

# CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-2-2

## Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 Hours

Maximum Marks: 80

### General Instructions

- (i) This question paper contains 38 questions. All questions are compulsory.
- (ii) There is no overall choice. An internal choice is provided in 8 questions (Q22, Q24, Q27, Q30, Q32, Q35, Q36, Q37); attempt only one of the alternatives in such questions.
- (iii) Marks for each question are indicated against it.

1. If  $\frac{d}{dx}f(x) = 2x + \frac{3}{x}$  and  $f(1) = 1$ , then  $f(x)$  is

[1 mark]

(A)  $x^2 + 3 \log |x| + 1$

(B)  $x^2 + 3 \log |x|$

(C)  $2 - \frac{3}{x^2}$

(D)  $x^2 + 3 \log |x| - 4$

**Correct Answer:** (B)  $x^2 + 3 \log |x|$

**Solution:**

**Concept:**

- An antiderivative is recovered by integrating the derivative; an additive constant is then determined by the given value.
- For  $x \neq 0$ ,  $\int x \, dx = x^2/2$  and  $\int dx/x = \log |x|$ .

**Step 1: Integrate the derivative** Integrate each term separately, retaining the constant of integration:

$$f(x) = \int (2x + 3/x) \, dx = x^2 + 3 \log |x| + C.$$

This has the required derivative on any interval excluding zero.

**Step 2: Use the initial value and test the options** At  $x = 1$ ,  $\log 1 = 0$ , so  $1 = 1 + C$ , giving  $C = 0$ . Options (A) and (D) have the right derivative but give initial values 2 and  $-3$ , respectively. Option (C) has derivative  $6/x^3$ , which is not the stated derivative. Option (B) satisfies both checks.

**Step 3: Conclusion**  $f(x) = x^2 + 3 \log |x|$  on the interval containing 1. Hence, the correct option is (B).

**Quick Tip to Solve:** First integrate, then substitute  $x = 1$ : the logarithm vanishes and immediately fixes the constant.

2. Degree of the differential equation  $\sin x + \cos \left( \frac{dy}{dx} \right) = y^2$  is

[1 mark]

(A) 2

(B) 1

(C) not defined

(D) 0

**Correct Answer:** (C) not defined

**Solution:**

**Concept:**

- The degree is the power of the highest-order derivative when the equation is polynomial in its derivatives.
- A derivative inside a transcendental function prevents the differential equation from being polynomial in derivatives.

**Quick Tip to Solve:** Before counting a derivative power, check whether any derivative lies inside a trigonometric or other transcendental function.

**Step 1: Identify how the derivative occurs** The highest derivative present is  $dy/dx$ , so the order is one. However, it occurs as  $\cos(dy/dx)$ , not as a polynomial power of  $dy/dx$ . The powers of  $y$  itself do not determine degree.

**Step 2: Apply the polynomial requirement** Rearranging gives  $\cos(dy/dx) = y^2 - \sin x$ , which still has a trigonometric function of the derivative. Thus the polynomial requirement for degree is not met. Option (A) mistakes the power of  $y$  for degree, (B) confuses order with degree, and (D) incorrectly treats an undefined degree as zero.

**Step 3: Conclusion** The degree is not defined. Hence, the correct option is (C).

**Quick Tip to Solve:** Before counting a derivative power, check whether any derivative lies inside a trigonometric or other transcendental function.

**3. The integrating factor of the differential equation  $(1 - y^2)\frac{dx}{dy} + yx = ay$ ,  $(-1 < y < 1)$  is** [1 mark]

- (A)  $\frac{1}{y^2 - 1}$   
(B)  $\frac{1}{\sqrt{y^2 - 1}}$   
(C)  $\frac{1}{1 - y^2}$   
(D)  $\frac{1}{\sqrt{1 - y^2}}$

**Correct Answer:** (D)  $\frac{1}{\sqrt{1 - y^2}}$

**Solution:**

**Concept:**

- For  $dx/dy + P(y)x = Q(y)$ , the integrating factor is  $e^{\int P(y) dy}$ .
- The independent variable here is  $y$ , and  $-1 < y < 1$  ensures  $1 - y^2 > 0$ .

**Step 1: Write the linear standard form** Divide throughout by  $1 - y^2$ , which is nonzero on the stated interval:

$$\frac{dx}{dy} + \frac{y}{1 - y^2}x = \frac{ay}{1 - y^2}.$$

Hence  $P(y) = y/(1 - y^2)$ ; the right-hand side does not enter the integrating factor.

**Step 2: Integrate the coefficient** Put  $u = 1 - y^2$ , so  $du = -2y dy$ . Therefore

$$\int \frac{y}{1 - y^2} dy = -\frac{1}{2} \log(1 - y^2).$$

Exponentiation gives  $(1 - y^2)^{-1/2}$ . Options (A) and (C) use power  $-1$  instead of  $-1/2$ ; option (B) has a negative radicand on the given real interval.

**Step 3: Conclusion** The integrating factor is  $\frac{1}{\sqrt{1 - y^2}}$ . Hence, the correct option is (D).

**Quick Tip to Solve:** In an equation for  $dx/dy$ , divide by its coefficient and integrate the coefficient of  $x$  with respect to  $y$ .

4. Unit vector along  $\overrightarrow{PQ}$ , where coordinates of  $P$  and  $Q$  respectively are  $(2, 1, -1)$  and  $(4, 4, -7)$ , is [1 mark]

- (A)  $2\hat{i} + 3\hat{j} - 6\hat{k}$   
(B)  $-2\hat{i} - 3\hat{j} + 6\hat{k}$   
(C)  $\frac{-2\hat{i}}{7} - \frac{3\hat{j}}{7} + \frac{6\hat{k}}{7}$   
(D)  $\frac{2\hat{i}}{7} + \frac{3\hat{j}}{7} - \frac{6\hat{k}}{7}$

**Correct Answer:** (D)  $\frac{2\hat{i}}{7} + \frac{3\hat{j}}{7} - \frac{6\hat{k}}{7}$

**Solution:**

**Concept:**

- The vector from  $P$  to  $Q$  is obtained by subtracting the coordinates of  $P$  from those of  $Q$ .
- A unit vector in the direction of a nonzero vector  $\vec{v}$  is  $\vec{v}/|\vec{v}|$ .

**Step 1: Find the directed displacement** Using terminal point minus initial point gives

$$\overrightarrow{PQ} = (4 - 2)\hat{i} + (4 - 1)\hat{j} + (-7 + 1)\hat{k} = 2\hat{i} + 3\hat{j} - 6\hat{k}.$$

Reversing this subtraction would give  $\overrightarrow{QP}$ , the opposite direction.

**Step 2: Normalize and compare the options** The magnitude is

$$|\overrightarrow{PQ}| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{49} = 7.$$

Divide all three components by seven. Options (A) and (B) have magnitude seven rather than one; option (C) is unit length but points from  $Q$  to  $P$ . Only (D) has both unit magnitude and the required direction.

**Step 3: Conclusion** The required unit vector is  $\frac{2\hat{i}}{7} + \frac{3\hat{j}}{7} - \frac{6\hat{k}}{7}$ . Hence, the correct option is (D).

**Quick Tip to Solve:** Use end point minus start point first, then divide by the magnitude without changing any component sign.

5. If in  $\triangle ABC$ ,  $\overrightarrow{BA} = 2\vec{a}$  and  $\overrightarrow{BC} = 3\vec{b}$ , then  $\overrightarrow{AC}$  is

[1 mark]

- (A)  $2\vec{a} + 3\vec{b}$
- (B)  $2\vec{a} - 3\vec{b}$
- (C)  $3\vec{b} - 2\vec{a}$
- (D)  $-2\vec{a} - 3\vec{b}$

**Correct Answer:** (C)  $3\vec{b} - 2\vec{a}$

**Solution:**

**Concept:**

- The triangle law adds consecutive directed displacements:  $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ .
- Reversing a vector changes its sign:  $\overrightarrow{AB} = -\overrightarrow{BA}$ .

**Step 1: Reverse the given first vector** The question gives the direction from  $B$  to  $A$ , whereas the path from  $A$  to  $C$  through  $B$  needs the direction from  $A$  to  $B$ . Thus

$$\overrightarrow{AB} = -2\vec{a}.$$

The second displacement is already in the needed direction:  $\overrightarrow{BC} = 3\vec{b}$ .

**Step 2: Add the consecutive displacements** Following  $A \rightarrow B \rightarrow C$  gives

$$\overrightarrow{AC} = -2\vec{a} + 3\vec{b}.$$

Option (A) fails to reverse  $\overrightarrow{BA}$ , option (B) gives the opposite displacement  $\overrightarrow{CA}$ , and option (D) also reverses  $\overrightarrow{BC}$  unnecessarily.

**Step 3: Conclusion**  $\overrightarrow{AC} = 3\vec{b} - 2\vec{a}$ . Hence, the correct option is (C).

**Quick Tip to Solve:** Write a continuous path from the requested starting point to its end; reverse and negate any supplied vector that runs against that path.

6. If  $|\vec{a} \times \vec{b}| = \sqrt{3}$  and  $\vec{a} \cdot \vec{b} = -3$ , then angle between  $\vec{a}$  and  $\vec{b}$  is

[1 mark]

- (A)  $\frac{2\pi}{3}$
- (B)  $\frac{\pi}{6}$
- (C)  $\frac{\pi}{3}$
- (D)  $\frac{5\pi}{6}$

**Correct Answer:** (D)  $\frac{5\pi}{6}$

**Solution:**

**Concept:**

- For the angle  $0 \leq \theta \leq \pi$ ,  $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \theta$  and  $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \theta$ .

**Quick Tip to Solve:** Use the cross-to-dot ratio for the reference angle, then use the sign of the dot product to decide whether the angle is acute or obtuse.

- A negative dot product makes the angle obtuse; use this sign to select the correct quadrant.

**Step 1: Find the tangent and quadrant** Dividing the cross-product magnitude by the dot product gives

$$\tan \theta = \frac{\sqrt{3}}{-3} = -\frac{1}{\sqrt{3}}.$$

Since the dot product is negative while the cross-product magnitude is positive, the angle lies strictly in the second quadrant.

**Step 2: Compare the four angles** The reference angle with tangent  $1/\sqrt{3}$  is  $\pi/6$ , so the second-quadrant angle is  $\pi - \pi/6 = 5\pi/6$ . Options (B) and (C) are acute and have positive cosine. Option (A), though obtuse, has tangent  $-\sqrt{3}$ , which does not match the calculated ratio.

**Step 3: Conclusion** The angle is  $\frac{5\pi}{6}$ . Hence, the correct option is (D).

**Quick Tip to Solve:** Use the cross-to-dot ratio for the reference angle, then use the sign of the dot product to decide whether the angle is acute or obtuse.

**7. Equation of line passing through origin and making  $30^\circ$ ,  $60^\circ$  and  $90^\circ$  with  $x, y, z$  axes respectively is** [1 mark]

- (A)  $\frac{2x}{\sqrt{3}} = \frac{y}{2} = \frac{z}{0}$   
 (B)  $\frac{2x}{\sqrt{3}} = \frac{2y}{1} = \frac{z}{0}$   
 (C)  $2x = \frac{2y}{\sqrt{3}} = \frac{z}{1}$   
 (D)  $\frac{2x}{\sqrt{3}} = \frac{2y}{1} = \frac{z}{1}$

**Correct Answer:** (B)  $\frac{2x}{\sqrt{3}} = \frac{2y}{1} = \frac{z}{0}$

**Solution:**

**Concept:**

- Direction cosines of a line are the cosines of its angles with the positive coordinate axes.
- A line through the origin with direction cosines  $(l, m, n)$  has parametric equations  $x = lt$ ,  $y = mt$ ,  $z = nt$ .

**Step 1: Find the direction cosines** The given angles yield

$$l = \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad m = \cos 60^\circ = \frac{1}{2}, \quad n = \cos 90^\circ = 0.$$

They satisfy  $l^2 + m^2 + n^2 = 3/4 + 1/4 = 1$ , as required.

**Step 2: Write the line and interpret the zero component** The line is  $x = \sqrt{3}t/2$ ,  $y = t/2$ ,  $z = 0$ . In

**Quick Tip to Solve:** Take the cosine of each axis angle; an angle of  $90^\circ$  makes that coordinate constant, here  $z = 0$ .

the printed symmetric notation this is

$$\frac{2x}{\sqrt{3}} = \frac{2y}{1} = \frac{z}{0}.$$

The zero denominator convention means  $z = 0$ ; it is not literal division by zero. Option (A) has the wrong  $y$ -direction ratio, and (C), (D) give a nonzero  $z$ -direction component.

**Step 3: Conclusion** The line is  $x = \sqrt{3}t/2$ ,  $y = t/2$ ,  $z = 0$ , equivalent to option (B) in the printed notation. Hence, the correct option is (B).

**Quick Tip to Solve:** Take the cosine of each axis angle; an angle of  $90^\circ$  makes that coordinate constant, here  $z = 0$ .

