CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-3-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) This question paper is divided into **five** Sections – **A, B, C, D and E**.

(iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculators is **not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $A = \begin{pmatrix} 1 & 4 & x \\ z & 2 & y \\ -3 & -1 & 3 \end{pmatrix}$ is a symmetric matrix, then the value of $x + y + z$ is: [1 mark]

(A) 10

(B) 6

(C) 8

(D) 0

Correct Answer: (D) 0

Solution:

Concept:

- A symmetric matrix satisfies $A^T = A$: entries reflected across its main diagonal are equal.
- Diagonal entries impose no restriction on the unknown off-diagonal entries.

Step 1: Compare corresponding entries Comparing positions $(1, 3)$ and $(3, 1)$ gives $x = -3$. Similarly, $(2, 3)$ and $(3, 2)$ give $y = -1$, while $(1, 2)$ and $(2, 1)$ give $z = 4$. These are equalities, not sign reversals; sign reversals would describe a skew-symmetric matrix.

Step 2: Evaluate the required sum Using all three values,

$$x + y + z = -3 - 1 + 4 = 0.$$

The values 10, 6 and 8 in (A), (B) and (C) do not equal this sum. In particular, taking absolute values of the negative entries would lose the signs required by symmetry.

Step 3: Conclusion $x + y + z = 0$. Hence, the correct option is (D).

Quick Tip to Solve: Pair each unknown with its reflected entry before adding; keep the two negative signs.

2. If $A \cdot (\text{adj } A) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, then the value of $|A| + |\text{adj } A|$ is equal to: [1 mark]

(A) 12

(B) 9

(C) 3

(D) 27

Correct Answer: (A) 12

Solution:

Concept:

- For a square matrix of order n, $A \,\text{adj}\, A = |A| I_n$.
- For order 3, $|\,\text{adj}\, A| = |A|^2$.

Step 1: Identify the determinant The given product is $3I_3$. Comparing it with $|A|I_3$, we obtain $|A| = 3$. The scalar on the diagonal is the determinant, rather than the determinant of the entire product.

Step 2: Evaluate the adjoint determinant Since A has order 3,

$$|\,\text{adj}\, A| = 3^2 = 9, \qquad |A| + |\,\text{adj}\, A| = 3 + 9 = 12.$$

Option (B) gives only the adjoint determinant, (C) gives only $|A|$, and (D) gives $|3I_3| = 27$, which is the determinant of the product rather than the requested sum.

Step 3: Conclusion The required sum is 12. Hence, the correct option is (A).

Quick Tip to Solve: Read the order from the identity matrix: the adjoint determinant uses exponent n minus 1.

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3. A and B are skew-symmetric matrices of same order. AB is symmetric, if: [1 mark]

(A) $AB = O$

(B) $AB = -BA$

(C) $AB = BA$

(D) $BA = O$

Correct Answer: (C) $AB = BA$

Solution:

Concept:

- Skew symmetry means $A^T = -A$ and $B^T = -B$.
- The transpose reverses multiplication order: $(AB)^T = B^T A^T$.

Step 1: Transpose the product Applying the transpose rule and both skew-symmetry conditions,

$$(AB)^T = B^T A^T = (-B)(-A) = BA.$$

Quick Tip to Solve: Transpose the product first; the reversed order immediately reveals the commutation condition.

The negative signs cancel, but the order of the two matrices is reversed.

Step 2: Apply the symmetry condition The product is symmetric precisely when $(AB)^T = AB$, so the necessary and sufficient condition is $BA = AB$. Anticommutation in (B) generally makes the product skew-symmetric instead. Conditions (A) and (D) are sufficient special cases: a zero product is symmetric, and its transpose is zero too. They are not the general characterization; symmetry also permits nonzero commuting products.

Step 3: Conclusion The general condition is $AB = BA$. Hence, the intended correct option is (C).

Quick Tip to Solve: Transpose the product first; the reversed order immediately reveals the commutation condition.

4. For what value of $x \in \left[0, \frac{\pi}{2}\right]$, is $A + A' = \sqrt{3}\, I$, where $A = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$? [1 mark]

(A) $\frac{\pi}{3}$
(B) $\frac{\pi}{6}$
(C) 0
(D) $\frac{\pi}{2}$

Correct Answer: (B) $\frac{\pi}{6}$

Solution:

Concept:

- The transpose A^T, also denoted $A\prime$, interchanges rows and columns.
- On $[0, \pi/2]$, each specified cosine value determines a unique angle.

Step 1: Add the matrix to its transpose The off-diagonal entries cancel on addition:

$$A + A^T = \begin{pmatrix} 2\cos x & 0 \\ 0 & 2\cos x \end{pmatrix} = 2\cos x\, I.$$

Thus the given matrix equation reduces to $2\cos x = \sqrt{3}$.

Step 2: Solve and check the choices We require $\cos x = \sqrt{3}/2$. On the stated interval this occurs at $x = \pi/6$. The alternatives $\pi/3$, 0, and $\pi/2$ produce scalar coefficients 1, 2, and 0, respectively, so none equals $\sqrt{3}$.

Step 3: Conclusion $x = \pi/6$. Hence, the correct option is (B).

Quick Tip to Solve: Cancel the sine entries in A plus its transpose, then compare the diagonal coefficient.

5. Let A be the area of a triangle having vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3). Which of the following is correct? [1 mark]

(A) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm A$

(B) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 2A$

(C) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm \frac{A}{2}$

(D) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = A^2$

Correct Answer: (B) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 2A$

Solution:

Concept:

- For vertices (x_i, y_i), set $D = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.
- The geometric area is $A = \frac{1}{2}|D|$; orientation controls the sign of D.

Step 1: Relate the determinant to area The coordinate formula gives

$$2A = |D|.$$

Consequently, depending on the order of the vertices, the determinant is either $2A$ or $-2A$. Interchanging two vertex rows changes its sign but cannot change the geometric area.

Step 2: Check each printed relation Thus $D = \pm 2A$, as in (B). Options (A) and (C) omit or reverse the required factor of two. Squaring the correct relation gives $D^2 = 4A^2$, whereas (D) claims $D^2 = A^2$. The determinant bars denote a determinant here, so an additional absolute value is needed in the area formula.

Step 3: Conclusion The determinant equals $\pm 2A$. Hence, the correct option is (B).

Quick Tip to Solve: The coordinate determinant gives signed double area; retain the factor two when squaring.

6. $\int 2^{x+2}\,dx$ **is equal to:** [1 mark]

(A) $2^{x+2}+C$

(B) $2^{x+2}\log 2+C$

(C) $\dfrac{2^{x+2}}{\log 2}+C$

(D) $2\cdot\dfrac{2^{x}}{\log 2}+C$

Correct Answer: (C) $\dfrac{2^{x+2}}{\log 2}+C$

Solution:

Concept:

- For $a>0$, $a\neq 1$, $\int a^x\,dx = a^x/\ln a + C$.
- Differentiating a^{x+2} multiplies it by $\ln a$, since $d(x+2)/dx = 1$.

Step 1: Integrate with the exponential rule Take $a=2$ and $u=x+2$, so $du=dx$. Then

$$\int 2^{x+2}\,dx = \frac{2^{x+2}}{\ln 2}+C.$$

Here the printed $\log 2$ is the natural logarithm used in calculus.

Step 2: Differentiate the alternatives The derivative of (C) is exactly 2^{x+2}. Option (A) has an extra factor $\ln 2$ in its derivative, while (B) has $(\ln 2)^2$. Option (D) differentiates to $2\cdot 2^x = 2^{x+1}$, only half of 2^{x+2}. Thus the shift in the exponent must remain intact.

Step 3: Conclusion $\int 2^{x+2}\,dx = \dfrac{2^{x+2}}{\log 2}+C$. Hence, the correct option is (C).

Quick Tip to Solve: Divide by the logarithm of the base and preserve x plus 2 in the exponent.

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7. $\int \dfrac{2\cos 2x-1}{1+2\sin x}\,dx$ **is equal to:** [1 mark]

(A) $x-2\cos x+C$

(B) $x+2\cos x+C$

(C) $-x-2\cos x+C$

(D) $-x+2\cos x+C$

Correct Answer: (B) $x+2\cos x+C$

Solution:

Concept:

- Use $\cos 2x = 1-2\sin^2 x$ to express the numerator in terms of $\sin x$.
- Cancellation is valid on intervals where the original denominator is nonzero.

Quick Tip to Solve: Factor 1 minus 4 sine squared before integrating; the cosine term has a positive sign.

Step 1: Factor and simplify the integrand We have

$$2\cos 2x - 1 = 1 - 4\sin^2 x = (1 - 2\sin x)(1 + 2\sin x).$$

Therefore the integrand is $1 - 2\sin x$ wherever $1 + 2\sin x \neq 0$.

Step 2: Integrate and check all signs Integrating gives

$$\int (1 - 2\sin x)\, dx = x + 2\cos x + C.$$

Its derivative is $1 - 2\sin x$. The derivatives of (A), (C), and (D) are respectively $1 + 2\sin x$, $-1 + 2\sin x$, and $-1 - 2\sin x$; each has at least one wrong sign. Constants apply separately on the intervals of definition.

Step 3: Conclusion $\displaystyle\int \frac{2\cos 2x - 1}{1 + 2\sin x}\, dx = x + 2\cos x + C$. Hence, the correct option is (B).

Quick Tip to Solve: Factor 1 minus 4 sine squared before integrating; the cosine term has a positive sign.

8. The solution of the differential equation $\dfrac{dx}{x} + \dfrac{dy}{y} = 0$ is: [1 mark]

(A) $\dfrac{1}{x} + \dfrac{1}{y} = C$

(B) $\log x - \log y = C$

(C) $xy = C$

(D) $x + y = C$

Correct Answer: (C) $xy = C$

Solution:

Concept:

- For nonzero variables, $\int dx/x = \ln|x|$ and $\int dy/y = \ln|y|$.
- An integration constant can be renamed after exponentiation on each connected domain.

Step 1: Integrate both differentials Integrating the given equation on a domain with $x \neq 0$ and $y \neq 0$ yields

$$\ln|x| + \ln|y| = K, \qquad \ln|xy| = K.$$

The plus sign is essential: it produces a product rather than a ratio.

Step 2: Express the constant relation Exponentiating shows that $|xy|$ is constant; the sign of xy is fixed on a connected domain, so this is written $xy = C$, with $C \neq 0$ for the original equation. Differentiating gives $y\,dx + x\,dy = 0$; division by xy recovers the equation. The reciprocal sum in (A), logarithmic difference in (B), and ordinary sum in (D) do not reproduce its differential.

Step 3: Conclusion The solution is $xy = C$. Hence, the correct option is (C).

Quick Tip to Solve: A sum of logarithmic differentials leads to a constant product, not a constant ratio.

9. What is the product of the order and degree of the differential equation $\frac{d^2y}{dx^2}\sin y + \left(\frac{dy}{dx}\right)^3 \cos y = \sqrt{y}$? [1 mark]

(A) 3

(B) 2

(C) 6

(D) not defined

Correct Answer: (B) 2

Solution:

Concept:

- The order is the order of the highest derivative present in the differential equation.
- The degree is the power of the highest-order derivative after the equation is expressed as a polynomial in its derivatives.

Step 1: Identify the highest derivative The equation contains dy/dx and d^2y/dx^2. The highest derivative is d^2y/dx^2, so its order is 2. The trigonometric factors depend on y, not on a derivative.

Step 2: Read its power and check the choices The equation is already polynomial in the derivatives, and d^2y/dx^2 occurs to power 1. Thus its degree is 1. The cube belongs to the first derivative, so it does not give degree 3; this excludes the products 3 and 6. Also, $\sqrt{y}$ does not make the degree undefined, because it contains no derivative.

Step 3: Conclusion The product is $2 \times 1 = 2$. Hence, the correct option is (B).

Quick Tip to Solve: Find the highest derivative first; take its power, not the largest power of any derivative.

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10. If a vector makes an angle of $\frac{\pi}{4}$ with the positive directions of both x-axis and y-axis, then the angle which it makes with positive z-axis is: [1 mark]

(A) $\frac{\pi}{4}$

(B) $\frac{3\pi}{4}$

(C) $\frac{\pi}{2}$

(D) 0

Correct Answer: (C) $\frac{\pi}{2}$

Solution:

Concept:

- For direction angles α, β, γ, the direction cosines satisfy $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.
- Direction angles with positive coordinate axes lie in $[0, \pi]$.

