CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-4-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.

(iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculators is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $x\begin{pmatrix}1\\2\end{pmatrix}+y\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}4\\9\end{pmatrix}$**, then:** [1 mark]

(A) $x=1,\ y=2$

(B) $x=2,\ y=1$

(C) $x=1,\ y=-1$

(D) $x=3,\ y=2$

Correct Answer: (B) $x=2,\ y=1$

Solution:

Concept:

- Equality of column matrices means equality of corresponding entries.
- A pair of simultaneous linear equations can be solved by elimination.

Step 1: Equate corresponding entries The first row gives $x+2y=4$, while the second row gives $2x+5y=9$. Both equations must hold for the same pair (x,y).

Step 2: Eliminate and check the choices Subtract twice the first equation from the second: $y=9-8=1$. Substitution gives $x=4-2=2$. Option (A) gives first entry 5, (C) gives -1 and (D) gives 7, so none satisfies the required first entry 4. Option (B) gives both entries 4 and 9.

Step 3: Conclusion $x=2,\ y=1$. Hence, the correct option is (B).

Quick Tip to Solve: For two column vectors, equate their entries before attempting any matrix operation.

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2. The product $\begin{pmatrix}a&b\\-b&a\end{pmatrix}\begin{pmatrix}a&-b\\b&a\end{pmatrix}$ **is equal to:** [1 mark]

(A) $\begin{pmatrix}a^2+b^2&0\\0&a^2+b^2\end{pmatrix}$

(B) $\begin{pmatrix}(a+b)^2&0\\(a+b)^2&0\end{pmatrix}$

(C) $\begin{pmatrix}a^2+b^2&0\\a^2+b^2&0\end{pmatrix}$

(D) $\begin{pmatrix}a&0\\0&b\end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix}a^2+b^2&0\\0&a^2+b^2\end{pmatrix}$

Solution:

Concept:

- The entry in row i, column j of a product is the dot product of row i of the first matrix and column j of the second.
- The identity matrix I_2 has diagonal entries 1 and off-diagonal entries 0.

Step 1: Multiply the entries The two diagonal entries are $a{\cdot}a+b{\cdot}b = a^2+b^2$ and $(-b)(-b)+a{\cdot}a = a^2+b^2$. The off-diagonal entries are $-ab+ab = 0$ and $-ab+ab = 0$.

Step 2: Compare with the options Thus the product is $(a^2+b^2)I_2$. Option (B) wrongly replaces the sum of squares by a square of a sum and has an incorrect lower row. Option (C) puts a nonzero entry off the diagonal; (D) omits the required squares and sum. Only (A) gives the product for arbitrary a, b.

Step 3: Conclusion $\begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$. Hence, the correct option is (A).

Quick Tip to Solve: Check the off-diagonal entries first: opposite signs make both cross terms cancel.

3. If A is a square matrix and $A^2 = A$, then $(I+A)^2 - 3A$ is equal to: [1 mark]

(A) I

(B) A

(C) $2A$

(D) $3I$

Correct Answer: (A) I

Solution:

Concept:

- The identity matrix I satisfies $IA = AI = A$ and $I^2 = I$.
- An idempotent matrix satisfies $A^2 = A$.

Step 1: Expand the square Using matrix multiplication, $(I+A)^2 = I^2+IA+AI+A^2 = I+2A+A^2$. The mixed terms combine because the identity commutes with every square matrix of its order.

Step 2: Apply the given identity Now $(I+A)^2 - 3A = I+2A+A^2-3A = I+A^2-A = I$. The expressions in (B) and (C) still depend on A, and cannot be the general identity obtained here; for example $A = 0$ rules them out. Option (D) has the wrong coefficient of I.

Step 3: Conclusion $(I+A)^2 - 3A = I$. Hence, the correct option is (A).

Quick Tip to Solve: Replace every occurrence of A squared by A immediately after expanding the square.

4. If a matrix $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, then the matrix AA' (where A' is the transpose of A) is: [1 mark]

(A) 14

(B) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

(D) $\begin{pmatrix} 14 \end{pmatrix}$

Correct Answer: (D) $\begin{pmatrix} 14 \end{pmatrix}$

Solution:

Concept:

- The transpose interchanges rows and columns; a 1×3 row matrix becomes a 3×1 column matrix.
- A 1×3 matrix times a 3×1 matrix is a 1×1 matrix.

Step 1: Find the transpose and order For $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, its transpose is $A' = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Thus AA' has only one entry; it cannot be either of the 3×3 matrices in (B) and (C).

Step 2: Compute the single entry Row-by-column multiplication gives $AA' = \begin{pmatrix} 1^2 + 2^2 + 3^2 \end{pmatrix} = \begin{pmatrix} 14 \end{pmatrix}$. Option (A) states the scalar 14, whereas the question asks for the matrix. Option (D) supplies the required one-entry matrix, including its brackets.

Step 3: Conclusion $AA' = \begin{pmatrix} 14 \end{pmatrix}$. Hence, the correct option is (D).

Quick Tip to Solve: Use the product order to distinguish a one-entry matrix from the scalar written without brackets.

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5. The value of $\begin{vmatrix} x+y & y+z & z+x \\ z & x & y \\ 1 & 1 & 1 \end{vmatrix}$ is: [1 mark]

(A) 0

(B) 1

(C) $x + y + z$

(D) $2(x + y + z)$

Correct Answer: (A) 0

Solution:

Concept:

- A determinant is zero if its rows are linearly dependent.
- Adding a multiple of another row to a row does not change the determinant.

Quick Tip to Solve: For cyclic-looking determinants, add two rows and look for a multiple of the row of ones.

Step 1: Identify a relation among rows Let $S = x + y + z$. Adding the first two rows entry by entry gives $(x + y + z, y + z + x, z + x + y) = (S, S, S)$. Therefore $R_1 + R_2 = SR_3$.

Step 2: Use the determinant property Apply $R_1 \to R_1 + R_2 - SR_3$, leaving a zero first row. Its determinant is zero for all x, y, z. This proves (A); (B) cannot be an identically zero expression. The expressions in (C) and (D) may be nonzero when $S \neq 0$, so neither is the general value.

Step 3: Conclusion The determinant equals 0. Hence, the correct option is (A).

Quick Tip to Solve: For cyclic-looking determinants, add two rows and look for a multiple of the row of ones.

6. The function $f(x) = |x|$ is [1 mark]

(A) continuous and differentiable everywhere.

(B) continuous and differentiable nowhere.

