CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-2-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This Question Paper contains **38** questions. All questions are compulsory.

(ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.

(iii) In **Section A** – Question Nos. **1** to **18** are Multiple Choice Questions (MCQs) and Question Nos. **19** & **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B** – Question Nos. **21** to **25** are Very Short Answer (VSA) type questions of **2** marks each.

(v) In **Section C** – Question Nos. **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D** – Question Nos. **32** to **35** are Long Answer (LA) type questions carrying **5** marks each.

(vii) In **Section E** – Question Nos. **36** to **38** are source based/case based/passage based/integrated units of assessment questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section **B**, **3** questions in Section **C**, **2** questions in Section **D** and **2** questions in Section **E**.

(ix) Use of calculators is **NOT** allowed.

Section A (20 marks)

(Multiple Choice Questions)
Each question carries 1 mark.

1. If the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{3}$ and $|\vec{a} \times \vec{b}| = 3\sqrt{3}$, then the value of $\vec{a} \cdot \vec{b}$ is [1 mark]

(A) 9
(B) 3
(C) $\frac{1}{9}$
(D) $\frac{1}{3}$

Correct Answer: (B) 3

Solution:

Concept:

- For vectors with angle θ, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$.
- The scalar product is $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.

Step 1: Find the product of magnitudes Using the given angle,

$$|\vec{a}||\vec{b}|\frac{\sqrt{3}}{2} = 3\sqrt{3} \quad \Longrightarrow \quad |\vec{a}||\vec{b}| = 6.$$

Both vectors are nonzero since their cross product has nonzero magnitude.

Step 2: Evaluate and compare the options Therefore

$$\vec{a} \cdot \vec{b} = 6\cos\frac{\pi}{3} = 6 \cdot \frac{1}{2} = 3.$$

The angle is acute, so the scalar product is positive. Option (A), 9, and the reciprocal values in (C) and (D) do not equal this product.

Step 3: Conclusion The value is 3; hence, the correct option is (B).

Quick Tip to Solve: Use $\vec{a} \cdot \vec{b} = |\vec{a} \times \vec{b}| \cot\theta$ directly when the angle is known.

2. The position vectors of three consecutive vertices of a parallelogram $ABCD$ are $A(4\hat{i}+2\hat{j}-6\hat{k})$, $B(5\hat{i}-3\hat{j}+\hat{k})$ and $C(12\hat{i}+4\hat{j}+5\hat{k})$. The position vector of D is given by [1 mark]

(A) $-3\hat{i}-5\hat{j}-10\hat{k}$

(B) $21\hat{i}+3\hat{j}$

(C) $11\hat{i}+9\hat{j}-2\hat{k}$

(D) $-11\hat{i}-9\hat{j}+2\hat{k}$

Correct Answer: (C) $11\hat{i}+9\hat{j}-2\hat{k}$

Solution:

Concept:

- The diagonals of a parallelogram bisect each other: $\vec{A}+\vec{C}=\vec{B}+\vec{D}$.
- For consecutive vertices, $\vec{D}=\vec{A}+\vec{C}-\vec{B}$; combine corresponding components.

Step 1: Apply the diagonal relation Let $\vec{A},\vec{B},\vec{C},\vec{D}$ denote the position vectors. Equality of the diagonal midpoints gives

$$(\vec{A}+\vec{C})/2=(\vec{B}+\vec{D})/2.$$

Thus the unknown vertex is obtained by subtracting $\vec{B}$ from $\vec{A}+\vec{C}$.

Step 2: Compute all three components With the printed vector $\vec{B}=5\hat{i}-3\hat{j}+\hat{k}$,

$$\begin{aligned}\vec{D}&=(4+12-5)\hat{i}+(2+4+3)\hat{j}\\&\quad+(-6+5-1)\hat{k}\\&=11\hat{i}+9\hat{j}-2\hat{k}.\end{aligned}$$

Option (D) reverses every sign; (A) and (B) also fail the diagonal-midpoint relation.

Step 3: Conclusion $\vec{D}=11\hat{i}+9\hat{j}-2\hat{k}$; hence, the correct option is (C).

Quick Tip to Solve: For consecutive vertices A, B, C, subtract the middle vertex B from A+C to find D.

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3. If for two events A and B, $P(A-B)=\frac{1}{5}$ and $P(A)=\frac{3}{5}$, then $P\left(\frac{B}{A}\right)$ is equal to [1 mark]

(A) $\frac{1}{2}$

(B) $\frac{3}{5}$

(C) $\frac{2}{5}$

(D) $\frac{2}{3}$

Correct Answer: (D) $\frac{2}{3}$

Solution:

Concept:

- $A-B=A\cap B^c$, and A splits into disjoint events $A-B$ and $A\cap B$.

Quick Tip to Solve: First subtract P(A-B) from P(A), then divide that intersection probability by P(A).

- For $P(A) > 0$, $P(B \mid A) = P(A \cap B)/P(A)$.

Step 1: Find the intersection probability By addition for disjoint events,

$$P(A \cap B) = P(A) - P(A - B) = \frac{3}{5} - \frac{1}{5} = \frac{2}{5}.$$

This is the joint probability, not yet the requested conditional probability.

Step 2: Condition on event A The printed notation $P(B/A)$ means the probability of B given A. Hence

$$P(B \mid A) = \frac{2/5}{3/5} = \frac{2}{3}.$$

Option (C) gives only the intersection and (B) gives the conditioning-event probability. Option (A) also fails the defining ratio.

Step 3: Conclusion $P(B \mid A) = \frac{2}{3}$; hence, the correct option is (D).

Quick Tip to Solve: First subtract P(A-B) from P(A), then divide that intersection probability by P(A).

4. If $\int_0^{2\pi} \cos^2 x\, dx = k \int_0^{\pi/2} \cos^2 x\, dx$, then the value of k is [1 mark]

(A) 4

(B) 2

(C) 1

(D) 0

Correct Answer: (A) 4

Solution:

Concept:

- The identity $\cos^2 x = (1 + \cos 2x)/2$ reduces the integrals to elementary terms.
- The factor k equals the left integral divided by the nonzero right integral.

Step 1: Evaluate the two integrals An antiderivative is

$$F(x) = \frac{x}{2} + \frac{\sin 2x}{4}.$$

Thus $\int_0^{2\pi} \cos^2 x\, dx = \pi$, whereas $\int_0^{\pi/2} \cos^2 x\, dx = \pi/4$; the sine terms vanish at each endpoint.

Step 2: Use the given equality The equation becomes

$$\pi = k\frac{\pi}{4} \quad \Longrightarrow \quad k = 4.$$

Option (B) counts only two equal quarter-period areas instead of four. Option (C) would require equal integrals, and (D) would force the positive left integral to vanish.

Step 3: Conclusion $k = 4$; hence, the correct option is (A).

Quick Tip to Solve: Split [0,2pi] into four quarters: symmetry makes each cos-squared integral equal to that on [0,pi/2].

5. If (a, b), (c, d) and (e, f) are the vertices of $\triangle ABC$ and Δ denotes the area of $\triangle ABC$, then $\begin{vmatrix} a & c & e \\ b & d & f \\ 1 & 1 & 1 \end{vmatrix}^2$ is equal to [1 mark]

(A) $2\Delta^2$

(B) $4\Delta^2$

(C) 2Δ

(D) 4Δ

Correct Answer: (B) $4\Delta^2$

Solution:

Concept:

- Triangle area equals half the absolute value of the coordinate determinant.
- A determinant and its transpose have the same value, and squaring removes its sign.

Step 1: Relate the determinant to area Write the displayed determinant as D. Its transpose has rows $(a, b, 1), (c, d, 1), (e, f, 1)$. The standard coordinate-area formula therefore gives

$$\Delta = \frac{1}{2}|D|, \qquad |D| = 2\Delta.$$

The absolute value accounts for either orientation of the vertices.

Step 2: Square the relation and check options Consequently

$$D^2 = |D|^2 = (2\Delta)^2 = 4\Delta^2.$$

Option (A) forgets to square the factor 2. Options (C) and (D) have an unsquared area, whereas the question asks for the squared determinant. The relation also holds when the vertices are collinear.

Step 3: Conclusion The expression is $4\Delta^2$; hence, the correct option is (B).

Quick Tip to Solve: Remember that the coordinate determinant gives twice the signed area; square the factor 2 as well.

6. The function $f(x) = x|x|$ is [1 mark]

(A) continuous and differentiable at $x = 0$.

(B) continuous but not differentiable at $x = 0$.

(C) differentiable but not continuous at $x = 0$.

(D) neither differentiable nor continuous at $x = 0$.

Correct Answer: (A) continuous and differentiable at $x = 0$.

Solution:

Concept:

- A function is continuous at zero if $\lim_{x\to 0} f(x) = f(0)$.
- The derivative at zero is $\lim_{h\to 0}[f(h) - f(0)]/h$, when this limit exists.

Step 1: Check continuity Here $f(0) = 0$ and

$$f(x) = \begin{cases} x^2, & x \geq 0, \\ -x^2, & x < 0. \end{cases}$$

Both one-sided limits at zero are zero, equal to the function value. Thus the function is continuous there.

Step 2: Check the derivative directly For $h \neq 0$,

$$\frac{f(h) - f(0)}{h} = \frac{h|h|}{h} = |h|.$$

Its two-sided limit is zero, so f is differentiable at zero with $f'(0) = 0$. Although $|x|$ alone has a corner, multiplying by x removes that obstruction here. Hence (B), (C) and (D) all contradict the computed limits.

Step 3: Conclusion The function is continuous and differentiable at $x = 0$; hence, the correct option is (A).

Quick Tip to Solve: At zero use the difference quotient directly: x|x| divided by x leaves |x|, whose limit is zero.

7. If $\tan\left(\frac{x+y}{x-y}\right) = k$, then $\frac{dy}{dx}$ is equal to [1 mark]

(A) $\frac{-y}{x}$

(B) $\frac{y}{x}$

(C) $\sec^2\left(\frac{y}{x}\right)$

(D) $-\sec^2\left(\frac{y}{x}\right)$

Correct Answer: (B) $\frac{y}{x}$

Solution:

Concept:

- Differentiate the constant relation using the chain rule: $d(\tan u)/dx = \sec^2 u\, du/dx$.
- The quotient rule applies to $u = (x+y)/(x-y)$, with $y = y(x)$.

Step 1: Differentiate the implicit equation At a point where the given expression is defined, $x - y \neq 0$, and $\sec^2 u$ is nonzero. Since k is constant,

$$\sec^2 u \frac{(1+y')(x-y) - (x+y)(1-y')}{(x-y)^2} = 0.$$

Therefore the numerator must vanish.

Step 2: Simplify and compare the options Expanding the numerator gives

$$x - y + xy' - yy' - x - y + xy' + yy' = 2xy' - 2y.$$

Hence $xy' = y$, so $y' = y/x$ wherever the curve is locally a differentiable function with $x \neq 0$. Option (A) has the wrong sign; (C) and (D) retain secant factors which cancel in the chain-rule equation.

Step 3: Conclusion $\frac{dy}{dx} = \frac{y}{x}$; hence, the correct option is (B).

Quick Tip to Solve: Cancel the nonzero secant-squared factor first, then simplify the quotient-rule numerator to 2xy prime minus 2y.

8. The objective function $Z = ax + by$ of an LPP has maximum value 42 at $(4, 6)$ and minimum value 19 at $(3, 2)$. Which of the following is true? [1 mark]

(A) $a = 9, b = 1$

(B) $a = 5, b = 2$

(C) $a = 3, b = 5$

(D) $a = 5, b = 3$

Correct Answer: (C) $a = 3, b = 5$

Solution:

Concept:

- The value of the objective function $Z = ax + by$ at a point is obtained by substituting its coordinates.
- The two stated values give simultaneous linear equations for the coefficients a and b.

Step 1: Form the equations At $(4, 6)$, the value is 42, so $4a + 6b = 42$, or $2a + 3b = 21$. At $(3, 2)$, the value is 19, so $3a + 2b = 19$. Both conditions must hold for the same coefficients.

Step 2: Solve and check the choices Multiply the first simplified equation by 3 and the second by 2. Subtracting $6a + 4b = 38$ from $6a + 9b = 63$ gives $5b = 25$, hence $b = 5$. Then $3a + 10 = 19$, so $a = 3$. These are option (C). Options (A), (B) and (D) give respectively 31, 19 and 21 at $(3, 2)$; (B) also gives 32 rather than 42 at $(4, 6)$.

Step 3: Conclusion $a = 3,\ b = 5$. Hence, the correct option is (C).

Quick Tip to Solve: Use both stated point values to form two equations; matching only the minimum can leave a wrong option.