Quick Tip to Solve: When the two given squared direction cosines already sum to one, the third direction cosine is zero.

Step 1: Substitute the given angles Here $\alpha = \beta = \pi/4$. Therefore

$$\cos^2\gamma = 1 - \cos^2(\pi/4) - \cos^2(\pi/4) = 1 - \frac{1}{2} - \frac{1}{2} = 0.$$

Step 2: Determine the angle and check options Thus $\cos\gamma = 0$. In the direction-angle interval $[0, \pi]$, this gives $\gamma = \pi/2$. The angles $\pi/4$ and $3\pi/4$ have squared cosine $1/2$, and 0 has squared cosine 1; none can satisfy the required identity.

Step 3: Conclusion The angle with the positive z-axis is $\pi/2$. Hence, the correct option is (C).

Quick Tip to Solve: When the two given squared direction cosines already sum to one, the third direction cosine is zero.

11. $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that the projection of $\vec{a}$ on $\vec{b}$ is 0. The angle between $\vec{a}$ and $\vec{b}$ is: [1 mark]

(A) $\frac{\pi}{2}$
(B) π
(C) $\frac{\pi}{4}$
(D) 0

Correct Answer: (A) $\frac{\pi}{2}$

Solution:

Concept:

- The scalar projection of $\vec{a}$ on non-zero $\vec{b}$ is $\vec{a}\cdot\vec{b}/|\vec{b}| = |\vec{a}|\cos\theta$.
- The angle between two non-zero vectors lies in $[0, \pi]$.

Step 1: Use the zero projection Since the projection is zero,

$$|\vec{a}|\cos\theta = 0.$$

The vector $\vec{a}$ is non-zero, so $|\vec{a}| > 0$, and division gives $\cos\theta = 0$.

Step 2: Check the possible angles Within $[0, \pi]$, the only angle whose cosine vanishes is $\pi/2$. At π, the projection is $-|\vec{a}|$; at $\pi/4$, it is $|\vec{a}|/\sqrt{2}$; and at 0, it is $|\vec{a}|$. Each is non-zero, so those three options are excluded.

Step 3: Conclusion The vectors are perpendicular and their angle is $\pi/2$. Hence, the correct option is (A).

Quick Tip to Solve: For two non-zero vectors, a zero projection immediately gives a zero dot product and perpendicularity.

12. In $\triangle ABC$, $\vec{AB} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{AC} = 3\hat{i} - \hat{j} + 4\hat{k}$. If D is mid-point of BC, then vector $\vec{AD}$ is equal to: [1 mark]

(A) $4\hat{i} + 6\hat{k}$

(B) $2\hat{i} - 2\hat{j} + 2\hat{k}$

(C) $\hat{i} - \hat{j} + \hat{k}$

(D) $2\hat{i} + 3\hat{k}$

Correct Answer: (D) $2\hat{i} + 3\hat{k}$

Solution:

Concept:

- With A as origin, the position vectors of B and C are $\vec{AB}$ and $\vec{AC}$.
- The midpoint position vector is the arithmetic mean of the endpoint position vectors.

Step 1: Apply the midpoint formula Because D is the midpoint of BC,

$$\vec{AD} = \frac{\vec{AB} + \vec{AC}}{2}.$$

This formula averages vectors measured from the same point A.

Step 2: Add and average components Substituting the given vectors gives

$$\vec{AD} = \frac{(1+3)\hat{i} + (1-1)\hat{j} + (2+4)\hat{k}}{2} = 2\hat{i} + 3\hat{k}.$$

Option (A) is the sum before division by two. Options (B) and (C) contain a non-zero $\hat{j}$ component, whereas the two given $\hat{j}$ components cancel.

Step 3: Conclusion $\vec{AD} = 2\hat{i} + 3\hat{k}$. Hence, the correct option is (D).

Quick Tip to Solve: For a median from A to the midpoint of BC, average AB and AC component by component.

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13. The value of λ for which the angle between the lines $\vec{r} = \hat{i} + \hat{j} + \hat{k} + p(2\hat{i} + \hat{j} + 2\hat{k})$ and $\vec{r} = (1+q)\hat{i} + (1+q\lambda)\hat{j} + (1+q)\hat{k}$ is $\frac{\pi}{2}$ is: [1 mark]

(A) -4

(B) 4

(C) 2

(D) -2

Correct Answer: (A) -4

Solution:

Concept:

- The direction vector of a line $\vec{r} = \vec{a} + t\vec{b}$ is the coefficient $\vec{b}$ of its parameter.
- Two non-zero direction vectors are perpendicular exactly when their dot product vanishes.

Quick Tip to Solve: Use only the coefficients of the line parameters for the dot product; the fixed position vectors do not determine the angle.

Step 1: Extract both direction vectors The first line has direction vector $\vec{b}_1 = 2\hat{i} + \hat{j} + 2\hat{k}$. Rewrite the second as

$$\vec{r} = \hat{i} + \hat{j} + \hat{k} + q(\hat{i} + \lambda\hat{j} + \hat{k}),$$

so its direction vector is $\vec{b}_2 = \hat{i} + \lambda\hat{j} + \hat{k}$.

Step 2: Impose perpendicularity and test choices The given angle is $\pi/2$, hence

$$\vec{b}_1 \cdot \vec{b}_2 = 2 + \lambda + 2 = 0.$$

For the four offered values $-4, 4, 2, -2$, this dot product is respectively $0, 8, 6, 2$. Only the first value satisfies perpendicularity.

Step 3: Conclusion $\lambda = -4$. Hence, the correct option is (A).

Quick Tip to Solve: Use only the coefficients of the line parameters for the dot product; the fixed position vectors do not determine the angle.

14. If $P(A \cap B) = \frac{1}{8}$ and $P(\bar{A}) = \frac{3}{4}$, then $P\left(\frac{B}{A}\right)$ is equal to: [1 mark]

(A) $\frac{1}{2}$

(B) $\frac{1}{3}$

(C) $\frac{1}{6}$

(D) $\frac{2}{3}$

Correct Answer: (A) $\frac{1}{2}$

Solution:

Concept:

- The complement rule gives $P(A) = 1 - P(\bar{A})$.
- For $P(A) > 0$, the conditional probability is $P(B \mid A) = P(A \cap B)/P(A)$.

Step 1: Find the conditioning probability The denominator must be the probability of the event after the conditioning bar. From the given complement,

$$P(A) = 1 - \frac{3}{4} = \frac{1}{4}.$$

This is positive, so the conditional probability is defined.

Step 2: Apply the conditional probability formula The notation $P(B/A)$ means $P(B \mid A)$. Therefore

$$P(B \mid A) = \frac{1/8}{1/4} = \frac{1}{8} \times 4 = \frac{1}{2}.$$

Dividing by $P(\bar{A}) = 3/4$ would give $1/6$, but conditions on the wrong event. The values $1/3$ and $2/3$

Quick Tip to Solve: Always compute the probability of the conditioning event first; here the given probability is its complement.

likewise do not equal the required intersection-to-conditioning ratio.

Step 3: Conclusion $P(B \mid A) = 1/2$. Hence, the correct option is (A).

Quick Tip to Solve: Always compute the probability of the conditioning event first; here the given probability is its complement.

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15. The value of k for which function $f(x) = \begin{cases} x^2, & x \geq 0 \\ kx, & x < 0 \end{cases}$ is differentiable at $x = 0$ is: [1 mark]

(A) 1

(B) 2

(C) any real number

(D) 0

Correct Answer: (D) 0

Solution:

Concept:

- Differentiability at an interior point requires equal, finite left and right derivatives.
- The derivative at zero can be checked from $\lim_{h\to 0}[f(h) - f(0)]/h$.

Step 1: Check continuity and the right derivative Here $f(0) = 0$. Both $\lim_{x\to 0^+} x^2$ and $\lim_{x\to 0^-} kx$ are zero for every real k, so continuity holds. The right derivative is

$$f'_+(0) = \lim_{h\to 0^+} \frac{h^2}{h} = \lim_{h\to 0^+} h = 0.$$

Step 2: Compare with the left derivative For $h < 0$, $f(h) = kh$, so

$$f'_-(0) = \lim_{h\to 0^-} \frac{kh}{h} = k.$$

Equality of the one-sided derivatives requires $k = 0$. The choices 1 and 2 give unequal slopes. Continuity for every real k does not imply differentiability for every real k.

Step 3: Conclusion The function is differentiable at zero precisely when $k = 0$. Hence, the correct option is (D).

Quick Tip to Solve: For a piecewise function at zero, compare the difference quotients on both sides after checking its value at zero.

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16. If $y = \dfrac{\cos x - \sin x}{\cos x + \sin x}$, then $\dfrac{dy}{dx}$ is: [1 mark]

(A) $\sec^2\left(\dfrac{\pi}{4} - x\right)$

(B) $\sec^2\left(\dfrac{\pi}{4} - x\right)$

(C) $\log\left|\sec\left(\dfrac{\pi}{4} - x\right)\right|$

(D) $-\log\left|\sec\left(\dfrac{\pi}{4} - x\right)\right|$

Correct Answer: (A) $\sec^2\left(\dfrac{\pi}{4} - x\right)$

Solution:

Concept:

- The tangent subtraction identity is $\tan(a-b) = (\sin a \cos b - \cos a \sin b)/(\cos a \cos b + \sin a \sin b)$.
- The chain rule gives $d(\tan u)/dx = \sec^2 u \, du/dx$.

Step 1: Rewrite the quotient as a tangent Taking $a = \pi/4$, where $\sin a = \cos a = 1/\sqrt{2}$, gives

$$y = \frac{\cos x - \sin x}{\cos x + \sin x} = \tan\left(\frac{\pi}{4} - x\right).$$

This identity holds wherever the original denominator is non-zero.

Step 2: Differentiate and check the options Let $u = \pi/4 - x$. Then $du/dx = -1$, so

$$\frac{dy}{dx} = -\sec^2\left(\frac{\pi}{4} - x\right).$$

Option (B) misses the negative chain-rule factor. Options (C) and (D) involve logarithms, which are not produced by differentiating a tangent; hence neither matches this derivative.

Step 3: Conclusion $dy/dx = -\sec^2(\pi/4 - x)$. Hence, the correct option is (A).

Quick Tip to Solve: Recognise the quotient as tan(pi/4 - x), then retain the minus sign from differentiating its argument.

17. The number of feasible solutions of the linear programming problem given as: Maximize $z = 15x + 30y$ subject to constraints: $3x + y \leq 12,\ x + 2y \leq 10,\ x \geq 0,\ y \geq 0$ is: [1 mark]

(A) 1

(B) 2

(C) 3

(D) infinite

Correct Answer: (D) infinite

Solution:

Concept:

- A feasible solution satisfies every constraint simultaneously, including non-negativity.
- The objective function determines the optimum; it does not restrict the set of feasible solutions.

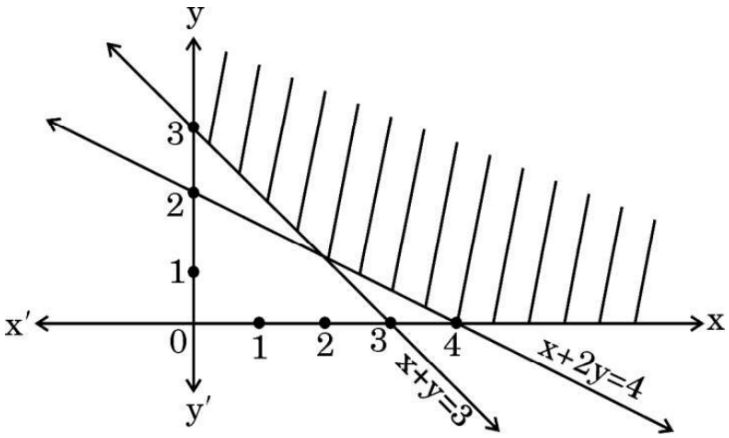
Step 1: Choose a family of points Take $y = 0$. The constraints become $3x \leq 12$, $x \leq 10$ and $x \geq 0$. Thus every point $(x, 0)$ with $0 \leq x \leq 4$ is feasible.

Step 2: Count feasible points The real interval $[0, 4]$ contains infinitely many values of x, so this family already gives infinitely many feasible solutions. Options (A), (B) and (C) incorrectly count only a finite number of points, possibly confusing corner points with all points in the feasible region.