(C) continuous everywhere, but differentiable everywhere except at $x = 0$.

(D) continuous everywhere, but differentiable nowhere.

Correct Answer: (C) continuous everywhere, but differentiable everywhere except at $x = 0$.

Solution:

Concept:

- $|x| = x$ for $x \geq 0$ and $|x| = -x$ for $x < 0$.
- Differentiability at a point requires equal finite left and right derivatives.

Step 1: Check continuity On each side of zero the function is linear, hence continuous. At zero, $\lim_{x \to 0^-} |x| = \lim_{x \to 0^+} |x| = 0 = f(0)$, so it is also continuous there.

Step 2: Check differentiability and the options For $x < 0$, $f'(x) = -1$; for $x > 0$, $f'(x) = 1$. At zero, the left derivative is -1 and the right derivative is 1, so the derivative does not exist. This rules out (A). Continuity everywhere rules out (B), and the derivatives at every nonzero point rule out (D).

Step 3: Conclusion The function is continuous everywhere and differentiable everywhere except at $x = 0$. Hence, the correct option is (C).

Quick Tip to Solve: For an absolute-value function, check one-sided derivatives at the point where its inner expression vanishes.

7. If $y = \sin^2(x^3)$, then $\frac{dy}{dx}$ is equal to: [1 mark]

(A) $2 \sin x^3 \cos x^3$

(B) $3x^3 \sin x^3 \cos x^3$

(C) $6x^2 \sin x^3 \cos x^3$

(D) $2x^2 \sin^2(x^3)$

Correct Answer: (C) $6x^2 \sin x^3 \cos x^3$

Solution:

Concept:

- The chain rule is $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$ for a composite function.
- $\frac{d}{du}(\sin^2 u) = 2 \sin u \cos u$ and $\frac{d}{dx}(x^3) = 3x^2$.

Step 1: Differentiate the outer function Put $u = x^3$, so $y = (\sin u)^2$. Differentiating the square and then the sine gives $\frac{dy}{du} = 2 \sin u \cos u$.

Step 2: Include the inner derivative Multiply by $du/dx = 3x^2$: $\frac{dy}{dx} = 2 \sin(x^3) \cos(x^3) \cdot 3x^2 = 6x^2 \sin(x^3) \cos(x^3)$. Option (A) omits the inner derivative; (B) has both the wrong power and coefficient. Option (D) does not correctly differentiate the sine square, so only (C) matches.

Step 3: Conclusion $\frac{dy}{dx} = 6x^2 \sin(x^3) \cos(x^3)$. Hence, the correct option is (C).

Quick Tip to Solve: Write the inner variable x cubed separately so that the factor 3x squared is not missed.

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8. $\int e^{5 \log x}\, dx$ is equal to: [1 mark]

(A) $\frac{x^5}{5} + C$

(B) $\frac{x^6}{6} + C$

(C) $5x^4 + C$

(D) $6x^5 + C$

Correct Answer: (B) $\frac{x^6}{6} + C$

Solution:

Concept:

- For $x > 0$, $e^{a \log x} = x^a$, where $\log$ denotes the natural logarithm.
- The power rule is $\int x^n\, dx = x^{n+1}/(n+1) + C$, for $n \neq -1$.

Step 1: Simplify the exponential The logarithm requires $x > 0$. Using the inverse relation between the exponential and natural logarithm,

$$e^{5 \log x} = e^{\log(x^5)} = x^5.$$

The coefficient 5 in the exponent becomes a power of x, not a multiplier outside the integral.

Quick Tip to Solve: First replace $e^{5 \log x}$ by x^5; then increase the power by one and divide by that new power.

Step 2: Integrate and check the options Therefore

$$\int e^{5\log x}\,dx = \int x^5\,dx = \frac{x^6}{6} + C.$$

Differentiating option (B) returns x^5. Option (A) differentiates to x^4, option (C) to $20x^3$, and option (D) to $30x^4$; none equals the integrand on its domain.

Step 3: Conclusion The integral is $\boxed{x^6/6 + C}$. Hence, the correct option is (B).

Quick Tip to Solve: First replace $e^{5\log x}$ by x^5; then increase the power by one and divide by that new power.

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9. If $\int_0^a 3x^2\,dx = 8$, then the value of a is: [1 mark]

(A) 2

(B) 4

(C) 8

(D) 10

Correct Answer: (A) 2

Solution:

Concept:

- The fundamental theorem gives $\int_0^a f(x)\,dx = F(a) - F(0)$, where F is an antiderivative of f.
- An antiderivative of $3x^2$ is x^3.

Step 1: Evaluate the definite integral Apply the power rule and substitute both limits:

$$\int_0^a 3x^2\,dx = [x^3]_0^a = a^3 - 0^3 = a^3.$$

No arbitrary constant remains in a definite integral, because its values at the two limits cancel.

Step 2: Solve and test the choices The given condition becomes $a^3 = 8$. The real cube root is $a = 2$, and cubing is one-to-one on the real line, so this real value is unique. Options (B), (C), and (D) give the integral values 64, 512, and 1000, respectively; none equals 8.

Step 3: Conclusion $\boxed{a = 2}$. Hence, the correct option is (A).

Quick Tip to Solve: For $\int_0^a 3x^2\,dx$, the integral immediately becomes a^3; take the real cube root of the given value.

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10. The integrating factor for solving the differential equation $x\dfrac{dy}{dx} - y = 2x^2$ is: [1 mark]

(A) e^{-y}

(B) e^{-x}

(C) x

(D) $\dfrac{1}{x}$

Correct Answer: (D) $\dfrac{1}{x}$

Solution:

Concept:

- For the linear equation $dy/dx + P(x)y = Q(x)$, an integrating factor is $e^{\int P(x)\,dx}$.
- The coefficient of dy/dx must first be made equal to 1.

Step 1: Write the standard linear form On any interval excluding $x = 0$, divide the given equation by x:

$$\frac{dy}{dx} - \frac{1}{x}y = 2x.$$

Thus the coefficient needed in the integrating-factor formula is $P(x) = -1/x$, not -1 and not a function of y.

Step 2: Calculate the integrating factor Now

$$\text{I.F.} = e^{\int -1/x\,dx} = e^{-\log|x|} = \frac{1}{|x|}.$$

An integrating factor may be multiplied by any nonzero constant. Hence on either interval $x > 0$ or $x < 0$, $1/x$ is a valid choice. Indeed, multiplying the standard equation by $1/x$ gives $(y/x)' = 2$. The other choices do not produce this product derivative.

