = 144$. Options (A), (B), (C), and (D) give $40, 96, 120, 136$, respectively; none equals 144. In particular, $136 = 232 - 96$ uses a value that is not the maximum.

Step 3: Conclusion The required difference is $\boxed{144}$. None of the printed options is correct.

Quick Tip to Solve: Tabulate every vertex value before subtracting the smallest from the largest.

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19. Assertion (A): Maximum value of $(\cos^{-1} x)^2$ is π^2.

Reason (R): Range of the principal value branch of $\cos^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Choose the correct option. [1 mark]

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).

(B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).

(C) (A) is true, but (R) is false.

(D) (A) is false, but (R) is true.

Correct Answer: (C) (A) is true, but (R) is false.

Solution:

Concept:

- For real $x \in [-1, 1]$, the principal value of $\cos^{-1} x$ lies in $[0, \pi]$.
- Squaring is increasing on the nonnegative interval $[0, \pi]$.

Step 1: Judge the assertion Write $t = \cos^{-1} x$. Since $0 \le t \le \pi$, we have $0 \le t^2 \le \pi^2$. Equality at the upper endpoint occurs for $x = -1$, because $\cos^{-1}(-1) = \pi$. Therefore the stated maximum π^2 is attained, and assertion (A) is true.

Step 2: Judge the reason The actual principal range of inverse cosine is $[0, \pi]$, not $[-\pi/2, \pi/2]$. The latter is the principal range of inverse sine. Thus reason (R) is false.

Step 3: Decide the explanatory relation A false reason cannot explain the true assertion. Options (A) and (B) both require a true reason, and option (D) requires a false assertion; all three are excluded.

Step 4: Conclusion The assertion is true and the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Remember inverse cosine takes values from zero to pi; inverse sine uses the symmetric interval.

20. Assertion (A): If a line makes angles α, β, γ with positive direction of the coordinate axes, then $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$.

Reason (R): The sum of squares of the direction cosines of a line is 1.

Choose the correct option. [1 mark]

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).

(B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).

(C) (A) is true, but (R) is false.

(D) (A) is false, but (R) is true.

Correct Answer: (A) Both (A) and (R) are true and (R) is the correct explanation of (A).

Solution:

Concept:

- The direction cosines of a line are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$. They obey $l^2 + m^2 + n^2 = 1$.
- For any angle θ, $\sin^2\theta = 1 - \cos^2\theta$.

Quick Tip to Solve: Replace each sine square by one minus the corresponding cosine square, then use their sum.

Step 1: Judge the assertion Using the identity for each of the three angles gives

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3 - 1 = 2.$$

Thus assertion (A) is true.

Step 2: Judge the reason The squares of the direction cosines of any line sum to 1. This follows by dividing the sum of the squares of its direction ratios by that same sum. Hence reason (R) is true.

Step 3: Check whether the reason explains the assertion The stated reason supplies exactly the sum of cosine squares used in the derivation of the assertion. It therefore explains the assertion directly. Option (B) incorrectly denies this link; options (C) and (D) incorrectly declare one of the statements false.

Step 4: Conclusion Both statements are true, and (R) is the correct explanation of (A). Hence, the correct option is (A).

Quick Tip to Solve: Replace each sine square by one minus the corresponding cosine square, then use their sum.

Section B (10 marks)

*This section comprises Very Short Answer Type questions (VSA) of **2** marks each.*

21(a). **Evaluate** $\sin^{-1}\left(\sin\dfrac{3\pi}{4}\right) + \cos^{-1}(\cos\pi) + \tan^{-1}(1)$**.** [2 marks]

Solution:

Concept:

- The principal ranges of $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ are respectively $[-\pi/2, \pi/2]$, $[0, \pi]$, and $(-\pi/2, \pi/2)$.
- An inverse trigonometric function returns the angle in its principal range having the required trigonometric value.

Step 1: Evaluate the inverse sine term The angle $3\pi/4$ lies outside the principal range of inverse sine. Use $\sin(3\pi/4) = \sin(\pi - 3\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$. Since $\pi/4$ lies in $[-\pi/2, \pi/2]$, we obtain $\sin^{-1}(\sin(3\pi/4)) = \pi/4$. Direct cancellation would give an inadmissible principal value.

Step 2: Evaluate the other terms and add The angle π belongs to $[0, \pi]$, so $\cos^{-1}(\cos\pi) = \cos^{-1}(-1) = \pi$. Also $\tan(\pi/4) = 1$, and $\pi/4$ belongs to the principal tangent interval, hence $\tan^{-1}(1) = \pi/4$. Therefore the requested sum is

$$\frac{\pi}{4} + \pi + \frac{\pi}{4} = \frac{3\pi}{2}.$$

Step 3: Conclusion The value of the expression is $\boxed{3\pi/2}$.

Quick Tip to Solve: Check each principal range before cancelling an inverse function with its trigonometric function.

OR

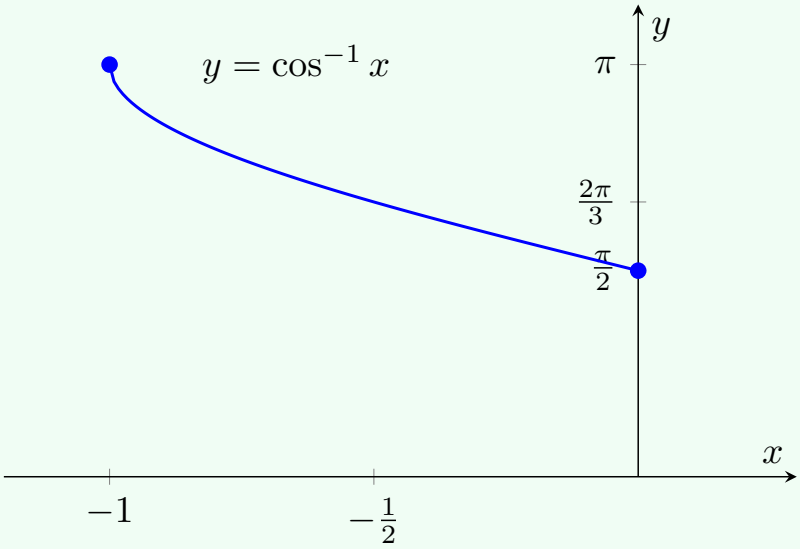
21(b). Draw the graph of $\cos^{-1} x$, where $x \in [-1, 0]$. Also, write its range. [2 marks]

Solution:

Concept:

- The function $y = \cos^{-1} x$ is the inverse of $x = \cos y$ on $0 \le y \le \pi$.
- Inverse cosine decreases as x increases; restrict its graph to the given closed interval $[-1, 0]$.

Step 1: Locate endpoints and an intermediate point At $x = -1$, $y = \cos^{-1}(-1) = \pi$, and at $x = 0$, $y = \cos^{-1}(0) = \pi/2$. For a point between them, $\cos(2\pi/3) = -1/2$, so the curve passes through $(-1/2, 2\pi/3)$. Draw the smooth decreasing inverse-cosine curve between the two endpoints, including both because the domain is closed.



Graph of $y = \cos^{-1} x$ for $-1 \le x \le 0$.

Step 2: Read the range from the graph For $-1 \le x \le 0$, inverse cosine continuously decreases from π to $\pi/2$. Thus its least value is $\pi/2$, achieved at $x = 0$, and its greatest value is π, achieved at $x = -1$. Every intermediate value occurs, so the range is the closed interval $[\pi/2, \pi]$. The curve should not be continued to positive x, since those inputs are outside the prescribed domain.

Step 3: Conclusion The graph is shown above and its range is $\boxed{[\pi/2, \pi]}$.

Quick Tip to Solve: For a restricted inverse-cosine graph, evaluate the endpoints first and retain only the required domain.

22. A particle moves along the curve $3y = ax^3 + 1$ such that at a point with x-coordinate 1, y-coordinate is changing twice as fast as x-coordinate. Find the value of a. [2 marks]

Solution:

Concept:

- The rates of change of the coordinates with time t are dx/dt and dy/dt. The chain rule relates these rates along the given curve.
- If the y-coordinate changes twice as fast as the x-coordinate, then $dy/dt = 2\,dx/dt$ at the specified point.

Step 1: Differentiate the equation with respect to time The particle satisfies $3y = ax^3 + 1$, where a is constant. Differentiating both sides with respect to t, using the chain rule for x^3, gives

$$3\frac{dy}{dt} = 3ax^2\frac{dx}{dt}, \qquad \frac{dy}{dt} = ax^2\frac{dx}{dt}.$$

Thus the ratio of the coordinate rates is determined by the slope ax^2 of the curve.

Step 2: Apply the given rate at the specified point At $x = 1$, substitute $dy/dt = 2\,dx/dt$ into the differentiated equation:

$$2\frac{dx}{dt} = a(1)^2\frac{dx}{dt}.$$

Since the coordinate is changing, its rate is non-zero. Cancelling dx/dt therefore gives $a = 2$. The value of y at the point is not needed to compare the two rates.