9. The corner points of the feasible region of a linear programming problem are $(0, 4)$, $(8, 0)$ and $\left(\frac{20}{3}, \frac{4}{3}\right)$. If $Z = 30x + 24y$ is the objective function, then (maximum value of Z - minimum value of Z) is equal to [1 mark]

(A) 40

(B) 96

(C) 120

(D) 136

Correct Answer: The calculated difference is 144; none of the printed options is correct.

Solution:

Concept:

- For a bounded polygonal feasible region, the maximum and minimum of a linear objective occur at corner points.
- Evaluate $Z = 30x + 24y$ at every supplied vertex before comparing the extreme values.

Step 1: Evaluate all corner points At $(0, 4)$, $Z = 30(0) + 24(4) = 96$. At $(8, 0)$, $Z = 30(8) + 24(0) = 240$. At $\left(\frac{20}{3}, \frac{4}{3}\right)$, $Z = 30\left(\frac{20}{3}\right) + 24\left(\frac{4}{3}\right) = 200 + 32 = 232$.

Quick Tip to Solve: List all three vertex values before subtracting; the fractional vertex need not give the maximum.

Step 2: Compare the extremes and options The three objective values are 96, 240 and 232. Thus the maximum is 240 and the minimum is 96. Their difference is $240 - 96 = 144$. The printed choices 40, 96, 120 and 136 do not equal 144. In particular, 96 is only the minimum, while 136 would be the difference between 232 and 96 and ignores the larger value at $(8, 0)$.

Step 3: Conclusion The required difference is $\boxed{144}$. None of the printed options (A)-(D) is correct.

Quick Tip to Solve: List all three vertex values before subtracting; the fractional vertex need not give the maximum.

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10. Number of symmetric matrices of order 3×3 with each entry 1 or -1 is [1 mark]

(A) 512

(B) 64

(C) 8

(D) 4

Correct Answer: (B) 64

Solution:

Concept:

- A symmetric matrix satisfies $A^T = A$, so entries on opposite sides of the main diagonal must agree.
- Each independent entry can be chosen in two ways, either 1 or -1.

Step 1: Count independent entries A symmetric matrix of order 3 has the form $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$. The three diagonal entries a, d, f and the three entries b, c, e above the diagonal are independent. The entries below the diagonal are fixed by symmetry.

Step 2: Apply the multiplication principle Each of these six independent entries has two choices. Consequently the number of matrices is $2^6 = 64$. The number 512 equals 2^9 and incorrectly treats all nine entries as independent. The number 8 counts only three independent choices, ignoring the off-diagonal pairs. The number 4 counts only two choices of entries and is also insufficient.

Step 3: Conclusion There are 64 such matrices. Hence, the correct option is (B).

Quick Tip to Solve: For a symmetric matrix of order n, count n diagonal and n(n-1)/2 upper-triangular entries.

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11. If $\begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix} = P + Q$**, where** P **is a symmetric and** Q **is a skew symmetric matrix, then** Q **is equal to** [1 mark]

(A) $\begin{pmatrix} 2 & 5/2 \\ 5/2 & 4 \end{pmatrix}$

(B) $\begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}$

(C) $\begin{pmatrix} 0 & 5/2 \\ -5/2 & 0 \end{pmatrix}$

(D) $\begin{pmatrix} 2 & -5/2 \\ 5/2 & 4 \end{pmatrix}$

Correct Answer: (B) $\begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}$

Solution:

Concept:

- Every real square matrix decomposes as $A = P + Q$, where $P = (A + A^T)/2$ is symmetric.
- The skew-symmetric component is $Q = (A - A^T)/2$, satisfying $Q^T = -Q$.

Step 1: Transpose and subtract Here $A = \begin{pmatrix} 2 & 0 \\ 5 & 4 \end{pmatrix}$ and $A^T = \begin{pmatrix} 2 & 5 \\ 0 & 4 \end{pmatrix}$. Thus $A - A^T = \begin{pmatrix} 0 & -5 \\ 5 & 0 \end{pmatrix}$. Diagonal entries cancel, as required for a skew-symmetric matrix.

Step 2: Find Q and compare the options Dividing every entry by 2 gives $Q = \begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}$. Its transpose equals $-Q$. Option (C) is the negative of the required component, so it has the wrong signs. Option (A) is the symmetric component P. Option (D) has nonzero diagonal entries and therefore cannot be skew-symmetric.

Step 3: Conclusion $Q = \begin{pmatrix} 0 & -5/2 \\ 5/2 & 0 \end{pmatrix}$. Hence, the correct option is (B).

Quick Tip to Solve: Use (A-A transpose)/2 for Q; subtract in that order to preserve the off-diagonal signs.

12. If $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & a & 1 \end{pmatrix}$ **is non-singular matrix and** $a \in A$**, then the set** A **is** [1 mark]

(A) $\mathbb{R}$

(B) $\{0\}$

(C) $\{4\}$

(D) $\mathbb{R} - \{4\}$

Correct Answer: (D) $\mathbb{R} - \{4\}$

Solution:

Concept:

- A square matrix is non-singular exactly when its determinant is nonzero.
- Subtracting one row from another does not change the determinant and simplifies its evaluation.

Step 1: Simplify the determinant Let the given matrix be M. Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$. Then $|M| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & a-2 & 0 \end{vmatrix}$. Only the first entry of the third column is nonzero.

Step 2: Impose the non-singularity condition Expanding along the third column, with positive cofactor sign at position $(1,3)$, gives $|M| = \begin{vmatrix} 1 & 1 \\ 2 & a-2 \end{vmatrix} = (a-2) - 2 = a - 4$. Thus $a \neq 4$. All other real values, including 0, are allowed. The set of all real numbers includes the forbidden value 4, while the singleton sets omit other admissible values.

Step 3: Conclusion $A = \mathbb{R} \setminus \{4\}$. Hence, the correct option is (D).

Quick Tip to Solve: Subtract the first row from the other two to create two zeros in the last column.

13. If $|A| = |kA|$**, where** A **is a square matrix of order 2, then sum of all possible values of** k **is** [1 mark]

(A) 1

(B) -1

(C) 2

(D) 0

Correct Answer: (D) 0

Solution:

Concept:

- For a square matrix of order n, $|kA| = k^n|A|$, because each of its n rows is multiplied by k.
- For order 2 this becomes $|kA| = k^2|A|$; cancellation requires $|A| \neq 0$.

Step 1: Use the determinant scaling rule The equality in the question gives $|A| = k^2|A|$, or $(1 - k^2)|A| = 0$. In the non-singular case, dividing by $|A|$ yields $k^2 = 1$, so the two possible values are $k = 1$ and $k = -1$.

Quick Tip to Solve: Apply the power equal to matrix order, and check that the determinant is nonzero before cancelling it.

Step 2: Sum the values and examine the singular case The sum of the two finite values is $1+(-1)=0$. Option (A) keeps only the positive value, option (B) keeps only the negative value, and option (C) adds their magnitudes instead of their signed values. Strictly, the printed question does not say that A is non-singular: if $|A|=0$, every real k satisfies the equality and a sum of all values is undefined. The finite answer represented among the options is therefore for the non-singular case.

Step 3: Conclusion For $|A|\neq 0$, the sum is 0, giving option (D). If A is singular, every real k is possible and no finite sum is defined.

Quick Tip to Solve: Apply the power equal to matrix order, and check that the determinant is nonzero before cancelling it.

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14. If $\frac{d}{dx}[f(x)]=ax+b$ and $f(0)=0$, then $f(x)$ is equal to [1 mark]

(A) $a+b$

(B) $\frac{ax^2}{2}+bx$

(C) $\frac{ax^2}{2}+bx+c$

(D) b

Correct Answer: (B) $\frac{ax^2}{2}+bx$

Solution:

Concept:

- Integration reverses differentiation: $\int(ax+b)\,dx=ax^2/2+bx+C$.
- An initial value such as $f(0)=0$ determines the constant of integration C.

Step 1: Integrate the given derivative Since $f\prime(x)=ax+b$, integrate each term with respect to x. This gives $f(x)=\frac{ax^2}{2}+bx+C$, where C is a constant. Differentiating this expression returns $ax+b$, so it has the required derivative.

Step 2: Use the initial condition and compare Substitution of $x=0$ gives $f(0)=C$. The given condition $f(0)=0$ therefore forces $C=0$. Thus the arbitrary constant in option (C) must be fixed, and the specified function is option (B). Options (A) and (D) are constant expressions with zero derivative, so they do not reproduce the general given derivative $ax+b$.

Step 3: Conclusion $f(x)=\frac{ax^2}{2}+bx$. Hence, the correct option is (B).

Quick Tip to Solve: Integrate first, then substitute x=0; all variable terms vanish and the constant is fixed immediately.

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15. Degree of the differential equation $\sin x + \cos\left(\frac{dy}{dx}\right) = y^2$ **is** [1 mark]

(A) 2

(B) 1

(C) not defined

(D) 0

Correct Answer: (C) not defined

Solution:

Concept:

- The degree of a differential equation is the highest power of its highest-order derivative, after the equation has been expressed as a polynomial in its derivatives.
- The order is determined by the highest derivative present; order and degree are distinct concepts.

Step 1: Inspect the derivative The equation contains only the first derivative dy/dx, but this derivative occurs inside the trigonometric function $\cos(dy/dx)$. Thus the equation is of first order.

Step 2: Apply the polynomial requirement The equation $\cos(dy/dx) = y^2 - \sin x$ is not a polynomial equation in its derivatives. Consequently no polynomial exponent can be assigned as its degree. The exponent 2 on y is irrelevant; option (A) confuses the dependent variable with a derivative. Option (B) confuses order with degree, and (D) does not represent an undefined degree.

Step 3: Conclusion The degree is not defined. Hence, the correct option is (C).

Quick Tip to Solve: Check whether derivatives occur inside a trigonometric function before assigning a degree.

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16. The integrating factor of the differential equation $(1-y^2)\frac{dx}{dy} + yx = ay$, $(-1 < y < 1)$ **is** [1 mark]

(A) $\frac{1}{y^2-1}$

(B) $\frac{1}{\sqrt{y^2-1}}$

(C) $\frac{1}{1-y^2}$

(D) $\frac{1}{\sqrt{1-y^2}}$

Correct Answer: (D) $\frac{1}{\sqrt{1-y^2}}$

Solution:

Concept:

- For $dx/dy + P(y)x = Q(y)$, the integrating factor is $e^{\int P(y)\,dy}$; here y is the independent variable.
- The restriction $-1 < y < 1$ gives $1 - y^2 > 0$, so its square root is real and nonzero.

Step 1: Write the standard linear form Dividing by $1 - y^2$, which is nonzero throughout the stated

Quick Tip to Solve: Divide by the coefficient of dx/dy first; the coefficient of x then determines the integrating factor.

interval, gives

$$\frac{dx}{dy} + \frac{y}{1-y^2}x = \frac{ay}{1-y^2}.$$

Therefore $P(y) = y/(1-y^2)$.

Step 2: Integrate the coefficient of x Using $u = 1-y^2$, we obtain

$$\int \frac{y}{1-y^2}\,dy = -\frac{1}{2}\log(1-y^2), \qquad \text{I.F.} = \frac{1}{\sqrt{1-y^2}}.$$

Options (A) and (C) have the wrong exponent. Option (B) has a negative radicand on this interval and is not the real integrating factor.

Step 3: Conclusion The integrating factor is $\frac{1}{\sqrt{1-y^2}}$. Hence, the correct option is (D).

Quick Tip to Solve: Divide by the coefficient of dx/dy first; the coefficient of x then determines the integrating factor.

17. Unit vector along $\overrightarrow{PQ}$, where coordinates of P and Q respectively are $(2,1,-1)$ and $(4,4,-7)$, is [1 mark]

(A) $2\hat{i} + 3\hat{j} - 6\hat{k}$

(B) $-2\hat{i} - 3\hat{j} + 6\hat{k}$

(C) $\frac{-2\hat{i}}{7} - \frac{3\hat{j}}{7} + \frac{6\hat{k}}{7}$

(D) $\frac{2\hat{i}}{7} + \frac{3\hat{j}}{7} - \frac{6\hat{k}}{7}$

Correct Answer: (D) $\frac{2\hat{i}}{7} + \frac{3\hat{j}}{7} - \frac{6\hat{k}}{7}$

Solution:

Concept:

- The vector from P to Q is obtained by subtracting the coordinates of P from those of Q.
- For a nonzero vector $\vec{v}$, its unit vector in the same direction is $\vec{v}/|\vec{v}|$.

Step 1: Find the directed displacement With $P = (2,1,-1)$ and $Q = (4,4,-7)$,

$$\overrightarrow{PQ} = (4-2)\hat{i} + (4-1)\hat{j} + (-7+1)\hat{k} = 2\hat{i} + 3\hat{j} - 6\hat{k}.$$

Step 2: Normalize and compare the options The magnitude is

$$|\overrightarrow{PQ}| = \sqrt{2^2 + 3^2 + (-6)^2} = \sqrt{49} = 7.$$

Division by 7 gives a vector of magnitude one in the required direction. Options (A) and (B) have magnitude 7, so neither is a unit vector. Option (C) is a unit vector but points from Q towards P.