Step 3: Conclusion A suitable integrating factor is $\boxed{1/x}$, for $x \neq 0$. Hence, the correct option is (D).

Quick Tip to Solve: Divide by x before reading the coefficient of y; integrate $-1/x$, then exponentiate.

11. The order and degree (if defined) of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 = x\sin\left(\frac{dy}{dx}\right)$ respectively are: [1 mark]

(A) $2,\ 2$

(B) $1,\ 3$

(C) $2,\ 3$

(D) 2, degree not defined

Correct Answer: (D) 2, degree not defined

Solution:

Concept:

- Order is the order of the highest derivative appearing in a differential equation.
- Degree is defined only when the differential equation is polynomial in its derivatives; it is then the power of the highest-order derivative.

Step 1: Identify the highest derivative The equation contains both dy/dx and d^2y/dx^2. Its highest derivative is the second derivative, even though the first derivative occurs inside the sine on the right-hand side. Therefore the order of the equation is 2.

Step 2: Check whether degree exists The term $x\sin(dy/dx)$ is a trigonometric function of a derivative, so the equation is not polynomial in its derivatives. Its degree is consequently not defined. The visible square of the second derivative does not by itself establish degree 2. Thus (A) and (C) wrongly assign a degree, while (B) also gives the wrong order.

Step 3: Conclusion Order $= \boxed{2}$; degree is not defined. Hence, the correct option is (D).

Quick Tip to Solve: Check for sine, exponential, or logarithm of any derivative before assigning degree; a visible exponent alone is insufficient.

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12. A unit vector along the vector $4\hat{i} - 3\hat{k}$ is: [1 mark]

(A) $\frac{1}{7}(4\hat{i} - 3\hat{k})$

(B) $\frac{1}{5}(4\hat{i} - 3\hat{k})$

(C) $\frac{1}{\sqrt{7}}(4\hat{i} - 3\hat{k})$

(D) $\frac{1}{\sqrt{5}}(4\hat{i} - 3\hat{k})$

Correct Answer: (B) $\frac{1}{5}(4\hat{i} - 3\hat{k})$

Solution:

Concept:

- The magnitude of $a\hat{i} + b\hat{j} + c\hat{k}$ is $\sqrt{a^2 + b^2 + c^2}$.
- A unit vector in the direction of a nonzero vector $\vec{v}$ is $\vec{v}/|\vec{v}|$.

Quick Tip to Solve: Use the 3-4-5 magnitude here; divide every component by 5 while retaining the negative sign of the k component.

Step 1: Find the magnitude For $\vec{v} = 4\hat{i} - 3\hat{k}$, the three components are $4, 0, -3$. Hence

$$|\vec{v}| = \sqrt{4^2 + 0^2 + (-3)^2} = \sqrt{25} = 5.$$

The negative sign of the third component disappears when computing its square.

Step 2: Normalize and verify Dividing by the positive magnitude preserves the direction:

$$\hat{v} = \frac{1}{5}(4\hat{i} - 3\hat{k}).$$

Its magnitude is $\sqrt{(4/5)^2 + (-3/5)^2} = 1$. The magnitudes of options (A), (C), and (D) are $5/7$, $5/\sqrt{7}$, and $5/\sqrt{5}$, respectively, so none is a unit vector.

Step 3: Conclusion The required vector is $\boxed{(4\hat{i} - 3\hat{k})/5}$. Hence, the correct option is (B).

Quick Tip to Solve: Use the 3-4-5 magnitude here; divide every component by 5 while retaining the negative sign of the k component.

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13. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a} \cdot \vec{b} \geq 0$ only when: [1 mark]

(A) $0 < \theta < \dfrac{\pi}{2}$

(B) $0 \leq \theta \leq \dfrac{\pi}{2}$

(C) $0 < \theta < \pi$

(D) $0 \leq \theta \leq \pi$

Correct Answer: (B) $0 \leq \theta \leq \dfrac{\pi}{2}$

Solution:

Concept:

- For two nonzero vectors, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, with $0 \leq \theta \leq \pi$.
- Because both magnitudes are positive, the sign of the dot product is the sign of $\cos\theta$.

Step 1: Convert the inequality to an angle condition Using the dot-product formula,

$$\vec{a} \cdot \vec{b} \geq 0 \quad \Longleftrightarrow \quad \cos\theta \geq 0.$$

On the allowed interval $[0, \pi]$, cosine is nonnegative exactly from 0 through $\pi/2$. The premise that an angle is given concerns nonzero vectors, for which that angle is defined.

Step 2: Include the boundary angles and compare At $\theta = 0$, the dot product is positive; at $\theta = \pi/2$, it is zero. Both satisfy the non-strict inequality, so option (A) omits valid boundary cases. Options (C) and (D) include obtuse angles, where cosine and the dot product are negative. The complete interval is therefore the one in option (B).

Step 3: Conclusion $\boxed{0 \leq \theta \leq \pi/2}$. Hence, the correct option is (B).

Quick Tip to Solve: The sign ≥ 0 includes perpendicular vectors, so retain $\pi/2$ as well as the parallel angle 0.

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14. Distance of the point (p, q, r) from y-axis is: [1 mark]

(A) q

(B) $|q|$

(C) $|q| + |r|$

(D) $\sqrt{p^2 + r^2}$

Correct Answer: (D) $\sqrt{p^2 + r^2}$

Solution:

Concept:

- Every point on the y-axis has coordinates $(0, t, 0)$, with $t \in \mathbb{R}$.
- The distance from a point to a line is its shortest distance to any point of that line.

Step 1: Locate the nearest point on the axis Let $P = (p, q, r)$ and let $A = (0, t, 0)$ be a general point on the y-axis. The squared distance is

$$PA^2 = p^2 + (q - t)^2 + r^2.$$

Since $(q - t)^2 \geq 0$, the minimum occurs at $t = q$, giving the perpendicular foot $(0, q, 0)$.

Step 2: Evaluate and compare the options The shortest distance is consequently

$$\sqrt{p^2 + 0^2 + r^2} = \sqrt{p^2 + r^2}.$$

The coordinate q measures position along the axis and does not contribute to distance from it. Thus q, $|q|$, and $|q| + |r|$ do not give the general distance. For instance, points on the axis have $p = r = 0$ and must have zero distance whatever q is.

Step 3: Conclusion The required distance is $\boxed{\sqrt{p^2 + r^2}}$. Hence, the correct option is (D).

Quick Tip to Solve: For distance from the y-axis, omit the y-coordinate and take the square root of the sum of squares of the other two.

