# CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-3-3

## Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

**Time Allowed: 3 hours** **Maximum Marks: 80**

### General Instructions

(i) This question paper contains **38** questions. **All questions are compulsory.**

(ii) This question paper is divided into **five** Sections – **A, B, C, D and E.**

(iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19 and 20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculators is not allowed.

## Section A (20 marks)

*This section comprises multiple choice questions (MCQs) of 1 mark each.*

**1. If the angle between the vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$ and $|\vec{a} \times \vec{b}| = 1$, then $\vec{a} \cdot \vec{b}$ is equal to:** [1 mark]

(A) $-1$

(B) 1

(C) $\frac{1}{\sqrt{2}}$

(D) $\sqrt{2}$

**Correct Answer:** (B) 1

**Solution:**

**Concept:**

- For angle $\theta$, $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ and $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.
- The dot product is positive when the angle is acute.

**Step 1:** Find the product of magnitudes Here $\theta = \pi/4$, so

$$1 = |\vec{a}||\vec{b}|\frac{1}{\sqrt{2}}, \qquad |\vec{a}||\vec{b}| = \sqrt{2}.$$

The cross product gives the sine component, not the full product of magnitudes.

**Step 2:** Evaluate and compare Therefore

$$\vec{a} \cdot \vec{b} = \sqrt{2}\cos(\pi/4) = \sqrt{2}\frac{1}{\sqrt{2}} = 1.$$

Option (A) has the wrong sign; (C) omits the product of magnitudes, and (D) gives that product before multiplying by the cosine.

**Step 3:** Conclusion The dot product is 1. Hence, the correct option is (B).

**Quick Tip to Solve:** At an angle of $\pi/4$, sine equals cosine, so the dot product equals the cross-product magnitude.

**2. $\vec{a}$ and $\vec{b}$ are two non-zero vectors such that the projection of $\vec{a}$ on $\vec{b}$ is $0$. The angle between $\vec{a}$ and $\vec{b}$ is:** [1 mark]

(A) $\frac{\pi}{2}$
(B) $\pi$
(C) $\frac{\pi}{4}$
(D) 0

**Correct Answer:** (A) $\frac{\pi}{2}$

**Solution:**

**Concept:**

- The scalar projection of $\vec{a}$ on non-zero $\vec{b}$ is $\vec{a} \cdot \vec{b}/|\vec{b}| = |\vec{a}| \cos\theta$.
- The angle between vectors lies in $[0, \pi]$.

**Step 1:** Use the zero projection The given projection is zero, hence

$$|\vec{a}| \cos\theta = 0.$$

Since $\vec{a}$ is non-zero, its magnitude is strictly positive and division gives $\cos\theta = 0$. This also means the two vectors are perpendicular.

**Step 2:** Identify the angle On $[0, \pi]$, the only zero of cosine is $\theta = \pi/2$. The other printed angles $\pi, \pi/4, 0$ have cosines $-1, 1/\sqrt{2}, 1$, respectively, and cannot give a zero projection of a non-zero vector.

**Step 3:** Conclusion The angle is $\frac{\pi}{2}$. Hence, the correct option is (A).

**Quick Tip to Solve:** Zero projection of a non-zero vector onto another non-zero vector means perpendicular directions.

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**3. In $\triangle ABC$, $\vec{AB} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{AC} = 3\hat{i} - \hat{j} + 4\hat{k}$. If $D$ is mid-point of $BC$, then vector $\vec{AD}$ is equal to:** [1 mark]

(A) $4\hat{i} + 6\hat{k}$
(B) $2\hat{i} - 2\hat{j} + 2\hat{k}$
(C) $\hat{i} - \hat{j} + \hat{k}$
(D) $2\hat{i} + 3\hat{k}$

**Correct Answer:** (D) $2\hat{i} + 3\hat{k}$

**Solution:**

**Concept:**

- The position vector of a midpoint is the average of the endpoint position vectors.
- Taking $A$ as reference gives $\vec{AD} = (\vec{AB} + \vec{AC})/2$.

**Step 1:** Apply the midpoint formula Since $D$ bisects $BC$, its vector from $A$ is

$$\vec{AD} = \frac{1}{2}\left[(\hat{i} + \hat{j} + 2\hat{k}) + (3\hat{i} - \hat{j} + 4\hat{k})\right].$$

**Quick Tip to Solve:** For a median from a vertex, average the two side vectors that start at that vertex.

Both vectors have the same initial point, so their average locates the midpoint.

**Step 2:** Combine the components The sum has components $(4, 0, 6)$; after division by two the components are $(2, 0, 3)$. Option (A) is the sum without division. Option (B) is the difference $\vec{AC} - \vec{AB}$, and (C) is half that difference; these concern the side $BC$, not the median from $A$.

**Step 3:** Conclusion $\vec{AD} = 2\hat{i} + 3\hat{k}$. Hence, the correct option is (D).

**Quick Tip to Solve:** For a median from a vertex, average the two side vectors that start at that vertex.

**4. The equation of a line passing through point $(2, -1, 0)$ and parallel to the line $\frac{x}{1} = \frac{y-1}{2} = \frac{2-z}{2}$ is:** [1 mark]

(A) $\frac{x+2}{1} = \frac{y-1}{2} = \frac{z}{2}$

(B) $\frac{x-2}{1} = \frac{y-1}{2} = \frac{z}{2}$

(C) $\frac{x+2}{1} = \frac{y-1}{2} = \frac{z}{-2}$

(D) $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z}{-2}$

**Correct Answer:** (D) $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z}{-2}$

**Solution:**

**Concept:**

- A line through $(x_0, y_0, z_0)$ with direction ratios $(a, b, c)$ has equation $(x - x_0)/a = (y - y_0)/b = (z - z_0)/c$.
- Parallel lines have proportional direction ratios.

**Step 1:** Read the direction carefully Writing the given common ratio as $t$, we get $x = t$, $y = 1 + 2t$, and $z = 2 - 2t$. Thus the direction ratios are $(1, 2, -2)$. The numerator $2 - z$ reverses the sign of the third direction ratio.

**Step 2:** Use the specified point Substitute $(2, -1, 0)$ into the point-direction form:

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z}{-2}.$$

Options (A) and (C) use the wrong point in the first numerator. Option (B) uses $y - 1$ instead of $y + 1$ and also has the wrong third direction ratio.

**Step 3:** Conclusion The required equation is $\frac{x-2}{1} = \frac{y+1}{2} = \frac{z}{-2}$. Hence, the correct option is (D).

**Quick Tip to Solve:** Convert a reversed numerator such as $2 - z$ to $z - 2$ by changing the denominator sign.

**5.** $X$ **and** $Y$ **are independent events such that** $P(X \cap \overline{Y}) = \frac{2}{5}$ **and** $P(X) = \frac{3}{5}$**. Then** $P(Y)$ **is equal to:** [1 mark]

(A) $\frac{2}{3}$

(B) $\frac{2}{5}$

(C) $\frac{1}{3}$

(D) $\frac{1}{5}$

**Correct Answer:** (C) $\frac{1}{3}$

**Solution:**

**Concept:**

- For independent events, $P(X \cap Y) = P(X)P(Y)$. Independence also gives $P(X \cap \overline{Y}) = P(X)P(\overline{Y})$.
- Complementary probabilities satisfy $P(Y) + P(\overline{Y}) = 1$.

**Step 1:** Calculate the complementary probability The printed intersection involves $\overline{Y}$. Using independence,

$$\frac{2}{5} = \frac{3}{5}P(\overline{Y}), \qquad P(\overline{Y}) = \frac{2/5}{3/5} = \frac{2}{3}.$$

This intermediate value concerns the complement and must not be reported as $P(Y)$.

**Step 2:** Take the complement and check We obtain

$$P(Y) = 1 - \frac{2}{3} = \frac{1}{3}.$$

Indeed, $(3/5)(1 - 1/3) = 2/5$, agreeing with the given intersection. Option (A) gives the complement instead; (B) repeats the intersection probability, while (D) does not satisfy the independence equation.

**Step 3:** Conclusion $P(Y) = \frac{1}{3}$. Hence, the correct option is (C).

**Quick Tip to Solve:** Check for the bar over $Y$: divide to obtain the complementary probability, then subtract from one.

**6. The value of $k$ for which function $f(x) = \begin{cases} x^2, & x \geq 0 \\ kx, & x < 0 \end{cases}$ is differentiable at $x = 0$ is:** [1 mark]

(A) 1

(B) 2

(C) any real number

(D) 0

**Correct Answer:** (D) 0

**Solution:**

**Concept:**

- Differentiability at a joining point requires equal finite left-hand and right-hand derivatives.
- At $x = 0$, $f(0) = 0$; both branches tend to zero for any finite real $k$, so continuity alone imposes no restriction.

**Step 1:** Compute the one-sided derivatives Using the difference quotient with $f(0) = 0$,

$$f'_+(0) = \lim_{h \to 0^+} \frac{h^2}{h} = 0, \qquad f'_-(0) = \lim_{h \to 0^-} \frac{kh}{h} = k.$$

The branch on the appropriate side must be used in each limit.

**Step 2:** Match the two slopes The two derivative limits agree precisely when $k = 0$. If $k = 1$ or $k = 2$, the left-hand slope differs from the right-hand slope. Thus the option allowing any real number confuses continuity with differentiability.

**Step 3:** Conclusion The required value is $k = 0$. Hence, the correct option is (D).

**Quick Tip to Solve:** At a piecewise join, compare derivative limits; matching function limits only checks continuity.

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**7. If $y = \dfrac{\cos x - \sin x}{\cos x + \sin x}$, then $\dfrac{dy}{dx}$ is:** [1 mark]

(A) $-\sec^2\left(\dfrac{\pi}{4} - x\right)$

(B) $\sec^2\left(\dfrac{\pi}{4} - x\right)$

(C) $\log\left|\sec\left(\dfrac{\pi}{4} - x\right)\right|$

(D) $-\log\left|\sec\left(\dfrac{\pi}{4} - x\right)\right|$

**Correct Answer:** (A) $-\sec^2\left(\dfrac{\pi}{4} - x\right)$

**Solution:**

**Concept:**

- The tangent subtraction identity is $\tan(A - B) = (\tan A - \tan B)/(1 + \tan A \tan B)$.
- The chain rule gives $d(\tan u)/dx = \sec^2 u \, du/dx$.

**Quick Tip to Solve:** Recognise the quotient as $\tan(\pi/4 - x)$, and retain the negative inner derivative.

**Step 1:** Rewrite the quotient Using $\sin(\pi/4 - x) = (\cos x - \sin x)/\sqrt{2}$ and $\cos(\pi/4 - x) = (\cos x + \sin x)/\sqrt{2}$,

$$y = \tan\left(\frac{\pi}{4} - x\right).$$

This is valid throughout the domain of the original quotient, where $\cos x + \sin x \neq 0$.

**Step 2:** Differentiate and compare the options The inner derivative is $-1$, so

$$\frac{dy}{dx} = -\sec^2\left(\frac{\pi}{4} - x\right).$$

Option (B) loses the negative chain-rule factor. Options (C) and (D) are logarithmic expressions and are not the derivative of this tangent function. A direct quotient rule also gives $-2/(\cos x + \sin x)^2$, the same result.

**Step 3:** Conclusion $\frac{dy}{dx} = -\sec^2\left(\frac{\pi}{4} - x\right)$. Hence, the correct option is (A).

**Quick Tip to Solve:** Recognise the quotient as $\tan(\pi/4 - x)$, and retain the negative inner derivative.

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**8. The number of feasible solutions of the linear programming problem given as: Maximize $z = 15x + 30y$ subject to constraints: $3x + y \leq 12,\ x + 2y \leq 10,\ x \geq 0,\ y \geq 0$ is:** [1 mark]

(A) 1

(B) 2

(C) 3

(D) infinite

**Correct Answer:** (D) infinite

**Solution:**

**Concept:**

- A feasible solution is any pair satisfying all constraints, irrespective of its objective value.
- A continuous line segment with distinct endpoints contains infinitely many points.

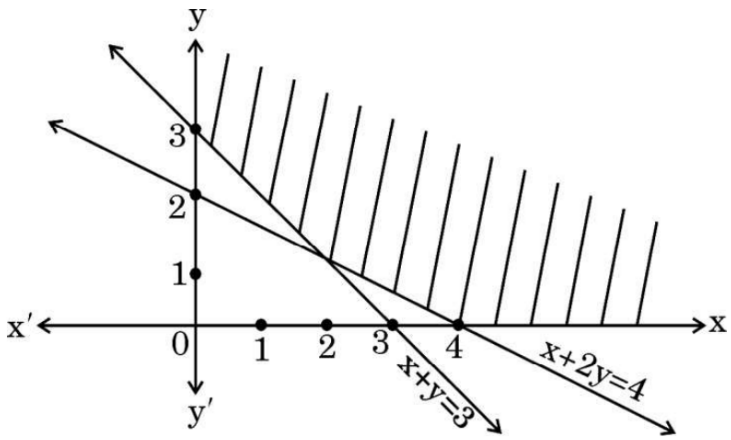
**Step 1:** Exhibit feasible points Set $y = 0$. The inequalities become $3x \leq 12$, $x \leq 10$, and $x \geq 0$. Thus every pair $(x, 0)$ with $0 \leq x \leq 4$ satisfies all four constraints. For example, $(0, 0)$, $(1, 0)$, and $(1/2, 0)$ are feasible.