**8. If  $A$  and  $B$  are two events such that  $P(A/B) = 2 \times P(B/A)$  and  $P(A) + P(B) = \frac{2}{3}$ , then  $P(B)$  is equal to** [1 mark]

- (A)  $\frac{2}{9}$
- (B)  $\frac{7}{9}$
- (C)  $\frac{4}{9}$
- (D)  $\frac{5}{9}$

**Correct Answer:**  $P(B)$  is not uniquely determined. The value  $2/9$  follows if  $P(A \cap B) > 0$ ; disjoint events also allow  $4/9$  or  $5/9$ .

**Solution:**

**Concept:**

- Conditional probability satisfies  $P(A | B) = P(A \cap B)/P(B)$  and  $P(B | A) = P(A \cap B)/P(A)$ , when the conditioning probabilities are positive.
- For a nonzero intersection probability, the common numerator cancels when the two conditional probabilities are compared.

**Step 1: Compare the conditional probabilities** Write  $p = P(A \cap B)$ . The given relation becomes  $p/P(B) = 2p/P(A)$ . When  $p > 0$ , this gives  $P(A) = 2P(B)$ . Thus the probability of  $A$  is twice that of  $B$ , rather than the reverse.

**Step 2: Use the sum and check the options** Putting  $P(A) = 2P(B)$  in the given sum gives  $3P(B) = 2/3$ , hence  $P(B) = 2/9$ . Option (C) reverses the required ratio; options (B) and (D) do not give the required ratio with the stated sum. If  $p = 0$ , both conditional probabilities vanish and the first equation imposes no ratio. For example, disjoint events with probabilities  $P(A) = 2/9$ ,  $P(B) = 4/9$ , or  $P(A) = 1/9$ ,  $P(B) = 5/9$ , satisfy both equations. Thus options (C) and (D) are also possible in the disjoint case; option (B) is impossible because it would make  $P(A) < 0$ .

**Step 3: Conclusion** The probability is not uniquely determined by the printed conditions. The value  $\frac{2}{9}$  follows only for a nonzero intersection; disjoint events permit other listed values.

**Quick Tip to Solve:** Cancel the common intersection probability before using the sum, checking that it is nonzero.

9. Equation of the line passing through the origin and making angles  $30^\circ$ ,  $60^\circ$  and  $90^\circ$  with  $x, y, z$  axes respectively is [1 mark]

(A)  $\frac{2x}{\sqrt{3}} = \frac{y}{2} = \frac{z}{0}$

(B)  $\frac{2x}{\sqrt{3}} = \frac{2y}{1} = \frac{z}{0}$

(C)  $2x = \frac{2y}{\sqrt{3}} = \frac{z}{1}$

(D)  $\frac{2x}{\sqrt{3}} = \frac{2y}{1} = \frac{z}{1}$

**Correct Answer:** (B)  $\frac{2x}{\sqrt{3}} = \frac{2y}{1} = \frac{z}{0}$

**Solution:**

**Concept:**

- The direction cosines of a line making angles  $\alpha, \beta, \gamma$  with the coordinate axes are  $\cos \alpha, \cos \beta, \cos \gamma$ .
- A line through the origin has parametric equations  $x = lt, y = mt, z = nt$ , where  $l, m, n$  are its direction cosines.

**Step 1: Find the direction cosines** Here  $l = \cos 30^\circ = \sqrt{3}/2$ ,  $m = \cos 60^\circ = 1/2$  and  $n = \cos 90^\circ = 0$ . These satisfy  $l^2 + m^2 + n^2 = 3/4 + 1/4 = 1$ , as required.

**Step 2: Write the line and compare** The parametric form is  $x = (\sqrt{3}/2)t, y = t/2, z = 0$ . Thus its conventional symmetric notation is  $2x/\sqrt{3} = 2y/1 = z/0$ ; the zero denominator means  $z = 0$ , not literal division by zero. Option (A) has the wrong direction ratio for  $y$ . Options (C) and (D) have a nonzero  $z$ -direction ratio and therefore cannot make an angle of  $90^\circ$  with the  $z$ -axis.

**Step 3: Conclusion** The line is  $\boxed{2x/\sqrt{3} = 2y/1 = z/0}$ , interpreted with  $z = 0$ . Hence, the correct option is (B).

**Quick Tip to Solve:** Convert axis angles to cosines first; a zero direction cosine fixes that coordinate for the line.

10. If  $(a, b)$ ,  $(c, d)$  and  $(e, f)$  are the vertices of  $\triangle ABC$  and  $\Delta$  denotes the area of  $\triangle ABC$ , then  $\begin{vmatrix} a & c & e \\ b & d & f \\ 1 & 1 & 1 \end{vmatrix}^2$  is equal to [1 mark]

- (A)  $2\Delta^2$   
 (B)  $4\Delta^2$   
 (C)  $2\Delta$   
 (D)  $4\Delta$

**Correct Answer:** (B)  $4\Delta^2$

**Solution:**

**Concept:**

- For vertices  $(a, b)$ ,  $(c, d)$ ,  $(e, f)$ , the triangle area is half the absolute value of their coordinate determinant.
- Transposing a matrix preserves its determinant; squaring removes the orientation sign.

**Step 1: Relate the determinant to the area** Let  $D = \begin{vmatrix} a & c & e \\ b & d & f \\ 1 & 1 & 1 \end{vmatrix}$ . Its transpose has rows consisting of each vertex and a final coordinate 1. The triangle-area formula therefore gives  $\Delta = \frac{1}{2}|D|$ , or  $|D| = 2\Delta$ .

**Step 2: Square the area relation** Squaring both sides yields  $D^2 = |D|^2 = (2\Delta)^2 = 4\Delta^2$ . Option (A) omits one factor of 2 when squaring. Options (C) and (D) depend linearly on area, whereas a squared coordinate determinant depends on the square of the area.

**Step 3: Conclusion** The determinant squared equals  $4\Delta^2$ . Hence, the correct option is (B).

**Quick Tip to Solve:** Remember that the determinant is twice the signed area; square both the factor two and the area.

11. The function  $f(x) = x|x|$  is

[1 mark]

- (A) continuous and differentiable at  $x = 0$ .  
 (B) continuous but not differentiable at  $x = 0$ .  
 (C) differentiable but not continuous at  $x = 0$ .  
 (D) neither differentiable nor continuous at  $x = 0$ .

**Correct Answer:** (A) continuous and differentiable at  $x = 0$ .

**Solution:**

**Concept:**

- Continuity at zero requires  $\lim_{x \rightarrow 0} f(x) = f(0)$ .
- Differentiability at zero is checked using  $\lim_{h \rightarrow 0} [f(h) - f(0)]/h$ , rather than differentiating  $|x|$  alone.

**Quick Tip to Solve:** At zero simplify the difference quotient directly: multiplication by  $x$  removes the cusp of the absolute value.

**Step 1: Check continuity** For  $x \geq 0$ ,  $f(x) = x^2$ , while for  $x < 0$ ,  $f(x) = -x^2$ . Both expressions tend to 0 at the origin, and  $f(0) = 0$ . Thus the left and right limits agree with the function value.

**Step 2: Check the derivative and the alternatives** For  $h \neq 0$ , the difference quotient is  $h|h|/h = |h|$ . Since  $\lim_{h \rightarrow 0} |h| = 0$ , the derivative exists and  $f'(0) = 0$ . Option (B) confuses this function with  $|x|$ , whose derivative does not exist at zero. Options (C) and (D) incorrectly deny continuity; differentiability itself also implies continuity.

**Step 3: Conclusion**  $f(x) = x|x|$  is continuous and differentiable at  $x = 0$ . Hence, the correct option is (A).

**Quick Tip to Solve:** At zero simplify the difference quotient directly: multiplication by  $x$  removes the cusp of the absolute value.

12. If  $\tan\left(\frac{x+y}{x-y}\right) = k$ , then  $\frac{dy}{dx}$  is equal to

[1 mark]

- (A)  $\frac{-y}{x}$   
(B)  $\frac{y}{x}$   
(C)  $\sec^2\left(\frac{y}{x}\right)$   
(D)  $\sec^2\left(\frac{x}{y}\right)$

**Correct Answer:** (B)  $\frac{y}{x}$

**Solution:**

**Concept:**

- For constant  $k$ , differentiate the given identity implicitly using the chain rule.
- The quotient rule gives  $(u/v)' = (u'v - uv')/v^2$ ; here  $y = y(x)$ .

**Step 1: Differentiate the identity** Let  $u = (x+y)/(x-y)$ , where  $x-y \neq 0$ . Differentiating  $\tan u = k$  gives  $\sec^2 u \cdot u' = 0$ . At every point where the original tangent is defined,  $\sec^2 u > 0$ , so  $u' = 0$ .

**Step 2: Simplify the quotient derivative** Writing  $y' = dy/dx$ , we obtain

$$(1 + y')(x - y) - (x + y)(1 - y') = 0.$$

Expanding and cancelling gives  $-2y + 2xy' = 0$ , hence  $y' = y/x$  wherever  $x \neq 0$ . Option (A) has the opposite sign. Options (C) and (D) retain a secant factor that cancels in the implicit equation, and neither gives the resulting slope.

**Step 3: Conclusion**  $\boxed{dy/dx = y/x}$  on the regular domain. Hence, the correct option is (B).

**Quick Tip to Solve:** The derivative of a constant is zero, so the positive secant-squared factor cancels immediately.

**13. The objective function  $Z = ax + by$  of an LPP has maximum value 42 at  $(4, 6)$  and minimum value 19 at  $(3, 2)$ . Which of the following is true?** [1 mark]

- (A)  $a = 9, b = 1$
- (B)  $a = 5, b = 2$
- (C)  $a = 3, b = 5$
- (D)  $a = 5, b = 3$

**Correct Answer:** (C)  $a = 3, b = 5$

**Solution:**

**Concept:**

- The value of the objective function  $Z = ax + by$  at a specified point follows by substitution.
- Two specified point-value pairs provide two simultaneous linear equations for  $a$  and  $b$ .

**Step 1: Form the two equations** At  $(4, 6)$ , the given maximum value implies  $4a + 6b = 42$ , or  $2a + 3b = 21$ . At  $(3, 2)$ , the given minimum gives  $3a + 2b = 19$ . Both equations must hold for the same coefficients.

**Step 2: Solve and test the choices** Multiply the first simplified equation by 3 and the second by 2:  $6a + 9b = 63$  and  $6a + 4b = 38$ . Subtraction gives  $5b = 25$ , so  $b = 5$ , and then  $3a + 10 = 19$  gives  $a = 3$ . The pairs in (A), (B) and (D) give respectively 42, 32, 38 at  $(4, 6)$ ; although (A) matches that value, it gives 29 at  $(3, 2)$ , not 19.

**Step 3: Conclusion**  $a = 3, b = 5$ . Hence, the correct option is (C).

**Quick Tip to Solve:** Test both specified point values; matching only the maximum can leave a tempting incorrect option.

**14. The corner points of the feasible region of a linear programming problem are  $(0, 4)$ ,  $(8, 0)$  and  $\left(\frac{20}{3}, \frac{4}{3}\right)$ . If  $Z = 30x + 24y$  is the objective function, then (maximum value of  $Z$  - minimum value of  $Z$ ) is equal to** [1 mark]

- (A) 40
- (B) 96
- (C) 120
- (D) 136

**Correct Answer:** The calculated difference is 144, which is absent from the printed options.

**Solution:**

**Concept:**

- A linear objective function attains its finite maximum and minimum at corner points of the feasible polygon.
- Evaluate  $Z = 30x + 24y$  at every listed vertex before subtracting the extreme values.

**Step 1: Evaluate the objective at each vertex** At  $(0, 4)$ ,  $Z = 30(0) + 24(4) = 96$ . At  $(8, 0)$ ,  $Z =$

**Quick Tip to Solve:** Compare all corner values before subtracting; the fractional vertex does not necessarily give the maximum.

$30(8) + 24(0) = 240$ . At  $(20/3, 4/3)$ ,  $Z = 30(20/3) + 24(4/3) = 200 + 32 = 232$ .

**Step 2: Select the extremes and compare the choices** The greatest vertex value is 240, and the least is 96. Their difference is therefore  $240 - 96 = 144$ . Option (A), 40, and option (C), 120, do not equal this difference. Option (B), 96, is the minimum itself. Option (D), 136, is  $232 - 96$ , which incorrectly treats the third vertex value as the maximum instead of 240.

**Step 3: Conclusion** The required difference is  $\boxed{144}$ . None of the printed options is correct.

**Quick Tip to Solve:** Compare all corner values before subtracting; the fractional vertex does not necessarily give the maximum.

**15. If  $A$  is a  $2 \times 3$  matrix such that  $AB$  and  $AB'$  both are defined, then order of the matrix  $B$  is** [1 mark]

(A)  $2 \times 2$

(B)  $2 \times 1$

(C)  $3 \times 2$

(D)  $3 \times 3$

**Correct Answer:** (D)  $3 \times 3$

**Solution:**

**Concept:**

- For matrix multiplication, the number of columns of the first matrix must equal the number of rows of the second matrix.
- If  $B$  has order  $m \times n$ , its transpose  $B'$  has order  $n \times m$ .