15. The solution set of the inequation $3x + 5y < 7$ is: [1 mark]

(A) whole xy-plane except the points lying on the line $3x + 5y = 7$.

(B) whole xy-plane along with the points lying on the line $3x + 5y = 7$.

(C) open half plane containing the origin except the points of line $3x + 5y = 7$.

(D) open half plane not containing the origin.

Correct Answer: (C) open half plane containing the origin except the points of line $3x + 5y = 7$.

Solution:

Concept:

- A strict linear inequality selects one open half-plane; its boundary line is excluded.
- Test a point off the boundary to identify the required side.

Step 1: Check the boundary and origin The boundary is $3x + 5y = 7$. Every point on this line gives

Quick Tip to Solve: Test the origin first: $0 < 7$ chooses its side; the strict sign excludes the boundary.

equality, so none satisfies $3x + 5y < 7$. At the origin, $3(0) + 5(0) = 0 < 7$; hence the required open half-plane contains the origin.

Step 2: Compare all four descriptions Options (A) and (B) include points on both sides of the boundary, whereas the inequality admits only one side. Option (B) also includes the forbidden boundary. Option (D) selects the opposite half-plane, since it excludes the origin. Option (C) correctly includes only the side containing the origin and excludes the line.

Step 3: Conclusion The solution set is the open half-plane containing the origin, excluding $3x+5y = 7$. Hence, the correct option is (C).

Quick Tip to Solve: Test the origin first: $0 < 7$ chooses its side; the strict sign excludes the boundary.

16. Which of the following points satisfies both the inequations $2x + y \leq 10$ and $x + 2y \geq 8$? [1 mark]

(A) $(-2, 4)$

(B) $(3, 2)$

(C) $(-5, 6)$

(D) $(4, 2)$

Correct Answer: (D) $(4, 2)$

Solution:

Concept:

- A point satisfies a pair of inequalities only when both substitutions are true.
- Compute $2x + y$ and $x + 2y$ separately for each option.

Step 1: Test options (A), (B) and (C) For $(-2, 4)$, the values are $0 \leq 10$ and $6 < 8$, so (A) fails the second inequality. For $(3, 2)$, they are $8 \leq 10$ and $7 < 8$, so (B) also fails. For $(-5, 6)$, they are $-4 \leq 10$ and $7 < 8$, so (C) fails as well.

Step 2: Test option (D) At $(4, 2)$,

$$2x + y = 2(4) + 2 = 10 \leq 10, \qquad x + 2y = 4 + 2(2) = 8 \geq 8.$$

Both conditions hold. Equality is permitted because both inequality signs are non-strict.

Step 3: Conclusion Only $(4, 2)$ satisfies both inequalities. Hence, the correct option is (D).

Quick Tip to Solve: Here all options pass the first inequality; checking $x + 2y \geq 8$ immediately isolates (D).

17. If the direction cosines of a line are $\left(\frac{1}{a}, \frac{1}{a}, \frac{1}{a}\right)$, then: [1 mark]

(A) $0 < a < 1$

(B) $a > 2$

(C) $a > 0$

(D) $a = \pm\sqrt{3}$

Correct Answer: (D) $a = \pm\sqrt{3}$

Solution:

Concept:

- Direction cosines l, m, n of a line satisfy $l^2 + m^2 + n^2 = 1$.
- The given reciprocals require $a \neq 0$, and squaring retains both signs of a.

Step 1: Apply the direction-cosine identity Substitute $l = m = n = 1/a$ into the identity:

$$\frac{1}{a^2} + \frac{1}{a^2} + \frac{1}{a^2} = 1 \quad \Longrightarrow \quad a^2 = 3.$$

Thus $a = \sqrt{3}$ or $a = -\sqrt{3}$. Both yield valid triples, corresponding to opposite orientations of the same line.

Step 2: Check the alternatives Neither value lies between 0 and 1, so (A) is false. The positive value $\sqrt{3}$ is less than 2, so (B) is false. Option (C) wrongly discards the admissible negative value. Option (D) gives the complete set of possible values.

Step 3: Conclusion $a = \pm\sqrt{3}$. Hence, the correct option is (D).

Quick Tip to Solve: When all three direction cosines are equal, use $3l^2 = 1$ and preserve both signs.

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18. The probability that A speaks the truth is $\frac{4}{5}$ and that of B speaking the truth is $\frac{3}{4}$. The probability that they contradict each other in stating the same fact is: [1 mark]

(A) $\frac{7}{20}$

(B) $\frac{1}{5}$

(C) $\frac{3}{20}$

(D) $\frac{4}{5}$

Correct Answer: (A) $\frac{7}{20}$

Solution:

Concept:

- Contradiction occurs when exactly one speaker tells the truth.
- Using the standard independent-speakers model, multiply probabilities within each case and add the two disjoint cases.

Quick Tip to Solve: For contradiction, add truth-lie and lie-truth; do not include truth-truth or lie-lie.

Step 1: Find the complementary probabilities Write T_A, T_B for truthful statements by A and B. Then

$$P(T_A^c) = 1 - \frac{4}{5} = \frac{1}{5}, \qquad P(T_B^c) = 1 - \frac{3}{4} = \frac{1}{4}.$$

The two contradictory cases are $T_A \cap T_B^c$ and $T_A^c \cap T_B$.

Step 2: Add the probabilities of the two cases

$$P(\text{contradiction}) = \frac{4}{5} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{3}{4} = \frac{1}{5} + \frac{3}{20} = \frac{7}{20}.$$

Option (B) counts only the first case, and (C) only the second. Option (D), $4/5$, is Aś truth probability, not the probability of disagreement.

Step 3: Conclusion The probability is $\frac{7}{20}$. Hence, the correct option is (A).

Quick Tip to Solve: For contradiction, add truth-lie and lie-truth; do not include truth-truth or lie-lie.

19. Assertion (A): All trigonometric functions have their inverses over their respective domains.
Reason (R): The inverse of $\tan^{-1} x$ exists for some $x \in \mathbb{R}$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (D) Assertion (A) is false and Reason (R) is true.

Solution:

Concept:

- A function has an inverse function only when it is one-one and onto its specified codomain.
- Trigonometric functions become invertible on suitable restricted domains; $\tan^{-1}$ denotes the principal inverse tangent.

Step 1: Judge Assertion (A) On their full natural domains, trigonometric functions repeat values and are not one-one. For example, $\sin 0 = \sin \pi = 0$. Their inverses therefore require restriction to appropriate intervals; they do not all have inverses over their unrestricted respective domains. Assertion (A) is false.