**Step 2:** Count feasible solutions The real interval $[0, 4]$ contains infinitely many choices of $x$, already giving infinitely many feasible pairs on a single segment. The question asks for feasible solutions, not corner points or the point maximizing $z$. Consequently the finite counts in (A), (B), and (C) cannot describe the feasible set.

**Step 3:** Conclusion There are infinitely many feasible solutions. Hence, the correct option is (D).

**Quick Tip to Solve:** To establish infinitely many feasible pairs, find a non-zero interval of allowed values on one axis.

**9. The feasible region of a linear programming problem is shown in the figure below:**
**Which of the following are the possible constraints?** [1 mark]




(A) $x + 2y \geq 4,\ x + y \leq 3,\ x \geq 0,\ y \geq 0$
(B) $x + 2y \leq 4,\ x + y \leq 3,\ x \geq 0,\ y \geq 0$
(C) $x + 2y \geq 4,\ x + y \geq 3,\ x \geq 0,\ y \geq 0$
(D) $x + 2y \geq 4,\ x + y \geq 3,\ x \leq 0,\ y \leq 0$

**Correct Answer:** (C) $x + 2y \geq 4,\ x + y \geq 3,\ x \geq 0,\ y \geq 0$

**Solution:**

**Concept:**

- A feasible region is the intersection of all constraint half-planes.
- A point on a boundary line satisfies equality; the shaded side determines the inequality.

**Step 1:** Read the two boundaries The labelled boundaries are $x + 2y = 4$ and $x + y = 3$. The shading lies in the first quadrant, so $x \geq 0$ and $y \geq 0$. The region extends away from the origin beyond both lines, rather than forming a bounded region near the axes.

**Step 2:** Test the shaded side and alternatives For example, the point $(4, 2)$ lies on the shaded side of both boundaries. It gives $x + 2y = 8 \geq 4$ and $x + y = 6 \geq 3$. Thus both inequalities must have the greater-than-or-equal sign. Option (A) puts the region below $x + y = 3$, and (B) puts it below both lines; neither matches. Option (D) places it in the third quadrant and is inconsistent with the graph.

**Step 3:** Conclusion The constraints are $x + 2y \geq 4,\ x + y \geq 3,\ x \geq 0,\ y \geq 0$. Hence, the correct option is (C).

**Quick Tip to Solve:** Use one clearly shaded point to choose the inequality signs on both boundary lines.

**10.** $A$ **and** $B$ **are square matrices of same order. If** $(A+B)^2 = A^2 + B^2$**, then:** [1 mark]

(A) $AB = BA$

(B) $AB = -BA$

(C) $AB = O$

(D) $BA = O$

**Correct Answer:** (B) $AB = -BA$

**Solution:**

**Concept:**

- Matrix multiplication is distributive, but $AB$ need not equal $BA$.
- $(A+B)^2 = A^2 + AB + BA + B^2$.

**Step 1:** Expand without assuming commutativity Multiplying the two factors in their given order gives

$$(A+B)(A+B) = A^2 + AB + BA + B^2.$$

The two cross-products must be retained separately; the scalar shortcut $2AB$ is not valid for arbitrary matrices.

**Step 2:** Use the given equality and compare options Comparing with $A^2 + B^2$ and subtracting the same matrices from both sides gives $AB + BA = O$, or $AB = -BA$. The equality does not require $AB = BA$, nor does it require either product individually to vanish. For example, $A = \text{diag}(1, -1)$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ have nonzero products which are negatives of one another. This also rules out (A), (C) and (D) as necessary conclusions.

**Step 3:** Conclusion $AB = -BA$. Hence, the correct option is (B).

**Quick Tip to Solve:** Keep AB and BA separate when expanding a square of a matrix sum.

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**11. If** $A \cdot (\text{adj}\, A) = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$**, then the value of** $|A| + |\text{adj}\, A|$ **is equal to:** [1 mark]

(A) 12

(B) 9

(C) 3

(D) 27

**Correct Answer:** (A) 12

**Solution:**

**Concept:**

- For a square matrix, $A\,\text{adj}\, A = |A|I$.
- For an order-$n$ matrix, $|\,\text{adj}\, A| = |A|^{n-1}$.

**Step 1:** Find the determinant of A The given product is $3I_3$. Comparing it with the identity

**Quick Tip to Solve:** Read the matrix order before applying the exponent n-1 to the determinant of A.

$A \operatorname{adj} A = |A| I_3$ gives $|A| = 3$. The displayed matrix is of order three, which is essential when using the adjoint determinant formula.

**Step 2:** Evaluate the adjoint determinant and sum For order three,

$$|\operatorname{adj} A| = |A|^2 = 3^2 = 9, \qquad |A| + |\operatorname{adj} A| = 3 + 9 = 12.$$

Option (B) gives only the adjoint determinant, and option (C) gives only $|A|$. Option (D), 27, is the determinant of $3I_3$, namely $|A||\operatorname{adj} A|$, not the requested sum.

**Step 3:** Conclusion The required value is 12. Hence, the correct option is (A).

**Quick Tip to Solve:** Read the matrix order before applying the exponent n-1 to the determinant of A.

**12. $A$ and $B$ are skew-symmetric matrices of same order. $AB$ is symmetric, if:** [1 mark]

(A) $AB = O$

(B) $AB = -BA$

(C) $AB = BA$

(D) $BA = O$

**Correct Answer:** (C) $AB = BA$

**Solution:**

**Concept:**

- A skew-symmetric matrix satisfies $A^T = -A$.
- Transpose reverses product order: $(AB)^T = B^T A^T$.

**Step 1:** Transpose the product As $A$ and $B$ are skew-symmetric, $A^T = -A$ and $B^T = -B$. Hence

$$(AB)^T = B^T A^T = (-B)(-A) = BA.$$

The two minus signs cancel, but the reversal of the product order remains.

**Step 2:** Apply symmetry and examine the choices The product $AB$ is symmetric exactly when $(AB)^T = AB$. The preceding calculation therefore gives the condition $BA = AB$, option (C). The relation $AB = -BA$ instead makes the product skew-symmetric, except in the zero case. Options (A) and (D) describe special zero-product cases: transpose shows either implies the other, and the zero matrix is symmetric. They are sufficient special cases, but the general necessary-and-sufficient condition asked for here is commutativity.

**Step 3:** Conclusion The general condition is $AB = BA$. Hence, the correct option is (C).

**Quick Tip to Solve:** Reverse the order first when transposing a product, then substitute the skew-symmetry signs.

**13. For what value of $x \in \left[0, \frac{\pi}{2}\right]$, is $A + A' = \sqrt{3}\, I$, where $A = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix}$?** [1 mark]

(A) $\frac{\pi}{3}$
(B) $\frac{\pi}{6}$
(C) 0
(D) $\frac{\pi}{2}$

**Correct Answer:** (B) $\frac{\pi}{6}$

**Solution:**

**Concept:**

- The prime in $A\prime$ denotes transpose.
- The off-diagonal entries cancel in $A + A^T$, giving $2\cos x\, I$.

**Step 1:** Add the matrix and its transpose We have

$$A^T = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix}, \quad A + A^T = \begin{pmatrix} 2\cos x & 0 \\ 0 & 2\cos x \end{pmatrix}.$$

Thus only the diagonal entries determine the required equality.

**Step 2:** Solve in the stated interval Comparing with $\sqrt{3}I$ gives $2\cos x = \sqrt{3}$, hence $\cos x = \sqrt{3}/2$. On $[0, \pi/2]$, cosine is decreasing and this value occurs uniquely at $x = \pi/6$. The alternatives $\pi/3, 0, \pi/2$ yield diagonal entries $1, 2, 0$, respectively, none of which equals $\sqrt{3}$.

**Step 3:** Conclusion $x = \frac{\pi}{6}$. Hence, the correct option is (B).

**Quick Tip to Solve:** Cancel the sine entries first; the matrix equation then reduces to one cosine equation.

**14. Let $A$ be the area of a triangle having vertices $(x_1, y_1)$, $(x_2, y_2)$ and $(x_3, y_3)$. Which of the following is correct?** [1 mark]

(A) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm A$

(B) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 2A$

(C) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm \frac{A}{2}$

(D) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = A^2$

**Correct Answer:** (B) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \pm 2A$

**Solution:**

**Concept:**

- The area is half the absolute value of the coordinate determinant.

**Quick Tip to Solve:** Remember that the coordinate determinant is twice the signed area, not the area itself.

- The determinant is signed; changing vertex order can reverse its sign without changing area.

**Step 1:** Write the area formula Let

$$D = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The coordinate-area formula gives $A = \frac{1}{2}|D|$, since area is nonnegative whereas the determinant carries the orientation of the ordered vertices.

**Step 2:** Relate the signed determinant to the area Multiplying by two gives $|D| = 2A$, so the signed determinant satisfies $D = \pm 2A$. This is the equality printed in option (B). Options (A) and (C) have the wrong scale factors. Squaring the correct relation gives $D^2 = 4A^2$, not the expression $D^2 = A^2$ in option (D). The sign accounts for clockwise or anticlockwise vertex order.

**Step 3:** Conclusion The determinant equals $\pm 2A$. Hence, the correct option is (B).

**Quick Tip to Solve:** Remember that the coordinate determinant is twice the signed area, not the area itself.

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**15.** $\int 2^{x+2}\,dx$ **is equal to:** [1 mark]

(A) $2^{x+2} + C$

(B) $2^{x+2}\log 2 + C$

(C) $\dfrac{2^{x+2}}{\log 2} + C$

(D) $2 \cdot \dfrac{2^x}{\log 2} + C$

**Correct Answer:** (C) $\dfrac{2^{x+2}}{\log 2} + C$

**Solution:**

**Concept:**

- For $a > 0$, $a \neq 1$, $\int a^x\,dx = a^x / \log a + C$.
- The derivative of $2^{x+2}$ is $2^{x+2}\log 2$, since the exponent has derivative one.

**Step 1:** Apply the exponential integral formula The shift by two in the exponent does not change its derivative. Therefore

$$\int 2^{x+2}\,dx = \frac{2^{x+2}}{\log 2} + C.$$

Here $\log$ denotes the natural logarithm, and $C$ is an arbitrary constant.

**Step 2:** Check the options by differentiation Differentiating option (C) cancels the factor $\log 2$ and returns exactly $2^{x+2}$. Option (A) introduces an unwanted factor $\log 2$, and option (B) introduces $(\log 2)^2$. Option (D) differentiates to $2 \cdot 2^x = 2^{x+1}$, which is only half the required integrand $2^{x+2} = 4 \cdot 2^x$.

**Step 3:** Conclusion $\int 2^{x+2}\,dx = \dfrac{2^{x+2}}{\log 2} + C$. Hence, the correct option is (C).

**Quick Tip to Solve:** For an exponential with constant base, divide by the natural logarithm of that base.

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**16.** $\int e^{-x}\left(\frac{x+1}{x^2}\right)dx$ **is equal to:** [1 mark]

(A) $\frac{e^{-x}}{x}+C$

(B) $\frac{e^{x}}{x}+C$

(C) $\frac{e^{x}}{x^2}+C$

(D) $-\frac{e^{-x}}{x}+C$

**Correct Answer:** (D) $-\frac{e^{-x}}{x}+C$

**Solution:**

**Concept:**

- Use the product rule to recognize an integrand as a derivative.
- $\frac{d}{dx}(e^{-x}/x) = -e^{-x}(x+1)/x^2$, for $x \neq 0$.

**Step 1:** Differentiate the likely product Writing $e^{-x}/x = e^{-x}x^{-1}$, the product rule gives

$$\frac{d}{dx}(e^{-x}x^{-1}) = -e^{-x}x^{-1} - e^{-x}x^{-2} = -e^{-x}\frac{x+1}{x^2}.$$

The given integrand is the negative of this derivative.

**Step 2:** Choose the sign and check the alternatives Thus an antiderivative is $-e^{-x}/x$, on any interval excluding zero. Option (A) differentiates to the negative of the required integrand. Option (B) gives $e^{x}(x-1)/x^2$, and option (C) gives $e^{x}(x-2)/x^3$; both have the wrong exponential and rational factors. Differentiating option (D) gives exactly $e^{-x}(x+1)/x^2$.