Step 3: Conclusion The required constant is $\boxed{a = 2}$.

Quick Tip to Solve: Differentiate with respect to time first, then insert the coordinate and the ratio of rates.

---

23. If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero unequal vectors such that $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, then find the angle between $\vec{a}$ and $\vec{b} - \vec{c}$. [2 marks]

Solution:

Concept:

- The dot product is distributive: $\vec{a} \cdot (\vec{b} - \vec{c}) = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$.
- For non-zero vectors $\vec{u}, \vec{v}$, $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$, where θ is their angle in $[0, \pi]$.

Step 1: Rewrite the given equality We are given $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$. Subtract the right-hand side from both sides and use distributivity:

$$\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = 0 \quad \Longrightarrow \quad \vec{a} \cdot (\vec{b} - \vec{c}) = 0.$$

This equality concerns the difference vector itself; it does not require $\vec{b}$ and $\vec{c}$ to have equal magnitudes.

Step 2: Use the formula for the angle Let θ be the angle between $\vec{a}$ and $\vec{b} - \vec{c}$. Since $\vec{a} \neq \vec{0}$ and the unequal vectors $\vec{b}, \vec{c}$ satisfy $\vec{b} - \vec{c} \neq \vec{0}$, both magnitudes in

$$|\vec{a}|\,|\vec{b} - \vec{c}| \cos\theta = 0$$

Quick Tip to Solve: Move equal dot products to one side and factor the common vector using distributivity.

are positive. Division by their product gives $\cos\theta = 0$. In the interval for the angle between vectors, this means $\theta = \pi/2$.

Step 3: Conclusion The vectors are perpendicular; the required angle is $\boxed{90^\circ}$.

Quick Tip to Solve: Move equal dot products to one side and factor the common vector using distributivity.

---

24. Find the coordinates of points on line $\frac{x}{1} = \frac{y-1}{2} = \frac{z+1}{2}$ which are at a distance of $\sqrt{11}$ units from origin. [2 marks]

Solution:

Concept:

- A common parameter in a symmetric line equation gives the coordinates of every point on the line.
- The distance of (x, y, z) from the origin is $\sqrt{x^2+y^2+z^2}$; square the required distance before substituting.

Step 1: Parameterise the line Put

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z+1}{2} = \lambda.$$

Hence every point on the line has coordinates

$$(x, y, z) = (\lambda,\ 1+2\lambda,\ -1+2\lambda).$$

The offsets in the second and third coordinates have opposite signs, so they must be retained when forming the distance equation.

Step 2: Solve the distance condition The required squared distance is 11, so

$$\begin{aligned} \lambda^2 + (1+2\lambda)^2 + (-1+2\lambda)^2 &= 11, \\ \lambda^2 + (1+4\lambda+4\lambda^2) + (1-4\lambda+4\lambda^2) &= 11, \\ 9\lambda^2 + 2 &= 11, \\ \lambda^2 &= 1. \end{aligned}$$

Thus $\lambda = 1$ or $\lambda = -1$. Substitution gives $(1, 3, 1)$ and $(-1, -1, -3)$, respectively. Each point has squared distance $1+9+1 = 11$ or $1+1+9 = 11$, confirming both solutions.

Step 3: Conclusion The required points are $\boxed{(1, 3, 1) \text{ and } (-1, -1, -3)}$.

Quick Tip to Solve: Keep both signs when solving the quadratic parameter equation; each sign gives a different point.

25(a). **If $y = \sqrt{ax+b}$, prove that $y\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = 0$.** [2 marks]

Solution:

Concept:

- Squaring $y = \sqrt{ax+b}$ gives $y^2 = ax+b$; this form permits implicit differentiation without fractions.
- The product rule gives $\frac{d}{dx}(yy') = (y')^2 + yy''$, where primes denote derivatives with respect to x.

Step 1: Differentiate the squared relation once From the given relation,

$$y^2 = ax + b.$$

On any interval where the indicated derivatives exist, differentiate both sides with respect to x. Since a and b are constants, the chain rule gives

$$2y\frac{dy}{dx} = a.$$

Keeping this equation in product form avoids dividing by y, and directly prepares the expression required in the question.

Step 2: Differentiate again and simplify Differentiate $2y(dy/dx) = a$ once more. The derivative of the constant a is zero, while the product rule on the left yields

$$2\left[\left(\frac{dy}{dx}\right)^2 + y\frac{d^2y}{dx^2}\right] = 0.$$

Dividing by 2 and rearranging the two terms gives precisely the requested identity. The argument applies wherever the first and second derivatives in the statement are defined.

Step 3: Conclusion Therefore $\boxed{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0}$, as required.

Quick Tip to Solve: Square the radical and differentiate twice implicitly to obtain the identity without denominators.

OR

25(b). **If $f(x) = \begin{cases} ax+b & ;\ 0 < x \le 1 \\ 2x^2 - x & ;\ 1 < x < 2 \end{cases}$ is a differentiable function in $(0,2)$, then find the values of a and b.** [2 marks]

Solution:

Concept:

- Differentiability at an interior joining point requires continuity there and equality of the left-hand and right-hand derivatives.
- Each polynomial branch is differentiable within its own interval, so only the joining point $x = 1$

Quick Tip to Solve: At a piecewise join, first equate function values and then equate the one-sided slopes.

needs conditions.

Step 1: Impose continuity at the joining point At $x = 1$, the first branch gives $f(1) = a + b$. The limit from the right, using the second branch, is

$$\lim_{x \to 1^+} (2x^2 - x) = 2 - 1 = 1.$$

The left limit is also $a + b$. Since a differentiable function is continuous, we require

$$a + b = 1.$$

This condition is necessary before comparing derivatives of the branches.

Step 2: Match the derivatives and determine both constants The derivative of $ax + b$ is a, while the derivative of $2x^2 - x$ is $4x - 1$. With continuity established, the one-sided derivatives at 1 are therefore

$$f\prime(1^-) = a, \qquad f\prime(1^+) = 4(1) - 1 = 3.$$

Their equality gives $a = 3$. Substituting in $a + b = 1$ yields $b = -2$. These values make both limits equal to 1 and both derivatives equal to 3, so the joining point is indeed differentiable.

Step 3: Conclusion The required values are $\boxed{a = 3,\ b = -2}$.

Quick Tip to Solve: At a piecewise join, first equate function values and then equate the one-sided slopes.

Section C (18 marks)

*This section comprises Short Answer type questions (SA) of **3** marks each.*

26(a). Evaluate $\displaystyle\int_0^{\pi/4} \log(1 + \tan x)\, dx$**.** [3 marks]

Solution:

Concept:

- For an integrable function on $[0, a]$, the reflection property gives $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$. Here take $a = \pi/4$.
- The identities $\tan(\pi/4 - x) = (1 - \tan x)/(1 + \tan x)$ and $\log u + \log v = \log(uv)$, for positive u, v, simplify the reflected integrands.

Step 1: Reflect the interval Let $I = \int_0^{\pi/4} \log(1 + \tan x)\, dx$. The integrand is continuous on this interval, since $0 \le \tan x \le 1$. Substitute $x = \pi/4 - t$; the limits reverse and $dx = -dt$. Thus, renaming the variable,

$$I = \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) dx.$$

Quick Tip to Solve: On an interval ending at $\pi/4$, try reflecting x to $\pi/4 - x$ before attempting integration by parts.

Step 2: Simplify and add the two forms Using the tangent subtraction identity,

$$1+\tan\left(\frac{\pi}{4}-x\right)=1+\frac{1-\tan x}{1+\tan x}=\frac{2}{1+\tan x}.$$

Both logarithm arguments are positive, so addition of the original and reflected integrals gives

$$\begin{aligned}2I&=\int_0^{\pi/4}\left[\log(1+\tan x)+\log\frac{2}{1+\tan x}\right]dx\\&=\int_0^{\pi/4}\log 2\,dx=\frac{\pi}{4}\log 2.\end{aligned}$$

The variable terms cancel across the whole interval; no separate antiderivative of the logarithm is needed.

Step 3: Conclusion Hence, $\int_0^{\pi/4}\log(1+\tan x)\,dx=\boxed{\frac{\pi}{8}\log 2}$.

Quick Tip to Solve: On an interval ending at $\pi/4$, try reflecting x to $\pi/4-x$ before attempting integration by parts.

OR

26(b). **Find** $\displaystyle\int\frac{dx}{\sqrt{\sin^3 x\,\cos(x-\alpha)}}$. [3 marks]

Solution:

Concept:

- Use $\cos(x-\alpha)=\cos x\cos\alpha+\sin x\sin\alpha$, where α is a constant parameter.
- The substitution $t=\cot x$ has differential $dt=-\csc^2 x\,dx$. Also, the principal square root satisfies $\sqrt{\sin^4 x}=\sin^2 x$.