Step 2: Judge Reason (R) The function $f(x) = \tan^{-1} x$ maps $\mathbb{R}$ bijectively onto $(-\pi/2, \pi/2)$. Its inverse is $f^{-1}(t) = \tan t$ on that interval. Thus the inverse exists, and in particular the stated existence for some real input is true.

Step 3: Check the explanatory relationship Since Assertion (A) is false while Reason (R) is true, the alternatives asserting that both are true cannot apply. The reason does not make the unrestricted trigonometric functions one-one.

Quick Tip to Solve: Distinguish a trigonometric function on its full domain from its restriction used to define a principal inverse.

Step 4: Conclusion The assertion is false and the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Distinguish a trigonometric function on its full domain from its restriction used to define a principal inverse.

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20. Assertion (A): The lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are perpendicular, when $\vec{b_1} \cdot \vec{b_2} = 0$.

Reason (R): The angle θ between the lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by $\cos\theta = \dfrac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|}$.

[1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- Nonzero direction vectors $\vec{b_1}, \vec{b_2}$ determine the angle between two lines.
- The dot-product formula is $\cos\theta = (\vec{b_1} \cdot \vec{b_2})/(|\vec{b_1}||\vec{b_2}|)$.

Step 1: Judge Assertion (A) If $\vec{b_1} \cdot \vec{b_2} = 0$, the nonzero direction vectors are orthogonal. Hence the angle between the directions of the given lines is 90°, so the lines are perpendicular in the stated angle-between-lines sense. Assertion (A) is true.

Step 2: Judge Reason (R) For vector equations $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$, the direction vectors are $\vec{b_1}$ and $\vec{b_2}$. The printed cosine formula follows directly from their dot product, so Reason (R) is true.

Step 3: Establish that the reason explains the assertion Substituting the zero dot product into that formula gives

$$\cos\theta = \frac{0}{|\vec{b_1}||\vec{b_2}|} = 0 \quad \Longrightarrow \quad \theta = 90^\circ.$$

Thus the reason directly establishes the assertion; option (B) wrongly denies this explanation, and (C) and (D) wrongly classify one statement as false.

Step 4: Conclusion Both statements are true, and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

Quick Tip to Solve: Identify the coefficient vectors of the parameters; their dot product alone controls the angle.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21(a). Find the domain of $y = \sin^{-1}(x^2 - 4)$. [2 marks]

Solution:

Concept:

- The real-valued principal inverse sine $\sin^{-1} u$ is defined precisely for $-1 \le u \le 1$.
- For $c, d \ge 0$, $c \le x^2 \le d$ means $\sqrt{c} \le |x| \le \sqrt{d}$.

Step 1: Impose the inverse-sine condition Here the argument of inverse sine is $u = x^2 - 4$, not x itself. For a real value of y,

$$-1 \le x^2 - 4 \le 1.$$

Adding 4 throughout gives

$$3 \le x^2 \le 5.$$

These simultaneous inequalities exclude values near zero as well as values with excessively large magnitude.

Step 2: Solve on both sides of zero Since $x^2 = |x|^2$, the nonnegative square-root function gives

$$\sqrt{3} \le |x| \le \sqrt{5}.$$

For $x \ge 0$, this is $\sqrt{3} \le x \le \sqrt{5}$. For $x < 0$, negating reverses the inequalities and gives $-\sqrt{5} \le x \le -\sqrt{3}$. Both intervals must be included; taking only positive square roots would omit valid inputs.

Step 3: Check inclusion of the endpoints At $x = \pm\sqrt{3}$, the argument is -1, and at $x = \pm\sqrt{5}$, it is 1. Both are permitted inputs of inverse sine. Thus all four endpoints belong to the domain and square brackets are required.

Step 4: Conclusion The domain is $\boxed{[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]}$.

Quick Tip to Solve: First bound the entire inverse-sine argument by -1 and 1; then solve using $|x|$ to retain both intervals.

OR

21(b). Evaluate: $\cos^{-1}\left[\cos\left(-\frac{7\pi}{3}\right)\right]$. [2 marks]

Solution:

Concept:

- The principal value of $\cos^{-1} t$, for $-1 \le t \le 1$, lies in $[0, \pi]$.
- Cosine is even and has period 2π: $\cos(-\theta) = \cos\theta$ and $\cos(\theta + 2\pi) = \cos\theta$.

Step 1: Reduce the angle using cosine identities The printed angle is $-7\pi/3$. Since cosine is even,

Quick Tip to Solve: Reduce the inner cosine first, then choose its inverse angle in $[0, \pi]$.

first remove its negative sign. Then remove one complete revolution:

$$\cos\left(-\frac{7\pi}{3}\right) = \cos\left(\frac{7\pi}{3}\right) = \cos\left(2\pi + \frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}.$$

These identities preserve the cosine value even though the original angle is outside the principal interval.

Step 2: Select the principal inverse value We must choose the angle in $[0, \pi]$ whose cosine is $1/2$. The angle $\pi/3$ belongs to this interval and satisfies that condition. Thus

$$\cos^{-1}\left[\cos\left(-\frac{7\pi}{3}\right)\right] = \cos^{-1}\left(\frac{1}{2}\right).$$

Directly cancelling cosine and its inverse would be invalid for the original angle, since it does not lie in the principal interval.

Step 3: Conclusion The required value is $\boxed{\pi/3}$.

Quick Tip to Solve: Reduce the inner cosine first, then choose its inverse angle in $[0, \pi]$.

22. If $(x^2 + y^2)^2 = xy$, then find $\frac{dy}{dx}$. [2 marks]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x; hence $d(y^2)/dx = 2y\,dy/dx$.
- The chain rule differentiates the square on the left; the product rule gives $d(xy)/dx = y + x\,dy/dx$.

Step 1: Differentiate both sides Start with $(x^2 + y^2)^2 = xy$. Writing $y' = dy/dx$, differentiation with respect to x gives

$$2(x^2 + y^2)(2x + 2yy') = y + xy'.$$

The factor $2x + 2yy'$ is essential because both x^2 and y^2 vary with x. Expanding the left side gives

$$4x(x^2 + y^2) + 4y(x^2 + y^2)y' = y + xy'.$$

Step 2: Collect and isolate the derivative Move the terms involving y' to the left and the remaining terms to the right:

$$[4y(x^2 + y^2) - x]y' = y - 4x(x^2 + y^2).$$

Divide by the coefficient of y' wherever it is nonzero. At points where this coefficient vanishes, this quotient cannot by itself provide a finite derivative; the local curve must be examined separately.