**Step 3:** Conclusion $\int e^{-x}\frac{x+1}{x^2}\,dx = -\frac{e^{-x}}{x}+C$. Hence, the correct option is (D).

**Quick Tip to Solve:** Differentiate e to the power -x divided by x; the overall minus sign is decisive.

**17. The value of** $\int_0^{\pi/2} \log\tan x\,dx$ **is:** [1 mark]

(A) $\frac{\pi}{2}$

(B) 0

(C) $-\frac{\pi}{2}$

(D) 1

**Correct Answer:** (B) 0

**Solution:**

**Concept:**

- For an integrable function, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$.
- The identity $\tan(\pi/2 - x) = \cot x$ gives $\log\cot x = -\log\tan x$.

**Quick Tip to Solve:** Pair $x$ with $\pi/2 - x$: their logarithmic tangent values cancel.

**Step 1:** Apply reflection about the midpoint Let $I = \int_0^{\pi/2} \log \tan x \, dx$. The logarithmic singularities at the endpoints are integrable. Substituting $u = \pi/2 - x$ and reversing the limits gives

$$I = \int_0^{\pi/2} \log \cot u \, du = -\int_0^{\pi/2} \log \tan u \, du = -I.$$

**Step 2:** Compare all four choices Thus $2I = 0$, so $I = 0$. Option (B) agrees with this value. The nonzero values $\pi/2$, $-\pi/2$, and $1$ in (A), (C), and (D) cannot satisfy $I = -I$.

**Step 3:** Conclusion The value is $\boxed{0}$. Hence, the correct option is (B).

**Quick Tip to Solve:** Pair $x$ with $\pi/2 - x$: their logarithmic tangent values cancel.

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**18. What is the product of the order and degree of the differential equation $\frac{d^2y}{dx^2} \sin y + \left(\frac{dy}{dx}\right)^3 \cos y = \sqrt{y}$?** [1 mark]

(A) 3

(B) 2

(C) 6

(D) not defined

**Correct Answer:** (B) 2

**Solution:**

**Concept:**

- The order is the order of the highest derivative present in a differential equation.
- The degree is the exponent of the highest-order derivative after the equation is expressed as a polynomial in its derivatives.

**Step 1:** Identify the highest derivative In $y'' \sin y + (y')^3 \cos y = \sqrt{y}$, the highest derivative is $y'' = d^2y/dx^2$. Therefore the order is 2. The cube of $y'$ does not change the order.

**Step 2:** Determine the degree and check options The equation is already polynomial in its derivatives, and $y''$ has exponent 1. Thus the degree is 1 and the product is $2 \times 1 = 2$. The radical involves $y$, not a derivative, so the degree is defined. Options (A) and (C) incorrectly use the power 3 of the first derivative; option (D) is also incorrect.

**Step 3:** Conclusion The required product is $\boxed{2}$. Hence, the correct option is (B).

**Quick Tip to Solve:** Read the degree from the highest-order derivative, not the largest power anywhere in the equation.

**19. Assertion (A): Range of $[\sin^{-1} x + 2\cos^{-1} x]$ is $[0, \pi]$.**

**Reason (R): Principal value branch of $\sin^{-1} x$ has range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.** [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

**Correct Answer:** (D) Assertion (A) is false and Reason (R) is true.

**Solution:**

**Concept:**

- For $-1 \le x \le 1$, $\sin^{-1} x + \cos^{-1} x = \pi/2$.
- The principal range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$.

**Step 1:** Test the assertion Using the inverse-trigonometric identity,

$$\sin^{-1} x + 2\cos^{-1} x = \pi - \sin^{-1} x.$$

As $\sin^{-1} x$ varies from $-\pi/2$ to $\pi/2$, this expression ranges over $[\pi/2, 3\pi/2]$, not $[0, \pi]$. Both endpoints are attained at $x = 1$ and $x = -1$, respectively. Assertion (A) is false.

**Step 2:** Test the reason The stated range $[-\pi/2, \pi/2]$ is exactly the principal-value range of $\sin^{-1} x$. Reason (R) is true.

**Step 3:** Assess the explanation and code A true reason cannot explain the false asserted range. The combination is therefore false Assertion and true Reason, which is code (D); (A), (B), and (C) each require the Assertion to be true.

**Step 4:** Conclusion The assertion is false and the reason is true. Hence, the correct option is (D).

**Quick Tip to Solve:** Reduce mixed inverse sine and cosine expressions to one function before finding their range.

**20. Assertion (A): A line through the points $(4, 7, 8)$ and $(2, 3, 4)$ is parallel to a line through the points $(-1, -2, 1)$ and $(1, 2, 5)$.**

**Reason (R): Lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are parallel if $\vec{b_1} \cdot \vec{b_2} = 0$.** [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

**Correct Answer:** (C) Assertion (A) is true and Reason (R) is false.

**Solution:**

**Concept:**

- A direction vector of a line through points $P, Q$ is $\overrightarrow{PQ} = Q - P$.
- Two lines have parallel directions when their nonzero direction vectors are scalar multiples; zero dot product indicates perpendicular directions.

**Step 1:** Test the assertion The first line has direction vector $\vec{b_1} = (2-4, 3-7, 4-8) = (-2, -4, -4)$. The second has $\vec{b_2} = (1+1, 2+2, 5-1) = (2, 4, 4)$. Since $\vec{b_1} = -\vec{b_2}$, their directions are parallel. Assertion (A) is true.

**Step 2:** Test the reason The condition $\vec{b_1} \cdot \vec{b_2} = 0$ means the nonzero direction vectors are perpendicular, not parallel. Here their dot product is $-4 - 16 - 16 = -36$, even though the lines are parallel. Reason (R) is false.

**Step 3:** Assess the explanation and code The false Reason cannot explain the true Assertion. The combination true Assertion and false Reason is option (C). Options (A) and (B) incorrectly call the Reason true, while (D) incorrectly calls the Assertion false.

**Step 4:** Conclusion The assertion is true and the reason is false. Hence, the correct option is (C).

**Quick Tip to Solve:** Compare direction-vector ratios for parallel lines; use a dot product to test perpendicular directions.

## Section B (10 marks)

*This section comprises very short answer (VSA) type questions of 2 marks each.*

**21(a). If $y = x^{\frac{1}{x}}$, then find $\frac{dy}{dx}$ at $x = 1$.** [2 marks]

**Solution:**

**Concept:**

- Logarithmic differentiation is useful when both the base and exponent depend on $x$.
- For a positive differentiable function $y$, $d(\log y)/dx = y^{-1}dy/dx$.

**Step 1:** Take logarithms on the real domain For $x > 0$, the function $y = x^{1/x}$ is positive, so

**Quick Tip to Solve:** Take logarithms first when the exponent varies, then substitute the point after differentiating.

taking natural logarithms is valid. The variable exponent must also be differentiated; the ordinary fixed-exponent power rule alone does not apply. We obtain

$$\log y = \frac{\log x}{x}.$$

**Step 2:** Differentiate using the quotient rule Differentiating both sides with respect to $x$,

$$\frac{1}{y}\frac{dy}{dx} = \frac{x(1/x) - \log x}{x^2} = \frac{1 - \log x}{x^2}.$$

Hence

$$\frac{dy}{dx} = x^{1/x}\frac{1 - \log x}{x^2}.$$

This formula is valid in a neighbourhood of $x = 1$, where all the factors are defined.

**Step 3:** Evaluate at the specified point At $x = 1$, $y = 1^{1/1} = 1$ and $\log 1 = 0$. Therefore

$$\left.\frac{dy}{dx}\right|_{x=1} = 1 \times \frac{1 - 0}{1^2} = 1.$$

**Step 4:** Conclusion The required derivative is $\boxed{\left.\frac{dy}{dx}\right|_{x=1} = 1}$.

**Quick Tip to Solve:** Take logarithms first when the exponent varies, then substitute the point after differentiating.

OR

**21(b). If $x = a\sin 2t$, $y = a(\cos 2t + \log\tan t)$, then find $\frac{dy}{dx}$.** [2 marks]

**Solution:**

**Concept:**

- For parametric functions $x(t), y(t)$, $dy/dx = (dy/dt)/(dx/dt)$ wherever $dx/dt \neq 0$.
- The identities $\sin 2t = 2\sin t\cos t$ and $1 - \sin^2 2t = \cos^2 2t$ simplify this derivative.

**Step 1:** Differentiate the parameter functions From $x = a\sin 2t$,

$$\frac{dx}{dt} = 2a\cos 2t.$$

For $y = a(\cos 2t + \log\tan t)$, the chain rule gives

$$\frac{dy}{dt} = a\left(-2\sin 2t + \frac{\sec^2 t}{\tan t}\right).$$

We work where $\log\tan t$ is real and defined, and where the parametric derivative exists.

**Quick Tip to Solve:** Rewrite the logarithmic tangent derivative as $2/\sin 2t$ before forming the parametric quotient.

**Step 2:** Simplify the numerator Using $\sec^2 t/\tan t = 1/(\sin t \cos t) = 2/\sin 2t$,

$$\frac{dy}{dt} = 2a\left(\frac{1}{\sin 2t} - \sin 2t\right)$$
$$= \frac{2a(1-\sin^2 2t)}{\sin 2t} = \frac{2a\cos^2 2t}{\sin 2t}.$$

**Step 3:** Divide by the derivative of x For $a \neq 0$ and $\cos 2t \neq 0$, dividing gives

$$\frac{dy}{dx} = \frac{2a\cos^2 2t/\sin 2t}{2a\cos 2t} = \frac{\cos 2t}{\sin 2t} = \cot 2t.$$

Cancellation is justified only where $dx/dt \neq 0$; the formula describes the regular points of the curve.

**Step 4:** Conclusion The required derivative is $\boxed{\frac{dy}{dx} = \cot 2t}$ wherever the parametric derivative is defined.

**Quick Tip to Solve:** Rewrite the logarithmic tangent derivative as $2/\sin 2t$ before forming the parametric quotient.

**22. If $\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k}$, find the value of $(\vec{r} \times \hat{j}) \cdot (\vec{r} \times \hat{k}) - 12$.** [2 marks]

**Solution:**

**Concept:**

- The cross product is bilinear and follows the cyclic rules $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, and $\hat{k} \times \hat{i} = \hat{j}$. Reversing either order changes the sign.
- The dot product of vectors in an orthonormal basis is the sum of products of corresponding components: $(a, b, c) \cdot (p, q, s) = ap + bq + cs$.

**Step 1:** Compute both cross products Using $\vec{r} = 3\hat{i} - 2\hat{j} + 6\hat{k}$, expand each product separately. The term parallel to the second vector contributes zero. Thus

$$\vec{r} \times \hat{j} = 3\hat{k} + 6(-\hat{i}) = -6\hat{i} + 3\hat{k},$$
$$\vec{r} \times \hat{k} = 3(-\hat{j}) - 2\hat{i} = -2\hat{i} - 3\hat{j}.$$

**Step 2:** Evaluate the scalar expression Only the common $\hat{i}$-components contribute to the dot product, since distinct unit vectors are perpendicular. Therefore

$$(\vec{r} \times \hat{j}) \cdot (\vec{r} \times \hat{k}) = (-6)(-2) + (0)(-3) + (3)(0) = 12.$$

The required expression subtracts 12 from this dot product, giving $12 - 12 = 0$.

**Step 3:** Conclusion The required value is $\boxed{0}$.

**Quick Tip to Solve:** Expand the cross products cyclically, then multiply only matching components; remember to subtract the final constant.

**23. Find the direction cosines of the line whose Cartesian equations are $5x - 3 = 15y + 7 = 3 - 10z$.**

[2 marks]

**Solution:**

**Concept:**

- Direction ratios are the coefficients of a parameter when a line is written as $x = x_0 + at$, $y = y_0 + bt$, $z = z_0 + ct$.
- Direction cosines are obtained by normalising direction ratios: $(l, m, n) = (a, b, c)/\sqrt{a^2 + b^2 + c^2}$. Reversing the direction changes all three signs together.

**Step 1:** Introduce a common parameter Let the common value of the three given expressions be $t$. Then

$$5x - 3 = t, \qquad 15y + 7 = t, \qquad 3 - 10z = t.$$

Solving each equation gives

$$x = \frac{3}{5} + \frac{t}{5}, \qquad y = -\frac{7}{15} + \frac{t}{15}, \qquad z = \frac{3}{10} - \frac{t}{10}.$$

Hence direction ratios are $\frac{1}{5}, \frac{1}{15}, -\frac{1}{10}$. Multiplying by 30 gives the simpler proportional ratios $(6, 2, -3)$.

**Step 2:** Normalise the direction ratios The length of the direction vector $(6, 2, -3)$ is

$$\sqrt{6^2 + 2^2 + (-3)^2} = \sqrt{49} = 7.$$

Dividing each ratio by this positive length gives $l = 6/7$, $m = 2/7$, and $n = -3/7$. As a check, $l^2 + m^2 + n^2 = (36 + 4 + 9)/49 = 1$, as required for direction cosines.

**Step 3:** Conclusion The direction cosines are $\boxed{\left(\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}\right)}$. The opposite orientation has $\left(-\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}\right)$.

**Quick Tip to Solve:** The minus sign in $3 - 10z$ makes the third direction ratio negative; use the parameter coefficients, not the displayed multipliers.