Step 1: Factor the expression under the root Work on any interval where the given real integrand is defined, so $\sin x\neq 0$ and the radicand is positive. Expand the cosine and factor four powers of $\sin x$:

$$\begin{aligned}\sin^3 x\cos(x-\alpha)&=\sin^3 x(\cos x\cos\alpha+\sin x\sin\alpha)\\&=\sin^4 x(\cot x\cos\alpha+\sin\alpha).\end{aligned}$$

Consequently the denominator is $\sin^2 x\sqrt{\cot x\cos\alpha+\sin\alpha}$; the remaining expression in the square root is positive on the interval.

Step 2: Substitute and integrate Taking $t=\cot x$, we obtain

$$\begin{aligned}I&=\int\frac{\csc^2 x\,dx}{\sqrt{\cot x\cos\alpha+\sin\alpha}}\\&=-\int\frac{dt}{\sqrt{t\cos\alpha+\sin\alpha}}.\end{aligned}$$

For $\cos\alpha\neq 0$, put $u=t\cos\alpha+\sin\alpha$. Since $du=\cos\alpha\,dt$, integration of $u^{-1/2}$ gives $I=-2\sqrt{u}/\cos\alpha+$

Quick Tip to Solve: Factor $\sin^4 x$ under the root to expose $\csc^2 x\,dx$, the differential of $-\cot x$.

C. Differentiating this expression gives $\csc^2 x/\sqrt{\cot x \cos\alpha + \sin\alpha}$, which agrees with the original integrand.

Step 3: Account for the constant-parameter special case If $\cos\alpha = 0$, the substituted denominator is the constant $\sqrt{\sin\alpha}$. Whenever a real integrand exists, $\sin\alpha > 0$, and directly $I = -\cot x/\sqrt{\sin\alpha} + C$. If instead $\sin\alpha < 0$, the original radicand is negative wherever $\sin x \neq 0$, so there is no real interval of definition.

Step 4: Conclusion For $\cos\alpha \neq 0$, the required antiderivative is $\boxed{-\frac{2}{\cos\alpha}\sqrt{\cot x \cos\alpha + \sin\alpha} + C}$. For $\cos\alpha = 0$ on the real domain, it is $\boxed{-\cot x/\sqrt{\sin\alpha} + C}$.

Quick Tip to Solve: Factor $\sin^4 x$ under the root to expose $\csc^2 x\,dx$, the differential of $-\cot x$.

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27. Find $\displaystyle\int e^{\cot^{-1}x} \cdot \left(\frac{1-x+x^2}{1+x^2}\right) dx$. [3 marks]

Solution:

Concept:

- On its usual real branch, $d(\cot^{-1}x)/dx = -1/(1+x^2)$. By the chain rule, $d(e^{\cot^{-1}x})/dx = -e^{\cot^{-1}x}/(1+x^2)$.
- The product rule $(uv)\prime = u\prime v + uv\prime$ can identify an antiderivative when an integrand combines an exponential with its derivative.

Step 1: Separate the rational factor Let $E = e^{\cot^{-1}x}$. Rewrite the numerator as $(1+x^2) - x$, so that

$$\frac{1-x+x^2}{1+x^2} = 1 - \frac{x}{1+x^2}.$$

Thus the integral splits into $\int E\,dx - \int xE/(1+x^2)\,dx$. This form is useful because $E\prime = -E/(1+x^2)$; the second term is exactly $\int xE\prime\,dx$. The denominator $1+x^2$ is nonzero for every real x.

Step 2: Apply integration by parts and cancel Using integration by parts on $\int xE\prime\,dx$, choose $u = x$, $dv = E\prime dx$, giving $du = dx$, $v = E$. Therefore

$$\begin{aligned}\int E\left(1 - \frac{x}{1+x^2}\right) dx &= \int E\,dx + \int xE\prime\,dx \\ &= \int E\,dx + xE - \int E\,dx.\end{aligned}$$

The two remaining integrals cancel, so no separate integration of $e^{\cot^{-1}x}$ is required.

Step 3: Check by differentiation The product rule independently verifies the result:

$$\begin{aligned}\frac{d}{dx}\left(xe^{\cot^{-1}x}\right) &= e^{\cot^{-1}x} - \frac{xe^{\cot^{-1}x}}{1+x^2} \\ &= e^{\cot^{-1}x}\frac{1-x+x^2}{1+x^2}.\end{aligned}$$

Quick Tip to Solve: Rewrite the rational factor as $1 - x/(1+x^2)$, then test the derivative of $xe^{\cot^{-1}x}$.

This reproduces every term and the negative sign in the printed integrand.

Step 4: Conclusion Hence, $\int e^{\cot^{-1}x}\dfrac{1-x+x^2}{1+x^2}\,dx = \boxed{xe^{\cot^{-1}x}+C}$.

Quick Tip to Solve: Rewrite the rational factor as $1-x/(1+x^2)$, then test the derivative of $xe^{\cot^{-1}x}$.

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28. Evaluate $\displaystyle\int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{1}{(e^x+e^{-x})(e^x-e^{-x})}\,dx$**.** [3 marks]

Solution:

Concept:

- The difference of squares gives $(e^x+e^{-x})(e^x-e^{-x}) = e^{2x}-e^{-2x}$.
- For $t=e^{2x}$, $dt = 2e^{2x}dx$. Partial fractions use $1/(t^2-1) = \frac{1}{2}[1/(t-1)-1/(t+1)]$.

Step 1: Simplify and change the variable First simplify the denominator and multiply numerator and denominator by e^{2x}:

$$I = \int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{e^{2x}}{e^{4x}-1}\,dx.$$

Set $t=e^{2x}$, so $e^{2x}dx = dt/2$. At the lower limit $t = e^{2\log\sqrt{2}} = 2$, and at the upper limit $t = e^{2\log\sqrt{3}} = 3$. Hence

$$I = \frac{1}{2}\int_2^3 \frac{dt}{t^2-1}.$$

Since $2 \le t \le 3$, neither factor of the new denominator vanishes.

Step 2: Resolve into partial fractions The identity

$$\frac{1}{t^2-1} = \frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)$$

follows by putting the right side over the common denominator $(t-1)(t+1)$. Integrating,

$$I = \frac{1}{4}\left[\log|t-1| - \log|t+1|\right]_2^3 = \frac{1}{4}\left[\log\frac{t-1}{t+1}\right]_2^3.$$

The absolute values can be removed because both arguments are positive throughout the transformed interval.

Step 3: Insert the limits At $t=3$, the logarithm argument is $2/4 = 1/2$; at $t=2$, it is $1/3$. Thus the difference is $\log(1/2) - \log(1/3) = \log(3/2)$. The integrand is positive on the original interval, and this positive logarithmic difference provides a check on the order of the limits.

Step 4: Conclusion Therefore, the value of the definite integral is $\boxed{\frac{1}{4}\log\frac{3}{2}}$.

Quick Tip to Solve: After using the difference of squares, multiply by e^{2x} and set $t=e^{2x}$; change both limits immediately.

29(a). Find the general solution of the differential equation $(xy - x^2)\,dy = y^2\,dx$. [3 marks]

Solution:

Concept:

- A homogeneous differential equation has coefficients of the same degree. Here both $xy - x^2$ and y^2 have degree two, so the ratio substitution $x = vy$ reduces the equation to separable form.
- Treat x locally as a function of y. With $v = x/y$, the product rule gives $dx/dy = v + y\,dv/dy$. The integral $\int dy/y = \ln|y|$ is used on an interval where $y \neq 0$.

Step 1: Rewrite the equation and make the substitution From the given equation, for $y \neq 0$,

$$\frac{dx}{dy} = \frac{xy - x^2}{y^2} = \frac{x}{y} - \left(\frac{x}{y}\right)^2.$$

Put $x = vy$. Substituting the derivative gives

$$v + y\frac{dv}{dy} = v - v^2, \qquad y\frac{dv}{dy} = -v^2.$$

Thus the substitution removes the separate occurrences of x and leaves an equation in v and y.

Step 2: Separate variables and integrate For $v \neq 0$, separate the variables and integrate:

$$\frac{dv}{v^2} = -\frac{dy}{y}, \qquad -\frac{1}{v} = -\ln|y| + C_1.$$

Absorbing the sign of the arbitrary constant into C, this is $1/v = \ln|y| + C$. Since $v = x/y$, it becomes $y/x = \ln|y| + C$. The logarithm is well defined for either sign of nonzero y; the constant is fixed on each solution interval.

Step 3: Verify the family and check a divided-out solution Differentiating $y/x - \ln|y| = C$ with respect to x gives

$$\frac{xy' - y}{x^2} - \frac{y'}{y} = 0.$$

Multiplying by x^2y yields $(xy - x^2)y' = y^2$, exactly the original equation. The division by y excluded $y = 0$, which also satisfies the original equation identically.

Step 4: Conclusion The general solution for nonzero x, y is $\boxed{\frac{y}{x} - \ln|y| = C}$. The constant solution $\boxed{y = 0}$ also satisfies the differential equation.

Quick Tip to Solve: Use x as a function of y here: x=vy immediately cancels the linear v terms, leaving a separable equation.