Step 3: Conclusion There are infinitely many feasible solutions. Hence, the correct option is (D).

Quick Tip to Solve: To prove infinitely many feasible solutions, exhibit a whole line segment satisfying all constraints.

18. The feasible region of a linear programming problem is shown in the figure below:
Which of the following are the possible constraints? [1 mark]



(A) $x + 2y \geq 4,\ x + y \leq 3,\ x \geq 0,\ y \geq 0$

(B) $x + 2y \leq 4,\ x + y \leq 3,\ x \geq 0,\ y \geq 0$

(C) $x + 2y \geq 4,\ x + y \geq 3,\ x \geq 0,\ y \geq 0$

(D) $x + 2y \geq 4,\ x + y \geq 3,\ x \leq 0,\ y \leq 0$

Correct Answer: (C) $x + 2y \geq 4,\ x + y \geq 3,\ x \geq 0,\ y \geq 0$

Solution:

Concept:

- A boundary line divides the plane into two half-planes; a point in the shaded region determines the inequality sign.
- In the first quadrant, $x \geq 0$ and $y \geq 0$.

Step 1: Identify the shaded half-planes The boundaries are $x+2y=4$ and $x+y=3$. The hatching lies above both lines. For example, $(0,4)$ is on the shaded side and gives $x+2y=8 \geq 4$ and $x+y=4 \geq 3$. The solid boundary lines are included.

Step 2: Check the options The region shown is in the first quadrant, so both coordinates are non-negative. Option (A) selects the wrong side of $x+y=3$, and (B) selects the lower sides of both lines. Option (D) instead imposes non-positive coordinates, contrary to the figure.

Step 3: Conclusion The constraints are $x+2y \geq 4,\ x+y \geq 3,\ x \geq 0,\ y \geq 0$. Hence, the correct option is (C).

Quick Tip to Solve: Use a point well inside the hatching to decide each inequality, then check the coordinate signs.

19. Assertion (A): Range of $[\sin^{-1} x + 2\cos^{-1} x]$ is $[0, \pi]$.

Reason (R): Principal value branch of $\sin^{-1} x$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (D) Assertion (A) is false and Reason (R) is true.

Solution:

Concept:

- For $-1 \leq x \leq 1$, principal values satisfy $\sin^{-1} x + \cos^{-1} x = \pi/2$.
- The principal range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$.

Step 1: Judge Assertion (A) Using the principal-value identity,

$$\sin^{-1} x + 2\cos^{-1} x = \pi - \sin^{-1} x.$$

As $\sin^{-1} x$ varies from $-\pi/2$ to $\pi/2$, this expression ranges over $[\pi/2, 3\pi/2]$, not $[0, \pi]$. Thus A is false.

Step 2: Judge Reason (R) The stated principal range $[-\pi/2, \pi/2]$ of $\sin^{-1} x$ is correct. Therefore R is true.

Step 3: Choose the assertion-reason code Since A is false, there is no true assertion for R to explain. Codes (A) and (B) require A to be true, while (C) also requires R to be false; none fits these truth

Quick Tip to Solve: Reduce the expression to a single inverse function before using its principal range.

values.

Step 4: Conclusion Assertion is false and Reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Reduce the expression to a single inverse function before using its principal range.

20. Assertion (A): A line through the points $(4, 7, 8)$ and $(2, 3, 4)$ is parallel to a line through the points $(-1, -2, 1)$ and $(1, 2, 5)$.

Reason (R): Lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are parallel if $\vec{b_1} \cdot \vec{b_2} = 0$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (C) Assertion (A) is true and Reason (R) is false.

Solution:

Concept:

- Two lines are parallel when their nonzero direction vectors are scalar multiples.
- A zero dot product of nonzero direction vectors means perpendicular directions.

Step 1: Judge Assertion (A) The first line has direction vector $(2-4, 3-7, 4-8) = (-2, -4, -4)$. The second has direction vector $(1+1, 2+2, 5-1) = (2, 4, 4)$. These vectors are negatives of each other, so the lines are parallel and A is true.

Step 2: Judge Reason (R) The condition $\vec{b_1} \cdot \vec{b_2} = 0$ describes perpendicular directions, not parallel directions. In this example the dot product is $-4 - 16 - 16 = -36$, although the lines are parallel. Thus R is false.

Step 3: Choose the assertion-reason code A is true and R is false, so R cannot explain A. Options (A) and (B) both require a true reason, while (D) requires a false assertion and a true reason.

Step 4: Conclusion Assertion is true and Reason is false. Hence, the correct option is (C).

Quick Tip to Solve: For parallel lines compare direction ratios; reserve the zero dot-product test for perpendicular directions.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21. If $\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k}$, find the value of $(\vec{r} \times \hat{j}) \cdot (\vec{r} \times \hat{k}) - 12$. [2 marks]

Solution:

Concept:

- The unit vectors satisfy $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, and $\hat{k} \times \hat{i} = \hat{j}$. Reversing the order changes the sign.
- The dot product is the sum of products of corresponding components in an orthonormal basis.

Step 1: Calculate both cross products For $\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k}$, expand using distributivity:

$$\vec{r} \times \hat{j} = 3\hat{k} + 0 - 6\hat{i} = -6\hat{i} + 3\hat{k},$$
$$\vec{r} \times \hat{k} = -3\hat{j} - 2\hat{i} + 0 = -2\hat{i} - 3\hat{j}.$$

The terms involving a unit vector crossed with itself vanish.

Step 2: Take the dot product and subtract The two vectors have components $(-6, 0, 3)$ and $(-2, -3, 0)$. Hence

$$(\vec{r} \times \hat{j}) \cdot (\vec{r} \times \hat{k}) = (-6)(-2) + (0)(-3) + (3)(0) = 12.$$

The requested expression therefore becomes $12 - 12$. As a check, the identity $(\vec{a} \times \vec{b}) \cdot (\vec{a} \times \vec{c}) = (\vec{a} \cdot \vec{a})(\vec{b} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})(\vec{a} \cdot \vec{c})$ gives $0 - (-2)(6) = 12$ for the dot product.

Step 3: Conclusion The required value is $\boxed{0}$.

Quick Tip to Solve: Keep the order of each cross product fixed; reversing it changes the sign before taking the dot product.

22. If the angle between the lines $\dfrac{x-5}{\alpha} = \dfrac{y+2}{-5} = \dfrac{z+\frac{24}{5}}{\beta}$ and $\dfrac{x}{1} = \dfrac{y}{0} = \dfrac{z}{1}$ is $\dfrac{\pi}{4}$, find the relation between α and β. [2 marks]

Solution:

Concept:

- The denominators in symmetric equations give direction ratios; shifts in the numerators locate the line and do not affect its direction.
- For direction vectors $\vec{u}, \vec{v}$, the acute angle satisfies $\cos\theta = |\vec{u} \cdot \vec{v}|/(|\vec{u}||\vec{v}|)$.

Step 1: Identify the direction vectors The given lines have direction vectors $\vec{u} = (\alpha, -5, \beta)$ and $\vec{v} = (1, 0, 1)$. A zero direction ratio in the second line means its y-coordinate remains fixed; it is interpreted through the parametric form, rather than division by zero. Their dot product is $\alpha + \beta$, and their magnitudes are $\sqrt{\alpha^2 + 25 + \beta^2}$ and $\sqrt{2}$.

Quick Tip to Solve: Read direction ratios from denominators alone; numerator shifts do not enter the angle formula.

Step 2: Apply the angle formula Substitute $\theta = \pi/4$:

$$\frac{1}{\sqrt{2}} = \frac{|\alpha+\beta|}{\sqrt{\alpha^2+\beta^2+25}\sqrt{2}}.$$

Thus $|\alpha+\beta| = \sqrt{\alpha^2+\beta^2+25}$. Both sides are non-negative, so squaring gives

$$\alpha^2 + 2\alpha\beta + \beta^2 = \alpha^2 + \beta^2 + 25.$$

Cancelling the common squares yields $2\alpha\beta = 25$. Conversely this relation makes the squared cosine $1/2$, confirming the required acute angle.

Step 3: Conclusion The required relation is $\boxed{\alpha\beta = \frac{25}{2}}$.

Quick Tip to Solve: Read direction ratios from denominators alone; numerator shifts do not enter the angle formula.

23. If $f(x) = a(\tan x - \cot x)$, where $a > 0$, then find whether $f(x)$ is increasing or decreasing function in its domain. [2 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive throughout that interval.
- The derivatives are $\frac{d}{dx}(\tan x) = \sec^2 x$ and $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

Step 1: Identify the intervals of the domain Both $\tan x$ and $\cot x$ must be defined. Thus $\cos x \neq 0$ and $\sin x \neq 0$, so the domain is $\mathbb{R} \setminus \{n\pi/2 : n \in \mathbb{Z}\}$. Its connected intervals are $(n\pi/2, (n+1)\pi/2)$. The derivative test is applied separately on each such interval.

Step 2: Determine the sign of the derivative Differentiating gives

$$f\prime(x) = a\left(\sec^2 x + \operatorname{cosec}^2 x\right).$$

At every point in the domain, both squared reciprocal functions are positive, and $a > 0$. Therefore $f\prime(x) > 0$. In particular, for two points in any one domain interval, the mean value theorem gives a positive difference quotient. The function need not preserve this ordering across its excluded points.

Step 3: Conclusion Hence f is strictly increasing on every interval $(n\pi/2, (n+1)\pi/2)$, $n \in \mathbb{Z}$, of its domain. It is not globally increasing across these disconnected intervals.

Quick Tip to Solve: Check the domain before applying the derivative test: a positive derivative gives increase on each connected interval.

24(a). Evaluate: $3\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)+2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\cos^{-1}(0)$. [2 marks]

Solution:

Concept:

- Principal values of $\sin^{-1}x$ lie in $[-\pi/2,\pi/2]$, while those of $\cos^{-1}x$ lie in $[0,\pi]$.
- Use standard trigonometric values, ensuring that each angle belongs to the principal-value interval.

Step 1: Find each principal angle Since $\sin(\pi/4)=1/\sqrt{2}$ and $\pi/4\in[-\pi/2,\pi/2]$, $\sin^{-1}(1/\sqrt{2})=\pi/4$. Similarly, $\cos(\pi/6)=\sqrt{3}/2$ and $\cos(\pi/2)=0$, with both angles in $[0,\pi]$. Hence the two cosine inverse values are $\pi/6$ and $\pi/2$, respectively. We use radians consistently; the coefficients multiply these angles.

Step 2: Substitute and combine the angles The required expression becomes

$$3\left(\frac{\pi}{4}\right)+2\left(\frac{\pi}{6}\right)+\frac{\pi}{2}=\frac{3\pi}{4}+\frac{\pi}{3}+\frac{\pi}{2}$$
$$=\frac{9\pi+4\pi+6\pi}{12}.$$

The common denominator 12 accounts for all three terms without rounding any trigonometric value.

Step 3: Conclusion The value of the expression is $\boxed{\frac{19\pi}{12}}$.

Quick Tip to Solve: Write the three principal angles first, then use denominator 12 to add their weighted radian values.

OR

24(b). Draw the graph of $f(x)=\sin^{-1}x,\ x\in\left[-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right]$. Also, write range of $f(x)$. [2 marks]

Solution:

Concept:

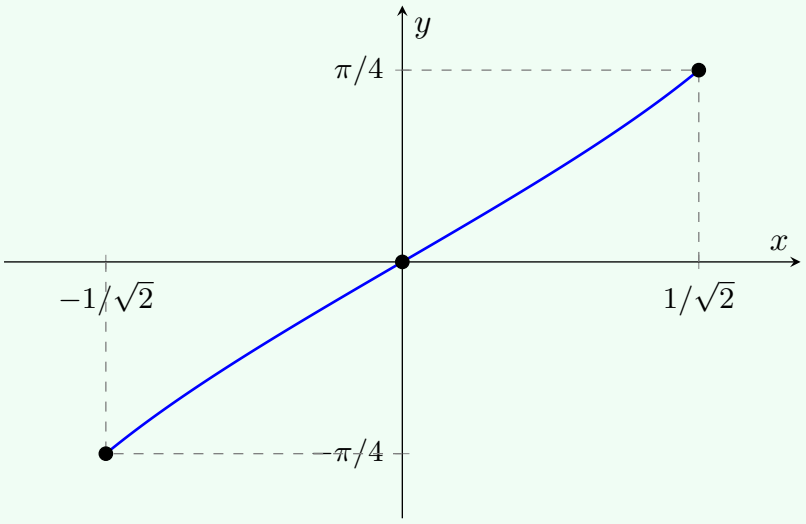
- The principal inverse sine is the inverse of sine restricted to $[-\pi/2,\pi/2]$. Its graph is $y=\sin^{-1}x$.
- On $(-1,1)$, $\frac{dy}{dx}=\frac{1}{\sqrt{1-x^2}}>0$; thus the endpoint values determine the range on the stated closed interval.