**Step 1: Use the condition on  $AB$**  The matrix  $A$  has three columns. Therefore, for  $AB$  to be defined,  $B$  must have three rows:  $m = 3$ . This eliminates options (A) and (B), both of which have only two rows.

**Step 2: Use the condition on  $AB'$**  The transpose has  $n$  rows, so  $AB'$  is defined only when  $n = 3$ . Thus  $B$  must also have three columns. Option (C) has order  $3 \times 2$ ; its transpose has only two rows, so it fails the second condition. A  $3 \times 3$  matrix meets both conditions, and both products have order  $2 \times 3$ .

**Step 3: Conclusion** The order of  $B$  is  $\boxed{3 \times 3}$ . Hence, the correct option is (D).

**Quick Tip to Solve:** Write the unknown order as  $m \times n$ : the products with  $B$  and  $B'$  fix  $m$  and  $n$  separately.

16. If  $\begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix} = P + Q$ , where  $P$  is a symmetric and  $Q$  is a skew symmetric matrix, then  $Q$  is equal to [1 mark]

(A)  $\begin{pmatrix} 2 & 5/2 \\ 5/2 & 4 \end{pmatrix}$

(B)  $\begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}$

(C)  $\begin{pmatrix} 0 & 5/2 \\ -5/2 & 0 \end{pmatrix}$

(D)  $\begin{pmatrix} 2 & 5/2 \\ 5/2 & 4 \end{pmatrix}$

**Correct Answer:** (B)  $\begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}$

**Solution:**

**Concept:**

- A symmetric matrix satisfies  $P^T = P$ , whereas a skew-symmetric matrix satisfies  $Q^T = -Q$ .
- For  $M = P + Q$ , subtracting its transpose gives  $Q = \frac{M - M^T}{2}$ .

**Step 1: Transpose the given matrix** Let  $M = \begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix}$ . Interchanging its rows and columns gives

$M^T = \begin{pmatrix} 2 & 5 \\ 0 & 4 \end{pmatrix}$ . Thus

$$M - M^T = \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}.$$

**Step 2: Find and check the skew-symmetric part** Halving this difference gives

$$Q = \begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}.$$

Its diagonal entries vanish and its off-diagonal entries have opposite signs, so  $Q^T = -Q$ . Options (A) and (D) are the symmetric part instead. Option (C) has the signs reversed and equals  $-Q$ , not the required skew-symmetric part of the given matrix.

**Step 3: Conclusion**  $Q = \begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}$ . Hence, the correct option is (B).

**Quick Tip to Solve:** For the skew part, subtract the transpose in the order  $M - M^T$ ; reversing this order reverses every sign.

17. If  $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & a & 1 \end{pmatrix}$  is non-singular matrix and  $a \in A$ , then the set  $A$  is

[1 mark]

- (A)  $\mathbb{R}$
- (B)  $\{0\}$
- (C)  $\{4\}$
- (D)  $\mathbb{R} - \{4\}$

**Correct Answer:** (D)  $\mathbb{R} - \{4\}$

**Solution:**

**Concept:**

- A square matrix is non-singular exactly when its determinant is non-zero.
- For expansion along a row, multiply each entry by its signed minor; the signs along the first row are  $+$ ,  $-$ ,  $+$ .

**Step 1: Evaluate the determinant** Denote the displayed matrix by  $M$ . Expanding along its first row,

$$\begin{aligned} |M| &= (3 - a) - 2(2 - 3) + (2a - 9) \\ &= 3 - a + 2 + 2a - 9 = a - 4. \end{aligned}$$

Each minor retains the entries of the other two rows and columns.

**Step 2: Apply the non-singularity condition** The condition is  $a - 4 \neq 0$ , so every real value except 4 is allowed. At  $a = 4$ , the third row equals twice the second row minus the first row, confirming singularity. Option (A) incorrectly includes this value; option (B) gives only one allowed value and omits the others; option (C) gives exactly the excluded value.

**Step 3: Conclusion** The set is  $\mathbb{R} \setminus \{4\}$ . Hence, the correct option is (D).

**Quick Tip to Solve:** Compute the determinant first, then exclude its zeros; non-singularity requires an inequality, not an equation.

18. If  $|A| = |kA|$ , where  $A$  is a square matrix of order 2, then sum of all possible values of  $k$  is [1 mark]

- (A) 1
- (B) -1
- (C) 2
- (D) 0

**Correct Answer:** No unique answer as stated: a singular matrix allows every real  $k$ . If  $A$  is non-singular, the intended answer is (D), zero.

**Solution:**

**Concept:**

- For a square matrix of order  $n$ ,  $|kA| = k^n|A|$ , since each of its  $n$  rows is multiplied by  $k$ .

**Quick Tip to Solve:** Before cancelling a determinant, check that it is non-zero; otherwise scalar identities can hold for every  $k$ .

- Cancelling  $|A|$  is valid only if  $|A| \neq 0$ ; a singular matrix has determinant zero.

**Step 1: Apply determinant scaling** Since the matrix has order two, the given equation becomes

$$|A| = k^2|A| \implies (k^2 - 1)|A| = 0.$$

If  $|A| \neq 0$ , this gives  $k = 1$  or  $k = -1$ , whose sum is zero. Under that additional condition, option (D) is the expected choice; options (A), (B) and (C) are not the sum.

**Step 2: Check the singular case permitted by the question** The question does not state that  $A$  is non-singular. If  $|A| = 0$ , the equality holds for every real  $k$ . For example, the zero matrix of order two satisfies it for all  $k$ . The sum of all such real values is not a defined finite sum. Consequently, no option supplies a universally valid answer under the printed assumptions.

**Step 3: Conclusion** As stated, there is no uniquely defined sum. With the additional assumption  $|A| \neq 0$ , the sum is  $\boxed{0}$ , corresponding to option (D).

**Quick Tip to Solve:** Before cancelling a determinant, check that it is non-zero; otherwise scalar identities can hold for every  $k$ .

**19. Assertion (A):** If a line makes angles  $\alpha, \beta, \gamma$  with positive direction of the coordinate axes, then  $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$ .

**Reason (R):** The sum of squares of the direction cosines of a line is 1.

Choose the correct option.

[1 mark]

- (A) Both (A) and (R) are true and (R) is the correct explanation of (A).
- (B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).
- (C) is true, but (R) is false.
- (D) is false, but (R) is true.

**Correct Answer:** (A) Both (A) and (R) are true and (R) is the correct explanation of (A).

**Solution:**

**Concept:**

- The direction cosines are  $l = \cos \alpha$ ,  $m = \cos \beta$ ,  $n = \cos \gamma$ , with  $l^2 + m^2 + n^2 = 1$ .
- The identity  $\sin^2 \theta = 1 - \cos^2 \theta$  converts the sum of squared sines into one involving direction cosines.

**Step 1: Judge the Assertion** Using the trigonometric identity for each angle gives

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 3 - (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = 3 - 1 = 2.$$

Therefore, the Assertion is true.

**Step 2: Judge the Reason** The direction cosines are components of a unit vector along the line. The square of its length is one, so the sum of their squares is one. Thus the Reason is true.

**Step 3: Check the explanatory link** The Reason supplies exactly the value of the squared-cosine sum

**Quick Tip to Solve:** Replace each squared sine by  $1 - \cos^2 \theta$ ; three constants minus the direction-cosine sum gives two.

used to prove the Assertion. Together with the elementary sine-cosine identity, it directly explains why the squared-sine sum is two. Option (B) denies this link, and options (C) and (D) incorrectly classify a statement as false.

**Step 4: Conclusion** Both statements are true, and the Reason correctly explains the Assertion. Hence, the correct option is (A).

**Quick Tip to Solve:** Replace each squared sine by  $1 - \cos^2 \theta$ ; three constants minus the direction-cosine sum gives two.

**20. Assertion (A):** Maximum value of  $(\cos^{-1} x)^2$  is  $\pi^2$ .

**Reason (R):** Range of the principal value branch of  $\cos^{-1} x$  is  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ .

Choose the correct option.

[1 mark]

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).

(B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).

(C) is true, but (R) is false.

(D) is false, but (R) is true.

**Correct Answer:** (C) is true, but (R) is false.

**Solution:**

**Concept:**

- For real  $x \in [-1, 1]$ , the principal range of  $\cos^{-1} x$  is  $[0, \pi]$ .
- Because this interval is non-negative, squaring its values gives the range  $[0, \pi^2]$ .

**Step 1: Judge the Assertion** Write  $t = \cos^{-1} x$ . Since  $0 \leq t \leq \pi$ , we have  $0 \leq t^2 \leq \pi^2$ . The upper bound is attained at  $x = -1$ , where  $\cos^{-1}(-1) = \pi$ . The maximum is therefore  $\pi^2$ , so the Assertion is true.

**Step 2: Judge the Reason** The stated interval  $[-\pi/2, \pi/2]$  is the principal range of  $\sin^{-1} x$ , not  $\cos^{-1} x$ . The latter has range  $[0, \pi]$ . In particular, its value at  $-1$  is  $\pi$ , outside the interval claimed by the Reason. Thus the Reason is false.

**Step 3: Select the truth combination** A false Reason cannot explain the true Assertion. Options (A) and (B) require a true Reason, while option (D) requires a false Assertion; all three fail. The required combination is a true Assertion and a false Reason.

**Step 4: Conclusion** The Assertion is true and the Reason is false. Hence, the correct option is (C).

**Quick Tip to Solve:** Check  $\cos^{-1}(-1) = \pi$  to remember the arccosine range and immediately obtain the maximum square.

**21.** If  $\vec{a}, \vec{b}, \vec{c}$  are three non-zero unequal vectors such that  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ , then find the angle between  $\vec{a}$  and  $\vec{b} - \vec{c}$ . [2 marks]

**Solution:**

**Concept:**

- The dot product distributes over subtraction:  $\vec{a} \cdot (\vec{b} - \vec{c}) = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c}$ .
- For non-zero vectors  $\vec{u}, \vec{v}$ ,  $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$ , where  $\theta \in [0, \pi]$  is their angle.

**Step 1: Convert the given equality to a dot product** The given relation is  $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$ . Subtracting the right-hand side from both sides and using distributivity yields

$$\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} = 0 \implies \vec{a} \cdot (\vec{b} - \vec{c}) = 0.$$

This identifies the two vectors whose angle is asked for.

**Step 2: Use the non-zero conditions** We are given  $\vec{a} \neq \vec{0}$ , and the unequal vectors  $\vec{b}$  and  $\vec{c}$  satisfy  $\vec{b} - \vec{c} \neq \vec{0}$ . Hence both magnitudes are positive and division by their product is valid. Therefore,

$$|\vec{a}| |\vec{b} - \vec{c}| \cos \theta = 0 \implies \cos \theta = 0.$$

In the angle interval  $[0, \pi]$ , this gives  $\theta = \pi/2$ . Thus the vector difference is perpendicular to  $\vec{a}$ ; equality of the original scalar products expresses precisely this orthogonality.

**Step 3: Conclusion** The required angle is  $\boxed{\pi/2 = 90^\circ}$ .

**Quick Tip to Solve:** Bring equal dot products to one side and factor out the common vector; a zero dot product of non-zero vectors means perpendicularity.

**22(a).** Evaluate  $\sin^{-1} \left( \sin \frac{3\pi}{4} \right) + \cos^{-1}(\cos \pi) + \tan^{-1}(1)$ . [2 marks]

**Solution:**

**Concept:**

- The principal range of  $\sin^{-1} x$  is  $[-\pi/2, \pi/2]$ , while that of  $\cos^{-1} x$  is  $[0, \pi]$ .
- The principal range of  $\tan^{-1} x$  is  $(-\pi/2, \pi/2)$ . An inverse trigonometric function returns the angle in its own principal range, not necessarily the angle originally supplied.

**Step 1: Find the principal inverse sine value** Since  $3\pi/4$  lies outside the principal range of inverse sine, it cannot be retained directly. Using the supplementary-angle identity,

$$\sin \frac{3\pi}{4} = \sin \left( \pi - \frac{3\pi}{4} \right) = \sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}.$$

The angle  $\pi/4$  belongs to  $[-\pi/2, \pi/2]$ , so  $\sin^{-1}(\sin(3\pi/4)) = \pi/4$ .

**Step 2: Evaluate the other terms and add** The angle  $\pi$  is an endpoint of the principal range of inverse cosine. Thus  $\cos^{-1}(\cos \pi) = \cos^{-1}(-1) = \pi$ . Also,  $\tan(\pi/4) = 1$  with  $\pi/4$  in the principal range,

**Quick Tip to Solve:** Before cancelling an inverse function with its trigonometric function, check whether the angle belongs to the principal range.

hence  $\tan^{-1}(1) = \pi/4$ . Therefore the required sum is

$$\frac{\pi}{4} + \pi + \frac{\pi}{4} = \frac{3\pi}{2}.$$

**Step 3: Conclusion** The required value is  $\boxed{3\pi/2}$ .

**Quick Tip to Solve:** Before cancelling an inverse function with its trigonometric function, check whether the angle belongs to the principal range.

