

CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-3-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)



Time Allowed: 3 hours

Maximum Marks: 80

General Instructions

- (i) This question paper contains **38** questions. **All** questions are **compulsory**.
- (ii) This question paper is divided into **five** Sections – **A, B, C, D and E**.
- (iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19 and 20** are Assertion-Reason based questions of **1** mark each.
- (iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.
- (v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.
- (vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.
- (vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.
- (viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.
- (ix) Use of calculators is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. $\int 2^{x+2} dx$ is equal to:

[1 mark]

- (A) $2^{x+2} + C$
- (B) $2^{x+2} \log 2 + C$
- (C) $\frac{2^{x+2}}{\log 2} + C$
- (D) $2 \cdot \frac{2^x}{\log 2} + C$

Correct Answer: (C) $\frac{2^{x+2}}{\log 2} + C$

Solution:

Concept:

- For a positive constant $a \neq 1$, $\frac{d}{dx} a^x = a^x \log a$. Here $\log$ denotes the natural logarithm.
- An indefinite integral is an antiderivative plus an arbitrary constant C .

Step 1: Use the exponential integration rule Since the derivative of $x + 2$ is 1, the chain rule gives $\frac{d}{dx} 2^{x+2} = 2^{x+2} \log 2$. Therefore, dividing by $\log 2$ supplies the required antiderivative:

$$\int 2^{x+2} dx = \frac{2^{x+2}}{\log 2} + C.$$

Step 2: Check the alternatives Option (A) differentiates to $2^{x+2} \log 2$, while (B) differentiates to $2^{x+2} (\log 2)^2$. Option (D) differentiates to $2 \cdot 2^x = 2^{x+1}$, only half the required integrand. Option (C) differentiates exactly to 2^{x+2} .

Step 3: Conclusion The antiderivative is $\frac{2^{x+2}}{\log 2} + C$. Hence, the correct option is (C).

Quick Tip to Solve: For an exponential with constant base, divide by its natural logarithm and verify by differentiation.

2. Let A be a skew-symmetric matrix of order 3. If $|A| = x$, then $(2023)^x$ is equal to:

[1 mark]

- (A) 2023
- (B) $\frac{1}{2023}$
- (C) $(2023)^2$
- (D) 1

Correct Answer: (D) 1

Solution:

Concept:

- A skew-symmetric matrix satisfies $A^T = -A$.
- Determinants satisfy $|A^T| = |A|$ and $|-A| = (-1)^n |A|$ for a matrix of order n .

Step 1: Find the determinant As the order is 3,

$$|A| = |A^T| = |-A| = (-1)^3 |A| = -|A|.$$

Thus $2|A| = 0$, so the given exponent $x = |A|$ is 0. This conclusion depends on the odd order, not on the individual entries.

Step 2: Evaluate the exponential and compare The base 2023 is nonzero, so $(2023)^0 = 1$. Option (A) would correspond to exponent 1, (B) to exponent -1 , and (C) to exponent 2; none agrees with the determinant just found.

Step 3: Conclusion $(2023)^x = 1$. Hence, the correct option is (D).

Quick Tip to Solve: An odd-order skew-symmetric matrix always has zero determinant; use this before evaluating the power.

3. $\int_0^2 \sqrt{4-x^2} dx$ equals:

[1 mark]

- (A) $2 \log 2$
- (B) $-2 \log 2$
- (C) $\frac{\pi}{2}$
- (D) π

Correct Answer: (D) π

Solution:

Concept:

- Use $x = 2 \sin \theta$ for $\sqrt{4-x^2}$, taking $0 \leq \theta \leq \pi/2$.
- The identity $\cos^2 \theta = (1 + \cos 2\theta)/2$ evaluates the resulting integral.

Step 1: Transform the integral With $x = 2 \sin \theta$, $dx = 2 \cos \theta d\theta$ and $\sqrt{4-x^2} = 2 \cos \theta$. The limits

Quick Tip to Solve: This integral is also the area of a quarter circle of radius 2, giving one quarter of its full area.

$x = 0, 2$ become $\theta = 0, \pi/2$. Thus

$$I = 4 \int_0^{\pi/2} \cos^2 \theta \, d\theta.$$

Step 2: Evaluate and compare the options Using the double-angle identity,

$$I = 2 \int_0^{\pi/2} (1 + \cos 2\theta) \, d\theta = [2\theta + \sin 2\theta]_0^{\pi/2} = \pi.$$

The negative option (B) cannot be an integral of a nonnegative function. Options (A) and (C) differ from the exact evaluation; (C) misses a factor of two.

Step 3: Conclusion The value is π . Hence, the correct option is (D).

Quick Tip to Solve: This integral is also the area of a quarter circle of radius 2, giving one quarter of its full area.

4. The solution of the differential equation $\frac{dx}{x} + \frac{dy}{y} = 0$ is:

[1 mark]

- (A) $\frac{1}{x} + \frac{1}{y} = C$
- (B) $\log x - \log y = C$
- (C) $xy = C$
- (D) $x + y = C$

Correct Answer: (C) $xy = C$

Solution:

Concept:

- For nonzero x, y , $\int dx/x = \log |x|$ and $\int dy/y = \log |y|$.
- Add logarithms using $\log |x| + \log |y| = \log |xy|$.

Step 1: Integrate the given differentials Integrating both terms gives

$$\log |x| + \log |y| = K, \quad \log |xy| = K.$$

Therefore $|xy| = e^K$. On a connected solution interval the sign is fixed, so it is absorbed into a nonzero constant C , giving $xy = C$.

Step 2: Verify and rule out other forms Differentiating $xy = C$ gives $y \, dx + x \, dy = 0$. Dividing by xy reproduces the printed equation. Option (B) instead gives $dx/x - dy/y = 0$; (A) gives reciprocal-square coefficients, and (D) gives $dx + dy = 0$. These do not reproduce the required differential relation.

Step 3: Conclusion The solution is $xy = C$, on intervals where $x, y \neq 0$. Hence, the correct option is (C).

Quick Tip to Solve: When the differentials are dx/x and dy/y with a plus sign, combine their integrals into a product.

5. What is the product of the order and degree of the differential equation $\frac{d^2y}{dx^2} \sin y + \left(\frac{dy}{dx}\right)^3 \cos y = \sqrt{y}$? [1 mark]

- (A) 3
- (B) 2
- (C) 6
- (D) not defined

Correct Answer: (B) 2

Solution:

Concept:

- The order is the highest derivative order occurring in the equation.
- The degree is the power of the highest-order derivative when the equation is polynomial in its derivatives. Coefficients may depend on x or y .

Step 1: Identify the highest derivative The equation contains d^2y/dx^2 and dy/dx . The highest derivative is the second derivative, so the order is 2. The cubed first derivative does not make the order equal to three.

Step 2: Determine the degree and compare The derivatives occur polynomially: the second derivative has power 1, with coefficient $\sin y$, and the first derivative has power 3, with coefficient $\cos y$. The term $\sqrt{y}$ involves the dependent variable, not a derivative, so it does not prevent the degree from being defined. Hence the degree is 1 and the product is $2 \times 1 = 2$. Options (A) and (C) incorrectly use the cube of the first derivative, while (D) incorrectly treats $\sqrt{y}$ as a radical in a derivative.

Step 3: Conclusion The product of order and degree is 2. Hence, the correct option is (B).

Quick Tip to Solve: For degree, inspect the power of the highest-order derivative, not the largest power of any derivative.

6. The direction cosines of vector $\vec{BA}$, where coordinates of A and B are $(1, 2, -1)$ and $(3, 4, 0)$ respectively, are: [1 mark]

- (A) $-2, -2, -1$
- (B) $-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}$
- (C) $2, 2, 1$
- (D) $\frac{2}{3}, \frac{2}{3}, \frac{1}{3}$

Correct Answer: (B) $-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}$

Solution:

Concept:

- For $\vec{BA}$, subtract the coordinates of B from those of A .
- Direction cosines are the components of a vector divided by its magnitude; their squares sum to 1.

Quick Tip to Solve: Write the terminal point minus the initial point first, then divide by the vector length.

Step 1: Form the directed vector As the vector points from B to A ,

$$\vec{BA} = (1 - 3)\hat{i} + (2 - 4)\hat{j} + (-1 - 0)\hat{k} = -2\hat{i} - 2\hat{j} - \hat{k}.$$

Reversing this subtraction would give $\vec{AB}$, which points in the opposite direction.

Step 2: Normalize and compare The magnitude is $\sqrt{(-2)^2 + (-2)^2 + (-1)^2} = 3$. Dividing each component by 3 gives the direction cosines $(-2/3, -2/3, -1/3)$. Options (A) and (C) are unnormalized components, while (D) has the signs of the opposite vector. The normalized values satisfy $4/9 + 4/9 + 1/9 = 1$.

Step 3: Conclusion The direction cosines are $\left(-\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3}\right)$. Hence, the correct option is (B).

Quick Tip to Solve: Write the terminal point minus the initial point first, then divide by the vector length.

7. $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that the projection of $\vec{a}$ on $\vec{b}$ is 0. The angle between $\vec{a}$ and $\vec{b}$ is: [1 mark]

- (A) $\frac{\pi}{2}$
- (B) π
- (C) $\frac{\pi}{4}$
- (D) 0

Correct Answer: (A) $\frac{\pi}{2}$

Solution:

Concept:

- The scalar projection of $\vec{a}$ on $\vec{b}$ is $\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = |\vec{a}| \cos \theta$.
- The angle between two nonzero vectors lies in $[0, \pi]$.

Step 1: Use the zero projection The given projection is zero, hence $|\vec{a}| \cos \theta = 0$. Since $\vec{a} \neq \vec{0}$, its magnitude is positive and can be cancelled. Thus $\cos \theta = 0$; equivalently, the two vectors have zero dot product.

Step 2: Determine the angle and test alternatives Within the allowed interval $[0, \pi]$, the only zero of cosine is $\pi/2$. For the other printed angles, $\cos \pi = -1$, $\cos(\pi/4) = 1/\sqrt{2}$, and $\cos 0 = 1$. Each would give a nonzero projection because $|\vec{a}| > 0$. The nonzero-vector condition also ensures the projection denominator is defined.

Step 3: Conclusion The angle is $\frac{\pi}{2}$. Hence, the correct option is (A).

Quick Tip to Solve: A zero projection of one nonzero vector on another immediately signals perpendicular directions.

8. In $\triangle ABC$, $\vec{AB} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{AC} = 3\hat{i} - \hat{j} + 4\hat{k}$. If D is mid-point of BC , then vector $\vec{AD}$ is equal to: [1 mark]

- (A) $4\hat{i} + 6\hat{k}$
(B) $2\hat{i} - 2\hat{j} + 2\hat{k}$
(C) $\hat{i} - \hat{j} + \hat{k}$
(D) $2\hat{i} + 3\hat{k}$

Correct Answer: (D) $2\hat{i} + 3\hat{k}$

Solution:

Concept:

- With A as the reference point, the midpoint formula is $\vec{AD} = \frac{1}{2}(\vec{AB} + \vec{AC})$.
- Add vectors component by component, then divide every component by 2.

Step 1: Apply the midpoint formula Since D is the midpoint of BC , its position vector relative to A is the average of the position vectors of B and C . Thus

$$\vec{AD} = \frac{1}{2}[(\hat{i} + \hat{j} + 2\hat{k}) + (3\hat{i} - \hat{j} + 4\hat{k})].$$

Step 2: Simplify and compare all options The $\hat{j}$ components cancel, leaving $\frac{1}{2}(4\hat{i} + 6\hat{k}) = 2\hat{i} + 3\hat{k}$. Option (A) is the sum before dividing by two. Option (B) is $\vec{AC} - \vec{AB} = \vec{BC}$, and (C) is half that difference, corresponding to $\vec{BD}$. Neither is the vector from A to the midpoint.

Step 3: Conclusion $\vec{AD} = 2\hat{i} + 3\hat{k}$. Hence, the correct option is (D).

Quick Tip to Solve: Average the two vectors from the same vertex to the endpoints; do not average their difference.

9. If the point $P(a, b, 0)$ lies on the line $\frac{x+1}{2} = \frac{y+2}{3} = \frac{z+3}{4}$, then (a, b) is: [1 mark]

- (A) $(1, 2)$
(B) $\left(\frac{1}{2}, \frac{2}{3}\right)$
(C) $\left(\frac{1}{2}, \frac{1}{4}\right)$
(D) $(0, 0)$

Correct Answer: (C) $\left(\frac{1}{2}, \frac{1}{4}\right)$

Solution:

Concept:

- A point on a line must satisfy all three equal ratios simultaneously.
- Write each ratio as a common parameter t ; then $x = 2t - 1$, $y = 3t - 2$, $z = 4t - 3$.