Step 3: Conclusion

$$\boxed{\frac{dy}{dx} = \frac{y - 4x(x^2 + y^2)}{4y(x^2 + y^2) - x}}$$

wherever $4y(x^2 + y^2) - x \neq 0$.

Quick Tip to Solve: Keep every term containing dy/dx on one side before dividing; this prevents sign errors.

23. Find the maximum and minimum values of the function given by $f(x) = 5 + \sin 2x$. [2 marks]

Solution:

Concept:

- For every real angle θ, $-1 \leq \sin\theta \leq 1$.
- Adding a constant shifts both endpoints of a function range by that constant; extrema are attained when the corresponding sine bounds are attained.

Step 1: Bound the function for all real inputs Because $2x$ is real whenever x is real, the usual bound for sine applies:

$$-1 \leq \sin 2x \leq 1.$$

Add 5 throughout the inequality:

$$4 \leq 5 + \sin 2x \leq 6.$$

Thus no function value can be smaller than 4 or larger than 6. This is a global range bound, so no further comparison between local extrema is needed.

Step 2: Show that both bounds are attained The upper bound is attained when $\sin 2x = 1$, that is,

$$2x = \frac{\pi}{2} + 2k\pi, \qquad x = \frac{\pi}{4} + k\pi.$$

The lower bound is attained when $\sin 2x = -1$, that is,

$$2x = \frac{3\pi}{2} + 2k\pi, \qquad x = \frac{3\pi}{4} + k\pi,$$

where $k \in \mathbb{Z}$. In particular, $x = \pi/4$ and $x = 3\pi/4$ respectively realise the upper and lower bounds, proving that they are actual extrema.

Step 3: Conclusion The maximum value is $\boxed{6}$, and the minimum value is $\boxed{4}$.

Quick Tip to Solve: For a constant plus a sine term, use the sine range and check that both endpoints are attainable.

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24. If the projection of the vector $\hat{i} + \hat{j} + \hat{k}$ on the vector $p\hat{i} + \hat{j} - 2\hat{k}$ is $\frac{1}{3}$, then find the value(s) of p. [2 marks]

Solution:

Concept:

- The scalar projection of $\vec{a}$ on a nonzero vector $\vec{b}$ is $\vec{a} \cdot \vec{b}/|\vec{b}|$.
- Here $\vec{a} = (1, 1, 1)$, $\vec{b} = (p, 1, -2)$; the positive given projection requires a positive dot product.

Step 1: Form the projection equation The dot product and magnitude are

$$\vec{a} \cdot \vec{b} = p + 1 - 2 = p - 1, \qquad |\vec{b}| = \sqrt{p^2 + 1 + 4} = \sqrt{p^2 + 5}.$$

Quick Tip to Solve: Record the sign of the numerator before squaring a scalar projection equation, then check every root.

The denominator is positive for every real p, so the prescribed projection gives

$$\frac{p-1}{\sqrt{p^2+5}} = \frac{1}{3}, \qquad 3(p-1) = \sqrt{p^2+5}.$$

In particular, the right side is positive, forcing $p > 1$. This sign condition must be retained before squaring.

Step 2: Solve and check the candidates Squaring yields

$$9(p-1)^2 = p^2+5 \quad \Longrightarrow \quad 8p^2 - 18p + 4 = 0.$$

Divide by 2 and factor:

$$4p^2 - 9p + 2 = (4p-1)(p-2) = 0.$$

The candidates are $p = 1/4$ and $p = 2$. The first violates $p > 1$ and gives projection $-1/3$, so it is extraneous. For $p = 2$, the projection is $(2-1)/\sqrt{4+5} = 1/3$, exactly the printed value.

Step 3: Conclusion The only admissible value is $\boxed{p = 2}$.

Quick Tip to Solve: Record the sign of the numerator before squaring a scalar projection equation, then check every root.

25(a). **Find the vector equation of the line passing through the point $(2, 1, 3)$ and perpendicular to both the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$.** [2 marks]

Solution:

Concept:

- A symmetric line equation supplies its direction vector through its denominators.
- The cross product of two nonparallel direction vectors is perpendicular to each. A line through position vector $\vec{a}$ in direction $\vec{d}$ has equation $\vec{r} = \vec{a} + \lambda\vec{d}$.

Step 1: Extract the two direction vectors From $(x-1)/1 = (y-2)/2 = (z-3)/3$, the first direction vector is $\vec{d_1} = \hat{i} + 2\hat{j} + 3\hat{k}$. From $x/(-3) = y/2 = z/5$, the second is $\vec{d_2} = -3\hat{i} + 2\hat{j} + 5\hat{k}$. A direction perpendicular to both is obtained from their cross product; the given point determines the position of the required line.

Step 2: Compute and verify a perpendicular direction Calculate

$$\vec{d_1} \times \vec{d_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & 2 & 5 \end{vmatrix} = 4\hat{i} - 14\hat{j} + 8\hat{k}.$$

Dividing by the nonzero scalar 2 gives the simpler direction $\vec{d} = 2\hat{i} - 7\hat{j} + 4\hat{k}$. Check that

$$\vec{d} \cdot \vec{d_1} = 2 - 14 + 12 = 0, \qquad \vec{d} \cdot \vec{d_2} = -6 - 14 + 20 = 0.$$

Quick Tip to Solve: Simplify the cross product by a common factor and check both dot products before writing the line.

Therefore it is perpendicular to both supplied directions. Use the position vector $2\hat{i}+\hat{j}+3\hat{k}$ of the required point.

Step 3: Conclusion The required vector equation is

$$\boxed{\vec{r}=(2\hat{i}+\hat{j}+3\hat{k})+\lambda(2\hat{i}-7\hat{j}+4\hat{k}),\quad \lambda\in\mathbb{R}.}$$

Quick Tip to Solve: Simplify the cross product by a common factor and check both dot products before writing the line.

OR

25(b). The equations of a line are $5x-3=15y+7=3-10z$. Write the direction cosines of the line and find the coordinates of a point through which it passes. [2 marks]

Solution:

Concept:

- Direction ratios are coefficients of a common parameter in the coordinates of a line; direction cosines are the corresponding ratios divided by their Euclidean length.
- For direction ratios (a,b,c), the direction cosines are $l=a/\sqrt{a^2+b^2+c^2}$, $m=b/\sqrt{a^2+b^2+c^2}$, and $n=c/\sqrt{a^2+b^2+c^2}$.