OR

**22(b). Draw the graph of  $\cos^{-1} x$ , where  $x \in [-1, 0]$ . Also, write its range.**

[2 marks]

**Solution:**

**Concept:**

- For the principal inverse cosine,  $y = \cos^{-1} x$  means  $x = \cos y$ , with  $0 \leq y \leq \pi$ .
- Cosine decreases from 0 to  $-1$  as its angle increases from  $\pi/2$  to  $\pi$ ; these endpoint values determine the restricted graph and range.

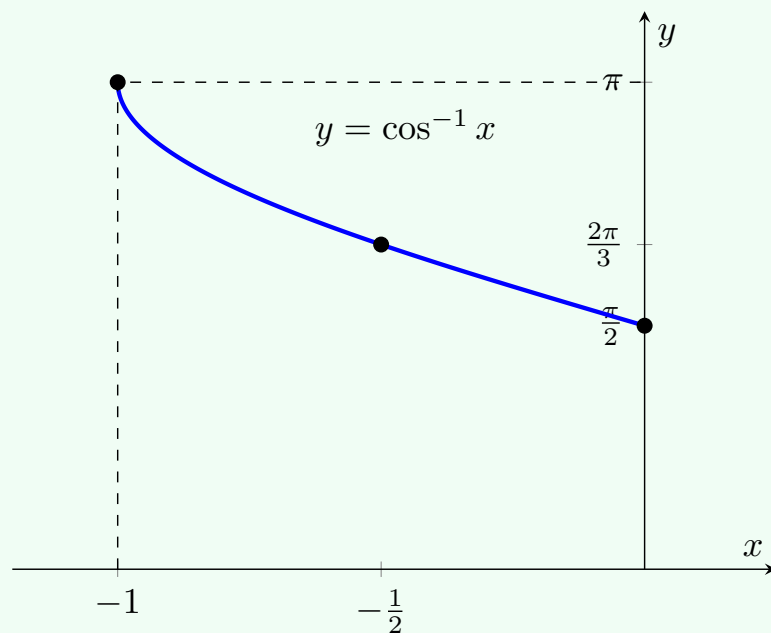
**Step 1: Locate the endpoints and an intermediate point** For the given domain  $[-1, 0]$ , the endpoint values are

$$\cos^{-1}(-1) = \pi, \quad \cos^{-1}(0) = \frac{\pi}{2}.$$

Also,  $\cos^{-1}(-1/2) = 2\pi/3$ , providing an intermediate point  $(-1/2, 2\pi/3)$ . Both endpoints must be included because the domain is a closed interval.

**Step 2: Draw the restricted inverse cosine curve** The graph is the decreasing portion of  $y = \cos^{-1} x$  from  $(-1, \pi)$  to  $(0, \pi/2)$ . Plot the three points and join them with the smooth inverse cosine curve shown below. The curve starts with a vertical tangent at  $x = -1$  and becomes less steep towards  $x = 0$ . As  $x$  runs through the entire interval  $[-1, 0]$ , the continuous decreasing function takes every value between  $\pi/2$  and  $\pi$ , including both endpoints.

**Quick Tip to Solve:** For a restricted inverse cosine graph, evaluate the domain endpoints first and retain the portion of the principal curve between them.



Graph of  $y = \cos^{-1} x$  for  $-1 \leq x \leq 0$ .

**Step 3: Conclusion** The required graph is shown above, and its range is  $[\pi/2, \pi]$ .

**Quick Tip to Solve:** For a restricted inverse cosine graph, evaluate the domain endpoints first and retain the portion of the principal curve between them.

**23. If the equation of a line is  $x = ay + b$ ,  $z = cy + d$ , then find the direction ratios of the line and a point on the line.** [2 marks]

**Solution:**

**Concept:**

- A line through a point  $(x_0, y_0, z_0)$  with direction ratios  $(l, m, n)$  has parametric equations  $x = x_0 + lt$ ,  $y = y_0 + mt$ ,  $z = z_0 + nt$ .
- Direction ratios are components of any nonzero vector parallel to a line. A common nonzero multiple of all three ratios gives the same direction.

**Step 1: Introduce a parameter** Take  $y = t$ . Substituting this in each of the given equations gives

$$x = b + at, \quad y = t, \quad z = d + ct.$$

Equivalently, the position vector of an arbitrary point on the line is

$$\vec{r} = (b, 0, d) + t(a, 1, c).$$

This expresses all three coordinates using the same freely varying real parameter.

**Step 2: Identify the direction and a point** When  $t$  increases by one, the corresponding changes in

**Quick Tip to Solve:** Choose the coordinate appearing on the right of both equations as the parameter; its coefficient in the direction vector is one.

the three coordinates are  $a$ , 1 and  $c$ . Hence the vector  $(a, 1, c)$  is parallel to the line and is nonzero, since its middle component is 1. Setting  $t = 0$  yields the point  $(b, 0, d)$ . Direct substitution verifies it:  $b = a(0) + b$  and  $d = c(0) + d$ . The result remains valid when either  $a$  or  $c$  is zero.

**Step 3: Conclusion** Direction ratios are  $\boxed{a, 1, c}$ , and one point on the line is  $\boxed{(b, 0, d)}$ .

**Quick Tip to Solve:** Choose the coordinate appearing on the right of both equations as the parameter; its coefficient in the direction vector is one.

**24(a).** If  $y = \sqrt{ax + b}$ , prove that  $y \left( \frac{d^2y}{dx^2} \right) + \left( \frac{dy}{dx} \right)^2 = 0$ .

[2 marks]

**Solution:**

**Concept:**

- For  $y = \sqrt{ax + b}$ , the equivalent identity is  $y^2 = ax + b$ , where  $a$  and  $b$  are constants.
- The chain rule gives  $\frac{d}{dx}(y^2) = 2yy'$ , and the product rule gives  $\frac{d}{dx}(yy') = (y')^2 + yy''$ .

**Step 1: Differentiate the squared identity once** Squaring the given relation yields

$$y^2 = ax + b.$$

Differentiating both sides with respect to  $x$ , treating  $y$  as a function of  $x$ , gives

$$2y \frac{dy}{dx} = a.$$

Here the derivative of  $b$  is zero and that of  $ax$  is the constant  $a$ . Squaring first avoids introducing fractions involving the square root.

**Step 2: Differentiate again using the product rule** On every interval where the derivatives exist (in particular where  $ax + b > 0$ ), differentiate the preceding identity once more. Since  $a$  is constant,

$$2 \left[ \left( \frac{dy}{dx} \right)^2 + y \frac{d^2y}{dx^2} \right] = 0.$$

Dividing by 2 gives the required relation. The product rule is essential: both  $y$  and its first derivative vary with  $x$ . If the function is identically zero, both derivatives are zero and the relation also holds directly.

**Step 3: Conclusion** Hence, wherever the derivatives exist,  $\boxed{y \frac{d^2y}{dx^2} + \left( \frac{dy}{dx} \right)^2 = 0}.$

**Quick Tip to Solve:** Square a square-root relation before differentiating; differentiating the resulting identity twice often proves the relation immediately.

**24(b).** If  $f(x) = \begin{cases} ax + b & ; 0 < x \leq 1 \\ 2x^2 - x & ; 1 < x < 2 \end{cases}$  is a differentiable function in  $(0, 2)$ , then find the values of  $a$  and  $b$ . [2 marks]

**Solution:****Concept:**

- Differentiability at an interior point requires continuity there and equality of the left and right derivatives.
- The derivative of  $ax + b$  is  $a$ ; the derivative of  $2x^2 - x$  is  $4x - 1$ .

**Step 1: Apply continuity at the joining point** Each polynomial branch is differentiable inside its own interval. Therefore only the joining point  $x = 1$  needs checking. The first branch includes this point, so  $f(1) = a + b$ . The limit from the second branch is  $2(1)^2 - 1 = 1$ . Since differentiability implies continuity, we must have

$$a + b = 1.$$

**Step 2: Equate the one-sided derivatives** Using continuity, the left difference quotient at 1 is

$$\lim_{h \rightarrow 0^-} \frac{a(1+h) + b - (a+b)}{h} = a.$$

The right branch has derivative  $4x - 1$ , whose value at 1 is 3. Thus differentiability gives  $a = 3$ . Substitution in the continuity equation gives  $3 + b = 1$ , so  $b = -2$ . These values make both the function values and the one-sided slopes agree, while the polynomial branches remain differentiable elsewhere in  $(0, 2)$ .

**Step 3: Conclusion** The required values are  $a = 3, b = -2$ .

**Quick Tip to Solve:** At a piecewise joining point, impose continuity first and then equate the one-sided slopes.

**25.** If the circumference of circle is increasing at the constant rate, prove that rate of change of area of circle is directly proportional to its radius. [2 marks]

**Solution:****Concept:**

- For a circle of radius  $r$ , circumference  $C = 2\pi r$  and area  $A = \pi r^2$ .
- Related rates are obtained by differentiating both geometric formulae with respect to the same time variable  $t$ .

**Step 1: Use the constant rate of circumference** Let the circumference increase at the constant rate  $k > 0$ , so  $dC/dt = k$ . Since the radius depends on time, differentiating  $C = 2\pi r$  with respect to  $t$  gives

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt} = k, \quad \frac{dr}{dt} = \frac{k}{2\pi}.$$

**Quick Tip to Solve:** Differentiate circumference first, then substitute its constant rate into the area derivative.

Thus the radius also increases at a constant rate. Here  $k$  is fixed throughout the motion, rather than depending on the instantaneous radius.

**Step 2: Compute the rate of area** Differentiating  $A = \pi r^2$  with respect to time and applying the chain rule gives

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r \left( \frac{k}{2\pi} \right) = kr.$$

For every positive radius,  $(dA/dt)/r = k$ , which is the same constant. Hence doubling the instantaneous radius doubles the instantaneous rate of increase of area under the given condition.

**Step 3: Conclusion**  $\frac{dA}{dt} = kr$ , with constant  $k = dC/dt$ ; therefore the rate of change of area is directly proportional to the radius.

**Quick Tip to Solve:** Differentiate circumference first, then substitute its constant rate into the area derivative.

**26. Evaluate**  $\int_{\log \sqrt{2}}^{\log \sqrt{3}} \frac{1}{(e^x + e^{-x})(e^x - e^{-x})} dx.$

[3 marks]

**Solution:**

**Concept:**

- The identity  $(a + b)(a - b) = a^2 - b^2$  simplifies the exponential product in the denominator.
- With  $t = e^{2x}$ , one has  $dt = 2e^{2x} dx$ , and  $\int dt/(t^2 - 1) = \frac{1}{2} \log |(t - 1)/(t + 1)| + C$ .

**Step 1: Simplify and transform the integral** Write the required integral as  $I$ . The product in its denominator is  $e^{2x} - e^{-2x}$ . Multiplying numerator and denominator by  $e^{2x}$ , we obtain

$$I = \int_{\log \sqrt{2}}^{\log \sqrt{3}} \frac{e^{2x}}{e^{4x} - 1} dx.$$

Set  $t = e^{2x}$ ; then  $e^{2x} dx = dt/2$ . At the lower endpoint,  $t = e^{2 \log \sqrt{2}} = 2$ , and at the upper endpoint  $t = 3$ . Therefore

$$I = \frac{1}{2} \int_2^3 \frac{dt}{t^2 - 1}.$$

The new interval lies entirely above 1, so there is no pole in the integration interval.

**Step 2: Integrate using partial fractions** Factor  $t^2 - 1 = (t - 1)(t + 1)$ . The partial fraction decomposition is

$$\frac{1}{t^2 - 1} = \frac{1}{2} \left( \frac{1}{t - 1} - \frac{1}{t + 1} \right).$$

Consequently,

$$\begin{aligned} I &= \frac{1}{4} [\log |t - 1| - \log |t + 1|]_2^3 \\ &= \frac{1}{4} \left( \log \frac{2}{4} - \log \frac{1}{3} \right) \\ &= \frac{1}{4} \log \frac{3}{2}. \end{aligned}$$

**Quick Tip to Solve:** Multiply by  $e^{2x}$  and use  $t = e^{2x}$ ; transform the bounds before integrating.

Both logarithm arguments in this interval are positive. The positive result also agrees with the positive original integrand over these limits.

**Step 3: Conclusion** The integral equals  $\boxed{\frac{1}{4} \log\left(\frac{3}{2}\right)}$ .

**Quick Tip to Solve:** Multiply by  $e^{2x}$  and use  $t = e^{2x}$ ; transform the bounds before integrating.

**27(a). Find the general solution of the differential equation  $(xy - x^2) dy = y^2 dx$ .**

[3 marks]

**Solution:**

**Concept:**

- A homogeneous differential equation  $dy/dx = F(y/x)$  can be reduced by substituting  $y = vx$ , so  $dy/dx = v + x dv/dx$ .
- Separation of variables gives an implicit solution; any solution excluded by dividing by a variable must be checked in the original equation.