OR

29(b). Find the general solution of the differential equation $(x^2+1)\dfrac{dy}{dx}+2xy=\sqrt{x^2+4}$. [3 marks]

Solution:

Concept:

- A first-order linear differential equation has the form $y\prime+P(x)y=Q(x)$, and its integrating factor is $\text{IF}=e^{\int P(x)\,dx}$.
- After multiplication by the integrating factor, the left side is the derivative of $y\,\text{IF}$. For $a>0$, $\int\sqrt{x^2+a^2}\,dx=\frac{1}{2}[x\sqrt{x^2+a^2}+a^2\ln|x+\sqrt{x^2+a^2}|]+C$.

Step 1: Find the integrating factor Since $x^2+1>0$ for every real x, divide the equation by x^2+1:

$$\frac{dy}{dx}+\frac{2x}{x^2+1}y=\frac{\sqrt{x^2+4}}{x^2+1}.$$

Hence $P(x)=2x/(x^2+1)$, and

$$\text{IF}=e^{\int\frac{2x}{x^2+1}\,dx}=e^{\ln(x^2+1)}=x^2+1.$$

Any nonzero constant factor in the integrating factor can be omitted.

Step 2: Apply the product rule and integrate Multiplying by x^2+1, the original left side is recognized as a product derivative:

$$\frac{d}{dx}\left[(x^2+1)y\right]=\sqrt{x^2+4}.$$

Integrating both sides and applying the standard radical integral with $a=2$,

$$(x^2+1)y=\frac{x}{2}\sqrt{x^2+4}+2\ln\left|x+\sqrt{x^2+4}\right|+C.$$

This integration introduces one arbitrary constant, as expected for a first-order equation.

Step 3: Verify the integral Write $s=\sqrt{x^2+4}$, so $s\prime=x/s$. The logarithmic derivative is

$$\frac{d}{dx}\ln(x+s)=\frac{1+x/s}{x+s}=\frac{1}{s}.$$

Therefore, the derivative of the integrated right-hand side is

$$\frac{s}{2}+\frac{x^2}{2s}+\frac{2}{s}=\frac{s}{2}+\frac{x^2+4}{2s}=s.$$

This reproduces the required integrand. Also $s>|x|$, so $x+s>0$ for all real x. Thus the logarithm and the solution are defined for every real x.

Step 4: Conclusion

$$\boxed{y=\frac{\frac{x}{2}\sqrt{x^2+4}+2\ln\left(x+\sqrt{x^2+4}\right)+C}{x^2+1}}.$$

Quick Tip to Solve: Notice directly that (x squared plus 1)y has derivative equal to the given left side; then integrate the radical.

30(a). Two balls are drawn at random one by one with replacement from an urn containing equal number of red balls and green balls. Find the probability distribution of number of red balls. Also, find the mean of the random variable. [3 marks]

Solution:

Concept:

- Let X denote the number of red balls in the two draws. Its possible values are $0, 1, 2$, since there are exactly two draws.
- Equal numbers of red and green balls give $P(R) = P(G) = 1/2$. Replacement restores the urn before the second draw, so the two draws are independent.
- The mean of a discrete random variable is $E(X) = \sum_x xP(X = x)$, where the sum runs over all its possible values.

Step 1: List the elementary colour outcomes Use R for red and G for green. Because the first ball is replaced, the probability of each colour remains $1/2$ on the second draw, regardless of the first result. Hence the ordered outcomes RR, RG, GR, GG each have probability $(1/2)(1/2) = 1/4$. The order matters when listing outcomes, even though X counts only how many reds occurred.

Step 2: Construct the probability distribution The event $X = 0$ is GG, the event $X = 1$ consists of the two mutually exclusive outcomes RG and GR, and the event $X = 2$ is RR. Therefore

x	0	1	2
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

All the probabilities are nonnegative, and their sum is $1/4+1/2+1/4 = 1$, so the distribution accounts for every possible result.

Step 3: Calculate the mean Weight each possible value by its probability:

$$E(X) = 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) = 1.$$

As a check, each draw contributes an expected red-ball count of $1/2$; adding the contributions of the two draws gives $1/2 + 1/2 = 1$. The mean is an average over repeated pairs of draws, not a claim that every pair contains one red ball.

Step 4: Conclusion The probability distribution is $\boxed{P(X = 0) = \frac{1}{4},\ P(X = 1) = \frac{1}{2},\ P(X = 2) = \frac{1}{4}}$, and the mean is $\boxed{E(X) = 1}$.

Quick Tip to Solve: With replacement, list RR, RG, GR and GG first; count both mixed outcomes when calculating the probability of one red.

OR

30(b). A **and** B **throw a die alternately till one of them gets a '6' and wins the game. Find their respective probabilities of winning, if** A **starts the game first.** [3 marks]

Solution:

Concept:

- For a fair die, a winning throw has probability $p = P(6) = 1/6$, and a nonwinning throw has probability $q = 5/6$. Successive throws are independent.
- A complete unsuccessful round consists of one failed throw by A and one by B, with probability $q^2 = 25/36$.
- The geometric-series identity $1 + r + r^2 + \cdots = 1/(1-r)$ applies for $|r| < 1$. Mutually exclusive winning histories have probabilities that can be added.

Step 1: Find the probability that A wins A can win on the first throw with probability p. To win on the next turn, both players must first fail, then A must throw a six, with probability q^2p. In general, a win after k unsuccessful full rounds has probability $(q^2)^k p$. These events are disjoint, so

$$P(A) = p\sum_{k=0}^{\infty}(q^2)^k = \frac{1/6}{1-25/36} = \frac{6}{11}.$$

Step 2: Find the probability that B wins For B to win on the first turn, A must fail and B must throw a six, giving qp. After k unsuccessful full rounds, the corresponding probability is $(q^2)^k qp$. Thus

$$P(B) = qp\sum_{k=0}^{\infty}(q^2)^k = \frac{(5/6)(1/6)}{1-25/36} = \frac{5}{11}.$$

The extra factor q reflects the fact that B gets a turn only if A has not already won.

Step 3: Check that the winning events exhaust the game The probability that neither player wins in the first n rounds is $(25/36)^n$, which tends to zero as n tends to infinity. Thus the game ends with probability one, and $P(A) + P(B) = 6/11 + 5/11 = 1$. The larger probability for A is consistent with the advantage of having the first opportunity to throw a six.

Step 4: Conclusion The respective probabilities of winning are $\boxed{P(A) = \frac{6}{11}}$ and $\boxed{P(B) = \frac{5}{11}}$.

Quick Tip to Solve: Group two failed throws into one round; the common geometric ratio is 25/36, and B needs one extra initial failure by A.

31. Solve the following linear programming problem graphically:

Minimize $Z = 5x + 10y$

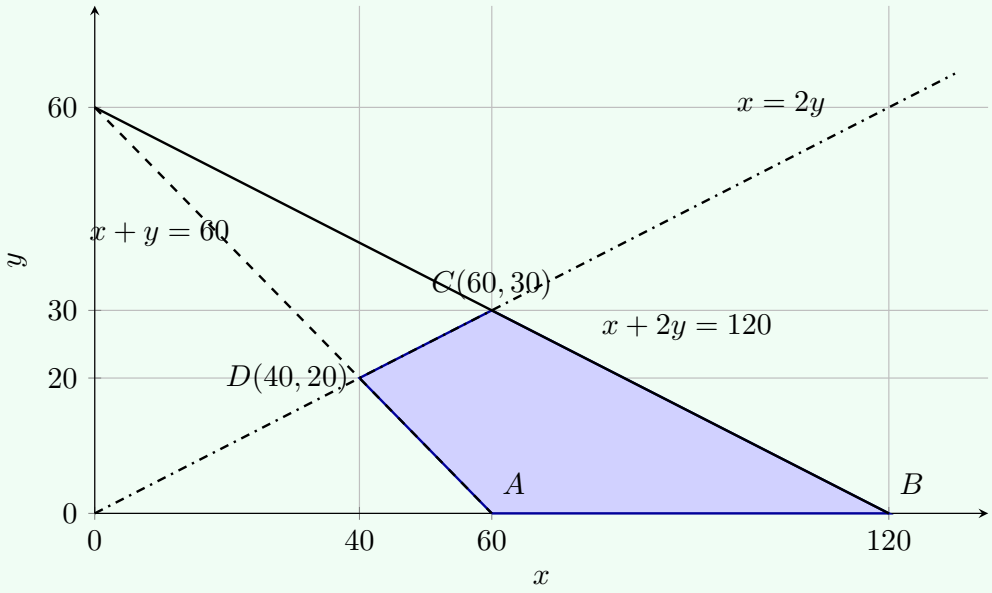
subject to constraints: $x + 2y \leq 120$, $x + y \geq 60$, $x - 2y \geq 0$, $x \geq 0$, $y \geq 0$. [3 marks]

Solution:

Concept:

- A linear inequality selects a half-plane. The feasible region is the intersection of all the permitted half-planes in the first quadrant.
- A linear objective attains its minimum at a vertex of a nonempty bounded polygon. Evaluate $Z = 5x + 10y$ at every feasible vertex.