Step 1: Introduce a common parameter Put $5x-3=15y+7=3-10z=t$. Solving each equality for its coordinate gives

$$x=\frac{3}{5}+\frac{t}{5},\qquad y=-\frac{7}{15}+\frac{t}{15},\qquad z=\frac{3}{10}-\frac{t}{10}.$$

Thus the direction ratios are $1/5, 1/15, -1/10$, proportional to $(6,2,-3)$. The negative sign in the third component is essential because the printed expression is $3-10z$.

Step 2: Normalise and identify a point The length of the direction-ratio vector is

$$\sqrt{6^2+2^2+(-3)^2}=\sqrt{49}=7.$$

Hence its unit direction vector has components $6/7, 2/7, -3/7$, whose squared sum is 1, as direction cosines require. Setting the common parameter $t=0$ gives the point $(3/5,-7/15,3/10)$. Substitution makes all three printed expressions zero, verifying that this point lies on the line. Reversing all three direction cosines describes the opposite orientation of the same line.

Step 3: Conclusion The direction cosines are $\boxed{(6/7,\ 2/7,\ -3/7)}$, and one point on the line is $\boxed{(3/5,\ -7/15,\ 3/10)}$.

Quick Tip to Solve: First set all three expressions equal to one parameter; use the coefficients of that parameter, not the coefficients of x, y, z, as direction ratios.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26. Find: $\int \frac{x^2+x+1}{(x+1)^2(x+2)}\,dx$. [3 marks]

Solution:

Concept:

- For a repeated linear factor, include both powers in the partial fractions: $A/(x+1)+B/(x+1)^2+C/(x+2)$.
- Use $\int dx/(x+a)=\log|x+a|$ and $\int(x+a)^{-2}dx=-(x+a)^{-1}$; the constant of integration is arbitrary.

Step 1: Resolve into partial fractions Write

$$\frac{x^2+x+1}{(x+1)^2(x+2)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{x+2}.$$

On multiplying by the denominator,

$$x^2+x+1=A(x+1)(x+2)+B(x+2)+C(x+1)^2.$$

Setting $x=-1$ yields $B=1$, while setting $x=-2$ yields $C=3$. Comparing coefficients of x^2 gives $A+C=1$, so $A=-2$. The coefficient of x checks as $3A+B+2C=-6+1+6=1$, and the constant checks as $2A+2B+C=-4+2+3=1$.

Step 2: Integrate each term and verify The integral therefore becomes

$$\begin{aligned} I&=\int\left(-\frac{2}{x+1}+\frac{1}{(x+1)^2}+\frac{3}{x+2}\right)dx\\ &=-2\log|x+1|-\frac{1}{x+1}+3\log|x+2|+C_0. \end{aligned}$$

Here C_0 is an arbitrary constant. Differentiating the expression gives $-2/(x+1)+1/(x+1)^2+3/(x+2)$, exactly the verified partial fractions. Absolute values in the logarithms ensure a real antiderivative on each interval avoiding the poles $x=-1,-2$. The repeated-factor term integrates to a rational term, rather than another logarithm.

Step 3: Conclusion $\boxed{I=3\log|x+2|-2\log|x+1|-\frac{1}{x+1}+C_0}$, valid on intervals excluding $x=-1,-2$.

Quick Tip to Solve: At a repeated factor, retain both $1/(x+1)$ and $1/(x+1)^2$; substituting the roots then finds two coefficients immediately.

27(a). Evaluate: $\displaystyle\int_{\pi/4}^{\pi/2} e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$**.** [3 marks]

Solution:

Concept:

- The double-angle identities $1-\cos 2x = 2\sin^2 x$ and $\sin 2x = 2\sin x\cos x$ convert the quotient to $\frac{1}{2}\csc^2 x - \cot x$.
- The product rule gives $\dfrac{d}{dx}(e^{2x}\cot x) = e^{2x}(2\cot x - \csc^2 x)$; evaluate an antiderivative at the upper and lower limits.

Step 1: Simplify the trigonometric quotient Let I denote the given integral. Throughout $[\pi/4, \pi/2]$, $\sin x \neq 0$, so the following simplification is legitimate:

$$\begin{aligned}\frac{1-\sin 2x}{1-\cos 2x} &= \frac{1-2\sin x\cos x}{2\sin^2 x}\\ &= \frac{1}{2}\csc^2 x - \cot x.\end{aligned}$$

Hence

$$I = \int_{\pi/4}^{\pi/2} e^{2x}\left(\frac{1}{2}\csc^2 x - \cot x\right)dx.$$

The numerator itself becomes zero at the lower limit, but the denominator is nonzero there, so there is no singularity in this definite integral.

Step 2: Recognise a product derivative and apply the limits Since $(e^{2x})' = 2e^{2x}$ and $(\cot x)' = -\csc^2 x$,

$$\frac{d}{dx}\left(-\frac{1}{2}e^{2x}\cot x\right) = e^{2x}\left(\frac{1}{2}\csc^2 x - \cot x\right).$$

Thus

$$\begin{aligned}I &= \left[-\frac{1}{2}e^{2x}\cot x\right]_{\pi/4}^{\pi/2}\\ &= -\frac{1}{2}e^{\pi}\cot(\pi/2) + \frac{1}{2}e^{\pi/2}\cot(\pi/4)\\ &= 0 + \frac{1}{2}e^{\pi/2}.\end{aligned}$$

We used $\cot(\pi/2) = 0$ and $\cot(\pi/4) = 1$; the lower-limit contribution has a positive sign because the antiderivative at that limit is subtracted.

Step 3: Conclusion The value of the integral is $\boxed{\dfrac{e^{\pi/2}}{2}}$.

Quick Tip to Solve: After using the double-angle identities, compare the integrand with the derivative of $e^{2x}\cot x$; this avoids integration by parts.

OR

27(b). Evaluate: $\displaystyle\int_{-2}^{2} \frac{x^2}{1+5^x}\,dx$. [3 marks]

Solution:

Concept:

- For an integrable function on a symmetric interval, $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} f(-x)\,dx$, by substituting $x = -t$.
- The complementary denominators satisfy $\dfrac{1}{1+5^x} + \dfrac{1}{1+5^{-x}} = 1$, while the numerator x^2 is even.