**Step 1: Reduce the homogeneous equation** For  $x \neq 0$  and  $y \neq x$ , the given equation becomes

$$\frac{dy}{dx} = \frac{y^2}{xy - x^2} = \frac{(y/x)^2}{y/x - 1}.$$

Put  $y = vx$ . Substituting  $dy/dx = v + x dv/dx$  gives

$$v + x \frac{dv}{dx} = \frac{v^2}{v - 1}, \quad x \frac{dv}{dx} = \frac{v}{v - 1}.$$

This follows because  $v^2 - v(v - 1) = v$ . The substitution is used on an interval where these denominators do not vanish.

**Step 2: Separate, integrate and return to the original variables** For  $v \neq 0$ , separation gives

$$\left(1 - \frac{1}{v}\right) dv = \frac{dx}{x}.$$

Integration yields

$$v - \log |v| = \log |x| + C.$$

Since  $v = y/x$  and  $\log |y/x| + \log |x| = \log |y|$ , the family simplifies to

$$\frac{y}{x} - \log |y| = C.$$

To verify, differentiate this relation:

$$\frac{xy' - y}{x^2} - \frac{y'}{y} = 0.$$

Multiplication by  $x^2y$  gives  $(xy - x^2)y' = y^2$ , exactly the original equation.

**Step 3: Check the solution lost during division** The separation divided by  $v$ , so examine  $v = 0$ , which

**Quick Tip to Solve:** After a homogeneous substitution, check any constant ratio excluded when separating variables.

means  $y = 0$ . In the original differential equation, both sides then vanish because  $dy = 0$  and  $y^2 = 0$ ; hence this is an additional solution. The line  $y = x$  is not a solution for nonzero  $x$ , because its left side is zero while its right side is  $x^2 dx$ .

**Step 4: Conclusion** The general implicit family is  $\boxed{\frac{y}{x} - \log |y| = C}$  on intervals with  $x \neq 0$ ,  $y \neq 0$ , and  $y \neq x$ . The additional solution is  $\boxed{y = 0}$ .

**Quick Tip to Solve:** After a homogeneous substitution, check any constant ratio excluded when separating variables.

OR

**27(b).** Find the general solution of the differential equation  $(x^2 + 1)\frac{dy}{dx} + 2xy = \sqrt{x^2 + 4}$ . [3 marks]

**Solution:**

**Concept:**

- A linear equation has the form  $y' + P(x)y = Q(x)$ . Its integrating factor is  $I = e^{\int P(x) dx}$ , and  $(Iy)' = IQ$ .
- The standard radical integral is  $\int \sqrt{x^2 + a^2} dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log |x + \sqrt{x^2 + a^2}| + C$ .

**Step 1: Put the equation in linear form** Since  $x^2 + 1 > 0$  for all real  $x$ , division is valid everywhere. We get

$$\frac{dy}{dx} + \frac{2x}{x^2 + 1}y = \frac{\sqrt{x^2 + 4}}{x^2 + 1}.$$

Thus  $P(x) = 2x/(x^2 + 1)$ . The integrating factor is

$$I = e^{\int 2x/(x^2+1) dx} = e^{\log(x^2+1)} = x^2 + 1.$$

**Step 2: Integrate the product derivative** Multiplication by the integrating factor gives

$$\frac{d}{dx} [(x^2 + 1)y] = \sqrt{x^2 + 4}.$$

Using the standard formula with  $a = 2$ ,

$$(x^2 + 1)y = \frac{x}{2}\sqrt{x^2 + 4} + 2 \log (x + \sqrt{x^2 + 4}) + C.$$

The logarithm is defined for every real  $x$ , since  $\sqrt{x^2 + 4} > |x|$ .

**Step 3: Check the antiderivative** Writing  $s = \sqrt{x^2 + 4}$ , we have  $s' = x/s$  and  $\frac{d}{dx} \log(x + s) = 1/s$ . Therefore the derivative of the right side is

$$\frac{s}{2} + \frac{x^2}{2s} + \frac{2}{s} = \frac{s^2 + x^2 + 4}{2s} = s.$$

This reproduces the original right-hand side exactly.

**Quick Tip to Solve:** Recognise the left side immediately as the derivative of  $(x^2 + 1)y$ , then use the radical integral with  $a = 2$ .

**Step 4: Conclusion** The general solution is

$$y = \frac{x\sqrt{x^2+4} + 4\log(x + \sqrt{x^2+4}) + C}{2(x^2+1)}.$$

Here  $C$  is an arbitrary real constant (absorbing a constant factor into it).

**Quick Tip to Solve:** Recognise the left side immediately as the derivative of  $(x^2+1)y$ , then use the radical integral with  $a=2$ .

**28. If  $A$  and  $B$  are two events such that  $P(A/B) = 2 \times P(B/A)$  and  $P(A) + P(B) = \frac{2}{3}$ , then  $P(B)$  is equal to** [3 marks]

- (A)  $\frac{2}{9}$
- (B)  $\frac{7}{9}$
- (C)  $\frac{4}{9}$
- (D)  $\frac{5}{9}$

**Correct Answer:** The stated conditions do not uniquely determine the probability. Option (A),  $2/9$ , follows if the intersection has positive probability; disjoint events also satisfy the printed condition.

**Solution:**

**Concept:**

- For events of positive probability,  $P(A/B) = P(A \cap B)/P(B)$  and  $P(B/A) = P(A \cap B)/P(A)$ .
- When  $P(A \cap B) > 0$ , dividing these two conditional probabilities gives  $P(A/B)/P(B/A) = P(A)/P(B)$ .

**Step 1: Express the given condition** Let  $a = P(A)$ ,  $b = P(B)$  and  $c = P(A \cap B)$ . The conditional probabilities require  $a, b > 0$ . The given relation becomes

$$\frac{c}{b} = \frac{2c}{a}, \quad ac = 2bc.$$

For the nonzero intersection case, cancellation of  $c$  yields  $a = 2b$ . The larger conditional probability has the smaller conditioning-event denominator.

**Step 2: Use the sum and compare the options** Using  $a + b = 2/3$  together with  $a = 2b$ , we obtain

$$2b + b = \frac{2}{3}, \quad 3b = \frac{2}{3}, \quad b = \frac{2}{9}.$$

In this case  $a = 4/9$ , so option (C) gives  $P(A)$ , not  $P(B)$ . Options (B) and (D) do not satisfy the derived ratio and sum; option (B) also exceeds the sum of the two probabilities.

**Step 3: Check the zero-intersection possibility** The printed condition does not explicitly state  $c > 0$ . If  $c = 0$ , both conditional probabilities are zero, so their given equality holds without requiring  $a = 2b$ .

**Quick Tip to Solve:** Write both conditional probabilities using the same intersection; check that it is nonzero before cancelling it.

For example, disjoint events with  $a = 1/9$  and  $b = 5/9$  satisfy the sum and the conditional-probability equality. Thus the usual cancellation requires an additional nonzero-intersection assumption; without it the probability is not uniquely determined.

**Step 4: Conclusion** The intended answer, for  $P(A \cap B) > 0$ , is  $b = 2/9$ , option (A). As literally stated, the condition also allows disjoint events and does not uniquely determine the required probability.

**Quick Tip to Solve:** Write both conditional probabilities using the same intersection; check that it is nonzero before cancelling it.

## 29. Solve the following linear programming problem graphically:

**Maximize:**  $Z = x + 2y$

**subject to constraints:**  $x + 2y \geq 100$ ,  $2x - y \leq 0$ ,  $2x + y \leq 200$ ,  $x \geq 0$ ,  $y \geq 0$ .

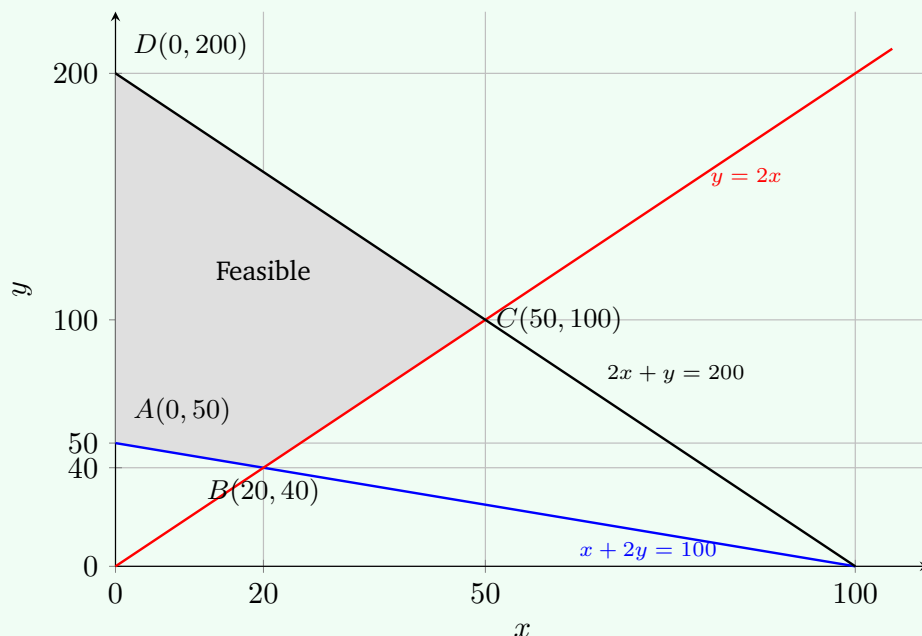
[3 marks]

### Solution:

#### Concept:

- The feasible region is the common intersection of all the constraint half-planes in the first quadrant.
- For a linear objective on a nonempty bounded polygon, an optimum occurs at a vertex. Evaluate  $Z = x + 2y$  at every feasible corner.

**Step 1: Draw the boundary lines and identify the half-planes** The boundary  $x + 2y = 100$  has intercepts  $(100, 0)$  and  $(0, 50)$ ; take its upper side  $y \geq 50 - x/2$ . The constraint  $2x - y \leq 0$  means  $y \geq 2x$ , above the line through  $(0, 0)$  and  $(50, 100)$ . Finally,  $2x + y \leq 200$  means  $y \leq 200 - 2x$ , below the line with intercepts  $(100, 0)$  and  $(0, 200)$ . Restrict the drawing to  $x, y \geq 0$ . The shaded quadrilateral is the common feasible region.



Feasible region ABCD in the first quadrant

**Step 2: Find all the corner points** On the  $y$ -axis the lower and upper corners are  $A = (0, 50)$  and

**Quick Tip to Solve:** Rewrite  $2x - y \leq 0$  as  $y \geq 2x$  before shading; reversing this side changes the feasible polygon.

$D = (0, 200)$ . Intersecting the lower two boundaries gives

$$x + 2(2x) = 100 \Rightarrow x = 20, y = 40,$$

so  $B = (20, 40)$ . Intersecting  $y = 2x$  with  $2x + y = 200$  gives

$$4x = 200 \Rightarrow x = 50, y = 100,$$

so  $C = (50, 100)$ . The region  $ABCD$  is bounded. Each listed point satisfies all five constraints.

**Step 3: Evaluate the objective at the feasible vertices** The objective values are

| Vertex       | $Z = x + 2y$ |
|--------------|--------------|
| $A(0, 50)$   | 100          |
| $B(20, 40)$  | 100          |
| $C(50, 100)$ | 250          |
| $D(0, 200)$  | 400          |

The greatest vertex value is attained at  $D$ . Also, using the upper constraint,  $Z = x + 2y \leq x + 2(200 - 2x) = 400 - 3x \leq 400$ , which independently confirms that no feasible point gives a larger value.

**Step 4: Conclusion** The maximum is  $Z_{\max} = 400$ , attained at  $(x, y) = (0, 200)$ .

**Quick Tip to Solve:** Rewrite  $2x - y \leq 0$  as  $y \geq 2x$  before shading; reversing this side changes the feasible polygon.

**30(a). Evaluate**  $\int_{-1}^1 |x^4 - x| dx$ .

[3 marks]

**Solution:**

**Concept:**

- The modulus satisfies  $|f(x)| = f(x)$  where  $f(x) \geq 0$ , and  $|f(x)| = -f(x)$  where  $f(x) < 0$ . Thus the interval must first be split wherever the sign changes.
- A definite integral is additive over adjacent intervals:  $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$ . The power rule gives  $\int x^n dx = x^{n+1}/(n+1)$  for  $n \neq -1$ .

**Step 1: Determine the sign on each interval** Factor the expression as

$$x^4 - x = x(x-1)(x^2 + x + 1).$$

The last factor is positive for every real  $x$ , since  $x^2 + x + 1 = (x + 1/2)^2 + 3/4$ . For  $-1 \leq x \leq 0$ , the product  $x(x-1)$  is non-negative, so the expression is non-negative. For  $0 < x < 1$ , it is negative, and at  $x = 1$  it is zero.

**Quick Tip to Solve:** Find the sign of the expression inside the modulus before integrating; here the only interior sign change is at  $x = 0$ .