Quick Tip to Solve: Use terminal point minus initial point, then divide all three components by the same magnitude.

Step 3: Conclusion The required unit vector is $\frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}-\frac{6}{7}\hat{k}$. Hence, the correct option is (D).

Quick Tip to Solve: Use terminal point minus initial point, then divide all three components by the same magnitude.

18. Equation of a line passing through point $(1,2,3)$ and equally inclined to the coordinate axis, is [1 mark]

(A) $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$

(B) $\frac{x}{1}=\frac{y}{1}=\frac{z}{1}$

(C) $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z-1}{3}$

(D) $\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{1}$

Correct Answer: (D) $\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{1}$

Solution:

Concept:

- A line through (x_0,y_0,z_0) with direction ratios a,b,c has symmetric equation $(x-x_0)/a=(y-y_0)/b=(z-z_0)/c$.
- Equal inclinations to the positive coordinate axes give equal direction cosines, so direction ratios may be chosen as $1,1,1$.

Step 1: Choose the direction ratios Writing the equal direction cosines as $l=m=n$, the relation $l^2+m^2+n^2=1$ gives $3l^2=1$. The direction ratios are therefore proportional to $(1,1,1)$; reversing all signs describes the same geometric line.

Step 2: Insert the given point Using $(x_0,y_0,z_0)=(1,2,3)$, the symmetric form has numerators $x-1,y-2,z-3$ and equal denominators. Option (A) passes through the point but has unequal direction ratios. Option (B) has equal direction ratios but does not pass through $(1,2,3)$. In option (C), this point gives ratios $0,1/2,2/3$, so that option also fails.

Step 3: Conclusion The line is $\frac{x-1}{1}=\frac{y-2}{1}=\frac{z-3}{1}$. Hence, the correct option is (D).

Quick Tip to Solve: Check both the point and the direction ratios; satisfying only one condition cannot identify the line.

19. Assertion (A): Maximum value of $(\cos^{-1} x)^2$ is π^2.

Reason (R): Range of the principal value branch of $\cos^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Choose the correct option. [1 mark]

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).

(B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).

(C) (A) is true, but (R) is false.

(D) (A) is false, but (R) is true.

Correct Answer: (C) (A) is true, but (R) is false.

Solution:

Concept:

- The principal branch of $\cos^{-1} x$ has domain $[-1, 1]$ and range $[0, \pi]$.
- Squaring preserves order for nonnegative numbers, so the largest square occurs at the largest principal angle.

Step 1: Judge the assertion For every admissible x, $0 \le \cos^{-1} x \le \pi$, giving $0 \le (\cos^{-1} x)^2 \le \pi^2$. Equality at the upper endpoint is attained at $x = -1$, since $\cos^{-1}(-1) = \pi$. Thus Assertion (A) is true.

Step 2: Judge the reason The range stated in Reason (R), $[-\pi/2, \pi/2]$, is the principal range of inverse sine, not inverse cosine. The correct inverse-cosine range is $[0, \pi]$, so Reason (R) is false.

Step 3: Check the explanatory link A false range statement cannot explain the true maximum in Assertion (A). Options (A) and (B) incorrectly call the reason true, while option (D) incorrectly calls the assertion false.

Step 4: Conclusion The assertion is true and the reason is false. Hence, the correct option is (C).

Quick Tip to Solve: Remember that inverse cosine takes principal values from zero to pi; test the endpoint x=-1.

20. Assertion (A): If a line makes angles α, β, γ with positive direction of the coordinate axes, then $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2$.

Reason (R): The sum of squares of the direction cosines of a line is 1.

Choose the correct option. [1 mark]

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).

(B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).

(C) (A) is true, but (R) is false.

(D) (A) is false, but (R) is true.

Correct Answer: (A) Both (A) and (R) are true and (R) is the correct explanation of (A).

Solution:

Concept:

- The direction cosines of a line are $l = \cos\alpha$, $m = \cos\beta$, and $n = \cos\gamma$, with $l^2 + m^2 + n^2 = 1$.
- For every angle θ, $\sin^2\theta = 1 - \cos^2\theta$.

Quick Tip to Solve: For three direction angles, sum sin squared as three minus the sum of cos squared.

Step 1: Judge the assertion Apply the trigonometric identity to each of the three angles:

$$\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3 - (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3 - 1 = 2.$$

Therefore Assertion (A) is true.

Step 2: Judge the reason The sum of the squares of the direction cosines is 1, as follows from normalizing any direction vector of the line. Thus Reason (R) is true.

Step 3: Check the explanatory link The statement in Reason (R) supplies exactly the value of the cosine-square sum used above to derive Assertion (A). It therefore explains the assertion. Option (B) denies this direct link, and options (C) and (D) each wrongly make one statement false.

Step 4: Conclusion Both (A) and (R) are true, and (R) correctly explains (A). Hence, the correct option is (A).

Quick Tip to Solve: For three direction angles, sum sin squared as three minus the sum of cos squared.

Section B (10 marks)

*This section comprises of Very Short Answer Type (VSA) questions, each of **2** marks.*

21. If points A, B and C have position vectors $2\hat{i}$, $\hat{j}$ and $2\hat{k}$ respectively, then show that $\triangle ABC$ is an isosceles triangle. [2 marks]

Solution:

Concept:

- If $\vec{a}, \vec{b}, \vec{c}$ are position vectors, then $\overrightarrow{AB} = \vec{b} - \vec{a}$, and similarly for the other sides.
- The length of $u\hat{i} + v\hat{j} + w\hat{k}$ is $\sqrt{u^2 + v^2 + w^2}$. A triangle is isosceles when two of its side lengths are equal.

Step 1: Obtain the side vectors The given position vectors specify $A = (2, 0, 0)$, $B = (0, 1, 0)$, and $C = (0, 0, 2)$. Subtracting the initial position vector from the terminal one gives

$$\overrightarrow{AB} = -2\hat{i} + \hat{j},$$
$$\overrightarrow{BC} = -\hat{j} + 2\hat{k},$$
$$\overrightarrow{AC} = -2\hat{i} + 2\hat{k}.$$

Step 2: Compare the side lengths Taking magnitudes,

$$AB = \sqrt{(-2)^2 + 1^2} = \sqrt{5},$$
$$BC = \sqrt{(-1)^2 + 2^2} = \sqrt{5},$$
$$AC = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}.$$

In particular, the sides meeting at B have equal lengths. The vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are not scalar multiples (one has a nonzero first component, the other a nonzero third component), so the three

Quick Tip to Solve: Compare squared side lengths first: equal squared lengths immediately establish equal sides.

points form a nondegenerate triangle.

Step 3: Conclusion Since $AB = BC = \sqrt{5}$, $\triangle ABC$ is isosceles, with equal sides AB and BC and base AC.

Quick Tip to Solve: Compare squared side lengths first: equal squared lengths immediately establish equal sides.

22(a). Evaluate $\sin^{-1}\left(\sin\frac{3\pi}{4}\right) + \cos^{-1}(\cos\pi) + \tan^{-1}(1)$**.** [2 marks]

Solution:

Concept:

- The principal ranges of $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$ are respectively $[-\pi/2, \pi/2]$, $[0, \pi]$ and $(-\pi/2, \pi/2)$.
- The identity $\sin(\pi - \theta) = \sin\theta$ converts a second-quadrant angle to the principal inverse-sine range.

Step 1: Evaluate the inverse sine We cannot directly cancel inverse sine and sine because $3\pi/4$ is outside the principal range. Instead,

$$\sin\frac{3\pi}{4} = \sin\left(\pi - \frac{3\pi}{4}\right) = \sin\frac{\pi}{4}.$$

Since $\pi/4 \in [-\pi/2, \pi/2]$, the first term is $\pi/4$.

Step 2: Evaluate the other terms and add The angle π lies in the principal range of inverse cosine. Thus $\cos^{-1}(\cos\pi) = \cos^{-1}(-1) = \pi$. Also, $\tan(\pi/4) = 1$ and $\pi/4$ lies in the principal range of inverse tangent, so $\tan^{-1}(1) = \pi/4$. The required sum is therefore

$$\frac{\pi}{4} + \pi + \frac{\pi}{4} = \frac{6\pi}{4}.$$

Step 3: Conclusion The required value is $\boxed{3\pi/2}$.

Quick Tip to Solve: Check the principal range before cancelling an inverse function with its trigonometric function.

OR

22(b). Draw the graph of $\cos^{-1} x$**, where** $x \in [-1, 0]$**. Also, write its range.** [2 marks]

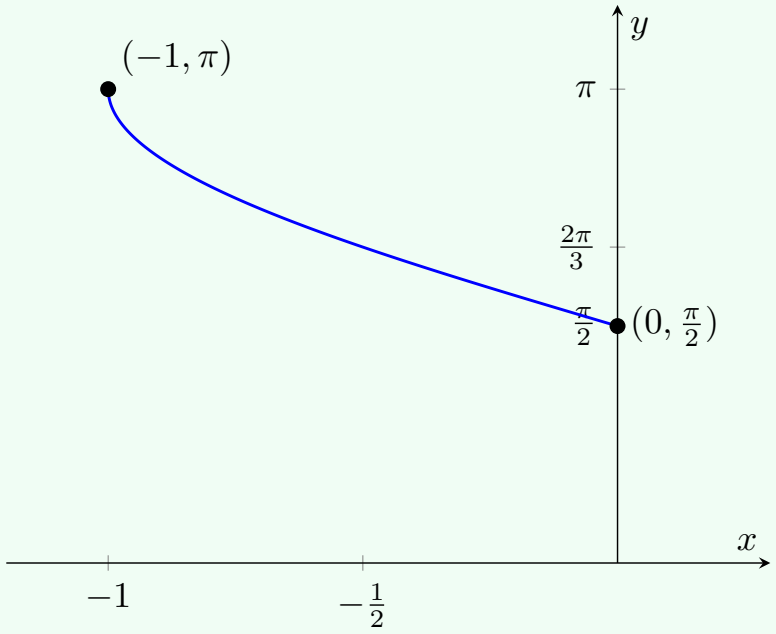
Solution:

Concept:

- The principal inverse cosine $y = \cos^{-1} x$ is defined for $-1 \le x \le 1$ and takes values in $[0, \pi]$.
- Cosine decreases on $[0, \pi]$, so its inverse is also decreasing. Restricting the input to $[-1, 0]$ selects only the corresponding part of this graph.

Quick Tip to Solve: Evaluate the inverse cosine at both endpoints, then write the smaller output first in the range.

Step 1: Find the endpoints and shape At the left endpoint, $\cos^{-1}(-1) = \pi$, whereas at the right endpoint, $\cos^{-1}(0) = \pi/2$. An intermediate point is $(-1/2, 2\pi/3)$, since $\cos(2\pi/3) = -1/2$. Hence draw the decreasing inverse-cosine curve from $(-1, \pi)$ to $(0, \pi/2)$, including both endpoints. Its tangent becomes vertical as x approaches -1 from the right.



Restricted graph of inverse cosine

Step 2: Read the range from the restricted graph As x increases continuously from -1 to 0, the value of $\cos^{-1} x$ decreases continuously from π to $\pi/2$. Thus every value between these endpoint ordinates is attained. Both endpoints belong to the domain, so both endpoint values belong to the range as well. The range is written in increasing order, even though the function decreases.

Step 3: Conclusion The graph is shown above and the range is $\boxed{[\pi/2, \pi]}$.

Quick Tip to Solve: Evaluate the inverse cosine at both endpoints, then write the smaller output first in the range.

23. If equal sides of an isosceles triangle with fixed base 10 cm are increasing at the rate of 4 cm/sec, how fast is the area of triangle increasing at an instant when all sides become equal? [2 marks]

Solution:

Concept:

- If each equal side is s cm and the altitude is h cm, the altitude bisects the fixed base of length 10 cm. Hence $h^2 = s^2 - 5^2$.
- The area is $A = \frac{1}{2}bh$, where b is the base length. When the base is fixed, differentiate with respect to time using the chain rule.

Quick Tip to Solve: Express the altitude using half the fixed base; do not differentiate the formula for an equilateral triangle.

Step 1: Express area in terms of the changing side By Pythagoras in either half of the triangle,

$$h = \sqrt{s^2 - 25}, \qquad A = \frac{1}{2}(10)\sqrt{s^2 - 25} = 5\sqrt{s^2 - 25}.$$

The positive square root is used because the altitude is a length. The given rate of change is $ds/dt = 4$ cm/sec.