Step 1: Graph the boundary lines Plot $x+2y = 120$ using $(120, 0)$ and $(0, 60)$; retain its side containing the origin. Plot $x + y = 60$ using $(60, 0)$ and $(0, 60)$; retain its side away from the origin. The third boundary is $y = x/2$, through the origin; retain the points on or below this line. Work only in the first quadrant. The common region is the shaded quadrilateral ABCD shown below, including its boundaries.



Shaded feasible region ABCD.

Step 2: Find all feasible vertices On $y = 0$, the constraints require $60 \leq x \leq 120$, giving $A = (60, 0)$ and $B = (120, 0)$. For the intersection of $x + 2y = 120$ and $x = 2y$, substitution gives $4y = 120$, hence $C = (60, 30)$. For the intersection of $x + y = 60$ and $x = 2y$, substitution gives $3y = 60$, hence $D = (40, 20)$. The intersection $(0, 60)$ of the first two boundary lines is excluded because it violates $x - 2y \geq 0$.

Step 3: Compare the objective values Substitution in $Z = 5x + 10y$ gives

$$Z(A) = 5(60) + 10(0) = 300,$$
$$Z(B) = 5(120) + 10(0) = 600,$$
$$Z(C) = 5(60) + 10(30) = 600,$$
$$Z(D) = 5(40) + 10(20) = 400.$$

Quick Tip to Solve: Reject any boundary-line intersection that fails another inequality before evaluating the objective there.

All four vertices satisfy every constraint. Since the region is bounded, the corner-point theorem applies; the least of these four values occurs at A.

Step 4: Conclusion The minimum is $\boxed{Z = 300}$, attained at $\boxed{x = 60,\ y = 0}$.

Quick Tip to Solve: Reject any boundary-line intersection that fails another inequality before evaluating the objective there.

Section D (20 marks)

*This section comprises Long Answer type questions (LA) of **5** marks each.*

32(a). If $A = \begin{pmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$, then find AB and use it to solve the following system of equations:

$x - 2y = 3$

$2x - y - z = 2$

$-2y + z = 3$ [5 marks]

Solution:

Concept:

- The entry in row i, column j of AB is the scalar product of row i of A and column j of B.
- If square matrices satisfy $AB = I$, then $B^{-1} = A$. Transposition reverses products, so $B^T A^T = I$ and $(B^T)^{-1} = A^T$.
- A system with coefficient matrix M, unknown column X and constant column C has the unique solution $X = M^{-1}C$ when M is invertible.

Step 1: Multiply the given matrices Take each row of A against each column of B. Keeping the signs of the negative entries explicit gives

$$AB = \begin{pmatrix} -3+4+0 & -6+2+4 & 0+4-4 \\ 2-2+0 & 4-1-2 & 0-2+2 \\ 2-2+0 & 4-1-3 & 0-2+3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus $AB = I_3$, establishing that B is invertible and that its inverse is A.

Step 2: Identify the coefficient matrix of the equations With the unknowns consistently ordered as x, y, z, the printed equations become

$$\begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}.$$

Notice that this coefficient matrix is B^T. Write the system as $B^T X = C$. Transposing $AB = I_3$ gives $B^T A^T = I_3$, so $(B^T)^{-1} = A^T$. This uses the product already found rather than computing a fresh

Quick Tip to Solve: Compare the coefficient matrix with both B and its transpose before using the computed product as an inverse.

inverse.

Step 3: Obtain the unknown column using the inverse Multiply the system on the left by A^T. Since A^T is the inverse of B^T, this gives $X = A^T C$. Therefore

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} -9+4+6 \\ -6+2+3 \\ -12+4+9 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Invertibility guarantees that this solution is unique.

Step 4: Check the result in all three equations Direct substitution checks the values independently of the matrix multiplication:

$$x - 2y = 1 - 2(-1) = 3,$$
$$2x - y - z = 2(1) - (-1) - 1 = 2,$$
$$-2y + z = -2(-1) + 1 = 3.$$

Each left-hand side equals its corresponding printed right-hand side.

Step 5: Conclusion $\boxed{AB = I_3}$. The system has the unique solution $\boxed{x = 1,\ y = -1,\ z = 1}$.

Quick Tip to Solve: Compare the coefficient matrix with both B and its transpose before using the computed product as an inverse.

OR

32(b). If $f(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$**, prove that** $f(\alpha) \cdot f(-\beta) = f(\alpha - \beta)$**.** [5 marks]

Solution:

Concept:

- A matrix product is obtained by taking the scalar product of each row of the first factor with each column of the second factor. The order of the factors must be preserved.
- Cosine is even and sine is odd: $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$.
- Use $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

Step 1: Evaluate the matrix at the negative angle Substituting $-\beta$ into the given definition and using the even and odd identities gives

$$f(-\beta) = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Quick Tip to Solve: Substitute the negative angle before multiplying and check both sine signs to obtain the angle difference correctly.

Thus the upper-right entry changes to positive $\sin\beta$, while the lower-left entry becomes negative $\sin\beta$. The cosine entries and the third diagonal entry remain unchanged. Keeping these signs correct is essential, because the question involves the difference of the two angles.

Step 2: Calculate the row-column products Define the following two expressions for compact working:

$$C = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$$
$$T = \sin\alpha\cos\beta - \cos\alpha\sin\beta.$$

The first row times the first column gives C. The first row times the second column gives $\cos\alpha\sin\beta - \sin\alpha\cos\beta = -T$. The second row times the first column gives T. The second row times the second column gives $\sin\alpha\sin\beta + \cos\alpha\cos\beta = C$. Each entry connecting the first two coordinates with the third is zero, since its row-column product contains only zero terms. The last diagonal entry is $0 + 0 + 1 = 1$. Hence the complete product is

$$f(\alpha)f(-\beta) = \begin{pmatrix} C & -T & 0 \\ T & C & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Step 3: Identify the resulting angle The angle-subtraction formulae give $C = \cos(\alpha - \beta)$ and $T = \sin(\alpha - \beta)$. Substituting these expressions into the product yields

$$\begin{pmatrix} \cos(\alpha-\beta) & -\sin(\alpha-\beta) & 0 \\ \sin(\alpha-\beta) & \cos(\alpha-\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Every entry agrees with the given definition evaluated at $\alpha - \beta$. The third row and column also agree, so the comparison proves equality of the entire matrices, for arbitrary real angles without any additional restriction.

Step 4: Conclusion Hence $\boxed{f(\alpha)\cdot f(-\beta) = f(\alpha-\beta)}$, as required.

Quick Tip to Solve: Substitute the negative angle before multiplying and check both sine signs to obtain the angle difference correctly.

33(a). **Find the equations of the diagonals of the parallelogram $PQRS$ whose vertices are $P(4, 2, -6)$, $Q(5, -3, 1)$, $R(12, 4, 5)$ and $S(11, 9, -2)$. Use these equations to find the point of intersection of diagonals.** [5 marks]

Solution:

Concept:

- The diagonals of parallelogram $PQRS$ join opposite vertices, so they are PR and QS.
- A line through position vector $\mathbf{a}$ with nonzero direction vector $\mathbf{d}$ has equation $\mathbf{r} = \mathbf{a} + t\mathbf{d}$, where $t \in \mathbb{R}$. Its symmetric form uses the direction ratios of $\mathbf{d}$.
- At the intersection the coordinates obtained from both line equations must agree. Diagonals of a

Quick Tip to Solve: Join opposite vertices first; after solving two coordinate equations, verify the third so that intersecting lines are distinguished from skew lines.

parallelogram also bisect each other, which provides an independent midpoint check.

Step 1: Find the equation of diagonal PR Using the opposite vertices $P(4, 2, -6)$ and $R(12, 4, 5)$, the direction vector is

$$\overrightarrow{PR} = (12 - 4, 4 - 2, 5 - (-6)) = (8, 2, 11).$$

Thus a vector equation and a symmetric equation of this diagonal are

$$\mathbf{r} = (4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}) + \lambda(8\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 11\hat{\mathbf{k}}),$$
$$\frac{x-4}{8} = \frac{y-2}{2} = \frac{z+6}{11}.$$

Equivalently, $x = 4 + 8\lambda$, $y = 2 + 2\lambda$, $z = -6 + 11\lambda$. These equations give P at $\lambda = 0$ and R at $\lambda = 1$.

Step 2: Find the equation of diagonal QS From $Q(5, -3, 1)$ and $S(11, 9, -2)$,

$$\overrightarrow{QS} = (11 - 5, 9 - (-3), -2 - 1) = (6, 12, -3).$$

Consequently, the equations of the other diagonal are

$$\mathbf{r} = (5\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}) + \mu(6\hat{\mathbf{i}} + 12\hat{\mathbf{j}} - 3\hat{\mathbf{k}}),$$
$$\frac{x-5}{6} = \frac{y+3}{12} = \frac{z-1}{-3}.$$

Its parametric coordinates are $x = 5 + 6\mu$, $y = -3 + 12\mu$, $z = 1 - 3\mu$. The negative third denominator reflects the decrease in the third coordinate from Q to S.