**24. Find the points on the curve $6y = x^3 + 2$ at which ordinate is changing $8$ times as fast as abscissa.**

[2 marks]

**Solution:**

**Concept:**

- If coordinates vary with time $t$, their rates are $dx/dt$ and $dy/dt$. The chain rule gives $dy/dt = (dy/dx)(dx/dt)$.
- The ordinate is $y$ and the abscissa is $x$. The stated rate relation is $dy/dt = 8\,dx/dt$, so the required slope is $dy/dx = 8$.

**Step 1:** Translate the rate condition into a slope Differentiate the curve equation $6y = x^3 + 2$ with

**Quick Tip to Solve:** Convert the relative coordinate rates to a slope first; keep both roots of $x^2 = 16$ before finding the ordinates.

respect to time:

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt}.$$

At a point where the abscissa is changing, substitute $dy/dt = 8\,dx/dt$ and cancel $dx/dt$. This gives $48 = 3x^2$, or $x^2 = 16$. Equivalently, differentiating with respect to $x$ gives $dy/dx = x^2/2$, which must equal 8.

**Step 2:** Find the corresponding ordinates The equation $x^2 = 16$ gives two abscissae, $x = 4$ and $x = -4$. Substitute each into the original curve rather than into the derivative:

$$x = 4: \quad y = \frac{4^3 + 2}{6} = \frac{66}{6} = 11,$$
$$x = -4: \quad y = \frac{(-4)^3 + 2}{6} = \frac{-62}{6} = -\frac{31}{3}.$$

At both values, the slope $x^2/2$ equals 8, verifying the rate condition.

**Step 3:** Conclusion The required points are $\boxed{(4, 11) \text{ and } \left(-4, -\frac{31}{3}\right)}$.

**Quick Tip to Solve:** Convert the relative coordinate rates to a slope first; keep both roots of $x^2 = 16$ before finding the ordinates.

---

**25(a). Evaluate:** $3\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + 2\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}(0)$. [2 marks]

**Solution:**

**Concept:**

- The principal range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$, while the principal range of $\cos^{-1} x$ is $[0, \pi]$. Inverse trigonometric notation denotes a principal angle.
- Standard angles give $\sin(\pi/4) = 1/\sqrt{2}$, $\cos(\pi/6) = \sqrt{3}/2$, and $\cos(\pi/2) = 0$. Each of these angles lies in the required principal range.

**Step 1:** Evaluate the principal inverse values For the first term, the principal angle with sine $1/\sqrt{2}$ is $\pi/4$. For the second term, the principal angle with cosine $\sqrt{3}/2$ is $\pi/6$. Finally, the principal angle with cosine zero is $\pi/2$. Therefore

$$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}, \qquad \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}, \qquad \cos^{-1}(0) = \frac{\pi}{2}.$$

**Step 2:** Apply the coefficients and add Substitute the three angles, keeping the given coefficients 3 and 2:

$$3\left(\frac{\pi}{4}\right) + 2\left(\frac{\pi}{6}\right) + \frac{\pi}{2} = \frac{3\pi}{4} + \frac{\pi}{3} + \frac{\pi}{2}$$
$$= \frac{9\pi + 4\pi + 6\pi}{12} = \frac{19\pi}{12}.$$

**Quick Tip to Solve:** Evaluate each inverse function in its principal range before applying coefficients; use denominator 12 for the sum.

The terms are principal inverse values individually; their weighted sum need not itself belong to either principal range.

**Step 3:** Conclusion The given expression evaluates to $\boxed{\frac{19\pi}{12}}$.

**Quick Tip to Solve:** Evaluate each inverse function in its principal range before applying coefficients; use denominator 12 for the sum.

OR

**25(b). Draw the graph of $f(x) = \sin^{-1} x,\ x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. Also, write range of $f(x)$.** [2 marks]

**Solution:**

**Concept:**

- The principal inverse sine $y = \sin^{-1} x$ takes values in $[-\pi/2, \pi/2]$. It is the inverse of sine restricted to that interval.
- The function $\sin^{-1} x$ is continuous and strictly increasing on $[-1, 1]$, so on a closed subinterval its smallest and largest values occur at its endpoints.

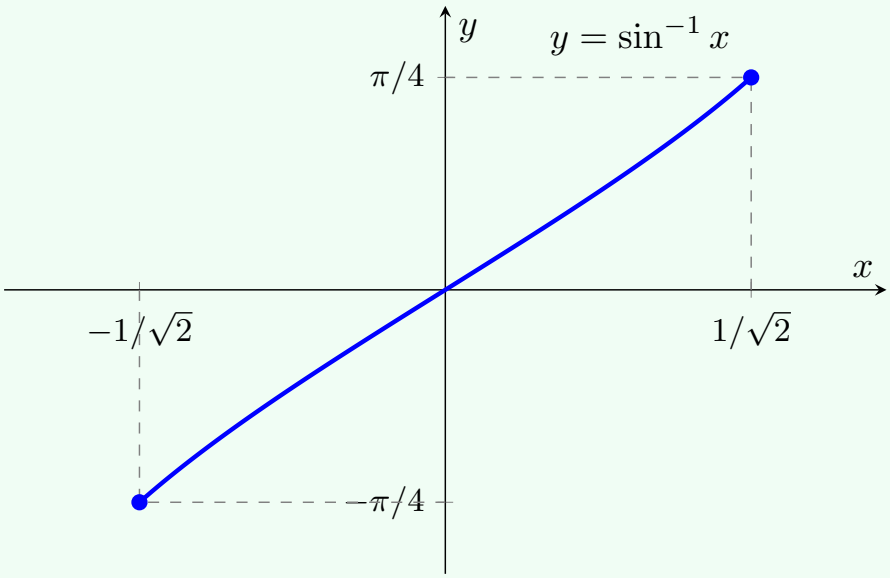
**Step 1:** Calculate the endpoint values Using principal values,

$$f\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4},$$
$$f(0) = 0,$$
$$f\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

Thus the required portion of the inverse-sine curve passes through the origin and has endpoints $(-1/\sqrt{2}, -\pi/4)$ and $(1/\sqrt{2}, \pi/4)$.

**Step 2:** Draw the restricted curve and identify its attained values Plot these three points and join them by the smooth increasing inverse-sine curve, retaining only the given domain. The endpoints are solid because both belong to the domain. The curve is symmetric about the origin since inverse sine is odd. Continuity ensures that every value between the two endpoint ordinates is attained; monotonicity ensures that no value outside them is attained.

**Quick Tip to Solve:** For an increasing inverse-trigonometric function on a closed interval, apply it to the two endpoints to obtain the range.




*Graph of inverse sine on the specified closed interval*

**Step 3:** Conclusion The required graph is shown above. The range is $\boxed{[-\pi/4, \pi/4]}$.

**Quick Tip to Solve:** For an increasing inverse-trigonometric function on a closed interval, apply it to the two endpoints to obtain the range.

## Section C (18 marks)

*This section comprises short answer (SA) type questions of 3 marks each.*

**26(a). A pair of dice is thrown simultaneously. If $X$ denotes the absolute difference of numbers obtained on the pair of dice, then find the probability distribution of $X$.** [3 marks]

**Solution:**

**Concept:**

- Two fair dice give $6 \times 6 = 36$ equally likely ordered outcomes $(a, b)$, where $a$ and $b$ are the numbers on the first and second die.
- The random variable is $X = |a - b|$. Its probability at each possible value is the number of favourable ordered outcomes divided by $36$.

**Step 1:** List the possible differences The smallest absolute difference is $0$, obtained when the two numbers agree. The largest is $5$, obtained from $(1, 6)$ or $(6, 1)$. Every integer between these is possible, so $X \in \{0, 1, 2, 3, 4, 5\}$. There are six outcomes with zero difference: $(1, 1), (2, 2), \ldots, (6, 6)$. Therefore $P(X = 0) = 6/36$.

**Step 2:** Count the outcomes for each positive difference For a fixed positive difference $k$, the outcomes with $b - a = k$ are $(1, 1 + k), (2, 2 + k), \ldots, (6 - k, 6)$, giving $6 - k$ outcomes. Reversing each pair gives a distinct outcome with $a - b = k$. Thus there are $2(6 - k)$ outcomes with $|a - b| = k$. In particular, the counts for $k = 1, 2, 3, 4, 5$ are respectively $10, 8, 6, 4, 2$. Order matters: for example, $(1, 2)$ and $(2, 1)$

**Quick Tip to Solve:** Count equal dice separately; for any positive difference k, count 6-k pairs and double the count for the two possible orders.

are separate equally likely results.

**Step 3:** Construct and check the probability distribution Dividing the counts by 36, the distribution is

| $x$ | 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| $P(X=x)$ | $\frac{1}{6}$ | $\frac{5}{18}$ | $\frac{2}{9}$ | $\frac{1}{6}$ | $\frac{1}{9}$ | $\frac{1}{18}$ |

All these probabilities are nonnegative, and

$$\frac{6+10+8+6+4+2}{36}=1.$$

Hence the table accounts for every possible outcome without overlap.

**Step 4:** Conclusion The probability distribution is the table above: $P(X=0)=1/6$, and $P(X=k)=(6-k)/18$ for $k=1,2,3,4,5$.

**Quick Tip to Solve:** Count equal dice separately; for any positive difference k, count 6-k pairs and double the count for the two possible orders.

OR

**26(b). There are two coins. One of them is a biased coin such that $P$(head) : $P$(tail) is $1:3$ and the other coin is a fair coin. A coin is selected at random and tossed once. If the coin showed head, then find the probability that it is a biased coin.** [3 marks]

**Solution:**

**Concept:**

- Let $B$ and $F$ denote selection of the biased coin and the fair coin, respectively, and let $H$ denote obtaining a head. Random selection gives $P(B)=P(F)=1/2$.
- Bayesťheorem updates a selection probability after observing an outcome: $P(B \mid H)=P(B)P(H \mid B)/P(H)$. Total probability gives $P(H)=P(B)P(H \mid B)+P(F)P(H \mid F)$.

**Step 1:** Find the head probability for each coin For the biased coin, the given ratio is $P(H \mid B) : P(T \mid B)=1:3$. Since head and tail exhaust the outcomes, their probabilities sum to one. Thus

$$P(H \mid B)=\frac{1}{1+3}=\frac{1}{4}, \qquad P(T \mid B)=\frac{3}{4}.$$

The fair coin has $P(H \mid F)=1/2$. The ratio therefore must be converted to a probability before applying Bayesťheorem.

**Step 2:** Calculate the total probability of a head A head can result from selecting either coin. These

**Quick Tip to Solve:** With equal prior chances of selecting the coins, compare their head probabilities: 1/4 to 1/2 gives posterior odds 1:2.

two routes are disjoint, so

$$\begin{aligned} P(H) &= P(B)P(H \mid B) + P(F)P(H \mid F) \\ &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{2} \\ &= \frac{1}{8} + \frac{1}{4} = \frac{3}{8}. \end{aligned}$$

The joint probability of selecting the biased coin and obtaining a head is $P(B \cap H) = 1/8$.

**Step 3:** Condition on the observed head Only trials producing a head are now relevant. Of their total probability $3/8$, the biased-coin route contributes $1/8$. Hence

$$P(B \mid H) = \frac{P(B \cap H)}{P(H)} = \frac{1/8}{3/8} = \frac{1}{3}.$$

As a check, the fair-coin route has conditional probability $(1/4)/(3/8) = 2/3$, and the two conditional probabilities sum to one. A head favours the fair coin because it is twice as likely to produce a head as the biased coin.

**Step 4:** Conclusion The probability that the selected coin is biased, given that it showed a head, is $\boxed{1/3}$.

**Quick Tip to Solve:** With equal prior chances of selecting the coins, compare their head probabilities: 1/4 to 1/2 gives posterior odds 1:2.

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**27(a). Find the general solution of the differential equation: $\frac{d}{dx}(xy^2) = 2y(1 + x^2)$.** [3 marks]

**Solution:**

**Concept:**

- Use the product rule: $\frac{d}{dx}(xy^2) = y^2 + 2xy\frac{dy}{dx}$.
- A linear equation $y\prime + P(x)y = Q(x)$ has integrating factor $e^{\int P(x)\,dx}$. Here $x \neq 0$ on the interval used.

**Step 1:** Expand the derivative and identify the linear equation The given equation becomes

$$y^2 + 2xy\frac{dy}{dx} = 2y(1 + x^2).$$

First observe that $y \equiv 0$ satisfies the original equation. For a nonzero solution, divide by $2xy$ to obtain

$$\frac{dy}{dx} + \frac{y}{2x} = \frac{1}{x} + x.$$

Thus $P(x) = 1/(2x)$ and $Q(x) = 1/x + x$. Dividing by $y$ requires retaining the zero solution separately.