Step 1: Find the endpoints and the central point At the left endpoint, $\sin^{-1}(-1/\sqrt{2})=-\pi/4$, since $\sin(-\pi/4)=-1/\sqrt{2}$. At the right endpoint, $\sin^{-1}(1/\sqrt{2})=\pi/4$. Also $\sin^{-1}0=0$. Thus the graph passes through $(-1/\sqrt{2},-\pi/4)$, $(0,0)$ and $(1/\sqrt{2},\pi/4)$, and the endpoint dots are included.

Step 2: Sketch the restricted inverse sine curve Draw only the part of the increasing inverse sine curve between these endpoints. The curve is odd, so it is symmetric about the origin. Its slope is positive throughout the specified interval; it is slightly concave down for negative x and concave up for positive x. Continuity ensures that every ordinate between the two endpoint ordinates is attained,

Quick Tip to Solve: Mark the two closed endpoints and the origin first; restrict the inverse sine curve to the given x interval.

and monotonicity excludes values outside them.



Graph of $y = \sin^{-1} x$ on the specified interval

Step 3: Conclusion The requested graph is shown above, and the range is $\boxed{[-\pi/4, \pi/4]}$.

Quick Tip to Solve: Mark the two closed endpoints and the origin first; restrict the inverse sine curve to the given x interval.

25(a). If $y = x^{\frac{1}{x}}$, then find $\frac{dy}{dx}$ at $x = 1$. [2 marks]

Solution:

Concept:

- For a variable base raised to a variable exponent, take natural logarithms before differentiating.
- For $y > 0$, $\frac{d}{dx}(\log y) = \frac{1}{y}\frac{dy}{dx}$; here $x > 0$ permits $\log x$.

Step 1: Take logarithms of the given equation For $y = x^{1/x}$, taking natural logarithms on both sides yields

$$\log y = \frac{\log x}{x}.$$

This transformation is valid near $x = 1$, since x and y are positive there. It turns the variable exponent into a quotient that can be differentiated using the quotient rule.

Step 2: Differentiate and evaluate at the specified point Differentiating implicitly gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{x(1/x) - \log x}{x^2} = \frac{1 - \log x}{x^2}.$$

Therefore

$$\frac{dy}{dx} = x^{1/x}\frac{1 - \log x}{x^2}.$$

At $x = 1$, the original equation gives $y = 1^1 = 1$, and $\log 1 = 0$. Substituting these exact values gives

Quick Tip to Solve: For a variable exponent, take logarithms first; substitute x=1 only after differentiating.

$\left.\frac{dy}{dx}\right|_{x=1} = 1(1-0)/1.$

Step 3: Conclusion Thus $\boxed{\left.\frac{dy}{dx}\right|_{x=1} = 1}$.

Quick Tip to Solve: For a variable exponent, take logarithms first; substitute x=1 only after differentiating.

OR

25(b). **If** $x = a\sin 2t$, $y = a(\cos 2t + \log\tan t)$, **then find** $\frac{dy}{dx}$. [2 marks]

Solution:

Concept:

- For parametric equations, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, wherever $dx/dt \neq 0$. Here t is the parameter and a is a constant.
- The chain rule gives $\frac{d}{dt}\log(\tan t) = \frac{\sec^2 t}{\tan t} = \frac{1}{\sin t\cos t}$. Use $\sin 2t = 2\sin t\cos t$.

Step 1: Differentiate both coordinates Treating a as constant and using the chain rule,

$$\frac{dx}{dt} = 2a\cos 2t, \qquad \frac{dy}{dt} = a\left(-2\sin 2t + \frac{\sec^2 t}{\tan t}\right).$$

The logarithmic term contributes a positive derivative. In particular, its derivative is not $\cot t$: the derivative of the inner function $\tan t$ must also be included.

Step 2: Simplify the quotient Since $\sec^2 t/\tan t = 2/\sin 2t$,

$$\frac{dy}{dt} = 2a\left(\frac{1}{\sin 2t} - \sin 2t\right)$$
$$= \frac{2a(1-\sin^2 2t)}{\sin 2t} = \frac{2a\cos^2 2t}{\sin 2t}.$$

Dividing by $dx/dt = 2a\cos 2t$, wherever the parametric derivative is defined, yields

$$\frac{dy}{dx} = \frac{2a\cos^2 2t}{\sin 2t\,(2a\cos 2t)} = \frac{\cos 2t}{\sin 2t}.$$

The real logarithm requires $\tan t > 0$, and the quotient rule for the parametric derivative additionally requires $a\cos 2t \neq 0$.

Step 3: Conclusion Hence, $\boxed{\frac{dy}{dx} = \cot 2t}$ wherever $dx/dt \neq 0$.

Quick Tip to Solve: Simplify the logarithmic derivative to $2/\sin 2t$ before dividing; then $1-\sin^2 2t = \cos^2 2t$ cancels the common factor.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26(a). Find the general solution of the differential equation: $\frac{d}{dx}(xy^2) = 2y(1+x^2)$. [3 marks]

Solution:

Concept:

- Expand $d(xy^2)/dx$ by the product and chain rules. For $y \neq 0$, the equation becomes linear in y.
- A linear equation $y' + P(x)y = Q(x)$ has integrating factor $e^{\int P(x)\,dx}$. Multiplication by this factor makes the left side an exact product derivative.

Step 1: Expand and reduce to linear form The given equation is

$$y^2 + 2xy\frac{dy}{dx} = 2y(1+x^2).$$

First observe that $y \equiv 0$ satisfies the original equation. For the nonzero branch, divide by $2xy$ on an interval with $x \neq 0$:

$$\frac{dy}{dx} + \frac{y}{2x} = \frac{1}{x} + x.$$

Thus $P(x) = 1/(2x)$ and $Q(x) = 1/x + x$. Dividing by y would lose the zero solution, so it has been recorded separately.

Step 2: Apply the integrating factor and integrate For $x > 0$,

$$\text{I.F.} = e^{\int dx/(2x)} = \sqrt{x}.$$

Hence

$$\frac{d}{dx}(\sqrt{x}\,y) = x^{-1/2} + x^{3/2}.$$

Integrating both sides gives

$$\sqrt{x}\,y = 2\sqrt{x} + \frac{2}{5}x^{5/2} + C.$$

Dividing by $\sqrt{x}$ expresses the arbitrary-constant family explicitly. On an interval with $x < 0$, use $\sqrt{|x|}$ as the integrating factor; the same particular solution $2 + 2x^2/5$ applies, while the complementary solution is $C/\sqrt{|x|}$.

Step 3: Check the resulting family Write $y = 2 + 2x^2/5 + C/\sqrt{|x|}$. Its derivative satisfies

$$y' + \frac{y}{2x} = \frac{1}{x} + x,$$

since the derivative of the complementary term cancels its contribution to $y/(2x)$. Multiplying this equality by $2xy$ recovers $2xyy' + y^2 = 2y(1+x^2)$, exactly the expanded original equation. This verification also shows that a member of the family remains valid at a point where its value is zero.

Step 4: Conclusion The general nonzero family is $\boxed{y = 2 + \frac{2}{5}x^2 + \frac{C}{\sqrt{|x|}}}$ on intervals with $x \neq 0$. For

Quick Tip to Solve: Before cancelling a dependent-variable factor, test whether its zero value solves the original equation; here cancellation of y loses $y \equiv 0$.

$x > 0$, this is equivalently $\sqrt{x}\,y = 2\sqrt{x} + \frac{2}{5}x^{5/2} + C$. The additional solution is $\boxed{y \equiv 0}$.

Quick Tip to Solve: Before cancelling a dependent-variable factor, test whether its zero value solves the original equation; here cancellation of y loses $y \equiv 0$.

OR

26(b). Solve the following differential equation: $xe^{\frac{y}{x}} - y + x\dfrac{dy}{dx} = 0$. [3 marks]

Solution:

Concept:

- A differential equation whose right side depends only on y/x is homogeneous. Substitute $v = y/x$, so $y = vx$ and $dy/dx = v + x\,dv/dx$.
- After the substitution, separate the variables and integrate. The identities $\int e^{-v}\,dv = -e^{-v}$ and $\int dx/x = \log|x|$ determine the signs.

Step 1: Introduce the homogeneous substitution For $x \neq 0$, the printed equation gives

$$\frac{dy}{dx} = \frac{y}{x} - e^{y/x}.$$

Both terms on the right depend only on the ratio y/x. Put $v = y/x$. Then $y = vx$, and the product rule yields

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substitution into the differential equation cancels the two terms involving v:

$$v + x\frac{dv}{dx} = v - e^{v}, \qquad x\frac{dv}{dx} = -e^{v}.$$

Step 2: Separate and integrate Because e^{v} is positive for every real v, dividing by it loses no constant solution. Rearrangement gives

$$e^{-v}\,dv = -\frac{dx}{x}.$$

Integrating on an interval that does not contain zero,

$$-e^{-v} = -\log|x| + C_1.$$

Absorbing the sign of the integration constant into a new arbitrary constant C gives

$$e^{-v} = \log|x| + C.$$

Substituting back $v = y/x$ produces the required implicit relation.

Quick Tip to Solve: After putting $y = vx$, cancel the two v terms first. Keep both minus signs during integration to obtain $e^{-v} = \log|x| + C$.

Step 3: Verify and state the real domain Differentiate the implicit relation with respect to x:

$$e^{-y/x}\left(-\frac{xy\prime - y}{x^2}\right) = \frac{1}{x}.$$

Multiplication by $x^2 e^{y/x}$ gives $-xy\prime + y = xe^{y/x}$, which rearranges to the original equation. For a real solution, $\log|x| + C$ must be positive, because it equals an exponential. If desired, taking its logarithm gives the explicit expression $y = -x\log(\log|x| + C)$.

Step 4: Conclusion $\boxed{e^{-y/x} = \log|x| + C}$, with $x \neq 0$ and $\log|x| + C > 0$, is the general solution.

Quick Tip to Solve: After putting $y = vx$, cancel the two v terms first. Keep both minus signs during integration to obtain $e^{-v} = \log|x| + C$.

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27. Evaluate: $\displaystyle\int_1^3 \frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}\,dx.$ [3 marks]

Solution:

Concept:

- The reflection property is $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$. Here $a+b=4$, so use $x \mapsto 4-x$.
- If the original integrand and its reflected integrand add to one, their two equal definite integrals add to the interval length $b-a$.

Step 1: Reflect the integral about the midpoint Let

$$I = \int_1^3 \frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}\,dx.$$

Both square roots are real and positive throughout $[1,3]$, so the denominator never vanishes. Put $u = 4-x$, giving $dx = -du$. At $x=1$, $u=3$; at $x=3$, $u=1$. Reversing the limits removes the minus sign:

$$I = \int_1^3 \frac{\sqrt{u}}{\sqrt{4-u}+\sqrt{u}}\,du.$$

The integration variable is a dummy variable, so it may now be renamed x.

Step 2: Add the original and reflected forms The two expressions represent the same number I. Thus

$$\begin{aligned} 2I &= \int_1^3 \left(\frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}} + \frac{\sqrt{x}}{\sqrt{4-x}+\sqrt{x}}\right)dx \\ &= \int_1^3 \frac{\sqrt{4-x}+\sqrt{x}}{\sqrt{x}+\sqrt{4-x}}\,dx \\ &= \int_1^3 1\,dx = [x]_1^3 = 3-1 = 2. \end{aligned}$$

The cancellation follows because both fractions have exactly the same denominator. No antiderivative of either square-root quotient is required; the reflection pairs complementary numerators. Dividing

Quick Tip to Solve: Check whether replacing x by the sum of the limits minus x swaps the numerator with the other denominator term; then add the two integrals.

the equality $2I = 2$ by two determines the value of the given integral.