Step 2: Differentiate and evaluate at the equilateral instant The base remains fixed throughout the motion. Differentiating the area expression gives

$$\frac{dA}{dt} = 5\frac{2s}{2\sqrt{s^2 - 25}}\frac{ds}{dt} = \frac{5s}{\sqrt{s^2 - 25}}\frac{ds}{dt}.$$

When all sides are equal, each equal side has length $s = 10$ cm and the altitude is $\sqrt{100 - 25} = 5\sqrt{3}$ cm. Substitution yields

$$\left.\frac{dA}{dt}\right|_{s=10} = \frac{5(10)(4)}{5\sqrt{3}} = \frac{40}{\sqrt{3}}\ \text{cm}^2/\text{sec}.$$

Step 3: Conclusion The area is increasing at $\boxed{\frac{40\sqrt{3}}{3}\ \text{cm}^2/\text{sec}}$.

Quick Tip to Solve: Express the altitude using half the fixed base; do not differentiate the formula for an equilateral triangle.

24. Find the coordinates of points on line $\frac{x}{1} = \frac{y-1}{2} = \frac{z+1}{2}$ which are at a distance of $\sqrt{11}$ units from origin. [2 marks]

Solution:

Concept:

- Introduce a common parameter in the symmetric equation of a line to express every point on it.
- The distance of (x, y, z) from the origin is $\sqrt{x^2 + y^2 + z^2}$; squaring the prescribed distance gives a quadratic equation in the parameter.

Step 1: Parametrize the line and impose the distance Let

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z+1}{2} = t.$$

Then a general point on the line is $(t, 1 + 2t, -1 + 2t)$. Its squared distance from the origin must equal 11, so

$$t^2 + (1 + 2t)^2 + (-1 + 2t)^2 = 11.$$

Using squared distances avoids unnecessary square roots and preserves both possible points.

Step 2: Solve the quadratic and find both points Expanding the left side, the linear terms cancel:

$$t^2 + (1 + 4t + 4t^2) + (1 - 4t + 4t^2) = 11.$$

Quick Tip to Solve: Keep both signs of the parameter after solving the squared-distance equation; each gives a point on the line.

Thus $9t^2 + 2 = 11$, which gives $t^2 = 1$ and $t = 1$ or $t = -1$. For $t = 1$, the point is $(1, 3, 1)$; for $t = -1$, it is $(-1, -1, -3)$. Their squared distances are respectively $1 + 9 + 1 = 11$ and $1 + 1 + 9 = 11$, and both satisfy the original line equation.

Step 3: Conclusion The required points are $\boxed{(1, 3, 1) \text{ and } (-1, -1, -3)}$.

Quick Tip to Solve: Keep both signs of the parameter after solving the squared-distance equation; each gives a point on the line.

25(a). If $y = \sqrt{ax + b}$, prove that $y\left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = 0$. [2 marks]

Solution:

Concept:

- Implicit differentiation: if $y^2 = ax + b$, differentiate both sides with respect to x, treating a, b as constants.
- The product rule is $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$. Apply it to $y\,dy/dx$.

Step 1: Square the given relation and differentiate Given $y = \sqrt{ax + b}$, we have $y^2 = ax + b$. At every interior point of its domain where the required derivatives exist, differentiation gives

$$2y\frac{dy}{dx} = a.$$

The derivative of the constant b is zero. This form avoids dividing by y, so the subsequent product-rule calculation is direct.

Step 2: Differentiate once more using the product rule Since a is constant, differentiating the previous identity yields

$$2\left[\frac{dy}{dx}\frac{dy}{dx} + y\frac{d^2y}{dx^2}\right] = 0.$$

Here the first term comes from differentiating the factor y, and the second from differentiating dy/dx. Dividing the entire identity by 2 gives the required relation wherever the two derivatives are defined.

Step 3: Conclusion

$$\boxed{y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0.}$$

Hence the stated identity is proved.

Quick Tip to Solve: Square the square-root relation first; two differentiations of y^2 give the desired expression without lengthy quotient calculations.

OR

25(b). **If** $f(x) = \begin{cases} ax + b & ; \ 0 < x \leq 1 \\ 2x^2 - x & ; \ 1 < x < 2 \end{cases}$ **is a differentiable function in** $(0, 2)$**, then find the values of** a **and** b**.** [2 marks]

Solution:

Concept:

- Differentiability at an interior point implies continuity there. Thus both branch limits at $x = 1$ must equal $f(1)$.
- The left and right derivatives at a joining point must agree. Away from $x = 1$, both branches are polynomials and hence differentiable.

Step 1: Apply continuity at the joining point For $0 < x \leq 1$, the function is $ax + b$; hence $f(1) = a + b$ and $\lim_{x \to 1^-} f(x) = a + b$. From the second branch,

$$\lim_{x \to 1^+} f(x) = 2(1)^2 - 1 = 1.$$

Continuity, which is necessary for differentiability, therefore requires

$$a + b = 1.$$

The equality uses the actual value from the first branch because that branch includes $x = 1$.

Step 2: Match the one-sided derivatives Using the continuity condition, the left difference quotient is

$$\lim_{h \to 0^-} \frac{a(1 + h) + b - (a + b)}{h} = a.$$

For the right quotient, $f(1) = 1$ gives

$$\lim_{h \to 0^+} \frac{2(1 + h)^2 - (1 + h) - 1}{h} = \lim_{h \to 0^+} (3 + 2h) = 3.$$

Thus differentiability requires $a = 3$, and the continuity equation then gives $b = 1 - 3 = -2$. These values make both the function values and the one-sided derivatives agree; the polynomial branches are differentiable everywhere else in $(0, 2)$.

Step 3: Conclusion

$$\boxed{a = 3, \qquad b = -2.}$$

Quick Tip to Solve: At a piecewise joining point, match the function values first and then the slopes; matching slopes alone does not ensure differentiability.

Section C (18 marks)

*The section comprises Short Answer (SA) type questions of **3** marks each.*

26. Solve the following Linear Programming problem graphically:

Maximize: $Z = 3x + 3.5y$

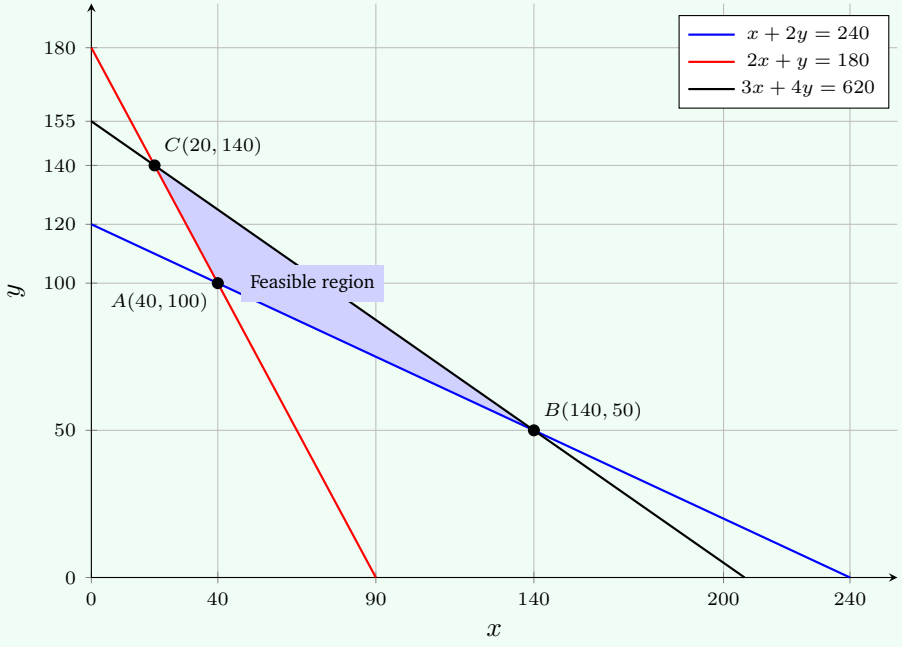
subject to constraints: $x + 2y \geq 240$, $3x + 1.5y \geq 270$, $1.5x + 2y \leq 310$, $x \geq 0$, $y \geq 0$. [3 marks]

Solution:

Concept:

- A linear constraint defines a half-plane. The feasible region is the common part of all the half-planes in the first quadrant.
- For a linear objective $Z = 3x + 3.5y$ on a nonempty bounded polygon, a maximum occurs at a corner point.

Step 1: Plot the boundary lines and choose the half-planes The boundary $x + 2y = 240$ has intercepts $(240, 0)$, $(0, 120)$; shade the side away from the origin. Rewrite the second boundary as $2x + y = 180$, with intercepts $(90, 0)$, $(0, 180)$; again choose the side away from the origin. The third boundary is $3x + 4y = 620$, with intercepts $(620/3, 0)$, $(0, 155)$; choose the side towards the origin. Together with $x, y \geq 0$, these half-planes enclose the shaded triangle shown below. No point on either coordinate axis satisfies all three constraints.



Graph of the constraints and the feasible triangle ABC.

Step 2: Find the vertices of the feasible triangle Intersecting the first two lines gives

$$x + 2y = 240, \quad 2x + y = 180$$
$$3y = 300, \qquad A = (40, 100).$$

Quick Tip to Solve: Clear decimals in the boundary equations before finding intersections, then test every corner of the common shaded region.

Intersecting the first and third gives

$$x + 2y = 240, \quad 1.5x + 2y = 310$$
$$0.5x = 70, \qquad B = (140, 50).$$

Intersecting the second and third gives

$$2x + y = 180, \quad 3x + 4y = 620$$
$$x = 20, \qquad C = (20, 140).$$

Each intersection lies in the first quadrant and satisfies the remaining inequality, so all three are feasible vertices.

Step 3: Evaluate the objective at every vertex Calculate $Z = 3x + 3.5y$ at the corners:

Corner	Z
$A(40, 100)$	$120 + 350 = 470$
$B(140, 50)$	$420 + 175 = 595$
$C(20, 140)$	$60 + 490 = 550$

The feasible region is a closed bounded triangle, so comparing these corner values determines the global maximum. The largest value is attained at B, where the first and third constraints are equalities and the second is satisfied.

Step 4: Conclusion The maximum is $\boxed{Z_{\max} = 595}$, attained at $\boxed{x = 140,\ y = 50}$.

Quick Tip to Solve: Clear decimals in the boundary equations before finding intersections, then test every corner of the common shaded region.

27(a). Find $\displaystyle\int \frac{x+2}{\sqrt{x^2-4x-5}}\,dx$**.** [3 marks]

Solution:

Concept:

- For a quadratic radicand, complete the square first: $x^2 - 4x - 5 = (x-2)^2 - 9$. The real integrand is defined on $x < -1$ or $x > 5$.
- The standard formula is $\displaystyle\int \frac{du}{\sqrt{u^2 - b^2}} = \log|u + \sqrt{u^2 - b^2}| + C$, where $b > 0$ and $u^2 > b^2$.

Step 1: Rewrite the numerator to match the derivative Let $Q = x^2 - 4x - 5$. Then $Q' = 2x - 4 = 2(x-2)$. The numerator is $x + 2 = (x - 2) + 4$, so the integral becomes

$$I = \int \frac{x-2}{\sqrt{x^2-4x-5}}\,dx + 4\int \frac{dx}{\sqrt{x^2-4x-5}}.$$

This split separates the part obtained by differentiating the square root from the remaining standard integral.

Step 2: Evaluate the two integrals For the first integral, put $t = x^2 - 4x - 5$, so $dt = 2(x-2)\,dx$.

Quick Tip to Solve: Split $x + 2$ as $(x - 2) + 4$, because $x - 2$ is half the derivative of the radicand.

Hence

$$\int \frac{x-2}{\sqrt{x^2-4x-5}}\,dx = \frac{1}{2}\int t^{-1/2}\,dt = \sqrt{x^2-4x-5}.$$

For the second, put $u = x-2$; then $du = dx$ and the radicand is u^2-9. Thus

$$\int \frac{dx}{\sqrt{x^2-4x-5}} = \log|x-2+\sqrt{x^2-4x-5}|.$$

Each expression is used on an interval where the original real integrand exists.

Step 3: Verify by differentiation Writing $S = \sqrt{x^2-4x-5}$, we have $S\prime = (x-2)/S$. Also $\frac{d}{dx}\log|x-2+S| = 1/S$. Therefore the derivative of the obtained primitive is $(x-2)/S + 4/S = (x+2)/S$, which reproduces the printed integrand.

Step 4: Conclusion

$$\boxed{\int \frac{x+2}{\sqrt{x^2-4x-5}}\,dx = \sqrt{x^2-4x-5} + 4\log\left|x-2+\sqrt{x^2-4x-5}\right| + C.}$$

Quick Tip to Solve: Split $x+2$ as $(x-2)+4$, because $x-2$ is half the derivative of the radicand.