Step 3: Use the line equations to find their intersection Equating the first two coordinates gives

$$8\lambda - 6\mu = 1,$$
$$2\lambda - 12\mu = -5.$$

Multiply the second equation by four and subtract it from the first. This yields $42\mu = 21$, hence $\mu = \frac{1}{2}$. Substituting back gives $\lambda = \frac{1}{2}$. The third coordinates also agree:

$$-6 + 11\left(\frac{1}{2}\right) = -\frac{1}{2} = 1 - 3\left(\frac{1}{2}\right).$$

Substitution into either diagonal gives $x = 8$, $y = 3$, $z = -\frac{1}{2}$. As a check, the midpoint of PR and the midpoint of QS both equal $(8, 3, -\frac{1}{2})$, confirming the expected bisection property.

Step 4: Conclusion The diagonals are $\boxed{\frac{x-4}{8} = \frac{y-2}{2} = \frac{z+6}{11}}$ and $\boxed{\frac{x-5}{6} = \frac{y+3}{12} = \frac{z-1}{-3}}$. Their point of intersection is $\boxed{(8, 3, -\frac{1}{2})}$.

Quick Tip to Solve: Join opposite vertices first; after solving two coordinate equations, verify the third so that intersecting lines are distinguished from skew lines.

OR

33(b). A line l passes through point $(-1, 3, -2)$ and is perpendicular to both the lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$. Find the vector equation of the line l. Hence, obtain its distance from origin. [5 marks]

Solution:

Concept:

- The direction ratios of a line in symmetric form are its denominators. A vector perpendicular to two nonparallel directions is their cross product.
- A line through position vector $\vec{a}$ in direction $\vec{d}$ is $\vec{r} = \vec{a} + \lambda\vec{d}$, where $\lambda \in \mathbb{R}$.
- The perpendicular distance from the origin to that line is $D = |\vec{a} \times \vec{d}|/|\vec{d}|$. The closest point has its position vector perpendicular to the line's direction.

Step 1: Identify the given directions The denominators in the two symmetric equations give

$$\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}, \qquad \vec{c} = -3\hat{i} + 2\hat{j} + 5\hat{k}.$$

Constants added to the numerators determine points on those lines; they do not alter the direction ratios. The desired direction must have zero scalar product with each of these two vectors. Their cross product supplies such a direction, since the two vectors are not parallel.

Step 2: Calculate and check the perpendicular direction Expanding the cross product by components gives

$$\begin{aligned}\vec{b} \times \vec{c} &= (2 \cdot 5 - 3 \cdot 2)\hat{i} + (3 \cdot (-3) - 1 \cdot 5)\hat{j} + (1 \cdot 2 - 2 \cdot (-3))\hat{k} \\ &= 4\hat{i} - 14\hat{j} + 8\hat{k}.\end{aligned}$$

Divide by 2 to take $\vec{d} = 2\hat{i} - 7\hat{j} + 4\hat{k}$. Direct verification gives

$$\vec{d} \cdot \vec{b} = 2 - 14 + 12 = 0, \qquad \vec{d} \cdot \vec{c} = -6 - 14 + 20 = 0.$$

Thus this direction meets both perpendicularity requirements. A nonzero scalar multiple would describe the same direction.

Step 3: Form the vector equation The specified point $A(-1, 3, -2)$ has position vector $\vec{a} = -\hat{i} + 3\hat{j} - 2\hat{k}$. Therefore the equation is

$$\vec{r} = (-\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k}), \quad \lambda \in \mathbb{R}.$$

Setting $\lambda = 0$ recovers the prescribed point, while its coefficient gives the direction already checked above.

Step 4: Obtain the distance from the origin Compute the vector product using the same point and direction:

$$\begin{aligned}\vec{a} \times \vec{d} &= (12 - 14)\hat{i} + (-4 + 4)\hat{j} + (7 - 6)\hat{k} \\ &= -2\hat{i} + \hat{k}, \\ |\vec{a} \times \vec{d}| &= \sqrt{5}, \qquad |\vec{d}| = \sqrt{4 + 49 + 16} = \sqrt{69}.\end{aligned}$$

Quick Tip to Solve: Reduce the cross-product direction before computing distance, and check its dot product with each given direction is zero.

Substitution in the perpendicular-distance formula gives $D = \sqrt{5}/\sqrt{69}$. To check independently, $\vec{a} \cdot \vec{d} = -31$ and $|\vec{a}|^2 = 14$. Removing the squared component along the line gives

$$D^2 = 14 - \frac{(-31)^2}{69} = \frac{966 - 961}{69} = \frac{5}{69}.$$

This confirms both the distance and its positivity.

Step 5: Conclusion The required line is $\boxed{\vec{r} = (-\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k}),\ \lambda \in \mathbb{R}}$, and its distance from the origin is $\boxed{\sqrt{\frac{5}{69}} \text{ units}}$.

Quick Tip to Solve: Reduce the cross-product direction before computing distance, and check its dot product with each given direction is zero.

34. Using integration, find the area of region bounded by line $y = \sqrt{3}\,x$, the curve $y = \sqrt{4 - x^2}$ and y-axis in first quadrant. [5 marks]

Solution:

Concept:

- The curve $y = \sqrt{4 - x^2}$ is the upper semicircle $x^2 + y^2 = 4$, with radius 2. In the first quadrant, both coordinates are nonnegative.
- The area between an upper curve $y = f(x)$ and lower curve $y = g(x)$, from $x = a$ to $x = b$, is $\int_a^b [f(x) - g(x)]\,dx$.
- A useful antiderivative is $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + C$, with inverse sine measured in radians.

Step 1: Locate the intersection in the first quadrant At the intersection of the line and semicircle,

$$\sqrt{3}\,x = \sqrt{4 - x^2}.$$

Both sides are nonnegative in the first quadrant, so squaring is valid:

$$3x^2 = 4 - x^2 \quad \Longrightarrow \quad 4x^2 = 4.$$

Since $x \geq 0$, choose $x = 1$; then $y = \sqrt{3}$. The other boundaries meet the y-axis at $(0, 0)$ for the line and $(0, 2)$ for the curve. These points identify the closed region specified in the question.

Step 2: Set up the area integral A vertical strip of the region extends from the line to the curve. For $0 \leq x \leq 1$,

$$4 - x^2 \geq 3x^2,$$

and hence $\sqrt{4 - x^2} \geq \sqrt{3}\,x$. Therefore the semicircle is the upper boundary throughout this interval. The y-axis supplies the left endpoint $x = 0$, and the intersection supplies the right endpoint $x = 1$.

Quick Tip to Solve: Find the positive intersection first, then subtract the line integral from the semicircle integral over the same interval.

Thus

$$A = \int_0^1 \left(\sqrt{4 - x^2} - \sqrt{3}\, x\right)\, dx.$$

Step 3: Integrate the semicircle contribution Applying the stated antiderivative with $a = 2$,

$$\begin{aligned}\int_0^1 \sqrt{4 - x^2}\, dx &= \left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}(x/2)\right]_0^1 \\ &= \frac{\sqrt{3}}{2} + 2\sin^{-1}(1/2) \\ &= \frac{\sqrt{3}}{2} + \frac{\pi}{3}.\end{aligned}$$

Here $\sin^{-1}(1/2) = \pi/6$, and both terms in the antiderivative vanish at the lower limit. This gives the whole area below the semicircle between $x = 0$ and $x = 1$.

Step 4: Subtract the area below the line The line contributes

$$\int_0^1 \sqrt{3}\, x\, dx = \left[\frac{\sqrt{3}}{2}x^2\right]_0^1 = \frac{\sqrt{3}}{2}.$$

Consequently, the strip integral evaluates to

$$A = \left(\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) - \frac{\sqrt{3}}{2} = \frac{\pi}{3}.$$

The radical terms cancel exactly. As a geometric check on this integration, the line makes angle $\pi/3$ with the positive x-axis, so the specified region is a circular sector of angle $\pi/2 - \pi/3 = \pi/6$ and radius 2; its area also equals $\frac{1}{2} \cdot 4 \cdot \pi/6$.

Step 5: Conclusion The area of the region bounded by the line, the curve and the y-axis in the first quadrant is $\boxed{\frac{\pi}{3} \text{ square units}}$.

Quick Tip to Solve: Find the positive intersection first, then subtract the line integral from the semicircle integral over the same interval.

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35. A function $f : [-4, 4] \to [0, 4]$ is given by $f(x) = \sqrt{16 - x^2}$. Show that f is an onto function but not a one-one function. Further, find all possible values of a for which $f(a) = \sqrt{7}$. [5 marks]

Solution:

Concept:

- A function is onto if every element of its codomain has at least one preimage in its domain.
- A function is one-one if distinct inputs always have distinct images. To disprove this, one pair of distinct inputs with equal images suffices.
- The principal square root is non-negative: $\sqrt{u} \geq 0$ for $u \geq 0$.