**Step 2:** Multiply by the integrating factor and integrate On any interval not containing zero, the

**Quick Tip to Solve:** Before dividing by y, record the zero solution; then use the integrating factor square root of the absolute value of x.

integrating factor is

$$\text{IF} = e^{\frac{1}{2}\log|x|} = \sqrt{|x|}.$$

Multiplying the linear equation by this factor gives

$$\frac{d}{dx}\left(y\sqrt{|x|}\right) = \sqrt{|x|}\left(\frac{1}{x} + x\right).$$

Since $\frac{d}{dx}\sqrt{|x|} = \frac{\sqrt{|x|}}{2x}$, integration gives

$$y\sqrt{|x|} = 2\sqrt{|x|} + \frac{2}{5}x^2\sqrt{|x|} + C.$$

The same expression holds on either side of zero.

**Step 3:** Verify the resulting family Dividing by $\sqrt{|x|}$, differentiate the resulting expression to check

$$y' = \frac{4}{5}x - \frac{C}{2x\sqrt{|x|}}.$$

Substitution into $y' + y/(2x)$ cancels the constant terms and leaves $x + 1/x$, as required. Multiplication by $2xy$ then recovers the original equation.

**Step 4:** Conclusion The general nonzero family is

$$\boxed{y = 2 + \frac{2}{5}x^2 + \frac{C}{\sqrt{|x|}}, \quad x \neq 0.}$$

In particular, for $x > 0$, replace $\sqrt{|x|}$ by $\sqrt{x}$. The additional solution $\boxed{y \equiv 0}$ must also be retained.

**Quick Tip to Solve:** Before dividing by y, record the zero solution; then use the integrating factor square root of the absolute value of x.

OR

**27(b). Solve the following differential equation:** $xe^{\frac{y}{x}} - y + x\frac{dy}{dx} = 0.$ [3 marks]

**Solution:**

**Concept:**

- An equation involving $y/x$ can be reduced using $v = y/x$, so $y = xv$ and $dy/dx = v + x\,dv/dx$.
- For a separable equation, collect all terms involving $v$ with $dv$ and all terms involving $x$ with $dx$.

**Step 1:** Apply the homogeneous substitution The printed equation is

$$xe^{y/x} - y + x\frac{dy}{dx} = 0, \qquad x \neq 0.$$

**Quick Tip to Solve:** With y equal to xv, the minus y and the xv part of x times dy/dx cancel immediately; separate the remaining equation.

Set $v = y/x$. Then $y = xv$, and the product rule gives $dy/dx = v + x\,dv/dx$. Substitution yields

$$xe^v - xv + x\left(v + x\frac{dv}{dx}\right) = 0.$$

The two terms involving $xv$ cancel, leaving $xe^v + x^2 dv/dx = 0$. This cancellation is the reason the substitution is useful.

**Step 2:** Separate variables and integrate Divide by $x^2$ and rearrange:

$$\frac{dv}{dx} = -\frac{e^v}{x}, \qquad e^{-v}\,dv = -\frac{dx}{x}.$$

Integrating both sides gives

$$-e^{-v} = -\log|x| + C_1.$$

Multiply by $-1$ and rename the arbitrary constant to obtain

$$e^{-v} = \log|x| + C.$$

The logarithm contains $|x|$ because the independent variable may be either positive or negative on a chosen interval.

**Step 3:** Restore the original variable and check As $v = y/x$, the implicit family is $e^{-y/x} = \log|x| + C$. Its right-hand side must be positive for a finite real $y$. Differentiating this relation gives

$$-e^{-y/x}\frac{xy\prime - y}{x^2} = \frac{1}{x}.$$

Hence $xy\prime - y = -xe^{y/x}$, which rearranges exactly to the differential equation in the question. No constant value of $v$ is lost, since $e^v$ never vanishes.

**Step 4:** Conclusion

$$\boxed{e^{-y/x} = \log|x| + C, \quad x \neq 0.}$$

Equivalently, $y = -x\log(\log|x| + C)$, on intervals where $\log|x| + C > 0$.

**Quick Tip to Solve:** With y equal to xv, the minus y and the xv part of x times dy/dx cancel immediately; separate the remaining equation.

**28. Evaluate:** $\displaystyle\int_0^{\pi/4} \log(1 + \tan x)\,dx$. [3 marks]

**Solution:**

**Concept:**

- The reflection property is $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$. Here take $a = \pi/4$.
- The tangent subtraction identity gives $\tan(\pi/4 - x) = (1 - \tan x)/(1 + \tan x)$. The logarithm converts a quotient to a difference.

**Quick Tip to Solve:** Reflect x to pi/4 minus x: the two logarithmic arguments multiply to 2, so their logarithms add to a constant.

**Step 1:** Reflect the interval about its midpoint Let

$$I = \int_0^{\pi/4} \log(1 + \tan x)\, dx.$$

The integrand is continuous throughout this interval because $0 \le \tan x \le 1$, so the logarithm has a positive argument. Use the substitution $x = \pi/4 - t$, reversing the transformed limits to get

$$I = \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - t\right)\right)\, dt.$$

Rename the dummy integration variable $t$ as $x$.

**Step 2:** Simplify the reflected integrand By the tangent subtraction formula,

$$1 + \tan\left(\frac{\pi}{4} - x\right) = 1 + \frac{1 - \tan x}{1 + \tan x} = \frac{2}{1 + \tan x}.$$

All quantities in this quotient are positive on the interval. Therefore the logarithm law applies directly and gives

$$I = \int_0^{\pi/4} [\log 2 - \log(1 + \tan x)]\, dx.$$

The second integral on the right is exactly the original integral $I$.

**Step 3:** Add the two expressions for the integral Adding the original and reflected forms eliminates the variable part of the integrand:

$$\begin{aligned} 2I &= \int_0^{\pi/4} \log 2\, dx \\ &= \log 2\, [x]_0^{\pi/4} \\ &= \frac{\pi}{4} \log 2. \end{aligned}$$

Divide both sides by two. This method evaluates the definite integral without requiring an antiderivative of its logarithmic integrand.

**Step 4:** Conclusion

$$\boxed{\int_0^{\pi/4} \log(1 + \tan x)\, dx = \frac{\pi}{8} \log 2.}$$

**Quick Tip to Solve:** Reflect x to pi/4 minus x: the two logarithmic arguments multiply to 2, so their logarithms add to a constant.

**29(a). Find:** $\int \frac{\cos x}{\sin 3x}\,dx$. [3 marks]

**Solution:**

**Concept:**

- The triple-angle identity is $\sin 3x = 3\sin x - 4\sin^3 x = \sin x(3 - 4\sin^2 x)$.
- Use $u = \sin x$, so $du = \cos x\,dx$, and then decompose the resulting rational function.

**Step 1:** Convert the trigonometric integral to a rational one Let

$$I = \int \frac{\cos x}{\sin 3x}\,dx.$$

Using the triple-angle identity, write the denominator as $\sin x(3 - 4\sin^2 x)$. Now put $u = \sin x$, with $du = \cos x\,dx$. Thus

$$I = \int \frac{du}{u(3 - 4u^2)}.$$

The original integrand is defined on intervals where $\sin 3x \neq 0$; hence neither factor of this denominator vanishes there.

**Step 2:** Decompose and integrate the rational expression Write

$$\frac{1}{u(3 - 4u^2)} = \frac{A}{u} + \frac{Bu}{3 - 4u^2}.$$

Multiplying by the common denominator gives $1 = A(3 - 4u^2) + Bu^2$. Comparing the constant and quadratic coefficients yields $A = 1/3$ and $B = 4/3$. Therefore

$$I = \frac{1}{3}\int \frac{du}{u} + \frac{4}{3}\int \frac{u\,du}{3 - 4u^2}.$$

For the second integral use $z = 3 - 4u^2$, so $dz = -8u\,du$. The result is

$$I = \frac{1}{3}\log|u| - \frac{1}{6}\log|3 - 4u^2| + C.$$

**Step 3:** Substitute back and verify the derivative Replace $u$ by $\sin x$. Differentiating the expression obtained gives

$$\frac{\cos x}{3\sin x} + \frac{4\sin x\cos x}{3(3 - 4\sin^2 x)}.$$

Combining these terms over their common denominator leaves

$$\frac{\cos x}{\sin x(3 - 4\sin^2 x)} = \frac{\cos x}{\sin 3x},$$

verifying the antiderivative. Absolute values ensure that each logarithm is valid on every interval of definition.

**Quick Tip to Solve:** Expand sin 3x in powers of sin x first; the numerator cos x then supplies exactly the differential for u equal to sin x.

**Step 4:** Conclusion

$$\boxed{\int \frac{\cos x}{\sin 3x}\,dx = \frac{1}{3}\log|\sin x| - \frac{1}{6}\log|3 - 4\sin^2 x| + C.}$$

**Quick Tip to Solve:** Expand sin 3x in powers of sin x first; the numerator cos x then supplies exactly the differential for u equal to sin x.

OR

**29(b). Find:** $\int x^2 \log(x^2+1)\,dx$. [3 marks]

**Solution:**

**Concept:**

- Integration by parts: $\int u\,dv = uv - \int v\,du$. Choose the logarithm as $u$, since differentiating it produces a rational function.
- Polynomial division reduces the rational remainder; $\int (1+x^2)^{-1}\,dx = \tan^{-1} x + C$, where $C$ is an arbitrary real constant.

**Step 1:** Apply integration by parts Let $I = \int x^2 \log(x^2+1)\,dx$. Take $u = \log(x^2+1)$ and $dv = x^2\,dx$, so that $du = \dfrac{2x}{x^2+1}\,dx$ and $v = \dfrac{x^3}{3}$. Hence

$$I = \frac{x^3}{3}\log(x^2+1) - \frac{2}{3}\int \frac{x^4}{x^2+1}\,dx.$$

The logarithm is defined for every real $x$, because $x^2 + 1 > 0$.

**Step 2:** Divide the polynomial and integrate The numerator satisfies $x^4 = (x^2+1)(x^2-1) + 1$. Therefore

$$\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1},$$
$$\int \frac{x^4}{x^2+1}\,dx = \frac{x^3}{3} - x + \tan^{-1} x.$$

Substituting this antiderivative gives

$$I = \frac{x^3}{3}\log(x^2+1) - \frac{2}{3}\left(\frac{x^3}{3} - x + \tan^{-1} x\right) + C.$$

**Step 3:** Check the antiderivative Differentiating the logarithmic term gives $x^2\log(x^2+1) + \dfrac{2x^4}{3(x^2+1)}$. The derivatives of the remaining nonconstant terms sum to $-\dfrac{2}{3}\left(x^2 - 1 + \dfrac{1}{x^2+1}\right) = -\dfrac{2x^4}{3(x^2+1)}$. These rational terms cancel, leaving exactly the given integrand.

**Quick Tip to Solve:** After integration by parts, use $x^4 = (x^2+1)(x^2-1) + 1$ to expose the inverse-tangent term.

**Step 4:** Conclusion

$$\boxed{\int x^2 \log(x^2+1)\,dx = \frac{x^3}{3}\log(x^2+1) - \frac{2x^3}{9} + \frac{2x}{3} - \frac{2}{3}\tan^{-1}x + C.}$$

**Quick Tip to Solve:** After integration by parts, use $x^4 = (x^2+1)(x^2-1)+1$ to expose the inverse-tangent term.

**30. Find:** $\displaystyle\int_1^4 \frac{1}{\sqrt{2x+1}-\sqrt{2x-1}}\,dx$. [3 marks]

**Solution:**

**Concept:**

- Multiply by the conjugate to rationalise a difference of square roots: $(a-b)(a+b) = a^2 - b^2$.
- For $u = 2x + c$, $du = 2\,dx$, so $\int \sqrt{2x+c}\,dx = \frac{1}{3}(2x+c)^{3/2} + C$. Apply the upper and lower limits to the antiderivative.

**Step 1:** Rationalise the denominator On $[1, 4]$, both radicands are positive and $\sqrt{2x+1} > \sqrt{2x-1}$, so the original integrand is well defined. Multiplying numerator and denominator by the conjugate gives

$$\begin{aligned}\frac{1}{\sqrt{2x+1}-\sqrt{2x-1}} &= \frac{\sqrt{2x+1}+\sqrt{2x-1}}{(2x+1)-(2x-1)} \\ &= \frac{1}{2}\left(\sqrt{2x+1}+\sqrt{2x-1}\right).\end{aligned}$$

The denominator becomes the constant 2, which makes direct integration possible.

**Step 2:** Integrate the two square-root terms Using the substitution formula separately for the two terms,

$$\begin{aligned}I &= \frac{1}{2}\int_1^4 \left(\sqrt{2x+1}+\sqrt{2x-1}\right)dx \\ &= \frac{1}{6}\left[(2x+1)^{3/2} + (2x-1)^{3/2}\right]_1^4.\end{aligned}$$

The factor $1/6$ combines the rationalisation factor $1/2$ with the integration factor $1/3$.