OR

27(b). Evaluate $\int_{-a}^{a} f(x)\,dx$, where $f(x) = \frac{9^x}{1+9^x}$. [3 marks]

Solution:

Concept:

- For symmetric limits, substitution $x = -t$ gives $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} f(-x)\,dx$.
- If $f(x)+f(-x) = k$, adding the two equal integrals reduces the calculation to integrating the constant k. Here $9^x > 0$, so all denominators are nonzero.

Step 1: Calculate the reflected function The given function is $f(x) = \frac{9^x}{1+9^x}$. Replacing x by $-x$ and multiplying numerator and denominator by 9^x gives

$$f(-x) = \frac{9^{-x}}{1+9^{-x}} = \frac{1}{1+9^x}.$$

Consequently,

$$f(x)+f(-x) = \frac{9^x+1}{1+9^x} = 1.$$

Although f itself is neither even nor odd, this sum is constant, which is the useful symmetry here.

Step 2: Apply the symmetric-limit property Let $I = \int_{-a}^{a} f(x)\,dx$. Under $x = -t$, the bounds $-a, a$

Quick Tip to Solve: For an exponential quotient on symmetric limits, test $f(x)+f(-x)$ before attempting direct integration.

become $a, -a$, while $dx = -dt$. Reversing the bounds cancels the minus sign, giving

$$I = \int_{-a}^{a} f(-t)\,dt = \int_{-a}^{a} f(-x)\,dx.$$

Add this identity to the original definition of I:

$$\begin{aligned} 2I &= \int_{-a}^{a} \big(f(x) + f(-x)\big)\,dx \\ &= \int_{-a}^{a} 1\,dx = [x]_{-a}^{a} = a - (-a) = 2a. \end{aligned}$$

The constant sum replaces the need to find an antiderivative of the exponential quotient.

Step 3: Isolate the required integral Dividing the identity $2I = 2a$ by 2 yields the value of the original integral. As a check, for $a = 0$ the interval collapses and both sides vanish; for positive a, the integrand lies between zero and one, consistent with an integral equal to half the interval length.

Step 4: Conclusion

$$\boxed{\int_{-a}^{a} \frac{9^x}{1+9^x}\,dx = a.}$$

Quick Tip to Solve: For an exponential quotient on symmetric limits, test $f(x) + f(-x)$ before attempting direct integration.

28. Find $\displaystyle\int e^{\cot^{-1} x} \cdot \left(\frac{1 - x + x^2}{1 + x^2}\right) dx$**.** [3 marks]

Solution:

Concept:

- On the standard real branch, $\dfrac{d}{dx}(\cot^{-1} x) = -\dfrac{1}{1+x^2}$. The inverse cotangent is differentiable for every real x.
- The product rule is $\dfrac{d}{dx}(uv) = u\prime v + uv\prime$. An integrand involving an exponential and the derivative of its exponent may be recognised as a product derivative.

Step 1: Separate the rational factor Let $E = e^{\cot^{-1} x}$. Since $1 - x + x^2 = (1 + x^2) - x$, the integrand can be written as

$$E\frac{1 - x + x^2}{1 + x^2} = E - \frac{xE}{1 + x^2}.$$

The denominator $1 + x^2$ is positive for all real x, so this rearrangement is valid throughout the real domain. The first term suggests differentiating xE, while the second suggests the chain-rule contribution.

Step 2: Differentiate the proposed product By the chain rule,

$$E\prime = e^{\cot^{-1} x}\frac{d}{dx}(\cot^{-1} x) = -\frac{E}{1 + x^2}.$$

Quick Tip to Solve: Rewrite the factor as $1 - x/(1 + x^2)$ and check the derivative of $xe^{\cot^{-1} x}$.

Now apply the product rule to xE:

$$\begin{aligned}\frac{d}{dx}(xE) &= E + xE\prime \\ &= E - \frac{xE}{1+x^2} \\ &= E\frac{1+x^2-x}{1+x^2}.\end{aligned}$$

This is exactly the given integrand, including the minus sign of its middle term. Thus the recognition supplies a primitive without introducing another substitution.

Step 3: Recover the indefinite integral An indefinite integral is the family of functions whose derivatives give the integrand. Since the product just differentiated matches the integrand for every real x, its additive constant accounts for all primitives on the real line. No logarithmic or inverse-trigonometric term has to be added separately: the inverse cotangent is already present inside the exponential.

Step 4: Conclusion

$$\boxed{\int e^{\cot^{-1}x}\left(\frac{1-x+x^2}{1+x^2}\right)dx = xe^{\cot^{-1}x} + C.}$$

Quick Tip to Solve: Rewrite the factor as $1 - x/(1+x^2)$ and check the derivative of $xe^{\cot^{-1}x}$.

29. Evaluate $\displaystyle\int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{1}{(e^x+e^{-x})(e^x-e^{-x})}\,dx$. [3 marks]

Solution:

Concept:

- Use $(a+b)(a-b) = a^2 - b^2$ to simplify the product in the denominator before substituting.
- For a definite integral, a change of variable must also change both limits. Here $t = e^{2x}$ gives $dt = 2e^{2x}\,dx$.
- The partial fraction identity $\dfrac{1}{t^2-1} = \dfrac{1}{2}\left(\dfrac{1}{t-1} - \dfrac{1}{t+1}\right)$ gives logarithmic antiderivatives.

Step 1: Simplify and choose a substitution Let the required integral be I. Multiplying out the two factors gives

$$I = \int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{dx}{e^{2x}-e^{-2x}} = \int_{\log\sqrt{2}}^{\log\sqrt{3}} \frac{e^{2x}}{e^{4x}-1}\,dx.$$

Set $t = e^{2x}$, so that $dt = 2e^{2x}\,dx$. At the lower limit $t = e^{2\log\sqrt{2}} = 2$, while at the upper limit $t = e^{2\log\sqrt{3}} = 3$. Thus the transformed interval lies entirely above $t = 1$; neither denominator vanishes on it.

Quick Tip to Solve: Multiply numerator and denominator by e^{2x}; then $t = e^{2x}$ changes the awkward exponential limits directly to 2 and 3.

Step 2: Integrate by partial fractions With the new limits,

$$\begin{aligned} I &= \frac{1}{2}\int_2^3 \frac{dt}{t^2-1} \\ &= \frac{1}{4}\int_2^3 \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt \\ &= \frac{1}{4}\left[\log\left|\frac{t-1}{t+1}\right|\right]_2^3. \end{aligned}$$

The fraction is positive at both endpoints, so the absolute value can be removed when substituting. We obtain

$$I = \frac{1}{4}\left(\log\frac{2}{4} - \log\frac{1}{3}\right) = \frac{1}{4}\log\left(\frac{1}{2}\times 3\right).$$

Both the original integrand and this logarithmic value are positive, providing a useful sign check.

Step 3: Conclusion The required value is $\boxed{\frac{1}{4}\log\frac{3}{2}}$.

Quick Tip to Solve: Multiply numerator and denominator by e^{2x}; then $t = e^{2x}$ changes the awkward exponential limits directly to 2 and 3.

30(a). Find the general solution of the differential equation $(xy - x^2)\,dy = y^2\,dx$. [3 marks]

Solution:

Concept:

- A homogeneous differential equation involves a ratio of the variables. Choose $x = vy$ when treating x as a function of y simplifies the algebra.
- Under $x = vy$, the product rule gives $\frac{dx}{dy} = v + y\frac{dv}{dy}$. Separate the resulting equation before integrating.
- Whenever division by a variable is used, check any excluded constant solution in the original differential equation.

Step 1: Express the equation in homogeneous form The given equation is

$$(xy - x^2)\,dy = y^2\,dx.$$

On an interval where $y \neq 0$, regard x as a function of y and divide by y^2:

$$\frac{dx}{dy} = \frac{x}{y} - \left(\frac{x}{y}\right)^2.$$

Put $x = vy$. Substituting $dx/dy = v + y\,dv/dy$ gives

$$v + y\frac{dv}{dy} = v - v^2, \qquad y\frac{dv}{dy} = -v^2.$$

Quick Tip to Solve: Treat x as the dependent variable and use $x = vy$; this reduces the right side to $v - v^2$, avoiding a more complicated separation.

The substitution has reduced the homogeneous equation to a separable one.

Step 2: Separate, integrate and restore the variables For $v \neq 0$, separate the variables and integrate:

$$\int v^{-2}\,dv = -\int \frac{dy}{y}, \qquad -\frac{1}{v} = -\log|y| + C_1.$$

Renaming the arbitrary constant gives

$$\frac{1}{v} = \log|y| + C, \qquad \frac{y}{x} = \log|y| + C.$$

To verify this relation where $x, y \neq 0$, differentiate with respect to x:

$$\frac{xy' - y}{x^2} = \frac{y'}{y}.$$

Multiplication by x^2y gives $(xy - x^2)y' = y^2$, exactly the original equation. The division by y omitted $y = 0$; substitution in the original equation confirms that it is a constant solution. The excluded value $v = 0$ gives the vertical integral curve $x = 0$, when the equation is viewed as a differential relation.

Step 3: Conclusion The general solution is $\boxed{\frac{y}{x} - \log|y| = C}$ on intervals with $xy \neq 0$. Also, $y = 0$ is a constant solution; $x = 0$ is a vertical integral curve.

Quick Tip to Solve: Treat x as the dependent variable and use $x = vy$; this reduces the right side to $v - v^2$, avoiding a more complicated separation.

OR

30(b). Find the general solution of the differential equation $(x^2+1)\dfrac{dy}{dx} + 2xy = \sqrt{x^2+4}$. [3 marks]

Solution:

Concept:

- A first-order linear equation has the form $y' + P(x)y = Q(x)$. Its integrating factor is $\text{IF} = e^{\int P(x)\,dx}$.
- Multiplication by the integrating factor makes the left side $\dfrac{d}{dx}(\text{IF}\, y)$. Integrate once to obtain the general solution.
- The standard integral is $\displaystyle\int \sqrt{x^2+a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2+a^2} + a^2\log\left|x + \sqrt{x^2+a^2}\right|\right] + C$.

Step 1: Find the integrating factor Since $x^2 + 1 > 0$ for every real x, divide the given equation by it:

$$\frac{dy}{dx} + \frac{2x}{x^2+1}y = \frac{\sqrt{x^2+4}}{x^2+1}.$$

Here $P(x) = 2x/(x^2+1)$. Therefore,

$$\text{IF} = e^{\int \frac{2x}{x^2+1}\,dx} = e^{\log(x^2+1)} = x^2 + 1.$$

Quick Tip to Solve: Notice directly that $(x^2+1)y$ has derivative $(x^2+1)y' + 2xy$; the original left side is already a product derivative.

The integrating factor is positive everywhere, so no real interval is excluded by this step.

Step 2: Integrate the product derivative Multiplication by x^2+1 converts the equation into

$$\frac{d}{dx}\left[(x^2+1)y\right] = \sqrt{x^2+4}.$$

Integrating and applying the standard square-root integral with $a=2$,

$$(x^2+1)y = \frac{x}{2}\sqrt{x^2+4} + 2\log\left|x+\sqrt{x^2+4}\right| + C.$$

For all real x, $\sqrt{x^2+4} > |x|$, hence the logarithm's argument is positive. Differentiating the right side yields $\sqrt{x^2+4}$, while differentiating the left side yields $(x^2+1)y' + 2xy$. Thus the result satisfies the original equation and contains one arbitrary constant, as required for a first-order equation.

Step 3: Conclusion The general solution is $\boxed{y = \dfrac{\frac{x}{2}\sqrt{x^2+4} + 2\log|x+\sqrt{x^2+4}| + C}{x^2+1}}$, where C is an arbitrary real constant.

Quick Tip to Solve: Notice directly that $(x^2+1)y$ has derivative $(x^2+1)y' + 2xy$; the original left side is already a product derivative.

31(a). Two numbers are selected from first six even natural numbers at random without replacement. If X denotes the greater of two numbers selected, find the probability distribution of X. [3 marks]

Solution:

Concept:

- The first six even natural numbers are $2, 4, 6, 8, 10, 12$. Without replacement, the selected numbers are distinct.
- For equally likely outcomes, $P(E) = \dfrac{\text{favourable outcomes}}{\text{total outcomes}}$. Every unordered pair is equally likely because each corresponds to two ordered selections.
- A probability distribution lists all possible values of X with their probabilities, whose sum must equal 1.

Step 1: Count the possible pairs There are $\binom{6}{2} = 15$ unordered pairs of distinct numbers. The greater number cannot be 2, since no smaller number is available. Thus the possible values of X are $4, 6, 8, 10, 12$. Using pairs avoids counting each selection twice, and does not change any probability since all pairs are equally likely.