Step 3: Conclusion Hence, $\boxed{\int_1^3 \frac{\sqrt{4-x}}{\sqrt{x}+\sqrt{4-x}}\,dx = 1}$.

Quick Tip to Solve: Check whether replacing x by the sum of the limits minus x swaps the numerator with the other denominator term; then add the two integrals.

28. Evaluate: $\int_1^e \frac{1}{\sqrt{4x^2-(x\log x)^2}}\,dx$. [3 marks]

Solution:

Concept:

- For positive x, $\sqrt{x^2} = x$. Factoring out x^2 from the radicand therefore preserves the sign of the denominator.
- The substitution $t = \log x$ gives $dt = dx/x$, where log is the natural logarithm.
- The standard antiderivative is $\int dt/\sqrt{a^2-t^2} = \sin^{-1}(t/a) + C$ for $|t| < a$.

Step 1: Simplify the denominator Let the integral be I. Since $1 \le x \le e$, x is positive and $0 \le \log x \le 1$. Thus

$$4x^2-(x\log x)^2 = x^2\big(4-(\log x)^2\big),$$
$$I = \int_1^e \frac{dx}{x\sqrt{4-(\log x)^2}}.$$

The remaining radicand is at least 3, so this simplification is valid throughout the interval and neither endpoint is singular.

Step 2: Change the variable and limits Put $t = \log x$, so $dt = dx/x$. When $x = 1$, $t = 0$; when $x = e$, $t = 1$. Changing both limits along with the variable gives

$$I = \int_0^1 \frac{dt}{\sqrt{4-t^2}} = \left[\sin^{-1}\left(\frac{t}{2}\right)\right]_0^1.$$

The inverse sine is its principal value, and both limits lie within the domain of this antiderivative.

Step 3: Evaluate at the endpoints At the upper endpoint, $\sin^{-1}(1/2) = \pi/6$, while at the lower endpoint $\sin^{-1}(0) = 0$. Hence

$$I = \sin^{-1}(1/2) - \sin^{-1}(0) = \frac{\pi}{6} - 0.$$

The positive value agrees with the fact that the original integrand is positive throughout the interval.

Step 4: Conclusion The given definite integral equals $\boxed{\frac{\pi}{6}}$.

Quick Tip to Solve: Factor x^2 out of the square root first: the resulting dx/x directly suggests $t = \log x$.

29(a). Find: $\int \frac{\cos x}{\sin 3x}\,dx$. [3 marks]

Solution:

Concept:

- Use the triple-angle identity $\sin 3x = 3\sin x - 4\sin^3 x$.
- Set $t = \sin x$, so $dt = \cos x\,dx$, then decompose the rational expression into simpler terms.
- Logarithmic antiderivatives use absolute values: $\int du/u = \log|u| + C$.

Step 1: Convert to a rational integral Let $I = \int \cos x/\sin 3x\,dx$. The triple-angle identity gives

$$I = \int \frac{\cos x\,dx}{\sin x(3 - 4\sin^2 x)}.$$

Put $t = \sin x$, with $dt = \cos x\,dx$. Then

$$I = \int \frac{dt}{t(3 - 4t^2)}.$$

This calculation applies on any interval where the original denominator is nonzero.

Step 2: Decompose and integrate We have

$$\frac{1}{t(3-4t^2)} = \frac{1}{3t} + \frac{4t}{3(3-4t^2)}.$$

Indeed, putting these terms over the common denominator gives numerator $(3 - 4t^2 + 4t^2)/3 = 1$. Therefore

$$\begin{aligned} I &= \frac{1}{3}\int \frac{dt}{t} + \frac{4}{3}\int \frac{t\,dt}{3-4t^2} \\ &= \frac{1}{3}\log|t| - \frac{1}{6}\log|3-4t^2| + C. \end{aligned}$$

For the second term use $u = 3 - 4t^2$, so $du = -8t\,dt$.

Step 3: Return to the original variable and verify Replacing t by $\sin x$ gives $\frac{1}{3}\log|\sin x| - \frac{1}{6}\log|3 - 4\sin^2 x| + C$. Differentiating its nonconstant part yields

$$\frac{\cos x}{3\sin x} + \frac{4\sin x\cos x}{3(3-4\sin^2 x)} = \frac{\cos x}{\sin x(3-4\sin^2 x)} = \frac{\cos x}{\sin 3x},$$

confirming the result wherever the integrand is defined.

Step 4: Conclusion $\boxed{\int \frac{\cos x}{\sin 3x}\,dx = \frac{1}{3}\log|\sin x| - \frac{1}{6}\log|3 - 4\sin^2 x| + C}$.

Quick Tip to Solve: Expand $\sin 3x$ in powers of $\sin x$; the numerator $\cos x$ then makes substitution immediate.

OR

29(b). Find: $\int x^2 \log(x^2+1)\,dx$. [3 marks]

Solution:

Concept:

- Integration by parts: $\int u\,dv = uv - \int v\,du$. Choose the logarithm as u, since its derivative is rational.
- Polynomial division reduces $x^4/(x^2+1)$ to a polynomial plus a standard inverse-tangent integrand.
- The standard formula $\int dx/(1+x^2) = \tan^{-1} x + C$ completes the remaining integral.

Step 1: Apply integration by parts Let $I = \int x^2 \log(x^2+1)\,dx$. Choose

$$u = \log(x^2+1), \qquad dv = x^2\,dx, \qquad du = \frac{2x}{x^2+1}\,dx, \qquad v = \frac{x^3}{3}.$$

The logarithm is defined for every real x, because $x^2+1 > 0$. Integration by parts gives

$$I = \frac{x^3}{3}\log(x^2+1) - \frac{2}{3}\int \frac{x^4}{x^2+1}\,dx.$$

Step 2: Divide and integrate the rational term The identity $x^4 = (x^2+1)(x^2-1)+1$ gives

$$\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}.$$

Consequently,

$$\int \frac{x^4}{x^2+1}\,dx = \int \left(x^2 - 1 + \frac{1}{x^2+1}\right)dx = \frac{x^3}{3} - x + \tan^{-1} x.$$

A separate integration constant here can be absorbed into the final constant.

Step 3: Substitute back and check the cancellation Combining the terms gives

$$I = \frac{x^3}{3}\log(x^2+1) - \frac{2x^3}{9} + \frac{2x}{3} - \frac{2}{3}\tan^{-1} x + C.$$

Differentiation of the first term gives $x^2\log(x^2+1) + 2x^4/[3(x^2+1)]$. The derivative of the other three terms is $-2x^2/3 + 2/3 - 2/[3(x^2+1)]$, which cancels that extra rational term by the division identity above. Thus the derivative is exactly the original integrand.

Step 4: Conclusion $\boxed{I = \frac{x^3}{3}\log(x^2+1) - \frac{2x^3}{9} + \frac{2x}{3} - \frac{2}{3}\tan^{-1} x + C}$.

Quick Tip to Solve: After integration by parts, divide x^4 by x^2+1 before integrating; this exposes the inverse-tangent term.

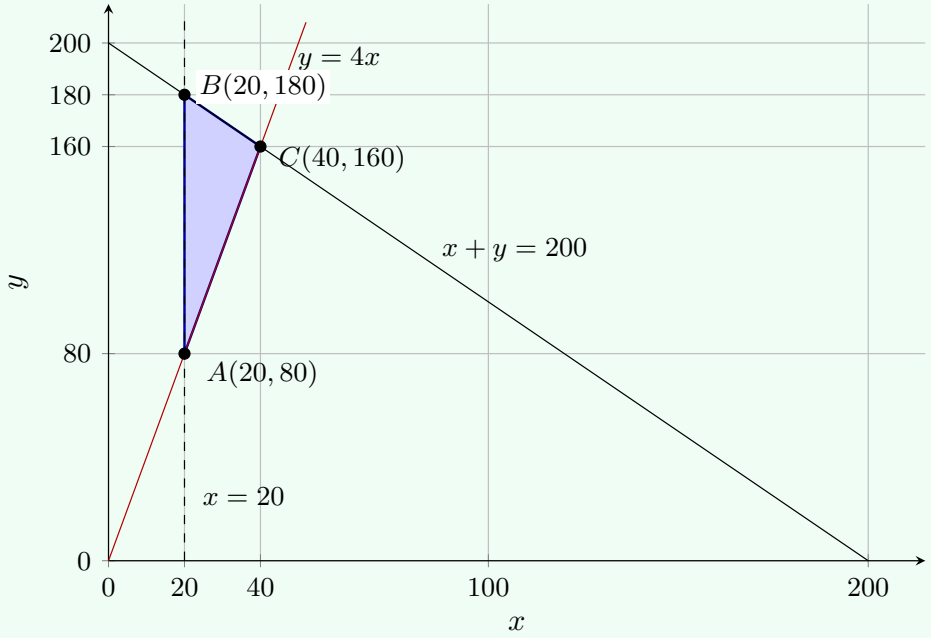
30. Determine graphically the minimum value of the following objective function: $z = 500x + 400y$ subject to constraints $x + y \leq 200,\ x \geq 20,\ y \geq 4x,\ y \geq 0$. [3 marks]

Solution:

Concept:

- The feasible region consists of points satisfying all four inequalities simultaneously. The boundary lines are obtained by replacing the inequality signs by equality signs.
- For a linear objective function on a nonempty bounded polygon, the minimum is attained at a vertex. Here the objective is $z = 500x + 400y$.

Step 1: Draw the boundary lines and shade the common region Plot $x + y = 200$ through $(0, 200)$ and $(200, 0)$, the vertical line $x = 20$, and $y = 4x$ through $(0, 0)$ and $(40, 160)$. Retain the region below $x + y = 200$, to the right of $x = 20$, and above $y = 4x$. Since $x \geq 20$ and $y \geq 4x$, every retained point already has $y \geq 80$, so the condition $y \geq 0$ is satisfied. The common region is the shaded closed triangle shown below.



Feasible triangle ABC for the given constraints

Step 2: Find all vertices of the feasible triangle At the intersection of $x = 20$ and $y = 4x$, we get $A = (20, 80)$. The intersection of $x = 20$ and $x + y = 200$ gives $B = (20, 180)$. For the remaining intersection, substitute $y = 4x$ in $x + y = 200$:

$$5x = 200 \quad \Rightarrow \quad x = 40, \qquad y = 160.$$

Thus $C = (40, 160)$. Each vertex satisfies all the given inequalities, and the shaded triangle includes its boundary.

Step 3: Evaluate the objective function at the vertices Compare the values at the three feasible

Quick Tip to Solve: First use $x \geq 20$ and $y \geq 4x$ to locate the lowest feasible vertex, then verify it by comparing all corner values.

vertices:

$$z(A) = 500(20) + 400(80) = 42000,$$
$$z(B) = 500(20) + 400(180) = 82000,$$
$$z(C) = 500(40) + 400(160) = 84000.$$

The feasible region is nonempty and bounded, so the vertex comparison applies. The smallest of these values is attained at A.

Step 4: Conclusion The minimum value is $\boxed{z_{\min} = 42000}$, attained at $\boxed{(x, y) = (20, 80)}$.

Quick Tip to Solve: First use $x \geq 20$ and $y \geq 4x$ to locate the lowest feasible vertex, then verify it by comparing all corner values.

31(a). A pair of dice is thrown simultaneously. If X denotes the absolute difference of numbers obtained on the pair of dice, then find the probability distribution of X. [3 marks]

Solution:

Concept:

- For two fair dice, the ordered outcomes (a, b), where $a, b \in \{1, 2, 3, 4, 5, 6\}$, form a sample space of $6 \times 6 = 36$ equally likely outcomes.
- The random variable is $X = |a - b|$. Thus its possible values are $0, 1, 2, 3, 4, 5$, and each probability equals the number of outcomes giving that difference divided by 36.

Step 1: Count the outcomes with zero difference The difference is zero precisely when the two dice show the same number. These outcomes are $(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)$. Hence there are six outcomes and

$$P(X = 0) = \frac{6}{36} = \frac{1}{6}.$$

These diagonal outcomes are counted once each because interchanging equal entries does not produce a new ordered outcome.