Quick Tip to Solve: For an onto proof, start with an arbitrary codomain value and construct a valid input; evenness supplies a quick counterexample to one-one behaviour.

Step 1: Check the domain and codomain For any $x \in [-4, 4]$, we have $0 \leq x^2 \leq 16$. Thus $0 \leq 16 - x^2 \leq 16$, and taking the principal square root gives $0 \leq f(x) \leq 4$. Hence the formula defines a real-valued function on the entire stated domain and its values lie in the stated codomain $[0, 4]$.

Step 2: Prove that every codomain value is attained Let y be an arbitrary element of $[0, 4]$. We seek an input satisfying $\sqrt{16 - x^2} = y$. Since $0 \leq y \leq 4$, the number $x = \sqrt{16 - y^2}$ is real and belongs to $[0, 4] \subseteq [-4, 4]$. Substituting this input gives

$$f\left(\sqrt{16 - y^2}\right) = \sqrt{16 - (16 - y^2)} = \sqrt{y^2} = y.$$

The last equality uses $y \geq 0$. This works at the endpoints too: $y = 0$ is attained at $x = 4$, and $y = 4$ at $x = 0$. Since y was arbitrary, the range equals the whole codomain, so f is onto.

Step 3: Disprove one-one behaviour Take $x_1 = 3$ and $x_2 = -3$, both in the domain. These inputs are distinct, but

$$f(3) = \sqrt{16 - 9} = \sqrt{7}, \qquad f(-3) = \sqrt{16 - 9} = \sqrt{7}.$$

Thus $f(x_1) = f(x_2)$ does not imply $x_1 = x_2$, so f is not one-one. More generally, the formula is even because $(-x)^2 = x^2$.

Step 4: Solve for all permissible inputs The condition is $\sqrt{16 - a^2} = \sqrt{7}$. Both sides are non-negative, so squaring is valid:

$$16 - a^2 = 7 \quad \Longrightarrow \quad a^2 = 9 \quad \Longrightarrow \quad a = 3 \text{ or } a = -3.$$

Both candidates belong to $[-4, 4]$, and substituting either gives $\sqrt{16 - 9} = \sqrt{7}$. Conversely, the squared equation has only these two real roots, so there are no additional possible inputs.

Step 5: Conclusion The function f is onto but not one-one, and all possible values are $\boxed{a = \pm 3}$.

Quick Tip to Solve: For an onto proof, start with an arbitrary codomain value and construct a valid input; evenness supplies a quick counterexample to one-one behaviour.

Section E (12 marks)

*This section comprises 3 source based/case-based/passage based/integrated units of assessment questions of **4** marks each.*

36. Engine displacement is the measure of the cylinder volume swept by all the pistons of a piston engine. The piston moves inside the cylinder bore.

The cylinder bore in the form of circular cylinder open at the top is to be made from a metal sheet of area 75π cm^2.

Based on the above information, answer the following questions: [4 marks]

36(i). If the radius of cylinder is r cm and height is h cm, then write the volume V of cylinder in terms of radius r. [1 mark]

Solution:

Concept:

- For an open-top cylinder, metal covers one circular base and the curved surface: $A = \pi r^2 + 2\pi rh$.
- The volume of a cylinder is $V = \pi r^2 h$, where r and h are its radius and height.

Step 1: Express height using the fixed sheet area The cylinder has no top, so the fixed area condition is

$$\pi r^2 + 2\pi rh = 75\pi.$$

Dividing by π and solving for the height gives

$$h = \frac{75 - r^2}{2r}.$$

A physical cylinder requires $r > 0$ and $h > 0$, hence $0 < r < \sqrt{75}$.

Step 2: Substitute in the volume formula Eliminate height by inserting this expression into $V = \pi r^2 h$:

$$V(r) = \pi r^2 \left(\frac{75 - r^2}{2r}\right) = \frac{\pi}{2}(75r - r^3).$$

The base term πr^2 must be included in the area; using curved surface area alone would give an incorrect height and volume.

Step 3: Conclusion $\boxed{V(r) = \frac{\pi}{2}(75r - r^3) \text{ cm}^3}$, for $0 < r < \sqrt{75}$.

Quick Tip to Solve: For an open-top cylinder, include exactly one base when eliminating height from the volume formula.

36(ii). Find $\frac{dV}{dr}$. [1 mark]

Solution:

Concept:

- With constant surface area, height varies with radius; first express volume entirely in terms of r.
- The power rule is $\frac{d}{dr}(r^n) = nr^{n-1}$; constant factors remain unchanged.

Step 1: Obtain the single-variable volume The open-top area condition gives $\pi r^2 + 2\pi rh = 75\pi$, so $h = (75 - r^2)/(2r)$. Therefore

$$V(r) = \pi r^2 h = \frac{\pi}{2}(75r - r^3).$$

This expression already accounts for the change in height as the radius changes.

Step 2: Differentiate term by term Keeping the sheet area fixed, apply the power rule:

$$\frac{dV}{dr} = \frac{\pi}{2}\left(\frac{d}{dr}(75r) - \frac{d}{dr}(r^3)\right) = \frac{\pi}{2}(75 - 3r^2).$$

Differentiating $\pi r^2 h$ while treating h as constant would not respect the fixed-area constraint.

Step 3: Conclusion $\boxed{\frac{dV}{dr} = \frac{\pi}{2}(75 - 3r^2) = \frac{3\pi}{2}(25 - r^2)}$.

Quick Tip to Solve: Eliminate the constrained variable before differentiating; here height is not a constant.

36(iii)(a). Find the radius of cylinder when its volume is maximum. [2 marks]

Solution:

Concept:

- An interior maximum of a differentiable volume function must satisfy $V\prime(r) = 0$.
- If $V\prime\prime(r) < 0$ at the critical radius, the second derivative test establishes a local maximum.

Step 1: Find the feasible critical radius For an open-top cylinder, $\pi r^2 + 2\pi rh = 75\pi$, giving $h = (75 - r^2)/(2r)$. Thus

$$V(r) = \frac{\pi}{2}(75r - r^3), \qquad 0 < r < \sqrt{75}.$$

Differentiating and setting the result to zero gives

$$V\prime(r) = \frac{3\pi}{2}(25 - r^2) = 0 \quad \Longrightarrow \quad r^2 = 25.$$

The negative root is not a physical radius, leaving $r = 5$, which lies inside the feasible interval.

Step 2: Verify that the stationary volume is maximum The second derivative is $V\prime\prime(r) = -3\pi r$. At $r = 5$, it equals $-15\pi < 0$, so the volume has a local maximum there. Also $V\prime(r) > 0$ for $0 < r < 5$, and $V\prime(r) < 0$ for $5 < r < \sqrt{75}$. Hence volume increases up to this radius and decreases throughout the remaining feasible interval, establishing the global maximum. The corresponding height is $(75 - 25)/(2 \cdot 5) = 5$ cm.

Quick Tip to Solve: After finding a positive stationary radius, check the derivative sign or second derivative to establish a maximum.

Step 3: Conclusion The cylinder has maximum volume when $\boxed{r = 5 \text{ cm}}$.

Quick Tip to Solve: After finding a positive stationary radius, check the derivative sign or second derivative to establish a maximum.

OR

36(iii)(b). For maximum volume, $h > r$. State true or false and justify. [2 marks]

Solution:

Concept:

- For an open-top cylinder of fixed area, $\pi r^2 + 2\pi rh = 75\pi$ and $V = \pi r^2 h$.
- Use the stationary point and second derivative test to find the maximizing dimensions before comparing h and r.

Step 1: Determine the radius at maximum volume Eliminating height gives

$$h = \frac{75 - r^2}{2r}, \qquad V(r) = \frac{\pi}{2}(75r - r^3).$$

Therefore

$$V\prime(r) = \frac{3\pi}{2}(25 - r^2).$$

Setting this to zero and taking the positive radius gives $r = 5$ cm. Since $V\prime\prime(5) = -15\pi < 0$, this is a maximum. The first derivative is positive before 5 and negative after 5 on the physical domain, so it is the maximum over all feasible cylinders.

Step 2: Compare the height with the radius Substitute $r = 5$ into the area constraint:

$$25 + 10h = 75 \quad \Longrightarrow \quad h = 5 \text{ cm}.$$

Thus the maximizing height and radius are equal. In general the stationary condition here is $75 = 3r^2$; inserting it into $r^2 + 2rh = 75$ yields $2rh = 2r^2$, and division by positive $2r$ yields $h = r$. A strict inequality $h > r$ is therefore not satisfied.

Step 3: Conclusion The statement is $\boxed{\text{False}}$: at maximum volume, $h = r = 5$ cm.

Quick Tip to Solve: Compare the optimized dimensions using the area constraint; an open-top cylinder has maximum volume at height equal to radius.