Step 1: Use the given third coordinate Since $P(a, b, 0)$ has $z = 0$, the third ratio gives $t = (0 + 3)/4 = 3/4$. The zero coordinate does not make the common ratio zero because the numerator is $z + 3$.

Quick Tip to Solve: Set the ratio containing the known zero coordinate first; keep its constant term.

Step 2: Calculate and check the first two coordinates

$$a = 2 \left(\frac{3}{4} \right) - 1 = \frac{1}{2}, \quad b = 3 \left(\frac{3}{4} \right) - 2 = \frac{1}{4}.$$

Thus the three ratios at this point are all $3/4$. Option (A) gives unequal ratios; option (B) has the correct first coordinate but an incorrect second coordinate; option (D) does not satisfy the line.

Step 3: Conclusion $(a, b) = (\frac{1}{2}, \frac{1}{4})$. Hence, the correct option is (C).

Quick Tip to Solve: Set the ratio containing the known zero coordinate first; keep its constant term.

10. For any two events A and B , if $P(\bar{A}) = \frac{1}{2}$, $P(\bar{B}) = \frac{2}{3}$ and $P(A \cap B) = \frac{1}{4}$, then $P\left(\frac{\bar{A}}{\bar{B}}\right)$ equals:

[1 mark]

- (A) $\frac{3}{8}$
- (B) $\frac{8}{9}$
- (C) $\frac{1}{8}$
- (D) $\frac{1}{4}$

Correct Answer: $\frac{5}{8}$; none of the printed options is correct.

Solution:

Concept:

- Conditional probability is $P(C | D) = P(C \cap D)/P(D)$, provided $P(D) > 0$.
- De Morgan's law gives $\bar{A} \cap \bar{B} = \overline{A \cup B}$, and $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Step 1: Calculate the intersection of the complements From the given complement probabilities, $P(A) = 1/2$ and $P(B) = 1/3$. Thus

$$P(A \cup B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = \frac{7}{12}, \quad P(\bar{A} \cap \bar{B}) = 1 - \frac{7}{12} = \frac{5}{12}.$$

Both events in the conditional probability printed in the paper carry bars.

Step 2: Apply the conditional probability formula

$$P(\bar{A} | \bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{5/12}{2/3} = \frac{5}{8}.$$

The four printed values $3/8$, $8/9$, $1/8$, and $1/4$ are all different from $5/8$. In particular, $3/8$ would be $P(A | \bar{B})$, with the numerator bar omitted; it does not answer the printed question.

Step 3: Conclusion $P(\bar{A} | \bar{B}) = \frac{5}{8}$. None of the printed options is correct.

Quick Tip to Solve: Check the bars on both events; use the complement of the union for their joint complement.

11. The value of k for which function $f(x) = \begin{cases} x^2, & x \geq 0 \\ kx, & x < 0 \end{cases}$ is differentiable at $x = 0$ is: [1 mark]

- (A) 1
- (B) 2
- (C) any real number
- (D) 0

Correct Answer: (D) 0

Solution:

Concept:

- Differentiability at zero requires equal finite left and right derivatives.
- Use the difference quotient $[f(h) - f(0)]/h$ separately for positive and negative h .

Step 1: Evaluate the right derivative Here $f(0) = 0$. For $h > 0$, $f(h) = h^2$, and hence

$$f'(0+) = \lim_{h \rightarrow 0+} \frac{h^2 - 0}{h} = \lim_{h \rightarrow 0+} h = 0.$$

Both branches tend to zero at the origin, so continuity imposes no further restriction on k .

Step 2: Compare with the left derivative For $h < 0$, $f(h) = kh$. Thus

$$f'(0-) = \lim_{h \rightarrow 0-} \frac{kh - 0}{h} = k.$$

Equality with the right derivative requires $k = 0$. Options (A) and (B) give slopes 1 and 2 on the left, respectively, whereas the right slope is zero. Option (C) confuses continuity for every real k with differentiability.

Step 3: Conclusion $k = 0$. Hence, the correct option is (D).

Quick Tip to Solve: At the joining point compare the slopes of the two branches, not only their values.

12. If $y = \frac{\cos x - \sin x}{\cos x + \sin x}$, then $\frac{dy}{dx}$ is:

[1 mark]

(A) $-\sec^2\left(\frac{\pi}{4} - x\right)$

(B) $\sec^2\left(\frac{\pi}{4} - x\right)$

(C) $\log\left|\sec\left(\frac{\pi}{4} - x\right)\right|$

(D) $-\log\left|\sec\left(\frac{\pi}{4} - x\right)\right|$

Correct Answer: (A) $-\sec^2\left(\frac{\pi}{4} - x\right)$

Solution:

Concept:

- The tangent subtraction identity is $\tan(u - v) = (\tan u - \tan v)/(1 + \tan u \tan v)$.
- The chain rule gives $d(\tan u)/dx = \sec^2 u \, du/dx$.

Step 1: Rewrite the trigonometric quotient Using the sine and cosine subtraction formulae,

$$y = \frac{\sin(\pi/4 - x)}{\cos(\pi/4 - x)} = \tan\left(\frac{\pi}{4} - x\right).$$

This identity holds wherever the given denominator $\cos x + \sin x$ is nonzero.

Step 2: Differentiate the composite function

$$\frac{dy}{dx} = \sec^2\left(\frac{\pi}{4} - x\right) \frac{d}{dx}\left(\frac{\pi}{4} - x\right) = -\sec^2\left(\frac{\pi}{4} - x\right).$$

Option (B) misses the inner derivative -1 . Options (C) and (D) are logarithmic expressions and do not arise from differentiating a tangent function; they therefore cannot be the required derivative.

Step 3: Conclusion $\frac{dy}{dx} = -\sec^2(\frac{\pi}{4} - x)$. Hence, the correct option is (A).

Quick Tip to Solve: Recognize the tangent of $\pi/4$ minus x , then retain the negative inner derivative.

13. The number of feasible solutions of the linear programming problem given as: Maximize $z = 15x + 30y$ subject to constraints: $3x + y \leq 12$, $x + 2y \leq 10$, $x \geq 0$, $y \geq 0$ is:

[1 mark]

(A) 1

(B) 2

(C) 3

(D) infinite

Correct Answer: (D) infinite

Solution:

Concept:

- A feasible solution is any ordered pair satisfying every constraint, whether or not it maximizes the objective.

Quick Tip to Solve: To prove infinitely many feasible points, find a whole admissible interval on one axis.

- An interval of distinct admissible points establishes infinitely many feasible solutions.

Step 1: Exhibit an interval of feasible points Choose $y = 0$. The inequalities become $3x \leq 12$, $x \leq 10$, and $x \geq 0$. Thus every point $(x, 0)$ with $0 \leq x \leq 4$ satisfies both original inequalities as well as the nonnegativity conditions.

Step 2: Distinguish feasible points from corner points The interval $[0, 4]$ contains infinitely many real numbers, so the feasible set already contains infinitely many distinct points on the horizontal axis. The objective $z = 15x + 30y$ is used to select an optimum from this set; it does not restrict feasibility. Options (A), (B), and (C) would give finite counts and overlook the continuously many admissible points between endpoints.

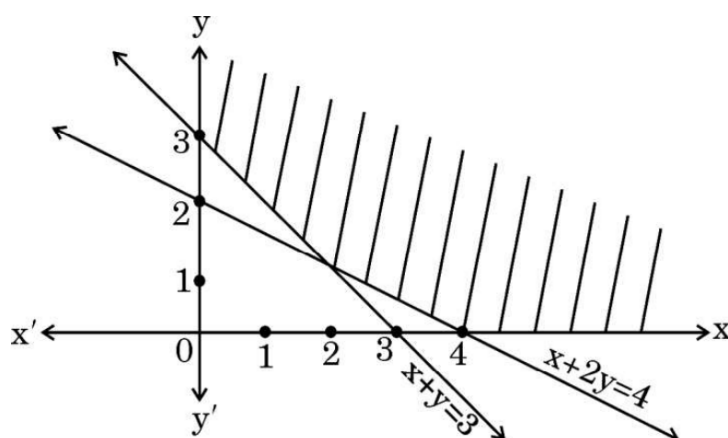
Step 3: Conclusion There are infinitely many feasible solutions. Hence, the correct option is (D).

Quick Tip to Solve: To prove infinitely many feasible points, find a whole admissible interval on one axis.

14. The feasible region of a linear programming problem is shown in the figure below:

Which of the following are the possible constraints?

[1 mark]



- (A) $x + 2y \geq 4$, $x + y \leq 3$, $x \geq 0$, $y \geq 0$
 (B) $x + 2y \leq 4$, $x + y \leq 3$, $x \geq 0$, $y \geq 0$
 (C) $x + 2y \geq 4$, $x + y \geq 3$, $x \geq 0$, $y \geq 0$
 (D) $x + 2y \geq 4$, $x + y \geq 3$, $x \leq 0$, $y \leq 0$

Correct Answer: (C) $x + 2y \geq 4$, $x + y \geq 3$, $x \geq 0$, $y \geq 0$

Solution:

Concept:

- The boundary lines are $x + 2y = 4$ and $x + y = 3$, as labelled in the figure.
- The shaded side of each boundary determines the inequality sign; points in the first quadrant satisfy $x, y \geq 0$.

Step 1: Determine the shaded half-planes The shaded region lies above both oblique lines. For example, the shaded point $(0, 4)$ gives $x + 2y = 8 > 4$ and $x + y = 4 > 3$. Therefore the two inequalities are $x + 2y \geq 4$ and $x + y \geq 3$; the boundary lines are included.

Quick Tip to Solve: Test a visibly shaded point in both boundary equations before choosing inequality signs.

Step 2: Check the quadrant and the options The shading is in the first quadrant, giving $x \geq 0$ and $y \geq 0$. Option (A) chooses the wrong side of $x + y = 3$, and option (B) chooses the lower side of both lines. Option (D) has the correct oblique inequality signs but requires nonpositive coordinates, contradicting the displayed region. All four constraints in option (C) match the shading.

Step 3: Conclusion The constraints are $x + 2y \geq 4$, $x + y \geq 3$, $x \geq 0$, $y \geq 0$. Hence, the correct option is (C).

Quick Tip to Solve: Test a visibly shaded point in both boundary equations before choosing inequality signs.

15. If $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then $B'A'$ is equal to:

[1 mark]

(A) $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$

(B) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

(D) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Correct Answer: (B) $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

Solution:

Concept:

- The prime denotes transpose, formed by interchanging rows and columns.
- Matrix multiplication uses each row of the first factor and each column of the second factor; preserve the printed order $B'A'$.

Step 1: Find the transposes

$$A' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

The matrix A is unchanged by transposition, whereas the off-diagonal 1 in B moves from its first row to its first column.

Step 2: Multiply in the specified order

$$B'A' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.$$

Both entries in the first column are 1 and both entries in the second column are 0. Option (A) is the untransposed matrix B ; option (C) incorrectly fills the second column with ones; option (D) incorrectly makes the nonzero first column vanish.

Quick Tip to Solve: Transpose each matrix first and preserve B-prime before A-prime when multiplying.

Step 3: Conclusion $B^t A^t = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. Hence, the correct option is (B).

Quick Tip to Solve: Transpose each matrix first and preserve B-prime before A-prime when multiplying.

16. If $A \cdot (\text{adj } A) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$, then the value of $|A| + |\text{adj } A|$ is equal to: [1 mark]

- (A) 12
- (B) 9
- (C) 3
- (D) 27

Correct Answer: (A) 12

Solution:

Concept:

- For a square matrix, $A \text{adj } A = |A|I$, where I is the identity matrix.
- For a matrix of order n , $|\text{adj } A| = |A|^{n-1}$.

Step 1: Read the determinant from the product The given product is $3I_3$. Comparing it with $A \text{adj } A = |A|I_3$ gives $|A| = 3$. The scalar on each diagonal entry is the determinant of A , not the determinant of the entire product.

Step 2: Evaluate the adjoint determinant and the sum Since A has order 3,

$$|\text{adj } A| = |A|^2 = 3^2 = 9, \quad |A| + |\text{adj } A| = 3 + 9 = 12.$$

Option (B) gives only the adjoint determinant, and option (C) gives only $|A|$. Option (D), 27, is the determinant of the product $3I_3$, which is a different quantity from the sum requested.