Step 2: Count pairs having a specified greater number If $X = 2k$, one selected number is $2k$ and its companion must be one of the $k-1$ smaller even numbers, where $k = 2, 3, 4, 5, 6$. Consequently,

$$P(X = 2k) = \frac{k-1}{15}.$$

For example, the maximum is 6 for exactly the pairs $\{2, 6\}$ and $\{4, 6\}$. Similarly, there is one pair with maximum 4, three with maximum 8, four with maximum 10, and five with maximum 12.

Quick Tip to Solve: Fix the larger number, count its possible smaller companions, and divide by the total number of unordered pairs.

Step 3: Write and verify the probability distribution Therefore the distribution is

x	4	6	8	10	12
$P(X=x)$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

Every probability is non-negative, and their sum is

$$\frac{1+2+3+4+5}{15} = 1.$$

The fifteen pairs have therefore been exhausted, with each pair assigned to its unique maximum.

Step 4: Conclusion X takes values $4, 6, 8, 10, 12$ with respective probabilities $\frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{4}{15}, \frac{5}{15}$, and has probability zero at all other values.

Quick Tip to Solve: Fix the larger number, count its possible smaller companions, and divide by the total number of unordered pairs.

OR

31(b). A fair coin and an unbiased die are tossed. Let A be the event, "Head appears on the coin" and B be the event, "3 comes on the die". Find whether A and B are independent events or not. [3 marks]

Solution:

Concept:

- Two events A and B are independent exactly when $P(A \cap B) = P(A)P(B)$.
- A fair coin has two equally likely outcomes H, T, and an unbiased die has six equally likely outcomes $1, 2, 3, 4, 5, 6$. Their joint experiment has twelve equally likely ordered outcomes.
- The intersection $A \cap B$ means both specified events occur in the same trial.

Step 1: Describe the sample space Write an outcome as (coin result, die result). The sample space is

$$S = \{(H,1),(H,2),(H,3),(H,4),(H,5),(H,6),$$
$$(T,1),(T,2),(T,3),(T,4),(T,5),(T,6)\}.$$

There are 12 equally likely outcomes. Each die result can occur with either face of the coin, so counting these joint outcomes gives the probabilities needed for the independence test.

Step 2: Compute the probabilities of the two events Event A, that a head appears, consists of the six outcomes $(H,1),(H,2),(H,3),(H,4),(H,5),(H,6)$. Thus

$$P(A) = \frac{6}{12} = \frac{1}{2}.$$

Event B, that the die shows 3, consists of $(H,3)$ and $(T,3)$. Hence

$$P(B) = \frac{2}{12} = \frac{1}{6}.$$

Quick Tip to Solve: Count the joint outcome $(H,3)$ and compare its probability with $\frac{1}{2} \times \frac{1}{6}$; independence requires equality.

Step 3: Test independence using the intersection Both events occur together only for $(H, 3)$. Therefore

$$P(A \cap B) = \frac{1}{12}, \qquad P(A)P(B) = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}.$$

The two quantities are equal, satisfying the defining criterion for independent events. The events are not mutually exclusive, because the common outcome $(H, 3)$ is possible; independence concerns the probability of their simultaneous occurrence, rather than the absence of common outcomes.

Step 4: Conclusion Since $P(A \cap B) = P(A)P(B) = \frac{1}{12}$, the events A and B are independent.

Quick Tip to Solve: Count the joint outcome $(H, 3)$ and compare its probability with $\frac{1}{2} \times \frac{1}{6}$; independence requires equality.

Section D (20 marks)

*This section comprises Long Answer type (LA) questions of **5** marks each.*

32. A function $f : [-4, 4] \to [0, 4]$ is given by $f(x) = \sqrt{16 - x^2}$. Show that f is an onto function but not a one-one function. Further, find all possible values of a for which $f(a) = \sqrt{7}$. [5 marks]

Solution:

Concept:

- A function is onto if every element of its codomain has at least one preimage in its domain. Thus, for every $y \in [0, 4]$, we must exhibit an $x \in [-4, 4]$ with $f(x) = y$.
- A function is one-one if distinct inputs always have distinct images. A single pair of distinct inputs with the same image disproves this property.
- The square root denotes the nonnegative root. For nonnegative u, $\sqrt{u^2} = u$; squaring an equation between nonnegative quantities preserves equivalence.

Step 1: Check the domain and the output bounds For $-4 \le x \le 4$, we have $0 \le x^2 \le 16$, so

$$0 \le 16 - x^2 \le 16 \quad \Longrightarrow \quad 0 \le \sqrt{16 - x^2} \le 4.$$

Hence the formula is defined on the entire stated domain and its values lie in the stated codomain. This establishes that the given rule is a function from $[-4, 4]$ to $[0, 4]$.

Step 2: Construct a preimage for every codomain element Let y be any element of $[0, 4]$. Choose

$$x = \sqrt{16 - y^2}.$$

Since $0 \le y^2 \le 16$, this input lies in $[0, 4]$, which is contained in $[-4, 4]$. Substitution gives

$$f(x) = \sqrt{16 - (16 - y^2)} = \sqrt{y^2} = y,$$

where the last equality follows from $y \ge 0$. This construction also covers the endpoints: $y = 0$ has preimage $x = 4$, and $y = 4$ has preimage $x = 0$. Therefore every codomain element is attained, proving that f is onto.

Quick Tip to Solve: To prove onto, start with an arbitrary output and construct its input; for this even function, opposite nonzero inputs immediately disprove one-one.

Step 3: Find distinct inputs with the same image Both -4 and 4 belong to the domain, and they are distinct. Nevertheless,

$$f(-4) = \sqrt{16-(-4)^2} = 0, \qquad f(4) = \sqrt{16-4^2} = 0.$$

Thus equal outputs need not have equal inputs, and f is not one-one. More generally, the dependence on x^2 gives $f(-x) = f(x)$ whenever both inputs are in the domain.

Step 4: Solve the specified output equation For an admissible input $a \in [-4, 4]$, the equation is

$$\sqrt{16-a^2} = \sqrt{7}.$$

Both sides are nonnegative, so squaring gives $16 - a^2 = 7$, or $a^2 = 9$. The candidates are $a = -3$ and $a = 3$. Both lie in the domain, and direct substitution gives $f(-3) = f(3) = \sqrt{16-9} = \sqrt{7}$. Since the squared equation has no other real roots, these are all possible inputs.

Step 5: Conclusion The function f is onto but not one-one, and all possible values are $\boxed{a = -3 \text{ or } a = 3}$.

Quick Tip to Solve: To prove onto, start with an arbitrary output and construct its input; for this even function, opposite nonzero inputs immediately disprove one-one.

33(a). If $A = \begin{pmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$ **and** $B = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{pmatrix}$**, then find** AB **and use it to solve the following system of equations:**

$x - 2y = 3$

$2x - y - z = 2$

$-2y + z = 3$ [5 marks]

Solution:

Concept:

- In matrix multiplication, the entry in row i, column j of AB is the scalar product of row i of A with column j of B.
- For square matrices, $AB = I$ implies that B is invertible and $B^{-1} = A$. Also, $(AB)^T = B^T A^T$, where the superscript T denotes transpose.
- A linear system with coefficient matrix C, unknown column X, and constant column D is written as $CX = D$. If C is invertible, its unique solution is $X = C^{-1}D$.

Step 1: Multiply the given matrices Multiplying each row of A by each column of B, we obtain

$$AB = \begin{pmatrix} -3+4+0 & -6+2+4 & 0+4-4 \\ 2-2+0 & 4-1-2 & 0-2+2 \\ 2-2+0 & 4-1-3 & 0-2+3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3.$$

Quick Tip to Solve: Compare the system's coefficient rows with both the given matrix and its transpose before applying the inverse obtained from the product.

For example, the entry in the first row and second column is $(-3)(2)+(-2)(-1)+(-4)(-1)=0$. All nine entries are evaluated above, so this is the full product rather than just a diagonal check. Since the matrices are square, the identity product implies $B^{-1}=A$, and hence $BA=I_3$ as well.

Step 2: Identify the system's coefficient matrix In the fixed order x, y, z, the equations have coefficient rows $(1,-2,0)$, $(2,-1,-1)$, and $(0,-2,1)$. Thus the given system is

$$\underbrace{\begin{pmatrix}1 & -2 & 0\\ 2 & -1 & -1\\ 0 & -2 & 1\end{pmatrix}}_{B^T}\underbrace{\begin{pmatrix}x\\ y\\ z\end{pmatrix}}_{X}=\underbrace{\begin{pmatrix}3\\ 2\\ 3\end{pmatrix}}_{D}.$$

It is essential to use B^T here: the rows of the coefficient matrix are the columns of the printed matrix B. The system is therefore $B^TX=D$.

Step 3: Use the computed product to obtain the unknowns From $B^{-1}=A$, we have $(B^T)^{-1}=A^T$. Equivalently, transposing $BA=I_3$ gives $A^TB^T=I_3$. Premultiplying the system by A^T,

$$X=A^TD=\begin{pmatrix}-3 & 2 & 2\\ -2 & 1 & 1\\ -4 & 2 & 3\end{pmatrix}\begin{pmatrix}3\\ 2\\ 3\end{pmatrix}=\begin{pmatrix}-9+4+6\\ -6+2+3\\ -12+4+9\end{pmatrix}=\begin{pmatrix}1\\ -1\\ 1\end{pmatrix}.$$

The invertibility of B^T also establishes uniqueness of this solution.

Step 4: Check the result in all three equations Substituting $x=1$, $y=-1$, and $z=1$, the left sides become

$$x-2y=1-2(-1)=3,$$
$$2x-y-z=2(1)-(-1)-1=2,$$
$$-2y+z=-2(-1)+1=3.$$

They agree with the respective right sides, confirming the matrix solution.

Step 5: Conclusion $\boxed{AB=I_3}$, and the unique solution of the system is $\boxed{x=1,\ y=-1,\ z=1}$.

Quick Tip to Solve: Compare the system's coefficient rows with both the given matrix and its transpose before applying the inverse obtained from the product.

OR

33(b). **If** $f(\alpha)=\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}$, **prove that** $f(\alpha)\cdot f(-\beta)=f(\alpha-\beta)$. [5 marks]

Solution:

Concept:

- Matrix multiplication uses the row-column rule: the entry in row i, column j of AB is the scalar product of row i of A with column j of B. The order of the factors must be preserved.

Quick Tip to Solve: Write f(-beta) explicitly before multiplying; its off-diagonal signs determine whether the resulting angle is a sum or a difference.

- Cosine is even and sine is odd: $\cos(-\beta) = \cos\beta$ and $\sin(-\beta) = -\sin\beta$.
- The difference identities are $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$ and $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$.

Step 1: Write the matrix at the negative argument Substitute $-\beta$ for the argument in the given definition. Using the parity of sine and cosine gives

$$f(-\beta) = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In particular, the upper-right entry changes from negative sine to positive sine, while the lower-left entry becomes negative sine. This sign change is essential for obtaining a difference of angles.

Step 2: Multiply the two matrices in the stated order For clarity, put $c_\alpha = \cos\alpha$, $s_\alpha = \sin\alpha$, $c_\beta = \cos\beta$ and $s_\beta = \sin\beta$. Applying the row-column rule yields

$$f(\alpha)f(-\beta) = \begin{pmatrix} c_\alpha c_\beta + s_\alpha s_\beta & c_\alpha s_\beta - s_\alpha c_\beta & 0 \\ s_\alpha c_\beta - c_\alpha s_\beta & s_\alpha s_\beta + c_\alpha c_\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

For example, the first entry is $c_\alpha c_\beta + (-s_\alpha)(-s_\beta)$. The entry in row two, column two is $s_\alpha s_\beta + c_\alpha c_\beta$. Each mixed entry involving the third row or column is zero, and the bottom-right entry is 1.

Step 3: Simplify every entry by the difference identities The two diagonal entries of the upper-left block equal $\cos(\alpha - \beta)$. Its lower-left entry equals $\sin(\alpha - \beta)$, while its upper-right entry is the negative of this sine. Therefore the product has the form

$$\begin{pmatrix} \cos(\alpha - \beta) & -\sin(\alpha - \beta) & 0 \\ \sin(\alpha - \beta) & \cos(\alpha - \beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This is exactly the given defining matrix evaluated at the argument $\alpha - \beta$; all nine entries agree.

Step 4: Conclusion Hence, for all real α and β, $\boxed{f(\alpha)\,f(-\beta) = f(\alpha - \beta)}$, as required.

Quick Tip to Solve: Write f(-beta) explicitly before multiplying; its off-diagonal signs determine whether the resulting angle is a sum or a difference.