37. Recent studies suggest that roughly 12% of the world population is left handed.
Depending upon the parents, the chances of having a left handed child are as follows:
A: When both father and mother are left handed: Chances of left handed child is 24%.
B: When father is right handed and mother is left handed: Chances of left handed child is 22%.
C: When father is left handed and mother is right handed: Chances of left handed child is 17%.
D: When both father and mother are right handed: Chances of left handed child is 9%.
Assuming that $P(A) = P(B) = P(C) = P(D) = \frac{1}{4}$ and L denotes the event that child is left handed.
Based on the above information, answer the following questions: [4 marks]

37(i). Find $P(L/C)$. [1 mark]

Solution:

Concept:

- Conditional probability $P(L \mid C)$ is the chance of a left-handed child when parental category C is already known.
- Convert a percentage to a probability by dividing it by 100.

Step 1: Identify the given parental category Category C means that the father is left handed and the mother is right handed. The passage gives the chance of a left-handed child in this category as 17%. This is conditional information, because it applies only to children with these parents.

Step 2: Express the given percentage as a probability The notation $P(L/C)$ means $P(L \mid C)$. Therefore the supplied percentage gives

$$P(L \mid C) = \frac{17}{100} = 0.17.$$

The prior probability $P(C) = 1/4$ is not a further factor here: multiplying by it would give the joint probability of L and C.

Step 3: Conclusion $\boxed{P(L/C) = 0.17}$.

Quick Tip to Solve: When the parents are already specified, read their conditional percentage directly; do not multiply it by the prior.

37(ii). Find $P(\overline{L}/A)$. [1 mark]

Solution:

Concept:

- Within a fixed parental category, L and $\overline{L}$ are complementary events.
- The conditional complement rule is $P(\overline{L} \mid A) = 1 - P(L \mid A)$.

Step 1: Read the probability for category A Category A means both parents are left handed. The passage states that the chance of their child being left handed is 24%. Thus $P(L \mid A) = 24/100 = 0.24$. The question asks for $\overline{L}$, meaning that the child is not left handed.

Quick Tip to Solve: Notice the bar over L: subtract the category A left-handed chance from 1 before writing the answer.

Step 2: Use the complement within the same category For children whose parents satisfy A, being left handed and not being left handed exhaust the possibilities. Their conditional probabilities therefore sum to 1:

$$P(\overline{L} \mid A) = 1 - 0.24 = 0.76.$$

No multiplication by $P(A)$ is needed, because the parental category is given.

Step 3: Conclusion $\boxed{P(\overline{L}/A) = 0.76}$, or 76%.

Quick Tip to Solve: Notice the bar over L: subtract the category A left-handed chance from 1 before writing the answer.

37(iii)(a). Find $P(A/L)$. [2 marks]

Solution:

Concept:

- The four parental categories A, B, C, D form a mutually exclusive and exhaustive partition.
- Total probability: $P(L) = \sum_{E \in \{A,B,C,D\}} P(E)P(L \mid E)$.
- Bayes theorem reverses the conditioning: $P(A \mid L) = P(A)P(L \mid A)/P(L)$.

Step 1: Find the overall probability of a left-handed child The priors for all four categories are $1/4$, while the conditional left-handed probabilities are $0.24, 0.22, 0.17, 0.09$. Hence

$$\begin{aligned} P(L) &= \frac{1}{4}(0.24 + 0.22 + 0.17 + 0.09) \\ &= \frac{1}{4}(0.72) = 0.18. \end{aligned}$$

The general introductory figure of roughly 12% is not the probability implied by the explicitly specified equal-prior model; use the given priors and conditional data.

Step 2: Apply Bayes theorem to the observed child We know that the child is left handed and seek the probability that both parents are left handed. The joint probability of A and L is $(1/4)(0.24) = 0.06$. Therefore

$$P(A \mid L) = \frac{P(A)P(L \mid A)}{P(L)} = \frac{0.06}{0.18} = \frac{1}{3}.$$

This differs from $P(L \mid A)$, because the known event and the event being estimated have exchanged roles.

Step 3: Conclusion $\boxed{P(A/L) = \frac{1}{3}}$.

Quick Tip to Solve: With equal category priors, Bayes reduces to 24 divided by the sum 24+22+17+9.

OR

37(iii)(b). Find the probability that a randomly selected child is left handed given that exactly one of the parents is left handed. [2 marks]

Solution:

Concept:

- Exactly one left-handed parent is the event $E = B \cup C$. These categories are disjoint.
- Conditional probability is $P(L \mid E) = P(L \cap E)/P(E)$.
- The weighted contribution from a category is its prior multiplied by its conditional left-handed probability.

Step 1: Identify and normalise the eligible parental categories Category B has a right-handed father and left-handed mother; category C has a left-handed father and right-handed mother. Both and only these categories have exactly one left-handed parent. Since $P(B) = P(C) = 1/4$,

$$P(E) = P(B) + P(C) = \frac{1}{2}.$$

Categories A and D are excluded from this conditioning event.

Step 2: Calculate the conditional probability Because B and C are disjoint, their left-handed-child contributions add:

$$\begin{aligned} P(L \cap E) &= P(B)P(L \mid B) + P(C)P(L \mid C) \\ &= \frac{1}{4}(0.22) + \frac{1}{4}(0.17) = 0.0975. \end{aligned}$$

Divide by the chance of the given parental event:

$$P(L \mid E) = \frac{0.0975}{0.5} = 0.195 = \frac{39}{200}.$$

Equal priors mean the two eligible categories carry equal conditional weight, so this also equals the average of 22% and 17%.

Step 3: Conclusion The required probability is $\boxed{\frac{39}{200} = 0.195}$, or 19.5%.

Quick Tip to Solve: For exactly one left-handed parent, retain only B and C and average their percentages because their priors are equal.

38. The use of electric vehicles will curb air pollution in the long run.

The use of electric vehicles is increasing every year and estimated electric vehicles in use at any time t is given by the function V:

$V(t) = \frac{1}{5}t^3 - \frac{5}{2}t^2 + 25t - 2$

where t represents the time and $t = 1, 2, 3, \ldots$ corresponds to year 2001, 2002, 2003, $\ldots$ respectively.

Based on the above information, answer the following questions: [4 marks]

38(i). Can the above function be used to estimate number of vehicles in the year 2000? Justify. [2 marks]

Solution:

Concept:

- The correspondence between time and year is $t = \text{year} - 2000$.
- A model for a number of vehicles must give a non-negative value on the years for which it is used.

Step 1: Find the time corresponding to the year 2000 The passage assigns $t = 1$ to 2001, $t = 2$ to 2002 and so on. Moving back one year from 2001 gives $t = 0$ for 2000. The stated sequence starts at $t = 1$, so 2000 is outside the years explicitly covered by the model. To check whether an extension is meaningful, substitute $t = 0$ in the printed formula.

Step 2: Test the value against the meaning of the model Using $V(t) = t^3/5 - 5t^2/2 + 25t - 2$, we obtain

$$V(0) = \frac{1}{5}(0)^3 - \frac{5}{2}(0)^2 + 25(0) - 2 = -2.$$

A number of vehicles cannot be negative. Although this polynomial is mathematically defined at zero, its value there cannot represent a valid vehicle count. Thus substituting the time for 2000 does not produce an admissible estimate.

Step 3: Conclusion No. The year 2000 corresponds to $t = 0$, giving $\boxed{V(0) = -2}$, which cannot be a number of vehicles.

Quick Tip to Solve: First translate the year to t; a polynomial value is a valid vehicle estimate only if it is non-negative.

38(ii). Prove that the function $V(t)$ is an increasing function. [2 marks]

Solution:

Concept:

- For a differentiable function, a strictly positive derivative throughout an interval proves that the function is increasing there.
- The power rule is $\frac{d}{dt}t^n = nt^{n-1}$.

Step 1: Differentiate the printed vehicle function Differentiate each term of $V(t) = t^3/5 - 5t^2/2 + 25t - 2$:

$$V\prime(t) = \frac{3}{5}t^2 - 5t + 25.$$

Quick Tip to Solve: Complete the square in $3t^2/5 - 5t + 25$; its positive constant term proves positivity for every t at once.

The cubic term gives $3t^2/5$, the quadratic term gives $-5t$, the linear term gives 25, and the constant term has derivative zero. We must establish that this expression is positive throughout the time domain, rather than check just a few years.

Step 2: Complete the square to prove positivity Write the derivative in a form whose sign is immediate:

$$\begin{aligned} V\prime(t) &= \frac{3}{5}\left(t^2 - \frac{25}{3}t + \frac{125}{3}\right) \\ &= \frac{3}{5}\left(t - \frac{25}{6}\right)^2 + \frac{175}{12}. \end{aligned}$$

The square is non-negative for every real t, and $175/12 > 0$. Hence $V\prime(t) > 0$ for all real t, in particular for the positive times represented in the model. The derivative criterion therefore establishes the required monotonicity.

Step 3: Conclusion $\boxed{V\prime(t) > 0}$ throughout the time domain; hence $V(t)$ is an increasing function.

Quick Tip to Solve: Complete the square in $3t^2/5 - 5t + 25$; its positive constant term proves positivity for every t at once.