Step 2: Count the outcomes for each positive difference For a difference k, where $1 \leq k \leq 5$, the outcomes with the second die larger are $(1, 1 + k), (2, 2 + k), \ldots, (6 - k, 6)$. There are $6 - k$ such outcomes. Their reversed pairs give another $6 - k$ distinct outcomes with the first die larger. Therefore

$$P(X = k) = \frac{2(6 - k)}{36}.$$

For $k = 1, 2, 3, 4, 5$, the corresponding counts are $10, 8, 6, 4, 2$. For example, difference 5 occurs only in $(1, 6)$ and $(6, 1)$.

Step 3: Tabulate and check the distribution The resulting probability distribution is

x	0	1	2	3	4	5
$P(X = x)$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

Quick Tip to Solve: For a positive absolute difference, count pairs in one order and double; for zero difference, count the six equal pairs only once.

All entries are nonnegative, and their sum is

$$\frac{6+10+8+6+4+2}{36} = 1.$$

The outcomes are thereby exhausted, so no other value of X has positive probability.

Step 4: Conclusion The distribution is $\boxed{P(X=0)=\frac{1}{6}, \quad P(X=k)=\frac{6-k}{18}\ (k=1,2,3,4,5)}$, as tabulated above.

Quick Tip to Solve: For a positive absolute difference, count pairs in one order and double; for zero difference, count the six equal pairs only once.

OR

31(b). There are two coins. One of them is a biased coin such that P(head) : P(tail) is $1:3$ and the other coin is a fair coin. A coin is selected at random and tossed once. If the coin showed head, then find the probability that it is a biased coin. [3 marks]

Solution:

Concept:

- Let B mean that the biased coin is selected, F that the fair coin is selected, and H that the toss shows a head. Random selection gives $P(B) = P(F) = \frac{1}{2}$.
- Bayes theorem reverses the conditioning: $P(B \mid H) = \dfrac{P(B)P(H \mid B)}{P(B)P(H \mid B) + P(F)P(H \mid F)}$. The denominator is the total probability of observing a head.

Step 1: Find the head probabilities for the two coins For the biased coin, the given ratio of head probability to tail probability is $1:3$. Write these probabilities as t and $3t$. Since a toss has only the two possible outcomes head and tail,

$$t + 3t = 1 \quad \Rightarrow \quad t = \frac{1}{4}.$$

Hence $P(H \mid B) = \frac{1}{4}$. For the fair coin, heads and tails are equally likely, so $P(H \mid F) = \frac{1}{2}$. The ratio $1:3$ must be normalised; the probability of a head is not $1/3$.

Step 2: Find the probability of the observed head The events B and F are mutually exclusive and together cover all selections. By the law of total probability,

$$\begin{aligned} P(H) &= P(B)P(H \mid B) + P(F)P(H \mid F) \\ &= \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2} \\ &= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}. \end{aligned}$$

The joint event that the biased coin was selected and a head appeared has probability $P(B \cap H) = \frac{1}{2}\cdot\frac{1}{4} = \frac{1}{8}$.

Quick Tip to Solve: With equally likely coin selections, normalise the head likelihoods $1/4$ and $1/2$: the biased share is $(1/4)/(1/4+1/2)$.

Step 3: Condition on the head using Bayes theorem The toss is known to have shown a head, so the relevant sample of possibilities is restricted to the event H. Thus

$$P(B \mid H) = \frac{P(B \cap H)}{P(H)} = \frac{\frac{1}{8}}{\frac{3}{8}} = \frac{1}{3}.$$

This posterior probability is smaller than the original selection probability $1/2$, as expected: a head is twice as likely from the fair coin as from the biased coin.

Step 4: Conclusion The probability that the selected coin was biased, given that a head appeared, is $\boxed{\frac{1}{3}}$.

Quick Tip to Solve: With equally likely coin selections, normalise the head likelihoods $1/4$ and $1/2$: the biased share is $(1/4)/(1/4 + 1/2)$.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32. Show that a function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \frac{5x - 3}{4}$ is both one-one and onto. [5 marks]

Solution:

Concept:

- A function is one-one if equal images imply equal inputs: for all a, b in its domain, $f(a) = f(b)$ implies $a = b$.
- A function is onto if every element of its codomain has a preimage in its domain. For each $y \in \mathbb{R}$, we must produce $x \in \mathbb{R}$ satisfying $f(x) = y$.
- A mapping which is both one-one and onto is bijective; every output then corresponds to exactly one input.

Step 1: Check the domain and function rule The function has domain $\mathbb{R}$ and codomain $\mathbb{R}$, with

$$f(x) = \frac{5x - 3}{4}.$$

For every real input the numerator is real, and the denominator is the fixed nonzero number 4. Thus the rule is defined and real-valued throughout the given domain. We establish injectivity and surjectivity separately: the first is a statement about uniqueness of inputs, while the second is a statement about the existence of an input for each possible output.

Step 2: Establish that the function is one-one Take arbitrary $a, b \in \mathbb{R}$, and assume $f(a) = f(b)$. Using

Quick Tip to Solve: For this affine function, equate two images to prove injectivity, then solve for x and substitute back to prove surjectivity.

the given formula gives

$$\begin{aligned}\frac{5a-3}{4} &= \frac{5b-3}{4},\\ 5a-3 &= 5b-3,\\ 5a &= 5b,\\ a &= b.\end{aligned}$$

Multiplication by 4 and division by 5 are valid because both numbers are nonzero. Thus equality of images forces equality of their original inputs. Since a and b were arbitrary, the implication holds for all elements of the domain and proves that f is one-one.

Step 3: Establish that the function is onto Let $y \in \mathbb{R}$ be any element of the codomain. To find a preimage, solve the equation $f(x) = y$:

$$\begin{aligned}\frac{5x-3}{4} &= y,\\ 5x-3 &= 4y,\\ x &= \frac{4y+3}{5}.\end{aligned}$$

This candidate is real for every real y, so it lies in the stated domain. We verify its image directly, rather than merely asserting the existence of the inverse expression:

$$f\left(\frac{4y+3}{5}\right) = \frac{5\left(\frac{4y+3}{5}\right)-3}{4} = \frac{4y+3-3}{4} = y.$$

Therefore every chosen codomain element has a preimage. There is no restriction on y; negative, zero and positive real values are all covered. This proves that the range equals the full codomain $\mathbb{R}$, and hence that f is onto.

Step 4: Conclusion The function $f(x) = (5x-3)/4$ is both one-one and onto, so it is bijective. Its inverse is $f^{-1}(y) = (4y+3)/5$.

Quick Tip to Solve: For this affine function, equate two images to prove injectivity, then solve for x and substitute back to prove surjectivity.

33. The area of the region bounded by the line $y = mx$ $(m > 0)$, the curve $x^2 + y^2 = 4$ and the x-axis in the first quadrant is $\frac{\pi}{2}$ units. Using integration, find the value of m. [5 marks]

Solution:

Concept:

- In the first quadrant the circle $x^2 + y^2 = 4$ has upper boundary $y = \sqrt{4-x^2}$, with $0 \le x \le 2$.
- Area above the x-axis is obtained by integrating the upper boundary. Where that boundary changes from a straight line to a circular arc, split the integral at the intersection.
- The standard antiderivative is $\int \sqrt{4-x^2}\,dx = \frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}(x/2) + C$.

Quick Tip to Solve: Split the area at the line-circle intersection; use the intersection equation to cancel the two algebraic terms before solving the inverse-sine equation.

Step 1: Locate the intersection and identify the boundaries Write the first-quadrant intersection of $y = mx$ and the circle as $P = (a, b)$. Since $m > 0$, both coordinates of this intersection are positive. Substituting $b = ma$ in the circle equation gives

$$a^2 + m^2a^2 = 4, \qquad a = \frac{2}{\sqrt{1+m^2}}, \qquad b = \frac{2m}{\sqrt{1+m^2}}.$$

In particular, $0 < a < 2$ and $b = \sqrt{4-a^2}$. The region bounded by all three given boundaries runs along the x-axis from the origin to $(2, 0)$, along the circular arc to P, and along the line back to the origin. Thus its upper boundary is mx between 0 and a, and $\sqrt{4-x^2}$ between a and 2.

Step 2: Calculate the area by integration Let the required area be A. The lower boundary is the x-axis throughout, hence

$$A = \int_0^a mx\,dx + \int_a^2 \sqrt{4-x^2}\,dx.$$

Evaluating both integrals using the stated antiderivative,

$$\begin{aligned} A &= \frac{ma^2}{2} + \left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_a^2 \\ &= \frac{ma^2}{2} + \pi - \frac{a}{2}\sqrt{4-a^2} - 2\sin^{-1}\left(\frac{a}{2}\right). \end{aligned}$$

At the intersection, $\sqrt{4-a^2} = ma$. Therefore the triangular term from the line integral cancels the algebraic term from the circle integral:

$$A = \pi - 2\sin^{-1}\left(\frac{a}{2}\right).$$

The inverse sine here is its principal value. Since $0 < a/2 < 1$, this angle lies strictly between 0 and $\pi/2$, as required by the first-quadrant geometry.

Step 3: Use the given area to determine the slope The printed area is $\pi/2$. Substitution into the integrated expression gives

$$\begin{aligned} \pi - 2\sin^{-1}(a/2) &= \frac{\pi}{2}, \\ \sin^{-1}(a/2) &= \frac{\pi}{4}, \\ a &= 2\sin\left(\frac{\pi}{4}\right) = \sqrt{2}. \end{aligned}$$

Now use $a^2(1+m^2) = 4$, the intersection equation already established:

$$2(1+m^2) = 4, \qquad m^2 = 1.$$

The given condition $m > 0$ selects the positive root. As a check, $a = b = \sqrt{2}$, and the area formula gives $\pi - 2(\pi/4) = \pi/2$, agreeing with the required value.

Step 4: Conclusion The required value of the positive slope is $\boxed{m = 1}$.

Quick Tip to Solve: Split the area at the line-circle intersection; use the intersection equation to cancel the two algebraic terms before solving the inverse-sine equation.

34(a). **If** $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$, **then show that** $A^3 - 6A^2 + 7A + 2I = O$. [5 marks]

Solution:

Concept:

- For matrix multiplication, the entry in row i, column j of AB is the dot product of row i of A and column j of B.
- Matrix powers mean repeated matrix products: $A^2 = AA$ and $A^3 = A^2A$. They are not obtained by squaring or cubing individual entries.
- The identity matrix I has diagonal entries equal to one and all other entries zero. To prove a matrix identity, verify that every entry of the difference is zero.

Step 1: Calculate the square of the matrix Multiplying the given matrix by itself, we obtain

$$A^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix}.$$

For example, the first row gives $1(1)+0(0)+2(2) = 5$, $1(0)+0(2)+2(0) = 0$, and $1(2)+0(1)+2(3) = 8$. In the second row, the first entry is $0(1) + 2(0) + 1(2) = 2$. Thus even the zero in the original matrix can give a nonzero entry in its square.

Step 2: Calculate the cube using the square Now multiply A^2 by A, preserving the order of rows and columns:

$$A^3 = \begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{pmatrix}.$$

The middle row is obtained from $2(1)+4(0)+5(2) = 12$, $2(0)+4(2)+5(0) = 8$, and $2(2)+4(1)+5(3) = 23$. The last diagonal entry is $8(2) + 0(1) + 13(3) = 55$.

Step 3: Substitute the powers and collect entries First subtract six times the square:

$$A^3 - 6A^2 = \begin{pmatrix} -9 & 0 & -14 \\ 0 & -16 & -7 \\ -14 & 0 & -23 \end{pmatrix}.$$

Adding $7A$ gives

$$A^3 - 6A^2 + 7A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Since $2I$ adds two only to each diagonal entry, the resulting matrix has zero in every position. In particular, the off-diagonal entries cancel before the identity term is added, while each diagonal entry becomes $-2 + 2 = 0$.

Step 4: Conclusion Hence $\boxed{A^3 - 6A^2 + 7A + 2I = O}$, where O is the 3×3 zero matrix.

Quick Tip to Solve: Calculate the second column first: its sparse form makes it easy to check the diagonal cancellation before working on the other columns.

OR

34(b). If $A = \begin{pmatrix} 3 & 2 \\ 5 & -7 \end{pmatrix}$, then find A^{-1} and use it to solve the following system of equations: $3x + 5y = 11,\ 2x - 7y = -3$. [5 marks]

Solution:

Concept:

- A square matrix is invertible when its determinant is nonzero. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- A system may be written using a column vector or a row vector. The coefficients must match the chosen multiplication exactly; matrix multiplication cannot be reversed arbitrarily.
- If $XA = B$, multiply on the right by A^{-1} to obtain $X = BA^{-1}$, because $AA^{-1} = I$.