**Step 3:** Evaluate the limits At $x = 4$, the two radicands are 9 and 7; at $x = 1$, they are 3 and 1. Thus

$$I = \frac{1}{6}\left(9^{3/2} + 7^{3/2} - 3^{3/2} - 1^{3/2}\right) = \frac{1}{6}\left(27 + 7\sqrt{7} - 3\sqrt{3} - 1\right).$$

Both lower-limit contributions must be subtracted. The resulting value is positive, as expected from the positive original integrand throughout the interval.

**Step 4:** Conclusion

$$\boxed{\int_1^4 \frac{dx}{\sqrt{2x+1}-\sqrt{2x-1}} = \frac{26+7\sqrt{7}-3\sqrt{3}}{6}.}$$

**Quick Tip to Solve:** Rationalise first: the squared radicands differ by 2, leaving two simple square-root integrals.

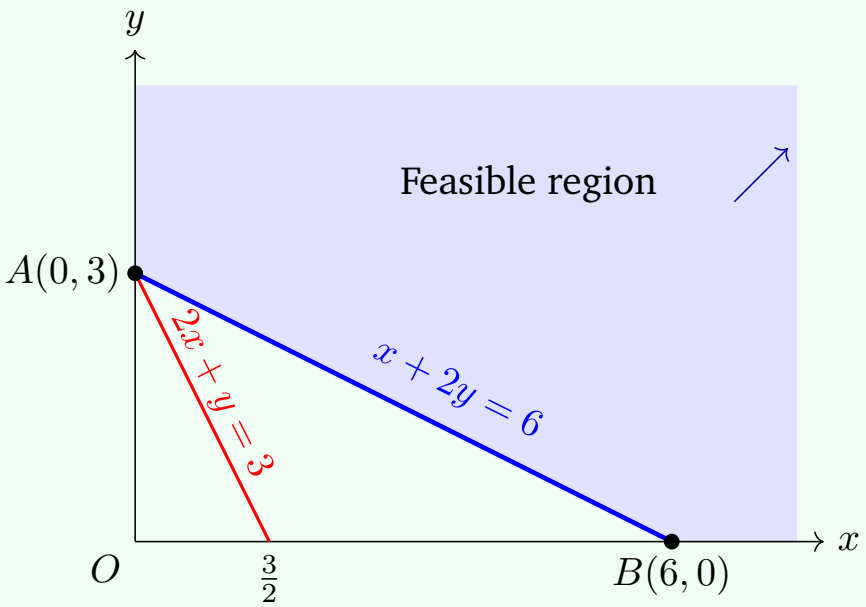
**31. Solve the following linear programming problem graphically: Minimize $z = x + 2y$ subject to the constraints $2x + y \geq 3,\ x + 2y \geq 6,\ x \geq 0,\ y \geq 0$.** [3 marks]

**Solution:**

**Concept:**

- The feasible region is the intersection of the half-planes specified by the inequalities, restricted to the first quadrant.
- The objective $z = x + 2y$ is constant along lines $x + 2y = k$. A minimum occurs where the smallest such line meets the feasible region; an entire edge can be optimal.

**Step 1:** Draw the constraint boundaries The boundary $2x + y = 3$ meets the axes at $(0, 3)$ and $(3/2, 0)$. The boundary $x + 2y = 6$ meets them at $(0, 3)$ and $(6, 0)$. Plot these intercepts and join each pair with a straight line. Since $(0, 0)$ satisfies neither inequality, choose the half-plane away from the origin for each boundary. The non-negativity constraints restrict the diagram to the first quadrant.




*The shaded first-quadrant region is feasible; every point of AB is optimal.*

**Step 2:** Identify the common feasible region The two lines intersect at $A = (0, 3)$: solving $2x + y = 3$ and $x + 2y = 6$ gives $x = 0$, $y = 3$. On the segment joining $A$ to $B = (6, 0)$, we have $y = 3 - x/2$, hence $2x + y = 3 + 3x/2 \geq 3$. In fact, for $x, y \geq 0$,

$$2x + y = \frac{1}{2}(x + 2y) + \frac{3}{2}x \geq 3 \quad \text{whenever } x + 2y \geq 6.$$

Thus the shaded feasible region lies on and above $AB$ in the first quadrant, and extends without bound.

**Step 3:** Locate the minimum objective line Every feasible point satisfies $z = x + 2y \geq 6$ directly from the second constraint. At the boundary vertices, $z(A) = 0 + 2(3) = 6$ and $z(B) = 6 + 2(0) = 6$. The objective line $x + 2y = 6$ coincides with the feasible boundary segment $AB$. Consequently every point on this segment attains the same least objective value. No feasible point can give a smaller value,

**Quick Tip to Solve:** When the objective equals a constraint left-hand side, test equality along its feasible boundary for multiple optimal points.

even though the feasible region is unbounded.

**Step 4:** Conclusion

$$\boxed{z_{\min} = 6.}$$

It is attained at every point of the segment from $(0,3)$ to $(6,0)$, namely $(x,y) = (t, 3 - t/2)$ for $0 \le t \le 6$.

**Quick Tip to Solve:** When the objective equals a constraint left-hand side, test equality along its feasible boundary for multiple optimal points.

## Section D (20 marks)

*This section comprises long answer (LA) type questions of 5 marks each.*

**32(a). Find the value of $b$ so that the lines $\frac{x-1}{2} = \frac{y-b}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ are intersecting lines. Also, find the point of intersection of these given lines.** [5 marks]

**Solution:**

**Concept:**

- A symmetric equation of a line gives a parametric representation by setting its three ratios equal to one parameter. Different lines require independent parameters.
- Two lines intersect when a point belongs to both lines, so their corresponding coordinates must agree simultaneously.
- Determine the line parameters from coordinates free of the unknown constant, then use the remaining coordinate to determine that constant.

**Step 1:** Express the two lines parametrically Let the common ratio of the first line be $t$, so that

$$(x, y, z) = (1 + 2t, b + 3t, 3 + 4t).$$

Let the common ratio of the second line be $s$, so that

$$(x, y, z) = (4 + 5s, 1 + 2s, s).$$

The last coordinate of the second line is $s$ because its final expression is simply $z$, or equivalently $(z - 0)/1$. The parameters must initially be independent, since they describe positions on different lines. At an intersection the three coordinates, rather than the two parameters, must be equal.

**Step 2:** Solve the x and z coordinate equations Equating the coordinates not involving $b$, we obtain

$$1 + 2t = 4 + 5s, \qquad 3 + 4t = s.$$

**Quick Tip to Solve:** Use the x and z equations first because neither contains b; then substitute their parameter values in the y equation.

Substituting the second equation into the first,

$$\begin{aligned} 1+2t &= 4+5(3+4t), \\ 1+2t &= 19+20t, \\ -18t &= 18, \\ t &= -1. \end{aligned}$$

Thus $s = 3+4(-1) = -1$. These parameter values are forced by the two equations, so any common point must correspond to them. The y coordinate now supplies the required condition on the position of the first line.

**Step 3:** Determine b and check the intersection Equating the y coordinates gives

$$b+3(-1) = 1+2(-1), \qquad b-3 = -1,$$

hence $b = 2$. With $t = -1$, the point on the first line is

$$(1-2, 2-3, 3-4) = (-1, -1, -1).$$

With $s = -1$, the point on the second line is $(4-5, 1-2, -1) = (-1, -1, -1)$. To verify directly, substitute the point and $b = 2$ in the given equations. For the first line, all three ratios are

$$\frac{-1-1}{2} = \frac{-1-2}{3} = \frac{-1-3}{4} = -1.$$

For the second line,

$$\frac{-1-4}{5} = \frac{-1-1}{2} = -1 = z.$$

Thus the point satisfies both complete line equations, establishing an actual intersection rather than merely agreement in two coordinates.

**Step 4:** Conclusion The required value is $\boxed{b = 2}$, and the point of intersection is $\boxed{(-1, -1, -1)}$.

**Quick Tip to Solve:** Use the x and z equations first because neither contains b; then substitute their parameter values in the y equation.

OR

**32(b). Find the equations of all the sides of the parallelogram $ABCD$ whose vertices are $A(4, 7, 8)$, $B(2, 3, 4)$, $C(-1, -2, 1)$ and $D(1, 2, 5)$. Also, find the coordinates of the foot of the perpendicular from $A$ to $CD$.** [5 marks]

**Solution:**

**Concept:**

- The line through $(x_0, y_0, z_0)$ with direction ratios $(l, m, n)$ has equation $(x-x_0)/l = (y-y_0)/m = (z-z_0)/n$.

**Quick Tip to Solve:** Use the reduced direction (1,2,2) for CD, then impose a zero dot product to find the foot with just one parameter.

- Direction ratios for a side are obtained by subtracting coordinates of its endpoints; any nonzero common multiple gives the same line.
- If $H$ is the foot from $A$ to a line with direction vector $\mathbf{d}$, then $H$ lies on the line and $(\overrightarrow{AH}) \cdot \mathbf{d} = 0$.

**Step 1:** Calculate the directions of all four sides Subtract the consecutive vertex coordinates:

$$\begin{aligned}
\overrightarrow{AB} &= (2-4, 3-7, 4-8) = (-2, -4, -4),\\
\overrightarrow{BC} &= (-1-2, -2-3, 1-4) = (-3, -5, -3),\\
\overrightarrow{CD} &= (1+1, 2+2, 5-1) = (2, 4, 4),\\
\overrightarrow{DA} &= (4-1, 7-2, 8-5) = (3, 5, 3).
\end{aligned}$$

Thus AB and CD have proportional directions, as do BC and DA, confirming the parallel pairs of the given parallelogram. Use the simpler ratios $(1, 2, 2)$ for AB and CD and $(3, 5, 3)$ for BC and DA. Reversing a direction does not change its line.

**Step 2:** Write the equations of the sides Using A for AB, B for BC, C for CD and D for DA, the required equations are

$$\begin{aligned}
AB: &\quad \frac{x-4}{1} = \frac{y-7}{2} = \frac{z-8}{2},\\
BC: &\quad \frac{x-2}{3} = \frac{y-3}{5} = \frac{z-4}{3},\\
CD: &\quad \frac{x+1}{1} = \frac{y+2}{2} = \frac{z-1}{2},\\
DA: &\quad \frac{x-1}{3} = \frac{y-2}{5} = \frac{z-5}{3}.
\end{aligned}$$

Each equation contains its chosen vertex. It also contains the other endpoint: the common parameter values are respectively $-2, -1, 2, 1$. Therefore these equations describe all four requested side lines.

**Step 3:** Impose the perpendicular condition for the foot on CD A general point H on CD, using its direction $(1, 2, 2)$, is

$$H = (-1+t, -2+2t, 1+2t).$$

Consequently $\overrightarrow{AH} = (t-5, 2t-9, 2t-7)$. Since AH is perpendicular to CD, its dot product with the direction of CD vanishes:

$$\begin{aligned}
(t-5) + 2(2t-9) + 2(2t-7) &= 0,\\
9t - 37 &= 0,\\
t &= \frac{37}{9}.
\end{aligned}$$

Substitution gives

$$H = \left(-1 + \frac{37}{9}, -2 + \frac{74}{9}, 1 + \frac{74}{9}\right) = \left(\frac{28}{9}, \frac{56}{9}, \frac{83}{9}\right).$$

As a check, $\overrightarrow{AH} = (-8, -7, 11)/9$, whose dot product with $(1, 2, 2)$ is $(-8 - 14 + 22)/9 = 0$. The parameter construction also guarantees H lies on CD. The foot is on the extended side line, which is allowed when dropping a perpendicular to CD.

**Step 4:** Conclusion The four side equations are the equations of AB, BC, CD and DA above. The foot of the perpendicular from A to CD is $\boxed{H(28/9, 56/9, 83/9)}$.

**Quick Tip to Solve:** Use the reduced direction (1,2,2) for CD, then impose a zero dot product to find the foot with just one parameter.

**33. Check whether a function $f : \mathbb{R} \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$ is one-one and onto or not.** [5 marks]

**Solution:**

**Concept:**

- A function is one-one if distinct inputs have distinct images. A pair of distinct inputs with the same image disproves this property.
- A function is onto when every element of its specified codomain has a preimage in its domain. Here the codomain is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.
- To find a preimage of $y$, solve $y = f(x)$. A quadratic with real coefficients has real roots when its discriminant is non-negative.

**Step 1:** Verify that the function takes values in its codomain For every real $x$, the denominator $1 + x^2$ is positive. The inequality $(|x| - 1)^2 \geq 0$ gives $2|x| \leq 1 + x^2$, hence

$$\left|\frac{x}{1+x^2}\right| \leq \frac{1}{2}.$$

Thus the rule is defined throughout the domain and every output lies in the stated closed interval. Both endpoints are attained since $f(1) = \frac{1}{2}$ and $f(-1) = -\frac{1}{2}$.

**Step 2:** Test the one-one property Take the distinct inputs $2$ and $\frac{1}{2}$. We obtain

$$f(2) = \frac{2}{5}, \qquad f\left(\frac{1}{2}\right) = \frac{1/2}{1+1/4} = \frac{2}{5}.$$

Equality of images therefore does not force equality of inputs, so the function is not one-one. In fact, every nonzero input $x$ has the same image as its reciprocal $1/x$; these inputs are distinct unless $x = \pm 1$. The displayed pair alone is enough to disprove injectivity.