**Step 2: Split the modulus integral and integrate** Therefore, with  $I$  denoting the required integral,

$$\begin{aligned} I &= \int_{-1}^0 (x^4 - x) dx + \int_0^1 (x - x^4) dx \\ &= \left[ \frac{x^5}{5} - \frac{x^2}{2} \right]_{-1}^0 + \left[ \frac{x^2}{2} - \frac{x^5}{5} \right]_0^1 \\ &= \left( 0 + \frac{1}{5} + \frac{1}{2} \right) + \left( \frac{1}{2} - \frac{1}{5} \right) \\ &= \frac{7}{10} + \frac{3}{10}. \end{aligned}$$

Notice that simply integrating  $x^4 - x$  over the entire interval would retain the negative contribution on  $(0, 1)$ ; the modulus reverses that contribution.

**Step 3: Conclusion** Hence,  $\int_{-1}^1 |x^4 - x| dx = \boxed{1}$ .

**Quick Tip to Solve:** Find the sign of the expression inside the modulus before integrating; here the only interior sign change is at  $x = 0$ .

OR

**30(b). Find**  $\int \frac{\sin^{-1} x}{(1 - x^2)^{3/2}} dx$ .

[3 marks]

**Solution:**

**Concept:**

- Integration by parts is  $\int u dv = uv - \int v du$ . When an inverse trigonometric function is multiplied by an algebraic derivative, choose the inverse trigonometric function as  $u$ .
- On  $-1 < x < 1$ ,  $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$  and  $\frac{d}{dx} \left( \frac{x}{\sqrt{1 - x^2}} \right) = \frac{1}{(1 - x^2)^{3/2}}$ .

**Step 1: Choose the two factors for integration by parts** Let

$$u = \sin^{-1} x, \quad dv = \frac{dx}{(1 - x^2)^{3/2}}.$$

Then

$$du = \frac{dx}{\sqrt{1 - x^2}}, \quad v = \frac{x}{\sqrt{1 - x^2}}.$$

The expression for  $v$  follows by differentiating  $x(1 - x^2)^{-1/2}$ , which gives  $(1 - x^2)^{-1/2} + x^2(1 - x^2)^{-3/2} = (1 - x^2)^{-3/2}$ .

**Step 2: Apply the formula and integrate the remaining rational term** Writing the required integral as  $I$ , we obtain

$$I = \frac{x \sin^{-1} x}{\sqrt{1 - x^2}} - \int \frac{x}{1 - x^2} dx.$$

**Quick Tip to Solve:** Take  $\sin^{-1} x$  as the first factor and recognize  $x/\sqrt{1 - x^2}$  as the primitive of the second factor.

Substitute  $t = 1 - x^2$ , so  $dt = -2x dx$ . Hence

$$\int \frac{x}{1-x^2} dx = -\frac{1}{2} \int \frac{dt}{t} = -\frac{1}{2} \ln |1-x^2|.$$

Since  $1 - x^2 > 0$  on the real domain of the original integrand, the absolute-value signs can be omitted here.

**Step 3: Verify by differentiating** The derivative of the first term is  $\sin^{-1} x / (1 - x^2)^{3/2} + x / (1 - x^2)$ . The derivative of  $\frac{1}{2} \ln(1 - x^2)$  is  $-x / (1 - x^2)$ . These rational terms cancel, leaving exactly the given integrand.

**Step 4: Conclusion** The required integral is

$$\boxed{\frac{x \sin^{-1} x}{\sqrt{1-x^2}} + \frac{1}{2} \ln(1-x^2) + C}, \quad -1 < x < 1.$$

**Quick Tip to Solve:** Take  $\sin^{-1} x$  as the first factor and recognize  $x/\sqrt{1-x^2}$  as the primitive of the second factor.

31. Find  $\int e^x \left( \frac{1 - \sin x}{1 - \cos x} \right) dx$ .

[3 marks]

**Solution:**

**Concept:**

- The product rule gives  $\frac{d}{dx}[e^x f(x)] = e^x[f(x) + f'(x)]$ . Thus a factor  $e^x$  suggests rewriting the remaining expression as a function plus its derivative.
- The half-angle identities are  $1 - \cos x = 2 \sin^2(x/2)$  and  $\sin x = 2 \sin(x/2) \cos(x/2)$ . Also,  $\frac{d}{dx} \cot(x/2) = -\frac{1}{2} \csc^2(x/2)$ .

**Step 1: Rewrite the trigonometric quotient** On any interval where  $1 - \cos x \neq 0$ ,

$$\begin{aligned} \frac{1 - \sin x}{1 - \cos x} &= \frac{1}{1 - \cos x} - \frac{\sin x}{1 - \cos x} \\ &= \frac{1}{2} \csc^2 \frac{x}{2} - \cot \frac{x}{2}. \end{aligned}$$

This decomposition separates the quotient into the negative of  $\cot(x/2)$  and the derivative of that negative function.

**Step 2: Recognize and verify a product derivative** Choose  $f(x) = -\cot(x/2)$ . Its derivative is

**Quick Tip to Solve:** With an  $e^x$  factor, look for  $f + f'$ ; the half-angle identities expose the derivative of  $-\cot(x/2)$ .

$f'(x) = \frac{1}{2} \csc^2(x/2)$ . Consequently,

$$\begin{aligned} \frac{d}{dx} \left[ -e^x \cot \frac{x}{2} \right] &= e^x \left[ -\cot \frac{x}{2} + \frac{1}{2} \csc^2 \frac{x}{2} \right] \\ &= e^x \frac{1 - \sin x}{1 - \cos x}. \end{aligned}$$

This equals the integrand exactly, so integrating recovers the displayed product, together with an arbitrary constant. The original denominator vanishes at  $x = 2k\pi$ , where  $k$  is an integer; the primitive therefore applies on each interval avoiding those points.

**Step 3: Conclusion** Hence,

$$\int e^x \frac{1 - \sin x}{1 - \cos x} dx = -e^x \cot \frac{x}{2} + C.$$

**Quick Tip to Solve:** With an  $e^x$  factor, look for  $f + f'$ ; the half-angle identities expose the derivative of  $-\cot(x/2)$ .

**32(a).** Find the equations of the diagonals of the parallelogram  $PQRS$  whose vertices are  $P(4, 2, -6)$ ,  $Q(5, -3, 1)$ ,  $R(12, 4, 5)$  and  $S(11, 9, -2)$ . Use these equations to find the point of intersection of diagonals. [5 marks]

**Solution:**

**Concept:**

- Diagonals join opposite vertices, so the diagonals of  $PQRS$  are  $PR$  and  $QS$ .
- A line through position vector  $\vec{a}$ , parallel to nonzero vector  $\vec{d}$ , has equation  $\vec{r} = \vec{a} + t\vec{d}$ , with  $t \in \mathbb{R}$ .

**Step 1: Determine the direction of each diagonal** Subtracting the coordinates of the opposite vertices gives

$$\begin{aligned} \overrightarrow{PR} &= (12 - 4, 4 - 2, 5 - (-6)) = (8, 2, 11), \\ \overrightarrow{QS} &= (11 - 5, 9 - (-3), -2 - 1) = (6, 12, -3). \end{aligned}$$

Use  $(2, 4, -1)$  for the direction of  $QS$ , dividing its original vector by 3.

**Step 2: Form the vector and Cartesian equations** Using  $P$  on the first diagonal, we obtain

$$\begin{aligned} \vec{r} &= (4\hat{i} + 2\hat{j} - 6\hat{k}) + t(8\hat{i} + 2\hat{j} + 11\hat{k}), \\ \frac{x - 4}{8} &= \frac{y - 2}{2} = \frac{z + 6}{11}. \end{aligned}$$

Using  $Q$  on the second diagonal gives

$$\begin{aligned} \vec{r} &= (5\hat{i} - 3\hat{j} + \hat{k}) + u(2\hat{i} + 4\hat{j} - \hat{k}), \\ \frac{x - 5}{2} &= \frac{y + 3}{4} = \frac{z - 1}{-1}. \end{aligned}$$

**Quick Tip to Solve:** Join opposite vertices and keep independent line parameters; verify the intersection using the midpoint of each diagonal.

**Step 3: Find the intersection from the equations** Equating the first two coordinates gives

$$\begin{aligned}4 + 8t &= 5 + 2u &\Rightarrow 8t - 2u &= 1, \\2 + 2t &= -3 + 4u &\Rightarrow 2t - 4u &= -5.\end{aligned}$$

The first equation gives  $u = 4t - \frac{1}{2}$ . Substituting into the second gives  $2t - 16t + 2 = -5$ , hence  $t = \frac{1}{2}$  and  $u = \frac{3}{2}$ . The third coordinates agree as well:

$$-6 + 11\left(\frac{1}{2}\right) = -\frac{1}{2} = 1 - \frac{3}{2}.$$

Substitution into the first line gives  $x = 4 + 8(\frac{1}{2}) = 8$  and  $y = 2 + 2(\frac{1}{2}) = 3$ .

**Step 4: Conclusion** The diagonal equations are  $\frac{x-4}{8} = \frac{y-2}{2} = \frac{z+6}{11}$  and  $\frac{x-5}{2} = \frac{y+3}{4} = \frac{z-1}{-1}$ . Their intersection is  $\left(8, 3, -\frac{1}{2}\right)$ .

**Quick Tip to Solve:** Join opposite vertices and keep independent line parameters; verify the intersection using the midpoint of each diagonal.

OR

**32(b).** A line  $l$  passes through point  $(-1, 3, -2)$  and is perpendicular to both the lines  $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$  and  $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$ . Find the vector equation of the line  $l$ . Hence, obtain its distance from origin. [5 marks]

**Solution:**

**Concept:**

- The cross product of two nonparallel direction vectors is perpendicular to both, and supplies the direction of the required line.
- For  $\vec{r} = \vec{a} + \lambda\vec{d}$ , the distance from the origin is  $D = |\vec{a} \times \vec{d}|/|\vec{d}|$ , the perpendicular height formed by the two vectors.

**Step 1: Extract the two given directions** The direction vectors are  $\vec{b}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$  and  $\vec{b}_2 = -3\hat{i} + 2\hat{j} + 5\hat{k}$ , read from the denominators. They are nonparallel.

**Step 2: Calculate a common perpendicular direction** Evaluate the cross product:

$$\begin{aligned}\vec{b}_1 \times \vec{b}_2 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & 2 & 5 \end{vmatrix} \\ &= 4\hat{i} - 14\hat{j} + 8\hat{k}.\end{aligned}$$

Dividing by 2, take  $\vec{d} = 2\hat{i} - 7\hat{j} + 4\hat{k}$ . Check directly that  $\vec{d} \cdot \vec{b}_1 = 2 - 14 + 12 = 0$  and  $\vec{d} \cdot \vec{b}_2 = -6 - 14 + 20 = 0$ .

**Quick Tip to Solve:** Take the cross product of the denominator vectors, simplify its direction, then use the same direction in the distance formula.

**Step 3: Write the line through the specified point** The position vector of  $A(-1, 3, -2)$  is  $\vec{a} = -\hat{i} + 3\hat{j} - 2\hat{k}$ . Therefore the required vector equation is

$$\vec{r} = (-\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k}), \quad \lambda \in \mathbb{R}.$$

**Step 4: Obtain the distance from the origin** Calculate the cross product of the point's position vector with the line direction:

$$\begin{aligned} \vec{a} \times \vec{d} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 3 & -2 \\ 2 & -7 & 4 \end{vmatrix} \\ &= -2\hat{i} + \hat{k}. \end{aligned}$$

Hence  $|\vec{a} \times \vec{d}| = \sqrt{5}$  and  $|\vec{d}| = \sqrt{4 + 49 + 16} = \sqrt{69}$ . Applying the perpendicular-distance formula gives  $D = \sqrt{5}/\sqrt{69}$ .

**Step 5: Conclusion** The line is  $\vec{r} = (-\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k})$ , with  $\lambda \in \mathbb{R}$ . Its distance from the origin is  $\sqrt{\frac{5}{69}}$  units.

**Quick Tip to Solve:** Take the cross product of the denominator vectors, simplify its direction, then use the same direction in the distance formula.

**33. Using integration, find the area of triangle whose vertices are  $(-1, 1)$ ,  $(0, 5)$  and  $(3, 2)$ . [5 marks]**

**Solution:**

**Concept:**

- The area between two curves is the integral of the upper ordinate minus the lower ordinate over the corresponding interval. A vertical strip of height  $y_u - y_l$  has area  $(y_u - y_l) dx$ .
- A side through two vertices follows from  $y - y_1 = m(x - x_1)$ , where  $m = (y_2 - y_1)/(x_2 - x_1)$ . Here  $x$  and  $y$  denote Cartesian coordinates.
- Split the integral wherever the upper or lower boundary changes. The two non-overlapping parts of a region have areas that can be added.