Step 1: Find the determinant and establish invertibility For the printed matrix,

$$A = \begin{pmatrix} 3 & 2 \\ 5 & -7 \end{pmatrix}, \qquad |A| = 3(-7) - 2(5) = -21 - 10 = -31.$$

As $|A| \neq 0$, the inverse exists. This also ensures that the system represented using this matrix has a unique solution. The negative determinant must be retained throughout the inverse calculation.

Step 2: Calculate the inverse Interchanging the diagonal entries and changing the signs of the off-diagonal entries gives

$$\text{adj } A = \begin{pmatrix} -7 & -2 \\ -5 & 3 \end{pmatrix}.$$

Therefore,

$$A^{-1} = \frac{1}{-31}\begin{pmatrix} -7 & -2 \\ -5 & 3 \end{pmatrix} = \frac{1}{31}\begin{pmatrix} 7 & 2 \\ 5 & -3 \end{pmatrix}.$$

A direct check is

$$AA^{-1} = \frac{1}{31}\begin{pmatrix} 31 & 0 \\ 0 & 31 \end{pmatrix} = I.$$

Thus the signs and placement of all four entries agree with the inverse formula.

Step 3: Express the equations using the given matrix The equations are $3x+5y = 11$ and $2x-7y = -3$. The coefficients in each equation are the columns of A, so use the row vector $X = \begin{pmatrix} x & y \end{pmatrix}$:

$$\begin{pmatrix} x & y \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 5 & -7 \end{pmatrix} = \begin{pmatrix} 11 & -3 \end{pmatrix}.$$

Quick Tip to Solve: Compare the equation coefficients with the columns of the given matrix: here a row-vector equation lets you use its inverse directly.

Multiplying this equation on the right by the inverse found above yields

$$\begin{pmatrix} x & y \end{pmatrix} = \frac{1}{31}\begin{pmatrix} 11 & -3 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 5 & -3 \end{pmatrix}$$
$$= \frac{1}{31}\begin{pmatrix} 77-15 & 22+9 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix}.$$

Step 4: Verify in both original equations Substitution gives $3(2)+5(1) = 6+5 = 11$ and $2(2)-7(1) = 4-7 = -3$. Both printed equations are satisfied, confirming that the row-vector representation used the coefficients in the correct order.

Step 5: Conclusion $\boxed{A^{-1} = \frac{1}{31}\begin{pmatrix} 7 & 2 \\ 5 & -3 \end{pmatrix}}$, and the unique solution is $\boxed{x = 2,\ y = 1}$.

Quick Tip to Solve: Compare the equation coefficients with the columns of the given matrix: here a row-vector equation lets you use its inverse directly.

35(a). Find the value of b so that the lines $\frac{x-1}{2} = \frac{y-b}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ are intersecting lines. Also, find the point of intersection of these given lines. [5 marks]

Solution:

Concept:

- Two lines intersect when their parametric forms give the same three coordinates for some values of their independent parameters.
- The symmetric equation $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n} = t$ gives the point (x_0+lt, y_0+mt, z_0+nt), where (l, m, n) is a direction vector.

Step 1: Write the lines in parametric form Let the common ratio for the first line be t, and that for the second line be s. These parameters are initially independent. Then

$$L_1: \quad x = 1+2t, \quad y = b+3t, \quad z = 3+4t,$$
$$L_2: \quad x = 4+5s, \quad y = 1+2s, \quad z = s.$$

The last expression in the printed equation of the second line is z, hence $z = s$, with no additive constant. For intersection, all three corresponding coordinates must agree:

$$1+2t = 4+5s, \qquad b+3t = 1+2s, \qquad 3+4t = s.$$

Step 2: Solve the equations independent of b Use the first and third coordinate equations to determine

Quick Tip to Solve: Equate the x and z coordinates first because neither contains b; then use the y coordinates to find b.

the parameters. Substitution of $s = 3 + 4t$ yields

$$\begin{aligned} 1 + 2t &= 4 + 5(3 + 4t), \\ 1 + 2t &= 19 + 20t, \\ -18t &= 18, \\ t &= -1. \end{aligned}$$

It follows that $s = 3 + 4(-1) = -1$. The nonzero coefficient of t means that this parameter pair is unique. No other pair can satisfy the first and third coordinate equations simultaneously.

Step 3: Determine b and verify the intersection The second-coordinate equation gives

$$b + 3(-1) = 1 + 2(-1), \qquad b - 3 = -1, \qquad b = 2.$$

For this value, the point on the first line at $t = -1$ is

$$(1 - 2, 2 - 3, 3 - 4) = (-1, -1, -1).$$

The point on the second line at $s = -1$ is also $(4 - 5, 1 - 2, -1) = (-1, -1, -1)$. To check against the symmetric equations directly, their respective ratios are

$$\frac{-1 - 1}{2} = \frac{-1 - 2}{3} = \frac{-1 - 3}{4} = -1,$$

and $\dfrac{-1 - 4}{5} = \dfrac{-1 - 1}{2} = -1 = z$. Thus the point satisfies both printed lines. Their direction vectors $(2, 3, 4)$ and $(5, 2, 1)$ are not proportional, so the distinct lines intersect at this unique point.

Step 4: Conclusion The required value is $\boxed{b = 2}$, and the point of intersection is $\boxed{(-1, -1, -1)}$.

Quick Tip to Solve: Equate the x and z coordinates first because neither contains b; then use the y coordinates to find b.

OR

35(b). Find the equations of all the sides of the parallelogram $ABCD$ whose vertices are $A(4, 7, 8)$, $B(2, 3, 4)$, $C(-1, -2, 1)$ and $D(1, 2, 5)$. Also, find the coordinates of the foot of the perpendicular from A to CD. [5 marks]

Solution:

Concept:

- The line through (x_0, y_0, z_0) with nonzero direction components (l, m, n) has equation $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$. Subtract endpoint coordinates to obtain a side's direction vector.
- If H is the foot of the perpendicular from A to a line with direction vector $\mathbf{d}$, then H lies on the line and $\overrightarrow{HA} \cdot \mathbf{d} = 0$. A scalar multiple of a direction vector defines the same line.

Quick Tip to Solve: Simplify the direction of CD to (1,2,2), put H on CD using one parameter, and set the dot product of HA with that direction to zero.

Step 1: Compute the directions of the four sides Subtract successive vertex coordinates:

$$\overrightarrow{AB} = (2-4, 3-7, 4-8) = (-2, -4, -4),$$
$$\overrightarrow{BC} = (-1-2, -2-3, 1-4) = (-3, -5, -3),$$
$$\overrightarrow{CD} = (1+1, 2+2, 5-1) = (2, 4, 4),$$
$$\overrightarrow{DA} = (4-1, 7-2, 8-5) = (3, 5, 3).$$

Thus AB and CD both have direction $(1, 2, 2)$, while BC and DA both have direction $(3, 5, 3)$. The opposite directed vectors are negatives of each other, consistent with the given parallelogram.

Step 2: Write the equations of all four sides Using a vertex on each side and its simplified direction vector, we obtain

$$AB: \quad \frac{x-4}{1} = \frac{y-7}{2} = \frac{z-8}{2},$$
$$BC: \quad \frac{x-2}{3} = \frac{y-3}{5} = \frac{z-4}{3},$$
$$CD: \quad \frac{x+1}{1} = \frac{y+2}{2} = \frac{z-1}{2},$$
$$DA: \quad \frac{x-1}{3} = \frac{y-2}{5} = \frac{z-5}{3}.$$

For example, taking the common ratio as -1 in the equation of BC produces C. Reversing a side's direction vector does not change its line. These are the supporting-line equations of the four sides; each side segment is bounded by its two given endpoints.

Step 3: Apply the perpendicularity condition on CD A general point H on CD is

$$H = (-1+t, -2+2t, 1+2t), \qquad t \in \mathbb{R}.$$

Since $A = (4, 7, 8)$, the vector from H to A is $(5-t, 9-2t, 7-2t)$. Its dot product with the direction $(1, 2, 2)$ must vanish:

$$(5-t) + 2(9-2t) + 2(7-2t) = 0,$$
$$37 - 9t = 0,$$
$$t = \frac{37}{9}.$$

Substitution gives

$$H = \left(\frac{28}{9}, \frac{56}{9}, \frac{83}{9}\right).$$

To verify the perpendicular, $\overrightarrow{HA} = \left(\frac{8}{9}, \frac{7}{9}, -\frac{11}{9}\right)$, whose dot product with $(1, 2, 2)$ is $(8 + 14 - 22)/9 = 0$. H also lies on CD by its parametric construction. Here the perpendicular meets the extension of the side CD, which is permitted when finding the foot on the line CD.

Quick Tip to Solve: Simplify the direction of CD to (1,2,2), put H on CD using one parameter, and set the dot product of HA with that direction to zero.

Step 4: Conclusion The four side equations are

$$AB : x-4 = \frac{y-7}{2} = \frac{z-8}{2},$$
$$BC : \frac{x-2}{3} = \frac{y-3}{5} = \frac{z-4}{3},$$
$$CD : x+1 = \frac{y+2}{2} = \frac{z-1}{2},$$
$$DA : \frac{x-1}{3} = \frac{y-2}{5} = \frac{z-5}{3}.$$

The foot of the perpendicular is $\boxed{\left(\frac{28}{9}, \frac{56}{9}, \frac{83}{9}\right)}$.

Quick Tip to Solve: Simplify the direction of CD to (1,2,2), put H on CD using one parameter, and set the dot product of HA with that direction to zero.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. An octagonal prism is a three-dimensional polyhedron bounded by two octagonal bases and eight rectangular side faces. It has 24 edges and 16 vertices.

The prism is rolled along the rectangular faces and number on the bottom face (touching the ground) is noted. Let X denote the number obtained on the bottom face and the following table gives the probability distribution of X.

X :	1	2	3	4	5	6	7	8
$P(X)$:	p	$2p$	$2p$	p	$2p$	p^2	$2p^2$	$7p^2+p$

Based on the above information, answer the following questions: [4 marks]

36(i). Find the value of p. [1 mark]

Solution:

Concept:

- For a probability distribution, all probabilities are non-negative and their sum is one.
- Disjoint outcomes are added: $P(X \in A) = \sum_{x \in A} P(X = x)$.

Step 1: Normalize the table The eight outcomes exhaust all possibilities. Therefore

$$p + 2p + 2p + p + 2p + p^2 + 2p^2 + (7p^2 + p) = 1.$$

Combining linear and quadratic terms gives $10p^2 + 9p - 1 = 0$.

Step 2: Select the admissible root Factorization gives $(10p-1)(p+1) = 0$, whose roots are $1/10$ and -1. A probability cannot be negative, and $P(X = 1) = p$, so the negative root is rejected. At the positive root all eight entries are non-negative and their total is exactly one.

Step 3: Conclusion $\boxed{p = \frac{1}{10}}$.

Quick Tip to Solve: Collect the linear terms as 9p and quadratic terms as 10p squared before factoring.

36(ii). Find $P(X > 6)$. [1 mark]

Solution:

Concept:

- For a probability distribution, all probabilities are non-negative and their sum is one.
- Disjoint outcomes are added: $P(X \in A) = \sum_{x \in A} P(X = x)$.

Step 1: Identify the event Adding all entries of the given table gives $9p+10p^2 = 1$, so $(10p-1)(p+1) = 0$. The root $p = -1$ would make $P(X = 1)$ negative. Thus the admissible value is $p = 1/10$, which makes every table entry non-negative. The strict inequality $X > 6$ includes only the outcomes 7 and 8; the outcome 6 must be excluded.

Step 2: Add the relevant entries These two outcomes are mutually exclusive, so

$$P(X > 6) = 2p^2 + (7p^2 + p) = 9p^2 + p.$$

Substituting $p = 1/10$ gives $9/100 + 1/10 = 19/100$. The event does not include 5 or 6, and treating the eight faces as equally probable would contradict the distribution supplied.

Step 3: Conclusion $\boxed{P(X > 6) = \dfrac{19}{100}}$.

Quick Tip to Solve: For a strict inequality, list the allowed outcomes before reading probabilities from the table.