**Step 3:** Establish that every codomain element has a preimage Let $y$ be any element of $\left[-\frac{1}{2}, \frac{1}{2}\right]$. To obtain a preimage, solve

$$y = \frac{x}{1+x^2} \quad \Longleftrightarrow \quad yx^2 - x + y = 0.$$

If $y = 0$, choose $x = 0$, giving $f(0) = 0$. If $y \neq 0$, the discriminant is

$$\Delta = 1 - 4y^2 \geq 0,$$

because $|y| \leq \frac{1}{2}$. There are therefore real roots

$$x = \frac{1 \pm \sqrt{1-4y^2}}{2y}.$$

Either root belongs to the domain and satisfies the original equation, as its denominator $1 + x^2$ cannot vanish. When $y = \frac{1}{2}$ the root is $x = 1$, and when $y = -\frac{1}{2}$ it is $x = -1$. Thus zero, the endpoints, and every interior point of the codomain are attained. This proves that the function is onto its specified codomain.

**Quick Tip to Solve:** Use reciprocal inputs to disprove injectivity, then use the preimage quadratic to prove surjectivity; handle $y = 0$ before dividing by $y$.

**Step 4:** Conclusion The function is $\boxed{\text{onto but not one-one}}$.

**Quick Tip to Solve:** Use reciprocal inputs to disprove injectivity, then use the preimage quadratic to prove surjectivity; handle $y = 0$ before dividing by $y$.

**34. The area of the region bounded by the line $y = mx$ $(m > 0)$, the curve $x^2 + y^2 = 4$ and the $x$-axis in the first quadrant is $\frac{\pi}{2}$ units. Using integration, find the value of $m$.** [5 marks]

**Solution:**

**Concept:**

- An area above the $x$-axis is the integral of the upper boundary. Split the integral where the boundary changes from a line to a circle.
- The first-quadrant circle is $y = \sqrt{4 - x^2}$. Its standard antiderivative is $\int \sqrt{4 - x^2}\, dx = \frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}(x/2) + C$.

**Step 1:** Find the intersection of the line and the circle Let the positive intersection be $P(a, b)$. Substituting $y = mx$ into the circle gives

$$x^2 + m^2x^2 = 4, \qquad a = \frac{2}{\sqrt{1 + m^2}}, \qquad b = ma.$$

Since $m > 0$, we have $0 < a < 2$ and $b > 0$. The bounded region has the line segment from the origin to $P$, the circular arc from $P$ to $(2, 0)$, and the segment of the $x$-axis back to the origin as its boundary.

**Step 2:** Express the area as two integrals For $0 \le x \le a$, the upper boundary is the line $y = mx$; for $a \le x \le 2$, it is the circle $y = \sqrt{4 - x^2}$. Thus the required area is

$$A = \int_0^a mx\, dx + \int_a^2 \sqrt{4 - x^2}\, dx.$$

Splitting at the intersection ensures that each vertical strip remains inside the region bounded by all three specified curves. Integrating the first part gives $ma^2/2$. For the circular part, use the standard antiderivative, or obtain it by the substitution $x = 2\sin t$.

**Step 3:** Evaluate and simplify the integrals Using the antiderivative of the circular boundary,

$$\begin{aligned} A &= \frac{ma^2}{2} + \left[\frac{x}{2}\sqrt{4 - x^2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_a^2 \\ &= \frac{ma^2}{2} + \pi - \frac{a}{2}\sqrt{4 - a^2} - 2\sin^{-1}\left(\frac{a}{2}\right). \end{aligned}$$

At the intersection, $\sqrt{4 - a^2} = b = ma$. Therefore the two algebraic terms cancel and

$$A = \pi - 2\sin^{-1}\left(\frac{a}{2}\right).$$

This expression uses the principal inverse sine; since $0 < a/2 < 1$, its value is between $0$ and $\pi/2$.

**Quick Tip to Solve:** Split the area at the line-circle intersection; use $\sqrt{4 - a^2} = ma$ to cancel the algebraic terms before solving for the slope.

**Step 4:** Use the given area to determine the positive slope The given area is $\pi/2$, so

$$\pi - 2\sin^{-1}\left(\frac{a}{2}\right) = \frac{\pi}{2} \quad \Longrightarrow \quad \sin^{-1}\left(\frac{a}{2}\right) = \frac{\pi}{4}.$$

Taking sine yields $a/2 = 1/\sqrt{2}$, hence $a = \sqrt{2}$. Substituting into $a^2(1+m^2) = 4$ gives

$$2(1+m^2) = 4, \qquad m^2 = 1.$$

The condition $m > 0$ selects the positive root. As a check, this gives the intersection $(\sqrt{2}, \sqrt{2})$, and the area formula returns $\pi - 2(\pi/4) = \pi/2$, exactly the stated area.

**Step 5:** Conclusion The required value is $\boxed{m = 1}$.

**Quick Tip to Solve:** Split the area at the line-circle intersection; use $\sqrt{4-a^2} = ma$ to cancel the algebraic terms before solving for the slope.

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**35(a).** **If** $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$**, then show that** $A^3 - 6A^2 + 7A + 2I = O$**.** [5 marks]

**Solution:**

**Concept:**

- Matrix multiplication uses the row-column rule: the entry in row $i$, column $j$ of $AB$ is the sum of products of corresponding entries of row $i$ of $A$ and column $j$ of $B$.
- Powers of a square matrix mean repeated multiplication: $A^2 = AA$ and $A^3 = A^2A$. The identity matrix $I$ has ones on its diagonal and zeros elsewhere.
- Matrix addition and scalar multiplication are performed entry by entry. To prove the required identity, every entry of the resulting matrix must be zero; checking just its determinant is insufficient.

**Step 1:** Calculate the square of the given matrix Applying the row-column rule gives

$$A^2 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix}.$$

For example, the first diagonal entry is $1\cdot 1 + 0\cdot 0 + 2\cdot 2 = 5$, while the entry in the second row and first column is $0\cdot 1 + 2\cdot 0 + 1\cdot 2 = 2$.

**Step 2:** Calculate the cube using the square Multiply the matrix just obtained by $A$:

$$A^3 = \begin{pmatrix} 5 & 0 & 8 \\ 2 & 4 & 5 \\ 8 & 0 & 13 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 21 & 0 & 34 \\ 12 & 8 & 23 \\ 34 & 0 & 55 \end{pmatrix}.$$

In particular, the middle row is $(2+10,\ 8,\ 4+4+15) = (12, 8, 23)$.

**Quick Tip to Solve:** Find $A^2$ once and use $A^3 = A^2A$; group $A^3 - 6A^2$ and $7A + 2I$ to spot exact entrywise cancellation.

**Step 3:** Substitute and simplify entry by entry Combining the terms in two stages keeps the subtraction clear:

$$A^3 - 6A^2 = \begin{pmatrix} -9 & 0 & -14 \\ 0 & -16 & -7 \\ -14 & 0 & -23 \end{pmatrix}.$$

Also,

$$7A + 2I = \begin{pmatrix} 9 & 0 & 14 \\ 0 & 16 & 7 \\ 14 & 0 & 23 \end{pmatrix}.$$

Each entry of the second displayed matrix is the negative of the corresponding entry of the first, so their sum has nine zero entries.

**Step 4:** Conclusion Hence $A^3 - 6A^2 + 7A + 2I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = O$, as required.

**Quick Tip to Solve:** Find $A^2$ once and use $A^3 = A^2A$; group $A^3 - 6A^2$ and $7A + 2I$ to spot exact entrywise cancellation.

OR

**35(b). If $A = \begin{pmatrix} 3 & 2 \\ 5 & -7 \end{pmatrix}$, then find $A^{-1}$ and use it to solve the following system of equations:** $3x + 5y = 11,\ 2x - 7y = -3$**.** [5 marks]

**Solution:**

**Concept:**

- For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its inverse exists when $ad - bc \neq 0$, and $A^{-1} = \dfrac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.
- A linear system is written $MX = B$, with coefficients in the same order as the variables in $X$. Its solution is $X = M^{-1}B$ when $M$ is invertible.
- Transposition interchanges rows and columns, and $(A^T)^{-1} = (A^{-1})^T$. Here the coefficient matrix of the printed equations is $A^T$.

**Step 1:** Find the determinant and inverse of the given matrix For the matrix in the question,

$$|A| = 3(-7) - 2(5) = -21 - 10 = -31 \neq 0.$$

Thus its inverse exists. Interchanging the diagonal entries and changing the signs of the off-diagonal entries gives

$$\text{adj } A = \begin{pmatrix} -7 & -2 \\ -5 & 3 \end{pmatrix}, \qquad A^{-1} = \frac{1}{31} \begin{pmatrix} 7 & 2 \\ 5 & -3 \end{pmatrix}.$$

**Step 2:** Express the equations with the correct coefficient matrix Writing the unknowns and constants

**Quick Tip to Solve:** Compare the equation coefficients with the rows of $A$ first: here they form $A^T$, so transpose $A^{-1}$ before multiplying by the constants.

as column matrices, the equations are

$$\begin{pmatrix} 3 & 5 \\ 2 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 11 \\ -3 \end{pmatrix}.$$

The coefficient matrix is $A^T$, since the first equation has coefficients $3, 5$, while the first row of the given $A$ is $3, 2$. Accordingly, use the transpose of the inverse already found:

$$(A^T)^{-1} = (A^{-1})^T = \frac{1}{31} \begin{pmatrix} 7 & 5 \\ 2 & -3 \end{pmatrix}.$$

**Step 3:** Multiply by the inverse and verify the solution Premultiplying the matrix equation by $(A^T)^{-1}$ yields

$$X = \frac{1}{31} \begin{pmatrix} 7 & 5 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} 11 \\ -3 \end{pmatrix} = \frac{1}{31} \begin{pmatrix} 77 - 15 \\ 22 + 9 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Substitution checks both equations: $3(2) + 5(1) = 11$ and $2(2) - 7(1) = -3$.

**Step 4:** Conclusion The required inverse is $A^{-1} = \frac{1}{31} \begin{pmatrix} 7 & 2 \\ 5 & -3 \end{pmatrix}$, and the system has the unique solution $x = 2,\ y = 1$.

**Quick Tip to Solve:** Compare the equation coefficients with the rows of $A$ first: here they form $A^T$, so transpose $A^{-1}$ before multiplying by the constants.

## Section E (12 marks)

*This section comprises 3 case study based questions of 4 marks each.*

**36. In order to set up a rain water harvesting system, a tank to collect rain water is to be dug. The tank should have a square base and a capacity of $250$ m$^3$. The cost of land is ₹$5{,}000$ per square metre and cost of digging increases with depth and for the whole tank, it is ₹$40{,}000\,h^2$, where $h$ is the depth of the tank in metres. $x$ is the side of the square base of the tank in metres.**

**Based on the above information, answer the following questions:** [4 marks]

**36(i). Find the total cost $C$ of digging the tank in terms of $x$.** [1 mark]

**Solution:**

**Concept:**

- The volume of a tank with square base of side $x$ and depth $h$ is $x^2h$.
- Total cost is the sum of land cost and the stated cost of digging the whole tank.

**Step 1:** Express depth in terms of the side The fixed capacity gives $x^2h = 250$, so $h = 250/x^2$, where $x > 0$. A larger base therefore requires a smaller depth.

**Step 2:** Add the two costs The base occupies $x^2$ square metres, so its land cost is $5000x^2$ rupees.

**Quick Tip to Solve:** Eliminate depth with the fixed-volume equation before adding the land and whole-tank digging costs.

Substituting the depth into the digging charge gives

$$C(x) = 5000x^2 + 40000\left(\frac{250}{x^2}\right)^2 = 5000x^2 + \frac{2500000000}{x^4}.$$

The digging charge already applies to the whole tank; it is not multiplied by the base area again.

**Step 3:** Conclusion $C(x) = 5000x^2 + 2500000000/x^4$ rupees, for $x > 0$.

**Quick Tip to Solve:** Eliminate depth with the fixed-volume equation before adding the land and whole-tank digging costs.

**36(ii). Find $\frac{dC}{dx}$.** [1 mark]

**Solution:**

**Concept:**

- For constant $a$, the power rule is $\frac{d}{dx}(ax^n) = anx^{n-1}$.
- The fixed volume yields $h = 250/x^2$; hence cost must first be expressed using only $x$.

**Step 1:** Write the cost using powers of x Using $x^2h = 250$, the total cost in rupees is

$$C(x) = 5000x^2 + 40000(250/x^2)^2 = 5000x^2 + 2500000000x^{-4}, \qquad x > 0.$$

Both the land cost and digging cost vary with the base side.

**Step 2:** Differentiate term by term The derivative of the first term is $10000x$. The derivative of the second term is $2500000000(-4)x^{-5}$. Thus

$$\frac{dC}{dx} = 10000x - 10000000000x^{-5}.$$

The negative sign accounts for the falling digging cost as the base side increases at fixed capacity.