Step 3: Conclusion $|A| + |\text{adj } A| = 12$. Hence, the correct option is (A).

Quick Tip to Solve: Read $|A|$ from the scalar identity matrix, then use exponent two for its 3 by 3 adjoint.

17. A and B are skew-symmetric matrices of same order. AB is symmetric, if:

[1 mark]

- (A) $AB = O$
- (B) $AB = -BA$
- (C) $AB = BA$
- (D) $BA = O$

Correct Answer: (C) $AB = BA$

Solution:

Concept:

- A skew-symmetric matrix satisfies $A^T = -A$. A symmetric matrix satisfies $M^T = M$.
- Transposition reverses products: $(AB)^T = B^T A^T$.

Step 1: Transpose the product Since both matrices are skew-symmetric,

$$(AB)^T = B^T A^T = (-B)(-A) = BA.$$

Thus symmetry of the product requires equality between AB and BA .

Step 2: Compare the conditions The general necessary and sufficient condition is $AB = BA$, option (C). The condition $AB = -BA$ instead makes the product skew-symmetric, except when it is zero. Options (A) and (D) describe special zero-product cases: either zero product implies the other is zero by transposition, so these also give symmetry, but do not express its general criterion.

Step 3: Conclusion The general symmetry criterion is $\boxed{AB = BA}$. Hence, the intended correct option is (C).

Quick Tip to Solve: Reverse the order when transposing a product; the two negative signs cancel.

18. For what value of $x \in \left[0, \frac{\pi}{2}\right]$, is $A + A' = \sqrt{3}I$, where $A = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$?

[1 mark]

- (A) $\frac{\pi}{3}$
- (B) $\frac{\pi}{6}$
- (C) 0
- (D) $\frac{\pi}{2}$

Correct Answer: (B) $\frac{\pi}{6}$

Solution:

Concept:

- The transpose interchanges rows and columns.
- The identity matrix I has diagonal entries 1 and off-diagonal entries 0.

Quick Tip to Solve: In a rotation matrix plus its transpose, off-diagonal sine terms cancel immediately.

Step 1: Add the transpose

$$A^T = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}, \quad A + A^T = \begin{pmatrix} 2\cos x & 0 \\ 0 & 2\cos x \end{pmatrix}.$$

The sine terms cancel.

Step 2: Solve and check the options Comparing diagonal entries with $\sqrt{3}I$, we obtain $2\cos x = \sqrt{3}$, so $\cos x = \sqrt{3}/2$. On $[0, \pi/2]$, cosine decreases strictly, giving the unique value $x = \pi/6$. At $\pi/3, 0, \pi/2$, the diagonal coefficient is respectively 1, 2, 0, none equal to $\sqrt{3}$.

Step 3: Conclusion $x = \pi/6$. Hence, the correct option is (B).

Quick Tip to Solve: In a rotation matrix plus its transpose, off-diagonal sine terms cancel immediately.

19. Assertion (A): A line through the points (4, 7, 8) and (2, 3, 4) is parallel to a line through the points (-1, -2, 1) and (1, 2, 5).

Reason (R): Lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are parallel if $\vec{b}_1 \cdot \vec{b}_2 = 0$.

[1 mark]

- (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).
(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).
(C) Assertion (A) is true and Reason (R) is false.
(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (C) Assertion (A) is true and Reason (R) is false.

Solution:

Concept:

- Direction vectors are obtained by subtracting coordinates of two points on a line.
- Two nonzero direction vectors define parallel lines when they are scalar multiples; a zero dot product indicates perpendicular directions.

Step 1: Judge the Assertion The first direction vector is $(2 - 4, 3 - 7, 4 - 8) = (-2, -4, -4)$. The second is $(1 - (-1), 2 - (-2), 5 - 1) = (2, 4, 4)$. The first equals minus the second, so the lines are parallel. Assertion (A) is true.

Step 2: Judge the Reason The condition $\vec{b}_1 \cdot \vec{b}_2 = 0$ means the direction vectors are perpendicular, not parallel. Here their dot product is $-4 - 16 - 16 = -36$, although the lines are parallel. Reason (R) is false.

Step 3: Check the explanation Since the Reason is false, it cannot explain the true Assertion. Options (A) and (B) require a true Reason, while (D) requires a false Assertion, so they are excluded.

Step 4: Conclusion Assertion is true and Reason is false. Hence, the correct option is (C).

Quick Tip to Solve: For parallelism compare coordinate ratios; reserve a zero dot product for perpendicularity.

20. Assertion (A): Range of $[\sin^{-1} x + 2 \cos^{-1} x]$ is $[0, \pi]$.

Reason (R): Principal value branch of $\sin^{-1} x$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

[1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (D) Assertion (A) is false and Reason (R) is true.

Solution:

Concept:

- For $-1 \leq x \leq 1$, $\sin^{-1} x + \cos^{-1} x = \pi/2$.
- The principal range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$.

Step 1: Judge the Assertion Let $t = \sin^{-1} x$. Then $\cos^{-1} x = \pi/2 - t$, so

$$\sin^{-1} x + 2 \cos^{-1} x = t + 2(\pi/2 - t) = \pi - t.$$

As t runs through $[-\pi/2, \pi/2]$, the expression runs through $[\pi/2, 3\pi/2]$. Thus the asserted range $[0, \pi]$ is false.

Step 2: Judge the Reason The stated range $[-\pi/2, \pi/2]$ is precisely the principal value branch of inverse sine, including both endpoints. Reason (R) is true.

Step 3: Choose the logical combination The Assertion is false whereas the Reason is true. Options (A), (B) and (C) all require a true Assertion, and therefore do not apply.

Step 4: Conclusion Assertion is false and Reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Replace inverse cosine by $\pi/2 - \sin^{-1} x$ before computing a combined range.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21. Consider the statement: "There exists at least one value of $b \in \mathbb{R}$ for which $f(x) = \frac{b}{x}$, $b \neq 0$ is strictly increasing in $\mathbb{R} - \{0\}$." State True or False. Justify.

[2 marks]

Solution:

Concept:

- Strict increase on a domain means $x_1 < x_2$ implies $f(x_1) < f(x_2)$ for every pair in that domain.
- A positive derivative proves increase on an interval; disconnected domains also require comparisons across the gap.

Step 1: Test positive values of b If $b > 0$, choose $x_1 = 1$ and $x_2 = 2$. Although $1 < 2$, $f(1) = b > b/2 = f(2)$, contradicting strict increase. Equivalently, $f'(x) = -b/x^2 < 0$ on each of the two intervals

Quick Tip to Solve: On a disconnected domain, test a pair on opposite sides of the missing point as well.

of the domain.

Step 2: Test negative values across zero If $b < 0$, the derivative $-b/x^2$ is positive, so the function increases separately on $(-\infty, 0)$ and $(0, \infty)$. However, take $x_1 = -1 < x_2 = 1$. Then $f(-1) = -b > 0$, whereas $f(1) = b < 0$. Thus $f(-1) > f(1)$, contradicting increase on the entire domain. These two cases cover every permitted nonzero real b .

Step 3: Conclusion The statement is False. No nonzero real value of b makes b/x strictly increasing on all of $\mathbb{R} \setminus \{0\}$.

Quick Tip to Solve: On a disconnected domain, test a pair on opposite sides of the missing point as well.

22(a). Evaluate: $3 \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + 2 \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}(0)$.

[2 marks]

Solution:

Concept:

- Principal inverse sine values lie in $[-\pi/2, \pi/2]$; principal inverse cosine values lie in $[0, \pi]$.
- Use standard trigonometric values and express all angles in radians before addition.

Step 1: Identify each principal angle Since $\sin(\pi/4) = 1/\sqrt{2}$ and $\pi/4$ lies in the principal inverse sine interval, $\sin^{-1}(1/\sqrt{2}) = \pi/4$. Similarly, $\cos(\pi/6) = \sqrt{3}/2$ and $\cos(\pi/2) = 0$, with both angles in $[0, \pi]$. Hence $\cos^{-1}(\sqrt{3}/2) = \pi/6$ and $\cos^{-1}(0) = \pi/2$.

Step 2: Substitute and add The required expression becomes

$$\begin{aligned} 3\left(\frac{\pi}{4}\right) + 2\left(\frac{\pi}{6}\right) + \frac{\pi}{2} &= \frac{3\pi}{4} + \frac{\pi}{3} + \frac{\pi}{2} \\ &= \frac{9\pi + 4\pi + 6\pi}{12} = \frac{19\pi}{12}. \end{aligned}$$

The coefficients multiply the angles after evaluation. Although the resulting sum exceeds π , it is a sum of principal values, not itself an inverse cosine value, so no further branch reduction is needed.

Step 3: Conclusion The value of the expression is $19\pi/12$.

Quick Tip to Solve: Evaluate each inverse function in its own principal interval before combining the angles.

OR

22(b). Draw the graph of $f(x) = \sin^{-1} x$, $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. Also, write range of $f(x)$.

[2 marks]

Solution:

Concept:

- The principal inverse sine function gives the unique angle $y \in [-\pi/2, \pi/2]$ for which $\sin y = x$.

Quick Tip to Solve: For a restricted increasing inverse sine graph, evaluate the two domain endpoints first and keep both endpoints filled.

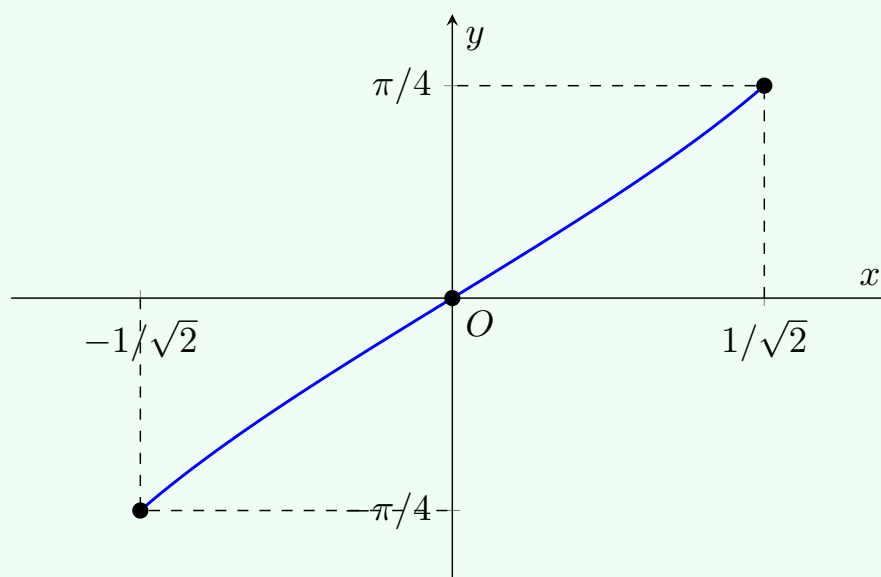
- On its principal interval, sine is strictly increasing. Hence $\sin^{-1} x$ is also strictly increasing, and its range on a closed interval is determined by its endpoint values.

Step 1: Calculate the endpoint and central values The given domain is $[-1/\sqrt{2}, 1/\sqrt{2}]$. Since $\sin(-\pi/4) = -1/\sqrt{2}$, $\sin 0 = 0$ and $\sin(\pi/4) = 1/\sqrt{2}$, the graph passes through

$$\left(-\frac{1}{\sqrt{2}}, -\frac{\pi}{4}\right), \quad (0, 0), \quad \left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right).$$

Both endpoints belong to the domain, so they are shown as filled points.

Step 2: Draw the restricted inverse sine curve and determine its range Draw a smooth increasing curve through these three points, retaining only the portion between the two specified endpoints. The graph is symmetric about the origin because inverse sine is odd. Its slope is $1/\sqrt{1-x^2} > 0$ throughout this interval. Thus every ordinate between the endpoint ordinates is attained, and no ordinate outside them occurs.



Graph of $y = \sin^{-1} x$ on the specified interval

Step 3: Conclusion The required graph is shown above, and the range is $[-\pi/4, \pi/4]$.

Quick Tip to Solve: For a restricted increasing inverse sine graph, evaluate the two domain endpoints first and keep both endpoints filled.

23(a). If $y = x^{\frac{1}{x}}$, then find $\frac{dy}{dx}$ at $x = 1$.

[2 marks]

Solution:

Concept:

- For a real variable exponent $y = u(x)^{v(x)}$ with $u(x) > 0$, take natural logarithms before differentiating: $\log y = v(x) \log u(x)$.