**Step 1: Determine the equations of the sides** Write  $A(-1, 1)$ ,  $B(0, 5)$  and  $C(3, 2)$ . The slope of  $AB$  is  $(5 - 1)/(0 + 1) = 4$ ; hence  $y - 1 = 4(x + 1)$ , giving  $y = 4x + 5$ . The slope of  $BC$  is  $(2 - 5)/(3 - 0) = -1$ ; hence  $y - 5 = -x$ , giving  $y = 5 - x$ . Finally, the slope of  $AC$  is  $(2 - 1)/(3 + 1) = 1/4$ ; hence

$$y - 1 = \frac{x + 1}{4}, \quad y = \frac{x + 5}{4}.$$

**Step 2: Set up the area integrals** The triangle extends from  $x = -1$  to  $x = 3$ . Its lower side is  $AC$  throughout. The upper side is  $AB$  from  $x = -1$  to  $x = 0$  and  $BC$  from  $x = 0$  to  $x = 3$ . Therefore the area is the sum

$$S = \int_{-1}^0 \left( 4x + 5 - \frac{x + 5}{4} \right) dx + \int_0^3 \left( 5 - x - \frac{x + 5}{4} \right) dx.$$

**Quick Tip to Solve:** Split at  $x = 0$ , the middle vertex's abscissa: the upper boundary changes there, while the lower side  $AC$  is unchanged.

The first integrand is  $15(x+1)/4$ , non-negative on  $[-1, 0]$ , and the second is  $5(3-x)/4$ , non-negative on  $[0, 3]$ .

**Step 3: Evaluate the left part** For the first interval, integrate the vertical distance from  $AC$  to  $AB$ :

$$S_1 = \frac{15}{4} \int_{-1}^0 (x+1) dx = \frac{15}{4} \left[ \frac{x^2}{2} + x \right]_{-1}^0 = \frac{15}{4} \left[ 0 - \left( \frac{1}{2} - 1 \right) \right] = \frac{15}{8}.$$

**Step 4: Evaluate the right part and add** On the second interval, the strip height is the distance from  $AC$  to  $BC$ :

$$S_2 = \frac{5}{4} \int_0^3 (3-x) dx = \frac{5}{4} \left[ 3x - \frac{x^2}{2} \right]_0^3 = \frac{5}{4} \left( 9 - \frac{9}{2} \right) = \frac{45}{8}.$$

Adding the two non-overlapping areas gives

$$S = S_1 + S_2 = \frac{15}{8} + \frac{45}{8} = \frac{60}{8}.$$

**Step 5: Conclusion** The required area, obtained using integration, is  $\boxed{\frac{15}{2} \text{ square units}}.$

**Quick Tip to Solve:** Split at  $x = 0$ , the middle vertex's abscissa: the upper boundary changes there, while the lower side  $AC$  is unchanged.

**34. Using integration, find the area of the triangle whose vertices are  $(-1, 1)$ ,  $(0, 5)$  and  $(3, 2)$ . [5 marks]**

**Solution:**

**Concept:**

- The area between two curves is the integral of the upper ordinate minus the lower ordinate over the corresponding interval. A vertical strip of height  $y_u - y_l$  has area  $(y_u - y_l) dx$ .
- A side through two vertices follows from  $y - y_1 = m(x - x_1)$ , where  $m = (y_2 - y_1)/(x_2 - x_1)$ . Here  $x$  and  $y$  denote Cartesian coordinates.
- Split the integral wherever the upper or lower boundary changes. The two non-overlapping parts of a region have areas that can be added.

**Step 1: Determine the equations of the sides** Write  $A(-1, 1)$ ,  $B(0, 5)$  and  $C(3, 2)$ . The slope of  $AB$  is  $(5-1)/(0+1) = 4$ ; hence  $y-1 = 4(x+1)$ , giving  $y = 4x+5$ . The slope of  $BC$  is  $(2-5)/(3-0) = -1$ ; hence  $y-5 = -x$ , giving  $y = 5-x$ . Finally, the slope of  $AC$  is  $(2-1)/(3+1) = 1/4$ ; hence

$$y-1 = \frac{x+1}{4}, \quad y = \frac{x+5}{4}.$$

**Step 2: Set up the area integrals** The triangle extends from  $x = -1$  to  $x = 3$ . Its lower side is  $AC$  throughout. The upper side is  $AB$  from  $x = -1$  to  $x = 0$  and  $BC$  from  $x = 0$  to  $x = 3$ . Therefore the area is the sum

$$S = \int_{-1}^0 \left( 4x+5 - \frac{x+5}{4} \right) dx + \int_0^3 \left( 5-x - \frac{x+5}{4} \right) dx.$$

**Quick Tip to Solve:** Split at  $x = 0$ , the middle vertex's abscissa: the upper boundary changes there, while the lower side  $AC$  is unchanged.

The first integrand is  $15(x+1)/4$ , non-negative on  $[-1, 0]$ , and the second is  $5(3-x)/4$ , non-negative on  $[0, 3]$ .

**Step 3: Evaluate the left part** For the first interval, integrate the vertical distance from  $AC$  to  $AB$ :

$$S_1 = \frac{15}{4} \int_{-1}^0 (x+1) dx = \frac{15}{4} \left[ \frac{x^2}{2} + x \right]_{-1}^0 = \frac{15}{4} \left[ 0 - \left( \frac{1}{2} - 1 \right) \right] = \frac{15}{8}.$$

**Step 4: Evaluate the right part and add** On the second interval, the strip height is the distance from  $AC$  to  $BC$ :

$$S_2 = \frac{5}{4} \int_0^3 (3-x) dx = \frac{5}{4} \left[ 3x - \frac{x^2}{2} \right]_0^3 = \frac{5}{4} \left( 9 - \frac{9}{2} \right) = \frac{45}{8}.$$

Adding the two non-overlapping areas gives

$$S = S_1 + S_2 = \frac{15}{8} + \frac{45}{8} = \frac{60}{8}.$$

**Step 5: Conclusion** The required area, obtained using integration, is  $\boxed{\frac{15}{2} \text{ square units}}.$

**Quick Tip to Solve:** Split at  $x = 0$ , the middle vertex's abscissa: the upper boundary changes there, while the lower side  $AC$  is unchanged.

**35(a).** If  $A = \begin{pmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$  and  $B = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$ , then find  $AB$  and use it to solve the following system of equations:

$$x - 2y = 3$$

$$2x - y - z = 2$$

$$-2y + z = 3$$

[5 marks]

**Solution:**

**Concept:**

- An entry of  $AB$  is the scalar product of the corresponding row of  $A$  and column of  $B$ .
- For square matrices,  $AB = I$  implies  $A = B^{-1}$ . Transposition reverses the order:  $(AB)^T = B^T A^T$ .
- If  $MX = C$  represents a linear system and  $M$  is invertible, its unique solution is  $X = M^{-1}C$ .

**Step 1: Calculate the product  $AB$**  Multiply the rows of  $A$  by the columns of  $B$ , retaining the indicated order of multiplication:

$$AB = \begin{pmatrix} -3+4+0 & -6+2+4 & 0+4-4 \\ 2-2+0 & 4-1-2 & 0-2+2 \\ 2-2+0 & 4-1-3 & 0-2+3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

All three diagonal entries are one and every off-diagonal entry is zero. Consequently  $B$  is invertible

**Quick Tip to Solve:** Check whether the coefficient matrix is  $B$  or  $B$  transpose before using  $AB$  equals  $I$ ; here multiply the constants by  $A$  transpose.

and its inverse is precisely the given matrix  $A$ .

**Step 2: Express the equations in matrix form** Let  $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$  and  $C = \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$ . Listing coefficients in the order  $x, y, z$  gives

$$\begin{pmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{pmatrix} X = C.$$

The coefficient matrix here is  $B^T$ . It is important to use this transpose, because the original  $B$  does not have the rows of coefficients of the stated equations. Transposing  $AB = I_3$  gives  $B^T A^T = I_3$ . Hence  $(B^T)^{-1} = A^T$ , and the matrix equation has solution  $X = A^T C$ .

**Step 3: Evaluate the solution column and verify it** Multiplying the transpose of  $A$  by the constants yields

$$X = \begin{pmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -9 + 4 + 6 \\ -6 + 2 + 3 \\ -12 + 4 + 9 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

These entries satisfy all three original equations:  $1 - 2(-1) = 3$ ,  $2(1) - (-1) - 1 = 2$ , and  $-2(-1) + 1 = 3$ . This direct substitution also checks that the transpose and the signs have been handled correctly. Since the coefficient matrix is invertible, there cannot be another solution column.

**Step 4: Conclusion**  $AB = I_3$ , and the unique solution is  $x = 1, y = -1, z = 1$ .

**Quick Tip to Solve:** Check whether the coefficient matrix is  $B$  or  $B$  transpose before using  $AB$  equals  $I$ ; here multiply the constants by  $A$  transpose.

OR

**35(b).** If  $f(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , prove that  $f(\alpha) \cdot f(\beta) = f(\alpha + \beta)$ .

[5 marks]

**Solution:**

**Concept:**

- Matrix multiplication uses the row-by-column rule; the entry  $(i, j)$  is  $\sum_{k=1}^3 a_{ik} b_{kj}$  for two matrices of order three.
- The angle addition identities are  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$  and  $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ .
- Two matrices are equal when they have the same order and all corresponding entries are equal.

**Step 1: Set up the product** For concise working, write  $c_\alpha = \cos \alpha$ ,  $s_\alpha = \sin \alpha$ ,  $c_\beta = \cos \beta$ , and

**Quick Tip to Solve:** Keep the negative sign in the upper-right entry when multiplying; the two angle addition identities then identify the whole upper block immediately.

$s_\beta = \sin \beta$ . Then the left-hand side of the required identity is

$$f(\alpha)f(\beta) = \begin{pmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_\beta & -s_\beta & 0 \\ s_\beta & c_\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Both factors have order three, so their product is defined and also has order three. We evaluate it directly without assuming any commutativity property for general matrices.

**Step 2: Calculate each entry by row and column** The first row gives  $c_\alpha c_\beta - s_\alpha s_\beta$ ,  $-c_\alpha s_\beta - s_\alpha c_\beta$ , and 0. The second row gives  $s_\alpha c_\beta + c_\alpha s_\beta$ ,  $-s_\alpha s_\beta + c_\alpha c_\beta$ , and 0. The third row gives 0, 0, 1: its only nonzero entry multiplies the last row of the second factor. Therefore

$$f(\alpha)f(\beta) = \begin{pmatrix} c_\alpha c_\beta - s_\alpha s_\beta & -(c_\alpha s_\beta + s_\alpha c_\beta) & 0 \\ s_\alpha c_\beta + c_\alpha s_\beta & c_\alpha c_\beta - s_\alpha s_\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, the upper-right entry of the two-dimensional block retains its negative sign.

**Step 3: Apply the angle addition identities** The two diagonal expressions in the upper block are each  $\cos(\alpha + \beta)$ . Its lower-left expression is  $\sin(\alpha + \beta)$ , while the upper-right expression is  $-\sin(\alpha + \beta)$ . Substituting these identities yields

$$f(\alpha)f(\beta) = \begin{pmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) & 0 \\ \sin(\alpha + \beta) & \cos(\alpha + \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is exactly the matrix obtained by replacing the argument  $\alpha$  by  $\alpha + \beta$  in the given definition of  $f$ . All nine corresponding entries agree; the identities used hold for arbitrary real angles, so no restriction on  $\alpha$  or  $\beta$  is needed.

**Step 4: Conclusion** Hence  $\boxed{f(\alpha) \cdot f(\beta) = f(\alpha + \beta)}$ , as required.

**Quick Tip to Solve:** Keep the negative sign in the upper-right entry when multiplying; the two angle addition identities then identify the whole upper block immediately.

36. Recent studies suggest that roughly 12% of the world population is left handed.

Depending upon the parents, the chances of having a left handed child are as follows:

A: When both father and mother are left handed: Chances of left handed child is 24%.

B: When father is right handed and mother is left handed: Chances of left handed child is 22%.

C: When father is left handed and mother is right handed: Chances of left handed child is 17%.

D: When both father and mother are right handed: Chances of left handed child is 9%.

Assuming that  $P(A) = P(B) = P(C) = P(D) = \frac{1}{4}$  and  $L$  denotes the event that child is left handed.

Based on the above information, answer the following questions:

[4 marks]

36(i). Find  $P(L/C)$ .

[1 mark]

**Solution:**

**Concept:**

- $P(L | C)$  is the probability of a left-handed child among families satisfying event  $C$ .
- A percentage  $p\%$  represents probability  $p/100$ .

**Step 1: Identify the given family type** Event  $C$  means that the father is left handed and the mother is right handed. The case study directly gives the chance of a left-handed child for this group as 17%.

**Step 2: Convert the conditional percentage** The slash in  $P(L/C)$  denotes conditioning on  $C$ . Since the family type is already specified, no multiplication by  $P(C) = 1/4$  is required. Thus the given percentage becomes  $17/100 = 0.17$ .

**Step 3: Conclusion**  $P(L/C) = 0.17$ .

**Quick Tip to Solve:** Read the percentage for group C directly; do not multiply it by the probability of group C.

36(ii). Find  $P(\bar{L}/A)$ .

[1 mark]

**Solution:**

**Concept:**

- Within a fixed conditioning event  $A$ , the events  $L$  and  $\bar{L}$  are complementary.
- $P(\bar{L} | A) = 1 - P(L | A)$ .

**Step 1: Read the probability for group A** Event  $A$  means that both parents are left handed. The given chance that their child is left handed is 24%, so  $P(L | A) = 24/100 = 0.24$ .