34(a). **Find the equations of the diagonals of the parallelogram $PQRS$ whose vertices are $P(4, 2, -6)$, $Q(5, -3, 1)$, $R(12, 4, 5)$ and $S(11, 9, -2)$. Use these equations to find the point of intersection of diagonals.** [5 marks]

Solution:

Concept:

- A diagonal of a parallelogram joins opposite vertices. Thus the diagonals of $PQRS$ are PR and QS.
- A line through (x_0, y_0, z_0) with nonzero direction ratios (a, b, c) has symmetric equation $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$. A direction vector is obtained by subtracting the coordinates of two points on the line.
- At the intersection of two lines their three coordinates must agree. The midpoint property of parallelogram diagonals can then provide an independent check.

Step 1: Find the equation of diagonal PR The given opposite vertices are $P(4, 2, -6)$ and $R(12, 4, 5)$. The direction vector from P to R is

$$\overrightarrow{PR} = (12 - 4,\ 4 - 2,\ 5 - (-6)) = (8, 2, 11).$$

Taking P as the point on the line, the equation of the first diagonal is

$$\boxed{\frac{x - 4}{8} = \frac{y - 2}{2} = \frac{z + 6}{11}}.$$

Equivalently, with a real parameter t, its coordinates are $x = 4 + 8t$, $y = 2 + 2t$, $z = -6 + 11t$. The values $t = 0$ and $t = 1$ give the two endpoints.

Step 2: Find the equation of diagonal QS Using $Q(5, -3, 1)$ and $S(11, 9, -2)$, the direction vector is

$$\overrightarrow{QS} = (11 - 5,\ 9 - (-3),\ -2 - 1) = (6, 12, -3).$$

These ratios can be divided by 3, giving $(2, 4, -1)$. The equation of the second diagonal is therefore

$$\boxed{\frac{x - 5}{2} = \frac{y + 3}{4} = \frac{z - 1}{-1}}.$$

To keep the endpoint parameters simple while finding the intersection, use the equivalent parametrisation $x = 5 + 6u$, $y = -3 + 12u$, $z = 1 - 3u$.

Step 3: Solve the diagonal equations simultaneously Equating the first and second coordinates of the two parametrisations gives

$$\begin{aligned} 4 + 8t = 5 + 6u \quad &\Rightarrow 8t - 6u = 1, \\ 2 + 2t = -3 + 12u \quad &\Rightarrow 2t - 12u = -5. \end{aligned}$$

Twice the first reduced equation is $16t - 12u = 2$. Subtracting the second reduced equation yields

Quick Tip to Solve: Join opposite vertices PR and QS, and check the intersection by their common midpoint after solving the line equations.

$14t = 7$, so $t = \frac{1}{2}$. Substitution gives $u = \frac{1}{2}$. The third coordinate agrees as well, since

$$-6 + 11\left(\frac{1}{2}\right) = -\frac{1}{2} = 1 - 3\left(\frac{1}{2}\right).$$

Hence the common coordinates obtained from the diagonal equations are $x = 8$, $y = 3$, $z = -\frac{1}{2}$. Checking the midpoints independently gives the same triple for both PR and QS, as expected for a parallelogram.

Step 4: Conclusion The diagonals are $\boxed{\frac{x-4}{8} = \frac{y-2}{2} = \frac{z+6}{11}}$ and $\boxed{\frac{x-5}{2} = \frac{y+3}{4} = \frac{z-1}{-1}}$. Their point of intersection is $\boxed{(8, 3, -\frac{1}{2})}$.

Quick Tip to Solve: Join opposite vertices PR and QS, and check the intersection by their common midpoint after solving the line equations.

OR

34(b). A line l passes through point $(-1, 3, -2)$ and is perpendicular to both the lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$. Find the vector equation of the line l. Hence, obtain its distance from origin. [5 marks]

Solution:

Concept:

- A line through position vector $\vec{a}$, with non-zero direction vector $\vec{d}$, has vector equation $\vec{r} = \vec{a} + \lambda\vec{d}$, where $\lambda \in \mathbb{R}$.
- A direction perpendicular to two non-parallel directions $\vec{b}$ and $\vec{c}$ is given by their cross product. Equivalently, it satisfies $\vec{d} \cdot \vec{b} = \vec{d} \cdot \vec{c} = 0$.
- The shortest distance from the origin to this line is $D = \frac{|\vec{a} \times \vec{d}|}{|\vec{d}|}$; it is the length of the perpendicular from the origin, not the distance to an arbitrary point of the line.

Step 1: Extract the two given directions From the denominators of the symmetric equations, the direction vectors are

$$\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}, \qquad \vec{c} = -3\hat{i} + 2\hat{j} + 5\hat{k}.$$

The translated numerators in the second line locate a point on that line; they do not change its direction. The required line passes through $A(-1, 3, -2)$, so $\vec{a} = -\hat{i} + 3\hat{j} - 2\hat{k}$.

Step 2: Determine and check a common perpendicular direction Taking the cross product gives

$$\begin{aligned}\vec{b} \times \vec{c} &= (2 \cdot 5 - 3 \cdot 2)\hat{i} - (1 \cdot 5 - 3(-3))\hat{j} + (1 \cdot 2 - 2(-3))\hat{k} \\ &= 4\hat{i} - 14\hat{j} + 8\hat{k}.\end{aligned}$$

Quick Tip to Solve: Simplify the cross product before calculating the distance; multiplying the direction by any non-zero scalar leaves the line and the distance unchanged.

Dividing by the non-zero scalar 2, choose $\vec{d} = 2\hat{i} - 7\hat{j} + 4\hat{k}$. Direct checks give

$$\vec{d} \cdot \vec{b} = 2 - 14 + 12 = 0, \qquad \vec{d} \cdot \vec{c} = -6 - 14 + 20 = 0.$$

Thus this direction satisfies both perpendicularity conditions, and substituting $\lambda = 0$ in $\vec{r} = \vec{a} + \lambda\vec{d}$ recovers the specified point.

Step 3: Calculate the perpendicular distance from the origin Compute the cross product using the point vector and the chosen direction:

$$\begin{aligned} \vec{a} \times \vec{d} &= (3 \cdot 4 - (-2)(-7))\hat{i} + ((-2) \cdot 2 - (-1) \cdot 4)\hat{j} \\ &\quad + ((-1)(-7) - 3 \cdot 2)\hat{k} \\ &= -2\hat{i} + \hat{k}. \end{aligned}$$

Therefore $|\vec{a} \times \vec{d}| = \sqrt{5}$ and $|\vec{d}| = \sqrt{4 + 49 + 16} = \sqrt{69}$. Substitution in the point-to-line distance formula gives $D = \sqrt{5}/\sqrt{69}$. As a check, $\vec{a} \cdot \vec{d} = -31$ and $|\vec{a}|^2 = 14$, so the squared perpendicular component is $14 - 31^2/69 = 5/69$, agreeing with the cross-product calculation.

Step 4: Conclusion The required vector equation and its distance from the origin are

$$\boxed{\vec{r} = (-\hat{i} + 3\hat{j} - 2\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k}), \quad \lambda \in \mathbb{R}}, \qquad \boxed{D = \sqrt{\frac{5}{69}} \text{ units}}.$$

Quick Tip to Solve: Simplify the cross product before calculating the distance; multiplying the direction by any non-zero scalar leaves the line and the distance unchanged.

35. Find the area of the smaller region bounded by the curves $\frac{x^2}{25}+\frac{y^2}{16}=1$ and $\frac{x}{5}+\frac{y}{4}=1$, using integration. [5 marks]

Solution:

Concept:

- The area between an upper curve $y=f(x)$ and a lower curve $y=g(x)$, from $x=a$ to $x=b$, is $\int_a^b[f(x)-g(x)]\,dx$.
- An ellipse $x^2/a^2+y^2/b^2=1$ has total area πab. Its upper branch is $y=b\sqrt{1-x^2/a^2}$.
- For an integral involving $\sqrt{a^2-x^2}$, the substitution $x=a\sin\theta$ changes the square root to $a\cos\theta$ on $0\le\theta\le\pi/2$.

Step 1: Find the intersection points and identify the smaller segment The line can be written as $y=4(1-x/5)$. Substitute this expression in the equation of the ellipse:

$$\begin{aligned}\frac{x^2}{25}+\left(1-\frac{x}{5}\right)^2&=1,\\ \frac{2x^2}{25}-\frac{2x}{5}&=0,\\ 2x(x-5)&=0.\end{aligned}$$

Thus the intersections are $(0,4)$ and $(5,0)$. The straight chord joins the positive intercepts, cutting off a segment in the first quadrant. This segment lies inside that quadrant of the ellipse, hence occupies less than a quarter of the full ellipse and is necessarily smaller than the remaining region.

Step 2: Express the area as an integral of the vertical separation For $0\le x\le 5$, the relevant ellipse branch is

$$y=4\sqrt{1-\frac{x^2}{25}}=\frac{4}{5}\sqrt{25-x^2}.$$

It lies above the chord: writing $t=x/5$, both ordinates are non-negative and $1-t^2-(1-t)^2=2t(1-t)\ge 0$. Therefore the required area is

$$A=\int_0^5\left[\frac{4}{5}\sqrt{25-x^2}-4+\frac{4}{5}x\right]dx.$$

This integral subtracts the triangular area below the chord from the first-quadrant area below the elliptical arc.

Step 3: Evaluate the elliptical area by trigonometric substitution Put $x=5\sin\theta$, so $dx=5\cos\theta\,d\theta$. As x increases from 0 to 5, θ increases from 0 to $\pi/2$. On this interval $\sqrt{25-x^2}=5\cos\theta$. Hence

$$\begin{aligned}\frac{4}{5}\int_0^5\sqrt{25-x^2}\,dx&=20\int_0^{\pi/2}\cos^2\theta\,d\theta\\ &=10\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}\\ &=5\pi.\end{aligned}$$

This agrees with one quarter of the ellipse area $\pi\cdot 5\cdot 4$.

Quick Tip to Solve: When the chord joins the positive intercepts of an ellipse, integrate the first-quadrant arc and subtract the intercept triangle; its area here is one half of 5×4.

Step 4: Subtract the area under the chord and check the region The chord contribution is

$$\int_0^5 \left(4 - \frac{4}{5}x\right) dx = \left[4x - \frac{2}{5}x^2\right]_0^5 = 20 - 10 = 10.$$

Thus the integral of the vertical separation equals $5\pi - 10$. The full ellipse has area 20π, and the other region consequently has area $20\pi - (5\pi - 10) = 15\pi + 10$, which confirms that the segment evaluated is the smaller one.

Step 5: Conclusion The area of the smaller region bounded by the given ellipse and line is

$$\boxed{5\pi - 10 = 5(\pi - 2) \text{ square units}}.$$

Quick Tip to Solve: When the chord joins the positive intercepts of an ellipse, integrate the first-quadrant arc and subtract the intercept triangle; its area here is one half of 5×4.

Section E (12 marks)

*This section comprises 3 source based case-based/passage based/integrated units of assessment questions of **4** marks each.*

36. Recent studies suggest that roughly 12% of the world population is left handed.
Depending upon the parents, the chances of having a left handed child are as follows:
A: When both father and mother are left handed: Chances of left handed child is 24%.
B: When father is right handed and mother is left handed: Chances of left handed child is 22%.
C: When father is left handed and mother is right handed: Chances of left handed child is 17%.
D: When both father and mother are right handed: Chances of left handed child is 9%.
Assuming that $P(A) = P(B) = P(C) = P(D) = \frac{1}{4}$ and L denotes the event that child is left handed.
Based on the above information, answer the following questions: [4 marks]

36(i). Find $P(L/C)$. [1 mark]

Solution:

Concept:

- Conditional probability $P(L \mid C)$ is the probability of a left-handed child when the father is left handed and the mother is right handed.
- A percentage is converted into probability by dividing by 100.

Step 1: Identify the conditioning event Event C specifies a left-handed father and a right-handed mother. Thus only the data for category C apply; the unconditional world-population percentage does not answer this conditional question.

Quick Tip to Solve: For a conditional chance already given in the table, use its category directly without multiplying by the prior.

Step 2: Read the given chance The chance stated for category C is 17%. Therefore

$$P(L \mid C) = \frac{17}{100} = 0.17.$$

The prior probability $P(C) = 1/4$ is not multiplied here, since the question already conditions on C.

Step 3: Conclusion $\boxed{P(L/C) = 0.17}$.

Quick Tip to Solve: For a conditional chance already given in the table, use its category directly without multiplying by the prior.

36(ii). Find $P(\bar{L}/A)$. [1 mark]

Solution:

Concept:

- Within a fixed conditioning event A, the events L and $\bar{L}$ are complementary.
- The conditional complement rule is $P(\bar{L} \mid A) = 1 - P(L \mid A)$.

Step 1: Use the correct parent category Event A means that both parents are left handed. The data give $P(L \mid A) = 24/100 = 0.24$. This is the chance for a child in category A, rather than the overall population percentage.