Quick Tip to Solve: When both base and exponent vary, take logarithms first; substitute the point only after obtaining the derivative.

- The chain rule gives $d(\log y)/dx = y^{-1}dy/dx$. Here the real logarithmic calculation applies for $x > 0$, including the requested point $x = 1$.

Step 1: Take logarithms and differentiate Given $y = x^{1/x}$, taking natural logarithms gives $\log y = (\log x)/x$. Differentiating both sides with respect to x , using the quotient rule on the right,

$$\frac{1}{y} \frac{dy}{dx} = \frac{x(1/x) - \log x}{x^2} = \frac{1 - \log x}{x^2}.$$

Both the base and the exponent depend on x ; treating the exponent as constant would omit an essential term.

Step 2: Evaluate the derivative at the specified point Multiplying by $y = x^{1/x}$ yields

$$\frac{dy}{dx} = x^{1/x} \frac{1 - \log x}{x^2}.$$

At $x = 1$, we have $y = 1^1 = 1$ and $\log 1 = 0$. Therefore the factor $(1 - \log x)/x^2$ takes the value $(1 - 0)/1 = 1$. The expression is well defined at the evaluation point.

Step 3: Conclusion Hence $\left. \frac{dy}{dx} \right|_{x=1} = 1$.

Quick Tip to Solve: When both base and exponent vary, take logarithms first; substitute the point only after obtaining the derivative.

OR

23(b). If $x = a \sin 2t$, $y = a(\cos 2t + \log \tan t)$, then find $\frac{dy}{dx}$.

[2 marks]

Solution:

Concept:

- For differentiable parametric equations $x = x(t)$, $y = y(t)$, the chain rule gives $dy/dx = (dy/dt)/(dx/dt)$ wherever $dx/dt \neq 0$.
- Using natural logarithms, $d(\log \tan t)/dt = \sec^2 t / \tan t = 1/(\sin t \cos t) = 2/\sin 2t$, on intervals where the given logarithm is defined.

Step 1: Differentiate both coordinates with respect to the parameter For $x = a \sin 2t$, the chain rule gives

$$\frac{dx}{dt} = 2a \cos 2t.$$

Similarly, from $y = a(\cos 2t + \log \tan t)$,

$$\begin{aligned} \frac{dy}{dt} &= a \left(-2 \sin 2t + \frac{2}{\sin 2t} \right) \\ &= \frac{2a(1 - \sin^2 2t)}{\sin 2t} = \frac{2a \cos^2 2t}{\sin 2t}. \end{aligned}$$

Quick Tip to Solve: Rewrite the logarithmic derivative as $2/\sin 2t$, then combine terms before dividing by dx/dt .

The factor 2 in the derivative of $\cos 2t$ comes from differentiating its argument.

Step 2: Form the ratio and simplify At regular parameter values, divide the two derivatives:

$$\frac{dy}{dx} = \frac{2a \cos^2 2t}{\sin 2t (2a \cos 2t)} = \frac{\cos 2t}{\sin 2t}.$$

This uses $1 - \sin^2 2t = \cos^2 2t$. The ratio calculation requires $a \neq 0$ and $\cos 2t \neq 0$; the original real logarithm requires $\tan t > 0$. These conditions specify the regular points where the parametric derivative formula applies.

Step 3: Conclusion Hence, at regular points of the given curve, $\boxed{\frac{dy}{dx} = \cot 2t}$.

Quick Tip to Solve: Rewrite the logarithmic derivative as $2/\sin 2t$, then combine terms before dividing by dx/dt .

24. If $\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k}$, find the value of $(\vec{r} \times \hat{j}) \cdot (\vec{r} \times \hat{k}) - 12$.

[2 marks]

Solution:

Concept:

- The unit vectors obey $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$; reversing the order changes the sign.
- For vectors expressed in the orthonormal Cartesian basis, their dot product is the sum of products of corresponding components. Cross products distribute over vector addition.

Step 1: Calculate the two cross products For $\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k}$, use $\hat{k} \times \hat{j} = -\hat{i}$ and $\hat{i} \times \hat{k} = -\hat{j}$:

$$\begin{aligned}\vec{r} \times \hat{j} &= 3\hat{k} + 0 - 6\hat{i} = -6\hat{i} + 3\hat{k}, \\ \vec{r} \times \hat{k} &= -3\hat{j} - 2\hat{i} + 0 = -2\hat{i} - 3\hat{j}.\end{aligned}$$

The terms parallel to the second vector have zero cross product. In particular, the negative coefficient of $\hat{j}$ gives $-2\hat{i}$ in the second line.

Step 2: Take the dot product and subtract the printed constant The component vectors are $(-6, 0, 3)$ and $(-2, -3, 0)$. Therefore

$$(\vec{r} \times \hat{j}) \cdot (\vec{r} \times \hat{k}) = (-6)(-2) + (0)(-3) + (3)(0) = 12.$$

The question asks for this dot product minus 12, so the final operation is $12 - 12$. Only matching coordinate components contribute; the $\hat{k}$ component of the first vector contributes nothing because the second has zero $\hat{k}$ component.

Step 3: Conclusion The required value is $\boxed{0}$.

Quick Tip to Solve: Write each cross product in component form before taking the dot product, and remember the final subtraction of 12.

25. Find the value of p , so that lines $\frac{x-1}{-2} = \frac{y-4}{3p} = \frac{z-3}{4}$ and $\frac{x-2}{4p} = \frac{y-5}{2} = \frac{1-z}{7}$ are perpendicular to each other.

[2 marks]

Solution:

Concept:

- The denominators in the symmetric form of a line give direction ratios when every numerator has the form coordinate minus constant.
- Two lines are perpendicular when their direction vectors have zero scalar product: if the ratios are (a_1, b_1, c_1) and (a_2, b_2, c_2) , then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

Step 1: Identify the direction ratios The first line has direction ratios $(-2, 3p, 4)$. For the second line, rewrite its last ratio as $\frac{1-z}{7} = \frac{z-1}{-7}$. Hence its direction ratios are $(4p, 2, -7)$. The negative sign in the third component is essential; the points through which the lines pass do not enter the perpendicularity condition.

Step 2: Apply the perpendicularity condition Using the scalar product of the direction vectors, we obtain

$$\begin{aligned} (-2)(4p) + (3p)(2) + (4)(-7) &= 0, \\ -8p + 6p - 28 &= 0, \\ -2p &= 28. \end{aligned}$$

As a check, for the resulting value the ratios become $(-2, -42, 4)$ and $(-56, 2, -7)$, whose scalar product is $112 - 84 - 28 = 0$. Both direction vectors are nonzero, so the condition is valid.

Step 3: Conclusion Therefore, $p = -14$.

Quick Tip to Solve: Before taking a dot product, rewrite $1 - z$ as $-(z - 1)$; this changes the corresponding denominator to -7 .

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26. Find: $\int \frac{e^x}{\sqrt{e^{2x} - 4e^x - 5}} dx$.

[3 marks]

Solution:

Concept:

- Use substitution when the numerator is the derivative of an expression in the denominator: for $t = e^x - 2$, we have $dt = e^x dx$.
- The standard integral is $\int \frac{dt}{\sqrt{t^2 - a^2}} = \log \left| t + \sqrt{t^2 - a^2} \right| + C$, where $a > 0$ and the radicand is positive.

Step 1: Complete the square Let the required integral be I . The expression under the square root can

Quick Tip to Solve: Complete the square in e^x first; shifting by 2 exposes the standard $\sqrt{t^2 - 3^2}$ integral immediately.

be rearranged without changing its value:

$$e^{2x} - 4e^x - 5 = (e^x - 2)^2 - 9.$$

Thus

$$I = \int \frac{e^x dx}{\sqrt{(e^x - 2)^2 - 3^2}}.$$

For a real integrand its denominator must be nonzero and real. Since $e^x > 0$, the factorisation $(e^x - 5)(e^x + 1) > 0$ gives $e^x > 5$, or $x > \log 5$.

Step 2: Substitute and use the standard form Put $t = e^x - 2$. Differentiating gives $dt = e^x dx$, so

$$\begin{aligned} I &= \int \frac{dt}{\sqrt{t^2 - 9}} \\ &= \log \left| t + \sqrt{t^2 - 9} \right| + C. \end{aligned}$$

Here $t > 3$ on the real domain. Replacing t by $e^x - 2$ yields the desired antiderivative in terms of the original variable.

Step 3: Verify by differentiation Write $s = \sqrt{t^2 - 9}$. Then $ds/dt = t/s$, and

$$\frac{d}{dt} \log |t + s| = \frac{1 + t/s}{t + s} = \frac{1}{s}.$$

Multiplying by $dt/dx = e^x$ gives $e^x / \sqrt{e^{2x} - 4e^x - 5}$, exactly the given integrand. The constant C accounts for all antiderivatives on this interval.

Step 4: Conclusion

$$I = \log \left| e^x - 2 + \sqrt{e^{2x} - 4e^x - 5} \right| + C$$

for $x > \log 5$.

Quick Tip to Solve: Complete the square in e^x first; shifting by 2 exposes the standard $\sqrt{t^2 - 3^2}$ integral immediately.

27(a). Find: $\int \frac{\cos x}{\sin 3x} dx$.

[3 marks]

Solution:

Concept:

- The triple-angle identity $\sin 3x = 3 \sin x - 4 \sin^3 x$ converts the denominator into a polynomial in $\sin x$.
- With $t = \sin x$, $dt = \cos x dx$; partial fractions then reduce the integral to logarithmic forms.

Quick Tip to Solve: For a numerator $\cos x$ and denominator $\sin 3x$, expand the triple-angle identity before setting $t = \sin x$.

Step 1: Use the triple-angle identity and substitute On any interval where $\sin 3x \neq 0$, write

$$I = \int \frac{\cos x \, dx}{\sin x(3 - 4 \sin^2 x)}.$$

Put $t = \sin x$, so that $dt = \cos x \, dx$. The integral becomes

$$I = \int \frac{dt}{t(3 - 4t^2)}.$$

This substitution uses the numerator directly and avoids expanding the triple-angle sine into functions of different arguments.

Step 2: Resolve into partial fractions and integrate The following identity is obtained by combining the right-hand terms:

$$\frac{1}{t(3 - 4t^2)} = \frac{1}{3t} + \frac{4t}{3(3 - 4t^2)}.$$

The combined numerator is $(3 - 4t^2 + 4t^2)/3 = 1$, confirming the decomposition. Hence

$$\begin{aligned} I &= \frac{1}{3} \int \frac{dt}{t} + \frac{4}{3} \int \frac{t \, dt}{3 - 4t^2} \\ &= \frac{1}{3} \log |t| - \frac{1}{6} \log |3 - 4t^2| + C. \end{aligned}$$

In the second integral, the derivative of $3 - 4t^2$ is $-8t$, which explains both the negative sign and the coefficient $1/6$.

Step 3: Return to the original variable Substitute $t = \sin x$ into the logarithms. Absolute values ensure the expression is real wherever the original integrand is defined. For verification, its derivative is

$$\frac{\cos x}{3 \sin x} + \frac{4 \sin x \cos x}{3(3 - 4 \sin^2 x)} = \frac{\cos x}{\sin x(3 - 4 \sin^2 x)} = \frac{\cos x}{\sin 3x}.$$

The antiderivative applies separately on intervals avoiding zeros of $\sin 3x$.

Step 4: Conclusion

$$I = \frac{1}{3} \log |\sin x| - \frac{1}{6} \log |3 - 4 \sin^2 x| + C$$

Quick Tip to Solve: For a numerator $\cos x$ and denominator $\sin 3x$, expand the triple-angle identity before setting $t = \sin x$.

27(b). Find: $\int x^2 \log(x^2 + 1) dx$.

[3 marks]

Solution:

Concept:

- Integration by parts gives $\int u dv = uv - \int v du$. Choose the logarithm as u , since its derivative is rational.
- Polynomial division simplifies a rational integrand; also $\int \frac{dx}{1+x^2} = \tan^{-1} x + C$.

Step 1: Choose the functions for integration by parts Let $I = \int x^2 \log(x^2 + 1) dx$, taking log as the natural logarithm. Set

$$u = \log(x^2 + 1), \quad dv = x^2 dx.$$

Then

$$du = \frac{2x}{x^2 + 1} dx, \quad v = \frac{x^3}{3}.$$

Integration by parts therefore yields

$$I = \frac{x^3}{3} \log(x^2 + 1) - \frac{2}{3} \int \frac{x^4}{x^2 + 1} dx.$$

Since $x^2 + 1 > 0$, the logarithm is defined for every real x .