36(iii)(a). Find $P(X = 3m)$, where m is a natural number. [2 marks]

Solution:

Concept:

- For a probability distribution, all probabilities are non-negative and their sum is one.
- Disjoint outcomes are added: $P(X \in A) = \sum_{x \in A} P(X = x)$.

Step 1: Find the admissible distribution Adding all entries of the given table gives $9p + 10p^2 = 1$, so $(10p - 1)(p + 1) = 0$. The root $p = -1$ would make $P(X = 1)$ negative. Thus the admissible value is $p = 1/10$, which makes every table entry non-negative. In particular, the table gives $P(X = 3) = 2p$ and $P(X = 6) = p^2$. These probabilities refer to different outcomes and must not be multiplied.

Step 2: List the multiples of three The support of X is $\{1, 2, 3, 4, 5, 6, 7, 8\}$. The values of the form $3m$, for natural m, that lie in this support are 3 and 6 (corresponding to $m = 1, 2$). All larger multiples lie outside the support and have probability zero. Hence the requested event is $\{X = 3\} \cup \{X = 6\}$.

Step 3: Add and substitute Since the outcomes are mutually exclusive,

$$P(X = 3m) = P(X = 3) + P(X = 6) = 2p + p^2 = \frac{2}{10} + \frac{1}{100} = \frac{21}{100}.$$

There is no contribution from 0 even under a convention that includes zero among natural numbers, because 0 is not a possible value of X.

Quick Tip to Solve: Intersect the positive multiples of three with the finite support; only 3 and 6 contribute.

Step 4: Conclusion $\boxed{P(X = 3m) = \frac{21}{100}}$.

Quick Tip to Solve: Intersect the positive multiples of three with the finite support; only 3 and 6 contribute.

OR

36(iii)(b). Find the mean $E(X)$. [2 marks]

Solution:

Concept:

- For a probability distribution, all probabilities are non-negative and their sum is one.
- The expectation of a discrete random variable is $E(X) = \sum_x xP(X = x)$, a probability-weighted mean.

Step 1: Determine the probabilities Adding all entries of the given table gives $9p + 10p^2 = 1$, so $(10p - 1)(p + 1) = 0$. The root $p = -1$ would make $P(X = 1)$ negative. Thus the admissible value is $p = 1/10$, which makes every table entry non-negative. The probabilities in increasing outcome order are $0.10, 0.20, 0.20, 0.10, 0.20, 0.01, 0.02, 0.17$; they sum to one.

Step 2: Form the weighted sum Multiply each possible value by its own probability, then add:

$$\begin{aligned} E(X) &= 1p + 2(2p) + 3(2p) + 4p + 5(2p) \\ &\quad + 6p^2 + 7(2p^2) + 8(7p^2 + p) \\ &= 33p + 76p^2. \end{aligned}$$

The last outcome contributes both $56p^2$ and $8p$; neither term can be omitted.

Step 3: Evaluate the mean With $p = 1/10$, the weighted sum is $33/10 + 76/100 = 330/100 + 76/100 = 406/100 = 203/50$. This lies between the smallest and largest possible values, 1 and 8, as an expectation must. The ordinary average of the eight labels is inappropriate because their probabilities are unequal.

Step 4: Conclusion $\boxed{E(X) = \frac{203}{50} = 4.06}$.

Quick Tip to Solve: Weight each label by its table entry; expand the outcome-8 contribution before collecting terms.

37. In order to set up a rain water harvesting system, a tank to collect rain water is to be dug. The tank should have a square base and a capacity of 250 m^3. The cost of land is ₹$5{,}000$ per square metre and cost of digging increases with depth and for the whole tank, it is ₹$40{,}000\,h^2$, where h is the depth of the tank in metres. x is the side of the square base of the tank in metres.

Based on the above information, answer the following questions: [4 marks]

37(i). Find the total cost C of digging the tank in terms of x. [1 mark]

Solution:

Concept:

- The volume of a tank with square base side x and depth h is x^2h. Here $x > 0$ and $x^2h = 250$.
- Total cost is land cost plus digging cost: $C = 5000x^2 + 40000h^2$, in rupees.

Step 1: Eliminate the depth The fixed capacity gives $x^2h = 250$, whence $h = 250/x^2$. The square base has area x^2, so its land cost is $5000x^2$ rupees. The stated digging cost applies to the whole tank and is already $40000h^2$.

Step 2: Express the full cost Add these two costs and substitute the depth:

$$C(x) = 5000x^2 + 40000\left(\frac{250}{x^2}\right)^2 = 5000x^2 + \frac{2500000000}{x^4}.$$

Squaring the depth squares both 250 and x^{-2}; thus the denominator is x^4. The domain is $x > 0$, since a side length is positive.

Step 3: Conclusion $\boxed{C(x) = 5000x^2 + \frac{2500000000}{x^4}}$ rupees, for $x > 0$.

Quick Tip to Solve: Use the volume condition to replace h before forming a cost function of x alone.

37(ii). Find $\frac{dC}{dx}$. [1 mark]

Solution:

Concept:

- The volume of a tank with square base side x and depth h is x^2h. Here $x > 0$ and $x^2h = 250$.
- Total cost is land cost plus digging cost: $C = 5000x^2 + 40000h^2$, in rupees.
- The power rule gives $\frac{d}{dx}x^n = nx^{n-1}$, for the powers used here and $x > 0$.

Step 1: Write cost using powers From $h = 250/x^2$, the total cost is $C(x) = 5000x^2 + 2500000000x^{-4}$. Writing the reciprocal term as a negative power permits direct use of the power rule. The depth is not constant as the side changes, because the volume remains fixed.

Step 2: Differentiate each term The derivative of the land term is $5000(2)x = 10000x$. The derivative of the digging term is $2500000000(-4)x^{-5}$. Therefore

$$\frac{dC}{dx} = 10000x - 10000000000x^{-5}.$$

Quick Tip to Solve: Write the digging term as a constant times x to the power minus four before differentiating.

The negative sign in the second term reflects the decreasing depth when the base is enlarged.

Step 3: Conclusion $\boxed{\dfrac{dC}{dx} = 10000x - \dfrac{10000000000}{x^5}}$, for $x > 0$.

Quick Tip to Solve: Write the digging term as a constant times x to the power minus four before differentiating.

37(iii)(a). Find the value of x for which cost C is minimum. [2 marks]

Solution:

Concept:

- The volume of a tank with square base side x and depth h is x^2h. Here $x > 0$ and $x^2h = 250$.
- Total cost is land cost plus digging cost: $C = 5000x^2 + 40000h^2$, in rupees.
- A stationary point satisfies $C\prime(x) = 0$. A positive second derivative throughout the domain ensures the stationary point is the unique minimum.

Step 1: Obtain the cost derivative Using $h = 250/x^2$,

$$C(x) = 5000x^2 + 2500000000x^{-4}, \qquad x > 0.$$

Term-by-term differentiation gives $C\prime(x) = 10000x - 10000000000x^{-5}$. Both land and digging cost must be included in this derivative.

Step 2: Solve the stationary equation Set the derivative equal to zero and multiply by the positive quantity x^5:

$$10000x^6 = 10000000000, \qquad x^6 = 1000000 = 10^6.$$

Only the positive real root is physically admissible, since x is a side length.

Step 3: Verify that the cost is minimized Differentiating again gives

$$C\prime\prime(x) = 10000 + 50000000000x^{-6} > 0 \quad (x > 0).$$

Thus the derivative is strictly increasing and the stationary point is the unique minimum. Equivalently, $C\prime(x) = 10000(x^6 - 10^6)/x^5$ is negative below the stationary value and positive above it, so the cost decreases and then increases over its entire domain.

Step 4: Conclusion The total cost is minimum when $\boxed{x = 10\,\text{m}}$.

Quick Tip to Solve: After setting the derivative to zero, multiply by x to the fifth power and check the second derivative.

OR

37(iii)(b). Check whether the cost function $C(x)$ expressed in terms of x is increasing or not, where $x > 0$. [2 marks]

Solution:

Concept:

- The volume of the tank is $x^2h = 250$, so its depth is $h = 250/x^2$, with $x > 0$.
- A differentiable function increases where its derivative is positive and decreases where it is negative. Test the sign throughout the given domain.

Step 1: Express the total cost The land cost is $5000x^2$ rupees and the digging cost is $40000h^2$ rupees. Substituting the depth gives

$$C(x) = 5000x^2 + 40000\left(\frac{250}{x^2}\right)^2 = 5000x^2 + \frac{2500000000}{x^4}.$$

Both contributions are needed: a wider base costs more land but needs less depth, which reduces the digging cost.

Step 2: Test the derivative on the domain Differentiating,

$$C\prime(x) = 10000x - \frac{10000000000}{x^5} = \frac{10000(x^6 - 1000000)}{x^5}.$$

Since $x^5 > 0$, the numerator determines the sign. Now $1000000 = 10^6$. Therefore $C\prime(x) < 0$ for $0 < x < 10$, $C\prime(10) = 0$, and $C\prime(x) > 0$ for $x > 10$. The function first decreases and then increases, so it cannot increase over the entire positive domain.

Step 3: Conclusion $C(x)$ is not increasing throughout $x > 0$: it decreases on $(0, 10)$ and increases on $(10, \infty)$.

Quick Tip to Solve: Multiply the derivative by the positive factor $x^5/10000$, then compare x^6 with 10^6.

38. A volleyball player serves the ball which takes a parabolic path given by the equation $h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1$, where $h(t)$ is the height of ball at any time t (in seconds), $(t \geq 0)$.
Based on the above information, answer the following questions: [4 marks]

38(i). Is $h(t)$ a continuous function? Justify. [2 marks]

Solution:

Concept:

- A polynomial is continuous at every real argument: sums and constant multiples of continuous functions remain continuous.
- At an interior point a, continuity means $\lim_{t\to a} h(t) = h(a)$. At the endpoint $t = 0$, use the right-hand limit within the stated domain.

Step 1: Identify the form of the height function The function $h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1$ is a quadratic

Quick Tip to Solve: Identify a polynomial first; at the initial time, explicitly use the right-hand limit on $t \geq 0$.

polynomial. The functions t^2, t, and the constant function are continuous. Multiplying them by real constants and adding them preserves continuity. There is no denominator, root restriction, or piecewise junction that could introduce a discontinuity.

Step 2: Verify the continuity condition For any $a > 0$, the laws of limits give

$$\lim_{t \to a} h(t) = -\frac{7}{2}a^2 + \frac{13}{2}a + 1 = h(a).$$

At the boundary of the stated domain,

$$\lim_{t \to 0^+} h(t) = 1 = h(0).$$

Thus the limit agrees with the function value at every admissible time, including the initial instant.

Step 3: Conclusion Yes. $h(t)$ is continuous for every $t \geq 0$, since it is a polynomial restricted to this domain.

Quick Tip to Solve: Identify a polynomial first; at the initial time, explicitly use the right-hand limit on $t \geq 0$.

38(ii). Find the time at which the height of the ball is maximum. [2 marks]

Solution:

Concept:

- At an interior maximum of a differentiable height function, the first derivative vanishes. Here the variable t denotes time in seconds.
- If the second derivative is negative at a stationary point, the point gives a local maximum. A downward-opening quadratic has its unique global maximum at its vertex.

Step 1: Find the stationary time Differentiate the given height function with respect to time:

$$h\prime(t) = -7t + \frac{13}{2}.$$

Solving $h\prime(t) = 0$ gives

$$7t = \frac{13}{2}, \qquad t = \frac{13}{14}.$$

This time is positive, so it belongs to the stated domain $t \geq 0$; it is an admissible candidate for the maximum.

Step 2: Confirm that the stationary point is a maximum Differentiate again to obtain $h\prime\prime(t) = -7 < 0$. Hence the height function is concave down and the stationary point is a maximum. More explicitly, $h\prime(t) > 0$ for $0 \leq t < 13/14$, while $h\prime(t) < 0$ for $t > 13/14$. The ball rises until that instant and falls afterwards. This sign change establishes the unique maximum throughout the given time domain, rather than just a stationary value.

Step 3: Conclusion The height of the ball is maximum at $\boxed{t = \frac{13}{14}\,\text{s}}$, approximately $0.929\,\text{s}$.

Quick Tip to Solve: Set the linear derivative equal to zero, then check the negative second derivative before declaring a maximum.