**Step 3:** Conclusion $\frac{dC}{dx} = 10000x - \frac{10000000000}{x^5}$, for $x > 0$.

**Quick Tip to Solve:** Write the reciprocal term as a negative power so the derivative sign and exponent are easy to track.

**36(iii)(a). Find the value of $x$ for which cost $C$ is minimum.** [2 marks]

**Solution:**

**Concept:**

- A differentiable cost function has a stationary point where $C\prime(x) = 0$.
- If $C\prime\prime(x) > 0$ throughout the domain, the stationary point gives the unique minimum.

**Quick Tip to Solve:** After obtaining the positive sixth root, use the positive second derivative to establish a minimum.

**Step 1:** Obtain the stationary side length The volume condition is $x^2h = 250$, so the total cost is

$$C(x) = 5000x^2 + 2500000000x^{-4}, \quad x > 0.$$

Differentiate and set the derivative to zero:

$$\begin{aligned} C\prime(x) &= 10000x - 10000000000x^{-5} = 0, \\ 10000x^6 &= 10000000000, \\ x^6 &= 1000000 = 10^6. \end{aligned}$$

The physical domain allows only the positive root.

**Step 2:** Verify that the cost is minimum The second derivative is

$$C\prime\prime(x) = 10000 + 50000000000x^{-6} > 0 \quad (x > 0).$$

Hence the derivative is strictly increasing and the stationary point is the unique minimum. Also, cost tends to infinity as $x$ approaches zero from the right and as $x$ tends to infinity, so this is the absolute minimum over the physical domain.

**Step 3:** Conclusion The total cost is minimum when the square-base side is $x = 10\,\text{m}$.

**Quick Tip to Solve:** After obtaining the positive sixth root, use the positive second derivative to establish a minimum.

OR

**36(iii)(b). Check whether the cost function $C(x)$ expressed in terms of $x$ is increasing or not, where $x > 0$.** [2 marks]

**Solution:**

**Concept:**

- A differentiable function increases where its derivative is positive and decreases where its derivative is negative.
- The physical domain is $x > 0$, since $x$ is the side length of a square base.

**Step 1:** Express the derivative in a form suitable for sign analysis From $x^2h = 250$, we have $h = 250/x^2$. Adding land and digging costs gives

$$C(x) = 5000x^2 + 2500000000x^{-4}.$$

Therefore

$$C\prime(x) = 10000x - 10000000000x^{-5} = \frac{10000}{x^5}(x^6 - 10^6).$$

Since $x > 0$, the factor $10000/x^5$ is positive, so only the bracket determines the sign.

**Step 2:** Check the intervals on either side of the stationary point For $0 < x < 10$, we have $x^6 < 10^6$, so $C\prime(x) < 0$. For $x > 10$, we have $x^6 > 10^6$, so $C\prime(x) > 0$. At $x = 10$, the derivative is zero. Thus the

**Quick Tip to Solve:** Factor the derivative by a positive quantity before checking its sign on the positive domain.

cost first decreases, reaches its minimum, and then increases. In particular, having a positive side length alone does not ensure an increasing cost function.

**Step 3:** Conclusion $C$ is not increasing on all of $(0, \infty)$: it decreases on $(0, 10)$ and increases on $(10, \infty)$.

**Quick Tip to Solve:** Factor the derivative by a positive quantity before checking its sign on the positive domain.

**37. An octagonal prism is a three-dimensional polyhedron bounded by two octagonal bases and eight rectangular side faces. It has 24 edges and 16 vertices.**

**The prism is rolled along the rectangular faces and number on the bottom face (touching the ground) is noted. Let $X$ denote the number obtained on the bottom face and the following table gives the probability distribution of $X$.**

| $X$ : | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $P(X)$ : | $p$ | $2p$ | $2p$ | $p$ | $2p$ | $p^2$ | $2p^2$ | $7p^2 + p$ |

**Based on the above information, answer the following questions:** [4 marks]

**37(i). Find the value of $p$.** [1 mark]

**Solution:**

**Concept:**

- The probabilities of all possible values of a discrete random variable sum to one.
- Every probability is nonnegative; since $P(X = 1) = p$, we must have $p \geq 0$.

**Step 1:** Use the normalization condition Adding all eight entries in the given distribution yields

$$p + 2p + 2p + p + 2p + p^2 + 2p^2 + (7p^2 + p) = 1.$$

Collecting like terms gives $10p^2 + 9p - 1 = 0$.

**Step 2:** Solve and check admissibility Factor the quadratic as $(10p - 1)(p + 1) = 0$. The candidates are $p = 1/10$ and $p = -1$. The latter makes $P(X = 1)$ negative and is impossible. With the positive candidate, all listed probabilities are nonnegative and their sum is one, as required.

**Step 3:** Conclusion $p = \frac{1}{10}$.

**Quick Tip to Solve:** Use both normalization and nonnegativity; a quadratic root is not automatically an admissible probability.

**37(ii). Find $P(X > 6)$.** [1 mark]

**Solution:**

**Concept:**

- For mutually exclusive values of $X$, the probability of their union is the sum of their probabilities.
- The distribution is normalized by $10p^2 + 9p = 1$, whose nonnegative solution is $p = 1/10$.

**Quick Tip to Solve:** For a strict inequality, list the permitted table values before adding their probabilities.

**Step 1:** Identify the outcomes in the event The variable takes integer values from 1 through 8. The strict inequality $X > 6$ therefore includes only $X = 7$ and $X = 8$; the outcome 6 is excluded.

**Step 2:** Add the corresponding table entries The given probabilities are $P(X = 7) = 2p^2$ and $P(X = 8) = 7p^2 + p$. Hence

$$P(X > 6) = 9p^2 + p = 9\left(\frac{1}{10}\right)^2 + \frac{1}{10} = \frac{9}{100} + \frac{10}{100}.$$

The value of $p$ follows by summing the entire distribution and rejecting the negative root.

**Step 3:** Conclusion $P(X > 6) = \frac{19}{100}$.

**Quick Tip to Solve:** For a strict inequality, list the permitted table values before adding their probabilities.

**37(iii)(a). Find $P(X = 3m)$, where $m$ is a natural number.** [2 marks]

**Solution:**

**Concept:**

- The event $X = 3m$, with $m$ a natural number, consists of the multiples of three in the support of $X$.
- Distinct values of $X$ are mutually exclusive, so their probabilities are added.

**Step 1:** Normalize the distribution The sum of the eight probabilities must equal one. Thus

$$p + 2p + 2p + p + 2p + p^2 + 2p^2 + (7p^2 + p) = 1,$$

or $10p^2 + 9p - 1 = 0$. Factoring gives $(10p - 1)(p + 1) = 0$. Since $p = P(X = 1) \geq 0$, the valid value is $p = 1/10$; the root $-1$ is rejected.

**Step 2:** List multiples of three and evaluate the event probability The support is $\{1, 2, 3, 4, 5, 6, 7, 8\}$. The values representable as $3m$ are 3 and 6, corresponding to $m = 1$ and $m = 2$. The next multiple, 9, is outside the support. Therefore

$$\begin{aligned} P(X = 3m) &= P(X = 3) + P(X = 6) \\ &= 2p + p^2 \\ &= \frac{2}{10} + \frac{1}{100} = \frac{21}{100}. \end{aligned}$$

Even under a convention including zero among natural numbers, zero adds no outcome because it is absent from the support.

**Step 3:** Conclusion $P(X = 3m) = \frac{21}{100}$, where the event includes the possible multiples 3 and 6.

**Quick Tip to Solve:** Intersect the multiples of three with the actual support before selecting entries from the table.

OR

**37(iii)(b). Find the mean $E(X)$.** [2 marks]

**Solution:**

**Concept:**

- For a probability distribution, all probabilities are non-negative and their sum is 1. This determines the parameter $p$.
- The mean of a discrete random variable is $E(X) = \sum_x xP(X = x)$: multiply each possible value by its corresponding probability and add.

**Step 1:** Determine the admissible value of p The probabilities in the table must add to unity. Hence

$$p + 2p + 2p + p + 2p + p^2 + 2p^2 + (7p^2 + p) = 1.$$

Thus

$$10p^2 + 9p - 1 = 0 \quad \Longrightarrow \quad (10p - 1)(p + 1) = 0.$$

The roots are $p = \frac{1}{10}$ and $p = -1$. Since $P(X = 1) = p \geq 0$, reject the negative root and use $p = \frac{1}{10}$.

**Step 2:** Form the weighted sum Use all eight entries, retaining both terms of the last probability:

$$\begin{aligned} E(X) &= 1(p) + 2(2p) + 3(2p) + 4(p) + 5(2p) \\ &\quad + 6(p^2) + 7(2p^2) + 8(7p^2 + p) \\ &= 33p + 76p^2. \end{aligned}$$

Substitution gives

$$E(X) = 33\left(\frac{1}{10}\right) + 76\left(\frac{1}{100}\right) = \frac{330 + 76}{100} = \frac{203}{50}.$$

This weighted mean lies between the smallest and largest possible values, 1 and 8, as required.

**Step 3:** Conclusion The mean is $\boxed{E(X) = \frac{203}{50} = 4.06}$.

**Quick Tip to Solve:** Group the weighted terms into coefficients of $p$ and $p^2$, then substitute $p = 0.1$ only once.

**38. A volleyball player serves the ball which takes a parabolic path given by the equation $h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1$, where $h(t)$ is the height of ball at any time $t$ (in seconds), $(t \geq 0)$.**

**Based on the above information, answer the following questions:** [4 marks]

**38(i). Is $h(t)$ a continuous function? Justify.** [2 marks]

**Solution:**

**Concept:**

- A function is continuous at an interior point $a$ when $\lim_{t \to a} h(t) = h(a)$; at an endpoint of its domain, the appropriate one-sided limit is used.

**Quick Tip to Solve:** Recognise the polynomial first; for the restricted domain $t \geq 0$, mention right continuity at the endpoint.

- Every polynomial is continuous on $\mathbb{R}$. Multiplication by constants and addition preserve continuity.

**Step 1:** Check continuity at every positive time The given height function

$$h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1$$

is a polynomial of degree two. For any $a > 0$, the functions $t^2$, $t$, and the constant $1$ are continuous at $a$. Applying the sum and constant-multiple laws for limits,

$$\lim_{t \to a} h(t) = -\frac{7}{2}a^2 + \frac{13}{2}a + 1 = h(a).$$

Thus there is no discontinuity at any positive time.

**Step 2:** Check the domain endpoint The stated domain is $t \geq 0$, so continuity at $t = 0$ is checked from the right. Direct substitution gives $h(0) = 1$, and

$$\lim_{t \to 0^+} \left( -\frac{7}{2}t^2 + \frac{13}{2}t + 1 \right) = 1 = h(0).$$

Therefore the function is also continuous at the initial time. Combining this endpoint check with the polynomial argument covers every point of the stated domain.

**Step 3:** Conclusion Yes, $\boxed{h(t) \text{ is continuous for every } t \geq 0}$, since it is a polynomial and its right-hand limit at $0$ equals $h(0)$.

**Quick Tip to Solve:** Recognise the polynomial first; for the restricted domain $t \geq 0$, mention right continuity at the endpoint.

**38(ii). Find the time at which the height of the ball is maximum.** [2 marks]

**Solution:**

**Concept:**

- For a differentiable height function, an interior maximum can occur at a stationary time where $h'(t) = 0$.
- If $h''(t) < 0$, the curve is concave down. A stationary point of this downward-opening quadratic is its unique maximum.

**Step 1:** Find the stationary time Differentiate the given function with respect to time:

$$h(t) = -\frac{7}{2}t^2 + \frac{13}{2}t + 1,$$
$$h'(t) = -7t + \frac{13}{2}.$$

Set the derivative equal to zero:

$$-7t + \frac{13}{2} = 0 \quad \Longrightarrow \quad 7t = \frac{13}{2} \quad \Longrightarrow \quad t = \frac{13}{14}.$$

**Quick Tip to Solve:** For $at^2 + bt + c$ with $a < 0$, check the maximum time using the vertex formula $t = -b/(2a)$.

This time is positive, so it belongs to the stated domain $t \geq 0$.

**Step 2:** Verify that the stationary height is maximum The second derivative is

$$h''(t) = -7 < 0.$$

Hence the stationary point gives a maximum height. More explicitly, $h'(t) > 0$ for $0 \leq t < \frac{13}{14}$ and $h'(t) < 0$ for $t > \frac{13}{14}$. The height therefore increases up to this time and decreases afterwards. This verifies that the point is the global maximum on the non-negative time domain, rather than merely a stationary point.

**Step 3:** Conclusion The height of the ball is maximum at $\boxed{t = \frac{13}{14}\,\text{s}}$ after the serve.

**Quick Tip to Solve:** For $at^2 + bt + c$ with $a < 0$, check the maximum time using the vertex formula $t = -b/(2a)$.