Step 2: Divide the polynomial and integrate Polynomial division gives

$$\frac{x^4}{x^2 + 1} = x^2 - 1 + \frac{1}{x^2 + 1},$$

because $(x^2 + 1)(x^2 - 1) + 1 = x^4$. Consequently,

$$\begin{aligned} \int \frac{x^4}{x^2 + 1} dx &= \int \left(x^2 - 1 + \frac{1}{x^2 + 1} \right) dx \\ &= \frac{x^3}{3} - x + \tan^{-1} x. \end{aligned}$$

Substitute this expression into the result of integration by parts and distribute the factor $-2/3$ carefully.

Step 3: Check the cancellation on differentiation The derivative of the first term is $x^2 \log(x^2 + 1) + 2x^4/[3(x^2 + 1)]$. The remaining terms have derivative $-2x^2/3 + 2/3 - 2/[3(x^2 + 1)]$, which equals $-2x^4/[3(x^2 + 1)]$ by the division identity above. These rational terms cancel, leaving exactly $x^2 \log(x^2 + 1)$. The result thus holds on the whole real line.

Step 4: Conclusion

$$I = \frac{x^3}{3} \log(x^2 + 1) - \frac{2x^3}{9} + \frac{2x}{3} - \frac{2}{3} \tan^{-1} x + C$$

Quick Tip to Solve: After choosing the logarithm for differentiation, divide x^4 by $x^2 + 1$ before doing the remaining integral.

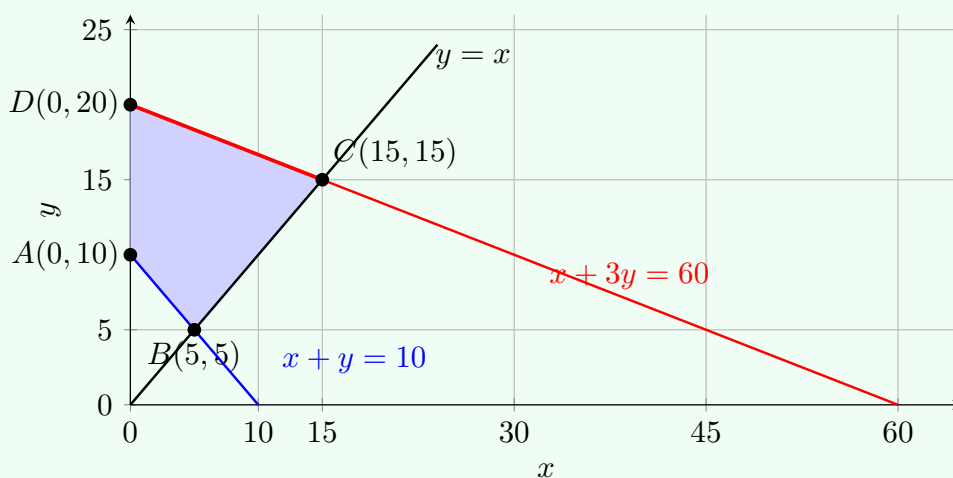
28. Solve the following linear programming problem graphically: Maximize $z = 3x + 9y$ subject to the constraints $x + y \geq 10$, $x + 3y \leq 60$, $x \leq y$, $x \geq 0$, $y \geq 0$. [3 marks]

Solution:

Concept:

- The feasible region is the common part of the half-planes specified by the inequalities. The restrictions $x \geq 0, y \geq 0$ keep it in the first quadrant.
- A linear objective on a bounded polygon attains its maximum at a vertex. Equal maximum values at adjacent vertices make the entire joining edge optimal.

Step 1: Plot the boundaries and shade the common region Draw $x + y = 10$ through $(10, 0)$, $(0, 10)$, retaining the side away from the origin. Draw $x + 3y = 60$ through $(60, 0)$, $(0, 20)$, retaining the side containing the origin. The restriction $x \leq y$ retains the side above $y = x$. Their intersection in the first quadrant is the shaded quadrilateral ABCD. Every boundary is included because the inequalities permit equality.



Shaded feasible region ABCD with optimal edge CD.

Step 2: Calculate the vertices The intersections on the y -axis are $A = (0, 10)$ and $D = (0, 20)$. On $y = x$, the lower boundary gives $2x = 10$, hence $B = (5, 5)$, while the upper boundary gives $4x = 60$, hence $C = (15, 15)$. Each vertex satisfies all the constraints. The intersection of $x + y = 10$ with $x + 3y = 60$ is $(-15, 25)$, which is outside the first quadrant and therefore excluded.

Step 3: Evaluate the objective and check the optimal edge The vertex values are

(x, y)	$(0, 10)$	$(5, 5)$	$(15, 15)$	$(0, 20)$
$3x + 9y$	90	60	180	180

The largest vertex value occurs at both C and D. Also, $z = 3(x + 3y) \leq 3 \times 60 = 180$ at every feasible point. Equality holds along the feasible segment CD, described by $y = 20 - x/3$, $0 \leq x \leq 15$. Thus the parallel objective line coincides with the full upper edge, giving infinitely many maximizers.

Step 4: Conclusion The maximum is 180, attained at every point of the segment joining $(0, 20)$ and $(15, 15)$, including its endpoints.

Quick Tip to Solve: Since the objective equals three times $x + 3y$, its upper bound is immediately 180; the graph identifies the feasible points where equality holds.

29(a). A pair of dice is thrown simultaneously. If X denotes the absolute difference of numbers obtained on the pair of dice, then find the probability distribution of X . [3 marks]

Solution:

Concept:

- Two fair dice produce $6 \times 6 = 36$ equally likely ordered outcomes (a, b) , with $a, b \in \{1, 2, 3, 4, 5, 6\}$.
- For a discrete random variable, $P(X = k)$ is the number of outcomes yielding k , divided by the total number of equally likely outcomes. These probabilities sum to 1.

Step 1: Determine the possible absolute differences Here $X = |a - b|$. Its smallest possible value is 0, when the dice show equal numbers, and its largest is $|6 - 1| = 5$. Every integer between these extremes occurs, so the range is $\{0, 1, 2, 3, 4, 5\}$. Since the dice can be distinguished by their positions, $(1, 2)$ and $(2, 1)$ are distinct outcomes even though their absolute differences agree.

Step 2: Count the outcomes for each value For $X = 0$, the outcomes are $(1, 1), (2, 2), \dots, (6, 6)$, giving 6 outcomes. For each $k \geq 1$, pairs with $b = a + k$ number $6 - k$, and the reversed pairs contribute another $6 - k$. Hence the count is $2(6 - k)$. The counts for $k = 1, 2, 3, 4, 5$ are therefore 10, 8, 6, 4, 2, respectively. For example, difference 5 arises only from $(1, 6)$ and $(6, 1)$.

Step 3: Form and verify the distribution Dividing each count by 36 gives

x	0	1	2	3	4	5
$P(X = x)$	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

The counts total $6 + 10 + 8 + 6 + 4 + 2 = 36$, so the probabilities add to $36/36 = 1$. Thus every possible value has been included and no ordered outcome has been counted twice.

Step 4: Conclusion The probability distribution for $X = 0, 1, 2, 3, 4, 5$ is respectively $\frac{1}{6}, \frac{5}{18}, \frac{2}{9}, \frac{1}{6}, \frac{1}{9}, \frac{1}{18}$.

Quick Tip to Solve: Count equal dice separately; for a positive difference k , use twice the number of pairs spaced k apart, namely $2(6 - k)$.

OR

29(b). There are two coins. One of them is a biased coin such that $P(\text{head}) : P(\text{tail})$ is $1 : 3$ and the other coin is a fair coin. A coin is selected at random and tossed once. If the coin showed head, then find the probability that it is a biased coin. [3 marks]

Solution:

Concept:

- The law of total probability gives $P(H) = P(B)P(H | B) + P(F)P(H | F)$, where B and F are mutually exclusive and exhaustive coin selections.
- Bayes' theorem reverses the conditioning: $P(B | H) = P(B)P(H | B)/P(H)$, provided $P(H) > 0$.

Step 1: Define the events and their probabilities Let B denote selection of the biased coin, F selection of the fair coin, and H obtaining a head on the toss. Since a coin is selected at random from two coins,

Quick Tip to Solve: With equal selection probabilities, compare head likelihoods $1/4 : 1/2 = 1 : 2$; the biased coin contributes one of the three posterior shares.

$P(B) = P(F) = 1/2$. For the biased coin, the ratio of head to tail probabilities is 1 : 3. Because their sum is 1, $P(H | B) = 1/(1 + 3) = 1/4$. For the fair coin, $P(H | F) = 1/2$.

Step 2: Find the overall probability of a head A head may arise from either coin. Since these selection events are disjoint, their contributions add:

$$\begin{aligned} P(H) &= P(B)P(H | B) + P(F)P(H | F) \\ &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}. \end{aligned}$$

The joint probability of selecting the biased coin and getting a head is $P(B \cap H) = 1/8$. The corresponding joint probability for the fair coin is $1/4$.

Step 3: Apply Bayes' theorem after observing the head Restrict the sample space to the event H, as the observed toss is known to be a head. Then

$$P(B | H) = \frac{P(B \cap H)}{P(H)} = \frac{1/8}{3/8} = \frac{1}{3}.$$

This is smaller than the original selection probability $1/2$, as expected: a head is twice as likely with the fair coin as with the biased coin. The complementary posterior probability is $P(F | H) = 2/3$, and the two posterior probabilities add to 1.

Step 4: Conclusion Given that the selected coin showed a head, the probability that it was the biased coin is $\boxed{\frac{1}{3}}$.

Quick Tip to Solve: With equal selection probabilities, compare head likelihoods $1/4 : 1/2 = 1 : 2$; the biased coin contributes one of the three posterior shares.

30(a). Find the general solution of the differential equation: $\frac{d}{dx}(xy^2) = 2y(1 + x^2)$. [3 marks]

Solution:

Concept:

- Expand the derivative of the product xy^2 using the product rule and the chain rule: $\frac{d}{dx}(xy^2) = y^2 + 2xy \frac{dy}{dx}$.
- A linear equation $y' + P(x)y = Q(x)$ has integrating factor $IF = e^{\int P(x) dx}$. Multiplication by this factor makes the left side a product derivative.
- Division by y excludes the identically zero solution, so that solution must be checked separately in the original equation.

Quick Tip to Solve: Expand $(xy^2)'$ before classifying the equation; after dividing by y , remember to check $y = 0$ in the original equation.

Step 1: Expand and reduce the equation Using the product and chain rules gives

$$y^2 + 2xy \frac{dy}{dx} = 2y(1 + x^2).$$

On an interval where $x \neq 0$ and $y \neq 0$, divide by $2xy$:

$$\frac{dy}{dx} + \frac{1}{2x}y = \frac{1 + x^2}{x} = \frac{1}{x} + x.$$

Thus the resulting equation is linear in y , with $P(x) = 1/(2x)$ and $Q(x) = 1/x + x$.

Step 2: Use the integrating factor On either interval $x > 0$ or $x < 0$, the integrating factor is

$$\text{IF} = e^{\frac{1}{2} \ln |x|} = \sqrt{|x|}.$$

Therefore

$$\frac{d}{dx}(y\sqrt{|x|}) = \sqrt{|x|} \left(\frac{1}{x} + x \right).$$

Integrating and absorbing the integration constant into C ,

$$y\sqrt{|x|} = 2\sqrt{|x|} + \frac{2}{5}x^2\sqrt{|x|} + C.$$

Indeed, differentiation of the first two terms on the right gives exactly $\sqrt{|x|}(1/x + x)$. For the usual interval $x > 0$, these terms arise directly from integrating $x^{-1/2} + x^{3/2}$.

Step 3: Check the excluded solution Dividing the integrated expression by $\sqrt{|x|}$ gives the one-parameter family. The reduction divided by y , so substitute $y = 0$ into the printed equation separately: both $\frac{d}{dx}(xy^2)$ and $2y(1 + x^2)$ vanish. Hence the identically zero function is also a solution. The linear-family expression satisfies the original equation even at a zero of y , since substituting its linear equation restores the expanded original identity.

Step 4: Conclusion The general nonzero family is $y = 2 + \frac{2}{5}x^2 + \frac{C}{\sqrt{|x|}}$, on intervals with $x \neq 0$. In particular, for $x > 0$, $y = 2 + \frac{2}{5}x^2 + \frac{C}{\sqrt{x}}$. The additional solution is $y \equiv 0$.