**Step 2: Use the complement within group A** The question asks for a child who is not left handed, while keeping the same parental condition  $A$ . The conditional probabilities of these two complementary outcomes add to one. Therefore  $P(\bar{L} | A) = 1 - 0.24 = 0.76$ .

**Step 3: Conclusion**  $P(\bar{L}/A) = 0.76$ .

**Quick Tip to Solve:** Subtract 24 percent from 100 percent, keeping the condition that both parents are left handed.

36(iii)(a). Find  $P(A/L)$ .

[2 marks]

**Solution:**

**Concept:**

- The four parental groups  $A, B, C, D$  are mutually exclusive and exhaustive, each with probability  $1/4$ .
- Total probability gives  $P(L) = \sum_{E=A,B,C,D} P(E)P(L | E)$ .
- Bayes theorem gives  $P(A | L) = P(A)P(L | A)/P(L)$ .

**Step 1: Calculate the model probability of L** Use all four parental groups with their stated equal probabilities. The opening population statistic is background information; the model specified in this question determines the required denominator. Thus

$$P(L) = \frac{1}{4}(0.24 + 0.22 + 0.17 + 0.09) = \frac{0.72}{4} = 0.18.$$

**Step 2: Find the joint probability and apply Bayes theorem** A left-handed child with both parents left handed belongs to  $A \cap L$ . Its probability is

$$P(A \cap L) = P(A)P(L | A) = \frac{1}{4} \times 0.24 = 0.06.$$

Conditioning on the observed left-handed child restricts the sample space to  $L$ . Hence

$$P(A | L) = \frac{P(A \cap L)}{P(L)} = \frac{0.06}{0.18} = \frac{1}{3}.$$

**Step 3: Conclusion**  $P(A/L) = \frac{1}{3}$ .

**Quick Tip to Solve:** For the reverse conditional probability, weight all four given child probabilities before using Bayes theorem.

OR

36(iii)(b). Find the probability that a randomly selected child is left handed given that exactly one of the parents is left handed.

[2 marks]

**Solution:**

**Concept:**

- Exactly one left-handed parent is the event  $E = B \cup C$ , where  $B$  and  $C$  are disjoint.
- Conditional probability is  $P(L | E) = P(L \cap E)/P(E)$ .
- For each parental group,  $P(L \cap B) = P(B)P(L | B)$ , and similarly for  $C$ .

**Step 1: Identify the conditioning event** Group  $B$  has a right-handed father and a left-handed mother, whereas group  $C$  has a left-handed father and a right-handed mother. These are exactly the two

**Quick Tip to Solve:** Exactly one left-handed parent includes only  $B$  and  $C$ ; their equal weights make the answer their average.

possibilities allowed. Groups  $A$  and  $D$  do not satisfy the condition. Since  $B, C$  are disjoint,

$$P(E) = P(B) + P(C) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

**Step 2: Calculate the conditional chance** The joint probability of a left-handed child and exactly one left-handed parent is

$$\begin{aligned} P(L \cap E) &= P(B)P(L | B) + P(C)P(L | C) \\ &= \frac{1}{4}(0.22) + \frac{1}{4}(0.17) = 0.0975. \end{aligned}$$

Dividing by the probability of the conditioning event gives

$$P(L | E) = \frac{0.0975}{0.5} = 0.195.$$

The equal group probabilities also make this the arithmetic mean of 0.22 and 0.17.

**Step 3: Conclusion** The required probability is  $\boxed{0.195}$ , or 19.5%.

**Quick Tip to Solve:** Exactly one left-handed parent includes only  $B$  and  $C$ ; their equal weights make the answer their average.

**37. Engine displacement is the measure of the cylinder volume swept by all the pistons of a piston engine. The piston moves inside the cylinder bore.**

The cylinder bore in the form of circular cylinder open at the top is to be made from a metal sheet of area  $75\pi \text{ cm}^2$ .

Based on the above information, answer the following questions:

[4 marks]

**37(i). If the radius of cylinder is  $r$  cm and height is  $h$  cm, then write the volume  $V$  of cylinder in terms of radius  $r$ .**

[1 mark]

**Solution:**

**Concept:**

- An open-top cylinder has sheet area  $\pi r^2 + 2\pi rh$ : one circular base plus its curved surface.
- The cylinder volume is  $V = \pi r^2 h$ , with  $r, h$  measured in centimetres.

**Step 1: Express height using the fixed area** There is no top circular face because the cylinder is open at the top. Consequently the metal-sheet condition is

$$\pi r^2 + 2\pi rh = 75\pi.$$

Dividing by  $\pi$  and solving for height gives

$$h = \frac{75 - r^2}{2r}.$$

**Quick Tip to Solve:** For an open-top cylinder count one circular base, then eliminate height before writing the volume.

**Step 2: Substitute into the cylinder volume** Using the base area times height formula,

$$V = \pi r^2 \left( \frac{75 - r^2}{2r} \right) = \frac{\pi}{2} (75r - r^3).$$

A physical cylinder has  $r > 0$  and  $h > 0$ , giving  $0 < r < \sqrt{75}$ .

**Step 3: Conclusion**  $V(r) = \frac{\pi}{2} (75r - r^3) \text{ cm}^3$ .

**Quick Tip to Solve:** For an open-top cylinder count one circular base, then eliminate height before writing the volume.

**37(ii). Find  $\frac{dV}{dr}$ .**

[1 mark]

**Solution:**

**Concept:**

- An open-top cylinder has sheet area  $\pi r^2 + 2\pi rh$ : one circular base plus its curved surface.
- The cylinder volume is  $V = \pi r^2 h$ , with  $r, h$  measured in centimetres.
- The power rule is  $\frac{d}{dr}(r^n) = nr^{n-1}$ .

**Step 1: Write volume as a function of radius** The fixed sheet area gives  $r^2 + 2rh = 75$ , so  $h = (75 - r^2)/(2r)$ . Substitution into  $V = \pi r^2 h$  gives

$$V(r) = \frac{\pi}{2} (75r - r^3).$$

This incorporates the dependence of height on radius.

**Step 2: Differentiate the polynomial** Treat  $\pi/2$  as a constant and differentiate each power of  $r$ :

$$\frac{dV}{dr} = \frac{\pi}{2} \left( 75 \frac{d(r)}{dr} - \frac{d(r^3)}{dr} \right) = \frac{\pi}{2} (75 - 3r^2).$$

Its unit is square centimetres because volume is differentiated with respect to a length.

**Step 3: Conclusion**  $\frac{dV}{dr} = \frac{\pi}{2} (75 - 3r^2)$ .

**Quick Tip to Solve:** Eliminate height using the fixed area first; differentiating with a constant height would be incorrect.

37(iii)(a). Find the radius of cylinder when its volume is maximum.

[2 marks]

**Solution:**

**Concept:**

- An open-top cylinder has sheet area  $\pi r^2 + 2\pi rh$ : one circular base plus its curved surface.
- The cylinder volume is  $V = \pi r^2 h$ , with  $r, h$  measured in centimetres.
- An interior maximum is a stationary point with  $V'(r) = 0$ ;  $V''(r) < 0$  confirms a local maximum.

**Step 1: Obtain the volume function and stationary radius** The area constraint gives  $h = (75 - r^2)/(2r)$ , hence

$$V(r) = \frac{\pi}{2}(75r - r^3), \quad 0 < r < \sqrt{75}.$$

Differentiating and setting the derivative equal to zero,

$$V'(r) = \frac{\pi}{2}(75 - 3r^2) = 0, \\ r^2 = 25.$$

A radius is positive, so the only admissible stationary radius is  $r = 5$  cm.

**Step 2: Verify maximum and feasibility** The second derivative is

$$V''(r) = -3\pi r, \quad V''(5) = -15\pi < 0.$$

In fact  $V'(r) > 0$  for  $0 < r < 5$  and  $V'(r) < 0$  for  $5 < r < \sqrt{75}$ , so this is the maximum over the whole physical domain. At this radius,

$$h = \frac{75 - 25}{2(5)} = 5 \text{ cm} > 0,$$

which satisfies the open-cylinder constraint.

**Step 3: Conclusion** The volume is maximum when  $r = 5$  cm.

**Quick Tip to Solve:** Keep only the positive stationary radius and check the second derivative or derivative sign change.

OR

37(iii)(b). For maximum volume,  $h > r$ . State true or false and justify.

[2 marks]

**Solution:**

**Concept:**

- An open cylinder has one circular base: its sheet area is  $\pi r^2 + 2\pi rh$ , where  $r > 0$  is radius and  $h > 0$  is height.
- Volume is  $V = \pi r^2 h$ . A stationary point is a maximum when the second derivative is negative.

**Step 1: Use the area constraint** The cylinder is open at the top, so

$$\pi r^2 + 2\pi rh = 75\pi, \quad h = \frac{75 - r^2}{2r}.$$

**Quick Tip to Solve:** Use one base for an open cylinder; its maximum-volume height equals its radius.

Thus  $0 < r < \sqrt{75}$  and

$$V(r) = \frac{\pi}{2}(75r - r^3).$$

Only one base is included; a second base would describe a different cylinder.

**Step 2: Find height at maximum volume** Differentiating,

$$V'(r) = \frac{\pi}{2}(75 - 3r^2), \quad V''(r) = -3\pi r.$$

The only stationary radius in the physical domain is  $r = 5$  cm, with  $V''(5) = -15\pi < 0$ . The first derivative is positive before 5 and negative after 5; hence this is the maximum throughout the permitted domain. Substituting in the area constraint gives

$$h = \frac{75 - 25}{2(5)} = 5 \text{ cm}.$$

**Step 3: Conclusion** The statement is false. At maximum volume,  $h = r = 5$  cm, so the strict inequality  $h > r$  does not hold.

**Quick Tip to Solve:** Use one base for an open cylinder; its maximum-volume height equals its radius.

**38. The use of electric vehicles is increasing every year and estimated electric vehicles in use at any time  $t$  is given by the function  $V$ :**

$$V(t) = \frac{1}{5}t^3 - \frac{5}{2}t^2 + 25t - 2$$

where  $t$  represents the time and  $t = 1, 2, 3, \dots$  corresponds to year 2001, 2002, 2003, ... respectively.

Based on the above information, answer the following questions:

[4 marks]

**38(i). Can the above function be used to estimate number of vehicles in the year 2000? Justify.**

[2 marks]

**Solution:**

**Concept:**

- The correspondence  $t = 1$  for 2001 gives  $t = \text{calendar year} - 2000$ .
- A vehicle-count model must give a nonnegative value and must be used within its specified time domain.

**Step 1: Identify the time for year 2000** The given correspondence assigns  $t = 1, 2, 3, \dots$  to years 2001, 2002, 2003, and so on. Therefore the year 2000 would correspond to  $t = 0$ . This value is outside the stated sequence of permissible times. Although the polynomial can be evaluated algebraically at zero, that alone does not make its value a valid count of vehicles.

**Step 2: Evaluate and interpret the extrapolation** If the formula is nevertheless substituted at this time, it gives

$$V(0) = \frac{1}{5}(0)^3 - \frac{5}{2}(0)^2 + 25(0) - 2 = -2.$$

A count of vehicles cannot be negative. Thus the formula would produce a physically impossible

**Quick Tip to Solve:** Convert the calendar year to the given time variable before judging whether a model produces a meaningful count.

estimate for 2000, as well as extrapolating outside the years specified in the question. The later increasing trend does not remove this failure at the earlier year.

**Step 3: Conclusion** No. The year 2000 corresponds to  $t = 0$ , for which  $V(0) = -2$ ; a negative number of vehicles is impossible.

**Quick Tip to Solve:** Convert the calendar year to the given time variable before judging whether a model produces a meaningful count.

**38(ii). Prove that the function  $V(t)$  is an increasing function.**

[2 marks]

**Solution:**

**Concept:**

- A differentiable function increases on an interval if its derivative is positive throughout that interval.
- Completing the square proves positivity of a quadratic without relying on a few sample values.

**Step 1: Differentiate the given function** For

$$V(t) = \frac{1}{5}t^3 - \frac{5}{2}t^2 + 25t - 2,$$

apply the power rule to each term. The constant contributes zero, giving

$$V'(t) = \frac{3}{5}t^2 - 5t + 25.$$

To establish increase over all the years in the model, we must show that this expression remains positive, rather than checking only particular integer times.

**Step 2: Prove positivity for every time** Complete the square exactly:

$$\begin{aligned} V'(t) &= \frac{3}{5} \left( t^2 - \frac{25}{3}t \right) + 25 \\ &= \frac{3}{5} \left( t - \frac{25}{6} \right)^2 + \frac{175}{12}. \end{aligned}$$

The squared expression is nonnegative for every real  $t$ , and  $175/12 > 0$ . Hence  $V'(t) > 0$  everywhere, in particular throughout  $t \geq 1$ . The derivative criterion therefore proves strict increase on the model interval; consequently each later integer year also has a larger estimated count.

**Step 3: Conclusion**  $V(t)$  is strictly increasing on its stated time domain because  $V'(t) > 0$  for all  $t \geq 1$ .

**Quick Tip to Solve:** Complete the square in the derivative: its positive constant term proves increase for every year at once.