Step 2: Apply the complement rule A child is either left handed or not left handed. These complementary possibilities exhaust category A, so

$$P(\bar{L} \mid A) = 1 - 0.24 = 0.76.$$

Multiplication by $P(A)$ would instead give a joint probability and is not required.

Step 3: Conclusion $\boxed{P(\bar{L}/A) = 0.76}$.

Quick Tip to Solve: Take the complement within the same parent category; do not change the conditioning event.

36(iii)(a). Find $P(A/L)$. [2 marks]

Solution:

Concept:

- The parent categories A, B, C, D form a mutually exclusive and exhaustive partition, with probability $1/4$ each.
- Total probability gives $P(L) = \sum_X P(X)P(L \mid X)$, where X runs over the four parent categories.
- Bayes' theorem reverses the conditioning: $P(A \mid L) = P(A)P(L \mid A)/P(L)$.

Step 1: Calculate the total probability Using the stated equal category probabilities and the four

Quick Tip to Solve: With equal prior probabilities, divide the category A chance by the sum of all four conditional chances.

conditional chances,

$$P(L) = \frac{1}{4}(0.24 + 0.22 + 0.17 + 0.09)$$
$$= \frac{1}{4}(0.72) = 0.18.$$

The descriptive world-population percentage is not substituted for this probability: the question explicitly specifies the parent-category model used in the calculation.

Step 2: Apply Bayes theorem The joint probability that both parents and their child are left handed is

$$P(A \cap L) = P(A)P(L \mid A) = \frac{1}{4}(0.24) = 0.06.$$

Hence, among left-handed children, the fraction belonging to category A is

$$P(A \mid L) = \frac{0.06}{0.18} = \frac{1}{3}.$$

This differs from $P(L \mid A) = 0.24$, because the conditioning event has been reversed.

Step 3: Conclusion $\boxed{P(A/L) = \frac{1}{3}}$.

Quick Tip to Solve: With equal prior probabilities, divide the category A chance by the sum of all four conditional chances.

OR

36(iii)(b). Find the probability that a randomly selected child is left handed given that exactly one of the parents is left handed. [2 marks]

Solution:

Concept:

- Exactly one left-handed parent means the event $E = B \cup C$. The two categories are mutually exclusive.
- Conditional probability is $P(L \mid E) = P(L \cap E)/P(E)$; joint probabilities use $P(L \cap X) = P(X)P(L \mid X)$.

Step 1: Find the probability of the conditioning event Category B has a right-handed father and a left-handed mother; category C has the opposite arrangement. Categories A and D do not satisfy the condition. Thus

$$P(E) = P(B) + P(C) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Within E, the two permitted categories therefore have equal conditional weights.

Quick Tip to Solve: Because B and C have equal priors, average their left-handed-child probabilities: (0.22+0.17)/2.

Step 2: Find the joint probability and divide Since B and C cannot occur together,

$$\begin{aligned} P(L \cap E) &= P(B)P(L \mid B) + P(C)P(L \mid C) \\ &= \frac{1}{4}(0.22) + \frac{1}{4}(0.17) \\ &= 0.0975. \end{aligned}$$

Consequently,

$$P(L \mid E) = \frac{0.0975}{0.5} = 0.195.$$

The numerator alone is a joint probability; division by the chance of exactly one left-handed parent gives the requested conditional chance.

Step 3: Conclusion The required probability is $\boxed{0.195 = \frac{39}{200}}$, or 19.5%.

Quick Tip to Solve: Because B and C have equal priors, average their left-handed-child probabilities: (0.22+0.17)/2.

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37. Engine displacement is the measure of the cylinder volume swept by all the pistons of a piston engine. The piston moves inside the cylinder bore.

The cylinder bore in the form of circular cylinder open at the top is to be made from a metal sheet of area 75π cm^2.

Based on the above information, answer the following questions: [4 marks]

37(i). If the radius of cylinder is r cm and height is h cm, then write the volume V of cylinder in terms of radius r. [1 mark]

Solution:

Concept:

- An open-top cylinder has metal area $\pi r^2 + 2\pi rh$: one circular base and the curved surface.
- The volume of a cylinder is $V = \pi r^2 h$, with r, h measured in centimetres.

Step 1: Express height using the area constraint The fixed sheet area gives

$$\pi r^2 + 2\pi rh = 75\pi.$$

Cancelling π and solving for height,

$$h = \frac{75 - r^2}{2r}.$$

For a non-degenerate cylinder, $r > 0$ and $h > 0$, so $0 < r < \sqrt{75}$.

Step 2: Substitute into the volume formula Replace h in $V = \pi r^2 h$ by its expression in terms of r:

$$V = \pi r^2 \frac{75 - r^2}{2r} = \frac{\pi}{2}(75r - r^3).$$

Quick Tip to Solve: For an open-top cylinder, count one base; solve the sheet-area equation for height before finding volume.

The area constraint eliminates height, leaving volume as a function of radius alone.

Step 3: Conclusion $\boxed{V(r) = \frac{\pi}{2}(75r - r^3)\ \text{cm}^3}$.

Quick Tip to Solve: For an open-top cylinder, count one base; solve the sheet-area equation for height before finding volume.

37(ii). Find $\frac{dV}{dr}$. [1 mark]

Solution:

Concept:

- The fixed-area constraint gives $V(r) = \frac{\pi}{2}(75r - r^3)$, for $0 < r < \sqrt{75}$.
- The power rule states $\frac{d}{dr}(r^n) = nr^{n-1}$, and constant multipliers remain outside differentiation.

Step 1: Write volume as a function of radius For this open-top cylinder, $r^2 + 2rh = 75$, so $h = (75 - r^2)/(2r)$. Substituting in $V = \pi r^2 h$ gives

$$V(r) = \frac{\pi}{2}(75r - r^3).$$

Height varies with radius to keep the metal area fixed.

Step 2: Differentiate term by term Treating $\pi/2$ and 75 as constants,

$$\frac{dV}{dr} = \frac{\pi}{2}\left(75\frac{dr}{dr} - \frac{dr^3}{dr}\right) = \frac{\pi}{2}(75 - 3r^2).$$

Holding height constant in the original volume formula would ignore the fixed-area condition.

Step 3: Conclusion $\boxed{\frac{dV}{dr} = \frac{\pi}{2}(75 - 3r^2)}$.

Quick Tip to Solve: Eliminate height before differentiating, because radius and height change together under the area constraint.

37(iii)(a). Find the radius of cylinder when its volume is maximum. [2 marks]

Solution:

Concept:

- For an open-top cylinder of area 75π cm^2, $V(r) = \frac{\pi}{2}(75r - r^3)$, with $0 < r < \sqrt{75}$.
- At an interior maximum, $V'(r) = 0$. A negative second derivative establishes a strict local maximum.

Step 1: Find the stationary radius From the base-plus-curved-area constraint, $h = (75 - r^2)/(2r)$. Therefore

$$V(r) = \frac{\pi}{2}(75r - r^3), \qquad V'(r) = \frac{\pi}{2}(75 - 3r^2).$$

Quick Tip to Solve: Solve the stationary equation for the positive radius, then check the derivative changes from positive to negative.

Setting the derivative to zero yields

$$75 - 3r^2 = 0 \quad \Longrightarrow \quad r^2 = 25.$$

Since a radius must be positive, the admissible stationary radius is $r = 5$ cm.

Step 2: Verify that it gives maximum volume The second derivative is

$$V''(r) = -3\pi r, \qquad V''(5) = -15\pi < 0.$$

Moreover, $V'(r) > 0$ for $0 < r < 5$ and $V'(r) < 0$ for $5 < r < \sqrt{75}$. Thus volume increases up to this radius and decreases thereafter throughout the admissible interval. The corresponding height is $(75 - 25)/(2 \cdot 5) = 5$ cm, which is positive and satisfies the area constraint.

Step 3: Conclusion The cylinder has maximum volume when $\boxed{r = 5 \text{ cm}}$.

Quick Tip to Solve: Solve the stationary equation for the positive radius, then check the derivative changes from positive to negative.

OR

37(iii)(b). For maximum volume, $h > r$. State true or false and justify. [2 marks]

Solution:

Concept:

- For a cylinder open at the top, the sheet covers one circular base and the curved surface, so $\pi r^2 + 2\pi rh = 75\pi$, where $r, h > 0$ are in centimetres.
- Its volume is $V = \pi r^2 h$. A stationary point with $V''(r) < 0$ gives a maximum.

Step 1: Express volume using the fixed sheet area Dividing the area relation by π and solving for height gives

$$h = \frac{75 - r^2}{2r}, \qquad 0 < r < \sqrt{75}.$$

Therefore the volume as a function of radius is

$$V(r) = \frac{\pi}{2}(75r - r^3).$$

The upper bound on radius ensures a positive height; the area relation includes no top disk because the cylinder is open.

Step 2: Find the maximizing dimensions and compare them Differentiating and setting the derivative to zero,

$$V'(r) = \frac{\pi}{2}(75 - 3r^2) = 0 \quad \Longrightarrow \quad r^2 = 25.$$

Since the radius is positive, $r = 5$ cm. Also $V''(r) = -3\pi r < 0$ throughout the admissible interval, so

Quick Tip to Solve: Include only one base for an open cylinder; its optimum height equals its radius.

this is the unique maximum. Substituting the maximizing radius into the area relation gives

$$h = \frac{75 - 25}{2(5)} = 5\,\text{cm}.$$

Step 3: Conclusion The statement is false. At maximum volume, $h = r = 5\,\text{cm}$, so $h > r$ does not hold.

Quick Tip to Solve: Include only one base for an open cylinder; its optimum height equals its radius.

38. The use of electric vehicles will curb air pollution in the long run.

The use of electric vehicles is increasing every year and estimated electric vehicles in use at any time is given by the function V:

$$V(t) = \frac{1}{5}t^3 - \frac{5}{2}t^2 + 25t - 2$$

where t represents the time and $t = 1, 2, 3, \ldots$ corresponds to year 2001, 2002, 2003, $\ldots$ respectively. Based on the above information, answer the following questions: [4 marks]

38(i). Can the above function be used to estimate number of vehicles in the year 2000? Justify. [2 marks]

Solution:

Concept:

- The year corresponding to time t is $2000 + t$, so the year 2000 corresponds to $t = 0$.
- A model for the number of vehicles must give a nonnegative value. A polynomial may be algebraically defined outside the intended time domain without giving a valid physical estimate there.

Step 1: Identify the time for the requested year The question assigns $t = 1$ to 2001, $t = 2$ to 2002 and so on. Moving back one year from 2001 therefore gives $t = 0$ for 2000. This value is outside the stated list $t = 1, 2, 3, \ldots$, so the given model is not specified for that year.

Step 2: Check the value obtained by extrapolation Even if we substitute $t = 0$ into the polynomial to test a possible estimate, we obtain

$$V(0) = \frac{1}{5}(0)^3 - \frac{5}{2}(0)^2 + 25(0) - 2 = -2.$$

The value is mathematically well defined, but a negative number of electric vehicles is impossible. Thus merely being able to substitute a value into the formula does not make the resulting estimate meaningful in the stated application.

Step 3: Conclusion No. The year 2000 corresponds to $t = 0$, outside the given time domain, and the formula gives $V(0) = -2$, which cannot represent a number of vehicles.

Quick Tip to Solve: Convert a calendar year to the model's time variable before substituting, then check that the estimate is physically possible.

38(ii). Prove that the function $V(t)$ is an increasing function. [2 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive everywhere on that interval.
- Completing the square shows the sign of a quadratic derivative: a squared real expression is nonnegative, so adding a positive constant makes the result strictly positive.

Step 1: Differentiate the vehicle model Treating time as a continuous variable when testing monotonicity, differentiate each term of

$$V(t) = \frac{1}{5}t^3 - \frac{5}{2}t^2 + 25t - 2.$$

The power rule gives

$$V'(t) = \frac{3}{5}t^2 - 5t + 25.$$

The constant term has derivative zero. To prove increasing behaviour, we must check the sign of this derivative throughout the time interval, rather than at just a few sample years.

Step 2: Show that the derivative is always positive Completing the square gives

$$\begin{aligned} V'(t) &= \frac{3}{5}\left(t^2 - \frac{25}{3}t\right) + 25 \\ &= \frac{3}{5}\left(t - \frac{25}{6}\right)^2 + \frac{175}{12}. \end{aligned}$$

The squared term is nonnegative for every real t, and $175/12 > 0$. Hence $V'(t) > 0$ everywhere, in particular for all positive times covered by the model. Strict increase on the continuous interval also implies increasing values at the integer times representing successive years.

Step 3: Conclusion Since $V'(t) > 0$ for every real t, the function $V(t)$ is strictly increasing, including on its stated time domain.

Quick Tip to Solve: Complete the square in the derivative to prove its sign for all times in one calculation.