Quick Tip to Solve: Expand $(xy^2)'$ before classifying the equation; after dividing by y , remember to check $y = 0$ in the original equation.

30(b). Solve the following differential equation: $xe^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0$.

[3 marks]

Solution:

Concept:

- For a homogeneous first-order equation involving y/x , put $v = y/x$, so $y = vx$ and $dy/dx = v + x dv/dx$.
- A separable equation is integrated after collecting all terms involving v on one side and all terms involving x on the other. Here $\int e^{-v} dv = -e^{-v}$ and $\int dx/x = \ln|x|$.

Step 1: Introduce the homogeneous substitution The printed equation is

$$xe^{y/x} - y + x\frac{dy}{dx} = 0, \quad x \neq 0.$$

Dividing by x makes its dependence on the ratio explicit:

$$\frac{dy}{dx} = \frac{y}{x} - e^{y/x}.$$

Set $v = y/x$, giving $y = vx$. Differentiating the product carefully,

$$\frac{dy}{dx} = v + x\frac{dv}{dx}.$$

Substitute this expression and $y/x = v$ into the equation.

Step 2: Separate and integrate After substitution, the two terms v cancel:

$$v + x\frac{dv}{dx} = v - e^v \implies x\frac{dv}{dx} = -e^v.$$

As e^v is always positive for real v , division by it excludes no real constant solution. Separating variables gives

$$e^{-v} dv = -\frac{dx}{x}.$$

Integrate on any interval that does not contain zero:

$$-e^{-v} = -\ln|x| + C_1.$$

Multiplying by -1 and renaming the arbitrary constant yields $e^{-v} = \ln|x| + C$.

Step 3: Restore the original variables and verify Substitution of $v = y/x$ gives the implicit solution. To check its sign and the integration, differentiate:

$$e^{-y/x} \left(-\frac{xy' - y}{x^2} \right) = \frac{1}{x}.$$

Hence $xy' - y = -xe^{y/x}$, which rearranges to the printed equation. Since the left side of the implicit solution is positive, the real solution is restricted to intervals where $\ln|x| + C > 0$. Equivalently one

Quick Tip to Solve: When using $y = vx$, retain both terms in $y' = v + xv'$; the v terms cancel and leave a separable equation.

can take its logarithm and write $y = -x \ln(\ln |x| + C)$.

Step 4: Conclusion $e^{-y/x} = \ln |x| + C$, with $x \neq 0$ and $\ln |x| + C > 0$. Equivalently, $y = -x \ln(\ln |x| + C)$.

Quick Tip to Solve: When using $y = vx$, retain both terms in $y' = v + xv'$; the v terms cancel and leave a separable equation.

31. Evaluate: $\int_{-\pi/2}^{\pi/2} \frac{\sin^{100} x}{\sin^{100} x + \cos^{100} x} dx$.

[3 marks]

Solution:

Concept:

- An even integrand satisfies $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$. Both $\sin^{100} x$ and $\cos^{100} x$ are even functions.
- The reflection property $\int_0^a f(x) dx = \int_0^a f(a-x) dx$ interchanges sine and cosine when $a = \pi/2$. Adding the two reflected integrals can simplify their numerator.

Step 1: Use evenness to halve the interval Write

$$f(x) = \frac{\sin^{100} x}{\sin^{100} x + \cos^{100} x}, \quad J = \int_0^{\pi/2} f(x) dx.$$

Since the powers are even, $\sin^{100}(-x) = \sin^{100} x$ and $\cos^{100}(-x) = \cos^{100} x$. Thus $f(-x) = f(x)$, and the required integral I equals $2J$. The denominator cannot vanish: sine and cosine are never both zero, and both even powers are nonnegative. Consequently the integrand is continuous over the whole integration interval.

Step 2: Reflect the first-quadrant integral Apply the substitution $x = \pi/2 - t$. Reversing the resulting limits removes the minus sign from $dx = -dt$. Since $\sin(\pi/2 - t) = \cos t$ and $\cos(\pi/2 - t) = \sin t$,

$$J = \int_0^{\pi/2} \frac{\cos^{100} t}{\cos^{100} t + \sin^{100} t} dt.$$

The integration variable is a dummy variable, so rename t to x before adding this identity to the original expression for J .

Step 3: Add the complementary fractions Both expressions have identical limits and denominator. Therefore

$$\begin{aligned} 2J &= \int_0^{\pi/2} \frac{\sin^{100} x + \cos^{100} x}{\sin^{100} x + \cos^{100} x} dx \\ &= \int_0^{\pi/2} 1 dx = \frac{\pi}{2}. \end{aligned}$$

Thus $J = \pi/4$. The original integral extends over both halves of the symmetric interval, so its value follows by multiplying this first-quadrant value by two.

Quick Tip to Solve: First use evenness, then reflect about $\pi/4$; the sine numerator and its cosine partner add to their common denominator.

Step 4: Conclusion The required integral is $I = 2J = \frac{\pi}{2}$.

Quick Tip to Solve: First use evenness, then reflect about $\pi/4$; the sine numerator and its cosine partner add to their common denominator.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32(a). If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$, then show that $A^3 - 6A^2 + 7A + 2I = O$.

[5 marks]

Solution:

Concept:

- Matrix multiplication uses row-by-column scalar products: entry (i, j) is obtained from row i of the first matrix and column j of the second.
- The powers are $A^2 = AA$ and $A^3 = A^2A$. Addition and scalar multiplication operate entry by entry; I is the identity matrix of order three.

Step 1: Calculate the square Multiplying each row of the given matrix by each column gives

$$A^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix}.$$

The first row comes from $1 + 4 = 5$, zero, and $2 + 6 = 8$. The second row comes from $0 + 0 + 2 = 2$, $0 + 4 + 0 = 4$, and $0 + 2 + 3 = 5$. The third row comes from $2 + 6 = 8$, zero, and $4 + 9 = 13$. In particular, the nonzero entry in position $(2, 1)$ must be retained even though the original matrix has zero in that position.

Step 2: Calculate the cube Multiplying the square by the original matrix gives

$$A^3 = \begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{pmatrix}.$$

For the first row the computations are $5 + 16 = 21$, zero, and $10 + 24 = 34$. For the second row they are $2 + 10 = 12$, 8, and $4 + 4 + 15 = 23$. For the third row they are $8 + 26 = 34$, zero, and $16 + 39 = 55$.

Step 3: Combine the matrices Subtract six times the square from the cube:

$$A^3 - 6A^2 = \begin{pmatrix} -9 & 0 & -14 \\ 0 & -16 & -7 \\ -14 & 0 & -23 \end{pmatrix}.$$

Quick Tip to Solve: Compute the powers first and combine the matrices in stages; adding the identity affects only the diagonal entries.

Now add seven times the original matrix:

$$A^3 - 6A^2 + 7A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix} = -2I.$$

Every off-diagonal entry has cancelled, and every diagonal entry is -2 . Adding $2I$ changes only the three diagonal entries, making each zero; the off-diagonal entries remain zero.

Step 4: Conclusion Hence $A^3 - 6A^2 + 7A + 2I = O$, as required.

Quick Tip to Solve: Compute the powers first and combine the matrices in stages; adding the identity affects only the diagonal entries.

OR

32(b). If $A = \begin{pmatrix} 3 & 2 \\ 5 & -7 \end{pmatrix}$, then find A^{-1} and use it to solve the following system of equations: $3x + 5y = 11$, $2x - 7y = -3$. [5 marks]

Solution:

Concept:

- A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible when $ad - bc \neq 0$; its inverse is $\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- The coefficients of the given equations form the columns of A . Thus the system can be written as $\begin{pmatrix} x & y \end{pmatrix} A = \begin{pmatrix} 11 & -3 \end{pmatrix}$, and solved by multiplying on the right by A^{-1} .

Step 1: Find the determinant and adjoint For the given matrix,

$$|A| = 3(-7) - 2(5) = -21 - 10 = -31 \neq 0.$$

Therefore the inverse exists and the system will have a unique solution. Interchange the diagonal entries and change the signs of the off-diagonal entries to obtain

$$\text{adj } A = \begin{pmatrix} -7 & -2 \\ -5 & 3 \end{pmatrix}.$$

The determinant is negative, so its sign must be applied to every entry when dividing the adjoint by the determinant.

Step 2: Calculate and check the inverse The inverse is

$$A^{-1} = \frac{1}{-31} \begin{pmatrix} -7 & -2 \\ -5 & 3 \end{pmatrix} = \frac{1}{31} \begin{pmatrix} 7 & 2 \\ 5 & -3 \end{pmatrix}.$$

Quick Tip to Solve: Check whether the equation coefficients are rows or columns of the given matrix before applying its inverse.

Indeed, multiplication gives

$$AA^{-1} = \frac{1}{31} \begin{pmatrix} 21+10 & 6-6 \\ 35-35 & 10+21 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This verifies both the signs and the placement of the entries of the inverse.

Step 3: Use the inverse to solve the equations Since the first equation is $3x + 5y = 11$, while the second is $2x - 7y = -3$, write them as a row-vector equation:

$$\begin{pmatrix} x & y \end{pmatrix} A = \begin{pmatrix} 11 & -3 \end{pmatrix}.$$

Multiplying both sides on the right by the inverse yields

$$\begin{aligned} \begin{pmatrix} x & y \end{pmatrix} &= \begin{pmatrix} 11 & -3 \end{pmatrix} A^{-1} \\ &= \frac{1}{31} \begin{pmatrix} 11 & -3 \end{pmatrix} \begin{pmatrix} 7 & 2 \\ 5 & -3 \end{pmatrix} \\ &= \frac{1}{31} \begin{pmatrix} 77-15 & 22+9 \end{pmatrix} = \begin{pmatrix} 2 & 1 \end{pmatrix}. \end{aligned}$$

As a check in the original equations, $3(2) + 5(1) = 11$ and $2(2) - 7(1) = -3$. Both equations are satisfied.

Step 4: Conclusion $A^{-1} = \frac{1}{31} \begin{pmatrix} 7 & 2 \\ 5 & -3 \end{pmatrix}$, and the unique solution is $x = 2, y = 1$.

Quick Tip to Solve: Check whether the equation coefficients are rows or columns of the given matrix before applying its inverse.

33(a). Find the value of b so that the lines $\frac{x-1}{2} = \frac{y-b}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ are intersecting lines. Also, find the point of intersection of these given lines. [5 marks]

Solution:

Concept:

- Two lines intersect when their parametric equations give the same values of all three coordinates for some values of their respective parameters.
- For a line $\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n} = t$, the parametric form is $x = x_0 + lt$, $y = y_0 + mt$, $z = z_0 + nt$. The numbers l, m, n are its direction ratios.

Step 1: Express both lines in parametric form Let the common ratio of the first line be t . Since the last expression for the second line is simply z , let its common ratio be s , so that $z = s$. Thus the two lines are

$$L_1 : (x, y, z) = (1 + 2t, b + 3t, 3 + 4t),$$

$$L_2 : (x, y, z) = (4 + 5s, 1 + 2s, s).$$

Quick Tip to Solve: First equate the coordinates that do not involve the unknown constant: here x and z determine the parameters, after which the y equation gives b .

The parameters t and s need not be equal at an intersection: each measures displacement along its own line. We must equate the coordinates, not the parameters.

Step 2: Use the x and z coordinates to determine the parameters At a common point, the first and third coordinates give

$$1 + 2t = 4 + 5s, \quad 3 + 4t = s.$$

Substitute the second equation into the first:

$$1 + 2t = 4 + 5(3 + 4t),$$

$$1 + 2t = 19 + 20t,$$

$$-18t = 18,$$

$$t = -1.$$

Therefore $s = 3 + 4(-1) = -1$. These two coordinates uniquely determine the candidate intersection; the remaining coordinate determines which value of b makes it an actual common point.

Step 3: Enforce equality of the y coordinates and verify The second coordinates must satisfy

$$b + 3t = 1 + 2s.$$

Using $t = s = -1$, we obtain

$$b - 3 = 1 - 2 = -1, \quad b = 2.$$

With this value, substitution in either parametric equation gives

$$L_1(-1) = (1 - 2, 2 - 3, 3 - 4) = (-1, -1, -1),$$

$$L_2(-1) = (4 - 5, 1 - 2, -1) = (-1, -1, -1).$$

For an additional check in the original symmetric forms, the three ratios on the first line are $-2/2 = -3/3 = -4/4 = -1$, and on the second they are $-5/5 = -2/2 = z = -1$. Thus the point satisfies both printed equations. Their direction ratios $(2, 3, 4)$ and $(5, 2, 1)$ are not proportional, so the lines have a unique intersection.

Step 4: Conclusion The required value is $b = 2$, and the point of intersection is $(-1, -1, -1)$.

Quick Tip to Solve: First equate the coordinates that do not involve the unknown constant: here x and z determine the parameters, after which the y equation gives b .

33(b). Find the equations of all the sides of the parallelogram $ABCD$ whose vertices are $A(4, 7, 8)$, $B(2, 3, 4)$, $C(-1, -2, 1)$ and $D(1, 2, 5)$. Also, find the coordinates of the foot of the perpendicular from A to CD . [5 marks]

Solution:

Concept:

- The line through $P(x_0, y_0, z_0)$ with direction ratios (l, m, n) has equation $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$. Direction ratios may be multiplied by any nonzero common factor.
- If $H = C + t\mathbf{d}$ is the foot of the perpendicular from A to the line through C in direction $\mathbf{d}$, then $(A - H) \cdot \mathbf{d} = 0$. Hence $t = \frac{(A - C) \cdot \mathbf{d}}{\mathbf{d} \cdot \mathbf{d}}$.

Step 1: Find the directions of the four sides Subtract the coordinates of consecutive vertices:

$$\begin{aligned}\overrightarrow{AB} &= (2 - 4, 3 - 7, 4 - 8) = (-2, -4, -4), \\ \overrightarrow{BC} &= (-1 - 2, -2 - 3, 1 - 4) = (-3, -5, -3), \\ \overrightarrow{CD} &= (1 + 1, 2 + 2, 5 - 1) = (2, 4, 4), \\ \overrightarrow{DA} &= (4 - 1, 7 - 2, 8 - 5) = (3, 5, 3).\end{aligned}$$

Thus AB and CD have direction ratios $(1, 2, 2)$, while BC and DA have direction ratios $(3, 5, 3)$. Opposite directions differ only by a nonzero scalar, as required for opposite sides of a parallelogram.

Step 2: Write the equations of all four sides Using a vertex on each side and the corresponding direction ratios, the required supporting lines are

$$\begin{aligned}AB : \quad \frac{x - 4}{1} &= \frac{y - 7}{2} = \frac{z - 8}{2}, \\ BC : \quad \frac{x - 2}{3} &= \frac{y - 3}{5} = \frac{z - 4}{3}, \\ CD : \quad \frac{x + 1}{1} &= \frac{y + 2}{2} = \frac{z - 1}{2}, \\ DA : \quad \frac{x - 1}{3} &= \frac{y - 2}{5} = \frac{z - 5}{3}.\end{aligned}$$

The direction chosen for a line need not point towards the next vertex. For example, the common ratio on AB is -2 at B , while that on BC is -1 at C . These substitutions confirm that each equation contains its two stated endpoints.

Step 3: Locate the perpendicular foot on CD A general point on CD , with direction vector $\mathbf{d} = (1, 2, 2)$, is

$$H = (-1 + t, -2 + 2t, 1 + 2t).$$

The vector from this point to A is

$$\overrightarrow{HA} = (5 - t, 9 - 2t, 7 - 2t).$$

Quick Tip to Solve: Reduce the direction vector of CD to $(1, 2, 2)$ before projecting; the squared length is then 9, which keeps the perpendicular-foot calculation short.

For AH to be perpendicular to CD , its dot product with $\mathbf{d}$ must vanish:

$$\begin{aligned}(5 - t) + 2(9 - 2t) + 2(7 - 2t) &= 0, \\ 37 - 9t &= 0, \\ t &= \frac{37}{9}.\end{aligned}$$

Therefore

$$H = \left(-1 + \frac{37}{9}, -2 + \frac{74}{9}, 1 + \frac{74}{9}\right) = \left(\frac{28}{9}, \frac{56}{9}, \frac{83}{9}\right).$$

To verify perpendicularity directly, $\overrightarrow{HA} = (8/9, 7/9, -11/9)$, whose dot product with $(1, 2, 2)$ is $(8 + 14 - 22)/9 = 0$. The common ratio in the equation of CD at H is $37/9$, verifying that the foot lies on the same line. Since this exceeds the ratio 2 at D , the foot lies on CD produced beyond D ; perpendicular projection is onto the line containing the side.

Step 4: Conclusion The four side equations are those displayed above, and the foot of the perpendicular from A to CD is $\left(\frac{28}{9}, \frac{56}{9}, \frac{83}{9}\right)$.

Quick Tip to Solve: Reduce the direction vector of CD to $(1, 2, 2)$ before projecting; the squared length is then 9, which keeps the perpendicular-foot calculation short.

34. Prove that a function $f : [0, \infty) \rightarrow [-5, \infty)$ defined as $f(x) = 4x^2 + 4x - 5$ is both one-one and onto. [5 marks]

Solution:

Concept:

- A function is one-one if $f(a) = f(b)$ implies $a = b$ for all domain elements a, b .
- A function is onto if every element y of its codomain has a preimage in its domain. Any inverse expression must respect the domain restriction.

Step 1: Check the stated codomain For every $x \geq 0$, both $4x^2$ and $4x$ are nonnegative. Consequently $f(x) = 4x^2 + 4x - 5 \geq -5$, so the output belongs to $[-5, \infty)$. The endpoint is attained because $f(0) = -5$. This verifies that the formula defines a function with the stated domain and codomain.

Step 2: Prove the function is one-one Let $a, b \in [0, \infty)$ satisfy $f(a) = f(b)$. Then

$$\begin{aligned}4a^2 + 4a - 5 &= 4b^2 + 4b - 5, \\ a^2 - b^2 + a - b &= 0, \\ (a - b)(a + b + 1) &= 0.\end{aligned}$$

Since $a, b \geq 0$, we have $a + b + 1 \geq 1$. The second factor cannot vanish, and therefore $a - b = 0$, giving $a = b$. This implication holds for arbitrary inputs in the domain and proves injectivity. The restriction to nonnegative inputs is essential when ruling out the second factor; one cannot simply assume that every quadratic function is one-one.

Quick Tip to Solve: Factor the equal-output equation as $(a - b)(a + b + 1) = 0$; for onto, complete the square and check that the inverse input is nonnegative.

Step 3: Construct a preimage for every codomain element Choose an arbitrary $y \in [-5, \infty)$. Solving $f(x) = y$ by completing the square gives

$$4x^2 + 4x - 5 = y \iff (2x + 1)^2 = y + 6.$$

An input $x \geq 0$ has $2x + 1 \geq 1$, so take the positive square root and set

$$x = \frac{\sqrt{y+6} - 1}{2}.$$

Because $y \geq -5$, we have $y + 6 \geq 1$ and $\sqrt{y+6} \geq 1$. Thus this input is real and nonnegative, and belongs to the domain. Direct substitution verifies

$$f(x) = (2x + 1)^2 - 6 = (\sqrt{y+6})^2 - 6 = y.$$

Every element of the codomain therefore has a preimage. In particular, $y = -5$ yields $x = 0$, so the endpoint is included as well. Since the choice of y was arbitrary, this proves surjectivity onto the entire stated codomain.

Step 4: Conclusion Hence $f : [0, \infty) \rightarrow [-5, \infty)$, $f(x) = 4x^2 + 4x - 5$, is both one-one and onto, so it is a bijection.

Quick Tip to Solve: Factor the equal-output equation as $(a - b)(a + b + 1) = 0$; for onto, complete the square and check that the inverse input is nonnegative.

35. The area of the region bounded by the line $y = mx$ ($m > 0$), the curve $x^2 + y^2 = 4$ and the x -axis in the first quadrant is $\frac{\pi}{2}$ units. Using integration, find the value of m . [5 marks]

Solution:

Concept:

- In the first quadrant the circle $x^2 + y^2 = 4$ has upper boundary $y = \sqrt{4 - x^2}$. Area between curves is the integral of upper ordinate minus lower ordinate.
- The line $y = mx$, the circular arc and the positive x -axis enclose a region whose upper boundary changes at the line-circle intersection.
- The substitution $x = 2 \sin \theta$ evaluates the integral involving $\sqrt{4 - x^2}$, with $0 \leq \theta \leq \pi/2$.

Step 1: Locate the line-circle intersection Write the first-quadrant intersection as $P = (a, ma)$. Substituting $y = mx$ into the circle gives

$$a^2 + m^2 a^2 = 4, \quad a = \frac{2}{\sqrt{1 + m^2}}.$$

As $m > 0$, we have $0 < a < 2$. The circular arc reaches the positive x -axis at $(2, 0)$, while the line reaches it at the origin. On $[0, a]$ the region is below the line, and on $[a, 2]$ it is below the circle. Both intervals have the x -axis as their lower boundary.

Quick Tip to Solve: Split the area at the line-circle intersection; the triangle term cancels the algebraic term from the circular integral.

Step 2: Express and evaluate the area by integration Therefore the required area A is

$$A = \int_0^a mx \, dx + \int_a^2 \sqrt{4-x^2} \, dx.$$

To evaluate the second integral, use $x = 2 \sin \theta$. Then $dx = 2 \cos \theta \, d\theta$, and the integrand becomes $4 \cos^2 \theta$. Hence

$$\begin{aligned} \int \sqrt{4-x^2} \, dx &= 2\theta + \sin 2\theta + C \\ &= 2 \sin^{-1}(x/2) + \frac{x}{2} \sqrt{4-x^2} + C. \end{aligned}$$

At $x = a$, the circle ordinate is $\sqrt{4-a^2} = ma$. Substituting both limits yields

$$\begin{aligned} A &= \frac{ma^2}{2} + \pi - 2 \sin^{-1}(a/2) - \frac{a}{2} \sqrt{4-a^2} \\ &= \pi - 2 \sin^{-1}(a/2). \end{aligned}$$

The two terms $ma^2/2$ cancel because the ordinate at the intersection is exactly ma . This cancellation uses the intersection condition rather than an assumption about the angle of the line.

Step 3: Use the given area to determine the slope Since $A = \pi/2$, the evaluated integral gives

$$\pi - 2 \sin^{-1}(a/2) = \frac{\pi}{2}, \quad \sin^{-1}(a/2) = \frac{\pi}{4}.$$

The inverse sine lies in $(0, \pi/2)$, so this implies $a/2 = 1/\sqrt{2}$, or $a = \sqrt{2}$. Returning to $a^2(1+m^2) = 4$, we get

$$2(1+m^2) = 4 \implies m^2 = 1.$$

The question states $m > 0$, which selects $m = 1$. For this slope the intersection is $(\sqrt{2}, \sqrt{2})$, and substitution in the area formula gives $\pi - 2(\pi/4) = \pi/2$, verifying the required area.

Step 4: Conclusion The required value of the positive slope is $\boxed{m = 1}$.

Quick Tip to Solve: Split the area at the line-circle intersection; the triangle term cancels the algebraic term from the circular integral.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. In order to set up a rain water harvesting system, a tank to collect rain water is to be dug. The tank should have a square base and a capacity of 250 m^3 . The cost of land is ₹ 5,000 per square metre and cost of digging increases with depth and for the whole tank, it is ₹ $40,000 h^2$, where h is the depth of the tank in metres. x is the side of the square base of the tank in metres.

Based on the above information, answer the following questions:

[4 marks]

36(i). Find the total cost C of digging the tank in terms of x .

[1 mark]

Solution:

Concept:

- The volume of a tank with square base of side x and depth h is x^2h .
- The total expenditure is the land cost plus the digging cost; here $x > 0$.

Step 1: Express the depth using the fixed capacity The capacity condition gives $x^2h = 250$, so $h = 250/x^2$ metres. This relation allows us to express both expenditures using the same independent variable.

Step 2: Add the two cost components The land area is x^2 square metres, so its cost is $5000x^2$ rupees. The digging cost for the whole tank is $40000h^2$, not this amount multiplied by the volume. Hence

$$C(x) = 5000x^2 + 40000 \left(\frac{250}{x^2} \right)^2 = 5000x^2 + \frac{2500000000}{x^4}.$$

Step 3: Conclusion $C(x) = 5000x^2 + \frac{2500000000}{x^4}$ rupees, $x > 0$.

Quick Tip to Solve: Use the volume constraint to eliminate depth before adding the two costs.

36(ii). Find $\frac{dC}{dx}$.

[1 mark]

Solution:

Concept:

- For a power x^n , $\frac{d}{dx}x^n = nx^{n-1}$.
- Differentiate the cost after eliminating h , since depth changes when the base side changes.

Step 1: Write the cost as powers of the side length The volume relation $x^2h = 250$ gives $h = 250/x^2$. Thus the cost in rupees is

$$C(x) = 5000x^2 + 2500000000x^{-4}, \quad x > 0.$$

Writing the reciprocal term with a negative exponent makes the power rule directly applicable.

Step 2: Differentiate each term The land-cost term contributes $5000(2x) = 10000x$. The digging-cost

Quick Tip to Solve: Rewrite the inverse fourth power as x to the power minus four before differentiating.

term contributes $2500000000(-4)x^{-5} = -10000000000x^{-5}$. Therefore

$$\frac{dC}{dx} = 10000x - 10000000000x^{-5}.$$

Step 3: Conclusion $\frac{dC}{dx} = 10000x - \frac{10000000000}{x^5}$, for $x > 0$.

Quick Tip to Solve: Rewrite the inverse fourth power as x to the power minus four before differentiating.

36(iii)(a). Find the value of x for which cost C is minimum.

[2 marks]

Solution:

Concept:

- A differentiable cost has a stationary point where $C'(x) = 0$.
- If $C''(x) > 0$ throughout the physical domain, the stationary point is a minimum.

Step 1: Find the stationary side length Using $h = 250/x^2$, the cost is $C(x) = 5000x^2 + 2500000000x^{-4}$. Differentiation gives

$$C'(x) = 10000x - \frac{10000000000}{x^5}.$$

Since $x > 0$, setting this derivative to zero and multiplying by x^5 gives

$$10000x^6 = 10000000000, \quad x^6 = 1000000 = 10^6.$$

The unique positive stationary value is therefore $x = 10$. Negative side lengths are outside the domain.

Step 2: Verify that the stationary point minimizes cost Differentiating once more,

$$C''(x) = 10000 + \frac{50000000000}{x^6} > 0 \quad (x > 0).$$

In particular, $C''(10) = 60000 > 0$. Also $C'(x) = 10000(x^6 - 10^6)/x^5$ is negative below 10 and positive above 10, so cost decreases up to this point and increases thereafter. It is the minimum over the whole physical domain. The corresponding depth is $250/10^2 = 2.5$ metres.

Step 3: Conclusion The total cost is minimum when $x = 10$ m.

Quick Tip to Solve: Multiply the stationary equation by the positive fifth power of x to obtain a sixth power.

36(iii)(b). Check whether the cost function $C(x)$ expressed in terms of x is increasing or not, where $x > 0$. [2 marks]

Solution:

Concept:

- A function is increasing on an interval where its derivative is positive and decreasing where its derivative is negative.
- For $x > 0$, multiplication or division by x^5 preserves the sign of an expression.

Step 1: Calculate and factor the derivative The volume constraint gives $h = 250/x^2$. Hence $C(x) = 5000x^2 + 2500000000/x^4$, and

$$C'(x) = 10000x - \frac{10000000000}{x^5} = \frac{10000(x^6 - 10^6)}{x^5}.$$

Throughout the domain $x > 0$, the denominator and the factor 10000 are positive. Therefore the sign depends only on $x^6 - 10^6$.

Step 2: Check the intervals on either side of the critical point For $0 < x < 10$, we have $x^6 < 10^6$, so $C'(x) < 0$ and cost decreases. At $x = 10$, the derivative is zero. For $x > 10$, we have $x^6 > 10^6$, so $C'(x) > 0$ and cost increases. Thus the function does not increase throughout the given domain; smaller base sizes first reduce digging expenditure enough to lower the total cost, while beyond the critical size the increasing land cost dominates.

Step 3: Conclusion $C(x)$ is not increasing on all of $x > 0$. It is decreasing on $(0, 10)$ and increasing on $(10, \infty)$.

Quick Tip to Solve: Factor the derivative and use positivity of x to compare x with 10.

37. An octagonal prism is a three-dimensional polyhedron bounded by two octagonal bases and eight rectangular side faces. It has 24 edges and 16 vertices.

The prism is rolled along the rectangular faces and number on the bottom face (touching the ground) is noted. Let X denote the number obtained on the bottom face and the following table gives the probability distribution of X .

$X :$	1	2	3	4	5	6	7	8
$P(X) :$	p	$2p$	$2p$	p	$2p$	p^2	$2p^2$	$7p^2 + p$

Based on the above information, answer the following questions:

[4 marks]

37(i). Find the value of p .

[1 mark]

Solution:

Concept:

- The probabilities in a discrete distribution sum to 1.
- Each probability is nonnegative; in particular the entry $P(X = 1) = p$ requires $p \geq 0$.

Quick Tip to Solve: Include the extra linear p in the eighth column when summing the distribution.

Step 1: Normalize the distribution Adding all eight entries of the table gives

$$p + 2p + 2p + p + 2p + p^2 + 2p^2 + (7p^2 + p) = 1.$$

Combining like terms yields $10p^2 + 9p - 1 = 0$. The linear contribution is $9p$, including the p in the last column.

Step 2: Select the admissible root Factor the equation as

$$(10p - 1)(p + 1) = 0.$$

Its roots are $p = 1/10$ and $p = -1$. The negative root would make $P(X = 1)$ negative, so it cannot represent a probability. For $p = 1/10$, all the tabulated entries are nonnegative and their sum is 1.

Step 3: Conclusion $p = \frac{1}{10}$.

Quick Tip to Solve: Include the extra linear p in the eighth column when summing the distribution.

37(ii). Find $P(X > 6)$.

[1 mark]

Solution:

Concept:

- Probabilities of mutually exclusive outcomes add.
- The event $X > 6$ contains only the outcomes 7 and 8 in this distribution.

Step 1: Obtain the distribution parameter Normalizing the table gives $10p^2 + 9p = 1$, or $(10p - 1)(p + 1) = 0$. Since p itself is a probability, the admissible root is $p = 1/10$; the negative root is rejected.

Step 2: Add the probabilities for the required outcomes The inequality is strict, so the outcome 6 is excluded. Using the columns for 7 and 8,

$$P(X > 6) = P(X = 7) + P(X = 8) = 2p^2 + (7p^2 + p) = 9p^2 + p.$$

Substituting $p = 1/10$ gives $9/100 + 1/10 = 19/100$.

Step 3: Conclusion $P(X > 6) = \frac{19}{100}$.

Quick Tip to Solve: For a strict inequality above 6, add only columns 7 and 8.

37(iii)(a). Find $P(X = 3m)$, where m is a natural number.

[2 marks]

Solution:

Concept:

- The event $X = 3m$, with m a natural number, means that the observed value is a positive multiple of 3.

Quick Tip to Solve: List the multiples of 3 that lie between 1 and 8 before reading their probabilities.

- For disjoint outcomes, the probability of their union is the sum of their individual probabilities.

Step 1: Identify the possible multiples in the support The distribution is supported on $\{1, 2, 3, 4, 5, 6, 7, 8\}$. The positive multiples of 3 in this set are 3 and 6, corresponding to $m = 1$ and $m = 2$. Values $m \geq 3$ give $3m \geq 9$, outside the support, and make no contribution. Thus the required event is $\{X = 3\} \cup \{X = 6\}$.

Step 2: Evaluate the sum using the normalized distribution The total probability condition is $10p^2 + 9p = 1$, giving $(10p - 1)(p + 1) = 0$. Nonnegativity of p selects $p = 1/10$. From the table, $P(X = 3) = 2p$ whereas $P(X = 6) = p^2$. These outcomes cannot occur together on one roll, so

$$P(X = 3m) = P(X = 3) + P(X = 6) = 2p + p^2 = \frac{2}{10} + \frac{1}{100} = \frac{21}{100}.$$

Step 3: Conclusion $P(X = 3m) = \frac{21}{100}$, where the event consists of the outcomes 3 and 6.

Quick Tip to Solve: List the multiples of 3 that lie between 1 and 8 before reading their probabilities.

OR

37(iii)(b). Find the mean $E(X)$.

[2 marks]

Solution:

Concept:

- For a discrete random variable, the mean is $E(X) = \sum_x xP(X = x)$: multiply each possible value by its probability and add.
- The probabilities must be non-negative and sum to 1. These conditions determine the permissible value of p .

Step 1: Determine the probability parameter Using all eight entries of the distribution,

$$p + 2p + 2p + p + 2p + p^2 + 2p^2 + (7p^2 + p) = 1.$$

Thus $10p^2 + 9p - 1 = 0$, or $(10p - 1)(p + 1) = 0$. The roots are $p = \frac{1}{10}$ and $p = -1$. The negative root would give negative probabilities, so only $p = \frac{1}{10}$ is valid.

Step 2: Calculate the weighted sum Include the contribution from every possible bottom face:

$$\begin{aligned} E(X) &= 1(p) + 2(2p) + 3(2p) + 4(p) + 5(2p) \\ &\quad + 6(p^2) + 7(2p^2) + 8(7p^2 + p) \\ &= 33p + 76p^2. \end{aligned}$$

Substituting the valid parameter gives

$$E(X) = 33 \left(\frac{1}{10} \right) + 76 \left(\frac{1}{100} \right) = \frac{330 + 76}{100}.$$

Quick Tip to Solve: Group the weighted terms into coefficients of p and p^2 , but remember that face 8 contributes to both groups.

The mean is a probability-weighted average, so it need not be an integer even though each observed face number is an integer.

Step 3: Conclusion The mean is $E(X) = \frac{203}{50} = 4.06$.

Quick Tip to Solve: Group the weighted terms into coefficients of p and p^2 , but remember that face 8 contributes to both groups.

38. A volleyball player serves the ball which takes a parabolic path given by the equation $h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1$, where $h(t)$ is the height of ball at any time t (in seconds), ($t \geq 0$).

Based on the above information, answer the following questions:

[4 marks]

38(i). Is $h(t)$ a continuous function? Justify.

[2 marks]

Solution:

Concept:

- A function is continuous at a point a in its domain when $\lim_{t \rightarrow a} h(t) = h(a)$; at a domain endpoint, use the limit from within the domain.
- Every polynomial is continuous on the real line. Sums and constant multiples of continuous functions remain continuous.

Step 1: Identify the form of the height function The given function is $h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1$, with $t \geq 0$. It is a polynomial of degree two in t . The functions t^2 , t , and the constant function are continuous everywhere, so their stated linear combination is continuous.

Step 2: Verify the continuity condition on the given domain For any $a > 0$, the limit laws give

$$\lim_{t \rightarrow a} h(t) = -\frac{7}{2}a^2 + \frac{13}{2}a + 1 = h(a).$$

At the endpoint $a = 0$, approach from the permitted side:

$$\lim_{t \rightarrow 0^+} h(t) = 1 = h(0).$$

Thus the continuity condition holds at every interior point and at the initial time. Restricting this polynomial to non-negative time introduces no break in its values.

Step 3: Conclusion Yes, $h(t)$ is continuous for every $t \geq 0$, because it is a polynomial.

Quick Tip to Solve: For polynomial height models, cite polynomial continuity and check the initial endpoint with a right-hand limit if the domain starts at zero.

38(ii). Find the time at which the height of the ball is maximum.

[2 marks]

Solution:

Concept:

- At an interior maximum of a differentiable function, its first derivative is zero. Here h denotes height and t denotes time in seconds.
- If $h''(t) < 0$, the curve is concave downward and a stationary point is a maximum. A change of h' from positive to negative also establishes a maximum.

Step 1: Find the stationary time Differentiate the given height polynomial with respect to time:

$$h'(t) = \frac{d}{dt} \left(-\frac{7}{2}t^2 + \frac{13}{2}t + 1 \right) = -7t + \frac{13}{2}.$$

At a stationary height,

$$-7t + \frac{13}{2} = 0 \implies t = \frac{13}{14}.$$

This value is positive and therefore belongs to the specified time domain.

Step 2: Confirm that the height is greatest at this time The second derivative is

$$h''(t) = -7 < 0.$$

Hence the stationary point is a maximum. More explicitly, $h'(t) > 0$ for $0 \leq t < \frac{13}{14}$, and $h'(t) < 0$ for $t > \frac{13}{14}$. The height therefore increases up to this time and decreases afterwards, establishing the greatest height on the non-negative time domain.

Step 3: Conclusion The height of the ball is maximum at $t = \frac{13}{14} \text{ s}$.

Quick Tip to Solve: Set the derivative of the height to zero, then use the negative quadratic coefficient or second derivative to confirm the maximum.