Step 1: Reflect the integration variable Let

$$I = \int_{-2}^{2} \frac{x^2}{1+5^x}\,dx.$$

Substitute $x = -t$, so $dx = -dt$. The old lower limit -2 becomes 2, and the old upper limit 2 becomes -2. Reversing the limits removes the negative sign and yields

$$I = \int_{-2}^{2} \frac{t^2}{1+5^{-t}}\,dt = \int_{-2}^{2} \frac{x^2 5^x}{1+5^x}\,dx.$$

Renaming the dummy variable does not change the value. The original integrand is continuous because $5^x > 0$, so all these definite-integral operations are valid.

Step 2: Add the equivalent integrals Add the original and reflected forms over their identical limits:

$$\begin{aligned} 2I &= \int_{-2}^{2} \left(\frac{x^2}{1+5^x} + \frac{x^2 5^x}{1+5^x} \right) dx \\ &= \int_{-2}^{2} x^2\,dx \\ &= \left[\frac{x^3}{3} \right]_{-2}^{2} \\ &= \frac{8}{3} - \left(-\frac{8}{3} \right) = \frac{16}{3}. \end{aligned}$$

The exponential cancels only after pairing the values at x and $-x$; the original integrand is not itself even. Finally divide this equality by 2 to obtain the requested integral.

Step 3: Conclusion The value of the integral is $\boxed{\frac{8}{3}}$.

Quick Tip to Solve: With symmetric limits and a denominator $1 + a^x$, replace x by $-x$ and add; an even numerator survives while the exponential denominator cancels.

28(a). Find: $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}\,dx$. [3 marks]

Solution:

Concept:

- Substitution replaces an expression and its differential together: $t = e^x$ gives $dt = e^x\,dx$, exactly the numerator here.
- Complete the square, then use $\int du/\sqrt{a^2-u^2} = \sin^{-1}(u/a) + C$, valid for $|u| < a$.

Step 1: Substitute the exponential Let the required integral be I. Put $t = e^x$, so $dt = e^x\,dx$. The numerator is absorbed entirely into the differential, and the integral becomes

$$I = \int \frac{dt}{\sqrt{5-4t-t^2}}.$$

The substitution converts the exponential expression into a quadratic polynomial in t, to which the standard inverse-trigonometric integral can be applied.

Step 2: Complete the square We have $5 - 4t - t^2 = 9 - (t+2)^2$. Set $u = t + 2$, so that $du = dt$. Thus

$$I = \int \frac{du}{\sqrt{3^2-u^2}} = \sin^{-1}\left(\frac{u}{3}\right) + C.$$

The real integrand is defined where $5 - 4e^x - e^{2x} > 0$. Since $e^x > 0$, this condition simplifies to $0 < e^x < 1$, or $x < 0$. On this interval $2 < u < 3$, so the inverse sine is within its real domain.

Step 3: Restore the variable and check by differentiation Replace u by $e^x + 2$. The chain rule gives

$$\frac{d}{dx}\sin^{-1}\left(\frac{e^x+2}{3}\right) = \frac{e^x}{\sqrt{9-(e^x+2)^2}} = \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}.$$

This verifies both the substitution and the absence of any extra numerical factor.

Step 4: Conclusion

$$\boxed{I = \sin^{-1}\left(\frac{e^x+2}{3}\right) + C, \quad x < 0.}$$

Quick Tip to Solve: Substitute the exponential first, then complete the square: the constant becomes 9, so the inverse-sine denominator is 3.

OR

28(b). Evaluate: $\int_0^{\pi/2} \sqrt{\sin x}\,\cos^5 x\,dx$. [3 marks]

Solution:

Concept:

- For an odd power of cosine, keep one factor $\cos x\,dx$ for the substitution $t = \sin x$ and express the remaining even power using $\cos^2 x = 1 - \sin^2 x$.
- In a definite integral, transform both limits along with the variable. The power rule is $\int t^n\,dt = t^{n+1}/(n+1)$ for $n \neq -1$.

Step 1: Rewrite the odd cosine power Denote the required integral by I. Since $\cos^5 x = (\cos^2 x)^2 \cos x$, the identity $\cos^2 x = 1 - \sin^2 x$ gives

$$I = \int_0^{\pi/2} \sqrt{\sin x}(1 - \sin^2 x)^2 \cos x\,dx.$$

The sine is nonnegative throughout the interval, so its square root is real. The remaining cosine factor is exactly the derivative required for a sine substitution.

Step 2: Transform the variable and limits Put $t = \sin x$, so $dt = \cos x\,dx$. When $x = 0$, $t = 0$; when $x = \pi/2$, $t = 1$. Therefore,

$$I = \int_0^1 t^{1/2}(1 - t^2)^2\,dt = \int_0^1 \left(t^{1/2} - 2t^{5/2} + t^{9/2}\right) dt.$$

The square must be expanded before integrating; the middle term carries the coefficient -2.

Step 3: Integrate each power and simplify Applying the power rule yields

$$\begin{aligned} I &= \left[\frac{2}{3}t^{3/2} - \frac{4}{7}t^{7/2} + \frac{2}{11}t^{11/2}\right]_0^1 \\ &= \frac{2}{3} - \frac{4}{7} + \frac{2}{11} \\ &= \frac{154 - 132 + 42}{231} = \frac{64}{231}. \end{aligned}$$

Every term at the lower limit is zero because all three powers are positive. The resulting value is positive, as expected from the nonnegative original integrand.

Step 4: Conclusion

$$\boxed{\int_0^{\pi/2} \sqrt{\sin x}\,\cos^5 x\,dx = \frac{64}{231}.}$$

Quick Tip to Solve: Keep one cosine factor with dx and convert the other four using cos squared; this turns the entire integral into three powers of sin x.

29(a). Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$, $y(1) = 0$. [3 marks]

Solution:

Concept:

- A linear differential equation has the form $dy/dx + P(x)y = Q(x)$. Its integrating factor is $\text{IF} = e^{\int P(x)\,dx}$.
- Multiplying by the integrating factor turns the left side into a product derivative. The given initial value fixes the integration constant on the interval containing the initial point.

Step 1: Express the equation in linear form The printed differential equation is

$$\frac{dy}{dx} = \frac{x+y}{x} = 1 + \frac{y}{x}, \qquad x \neq 0.$$

Moving the term containing y to the left gives

$$\frac{dy}{dx} - \frac{1}{x}y = 1.$$

Thus $P(x) = -1/x$ and $Q(x) = 1$. Since the initial condition is at $x = 1$, we solve on the interval $x > 0$, which avoids the singular point $x = 0$.

Step 2: Use the integrating factor On $x > 0$, the integrating factor is

$$\text{IF} = e^{\int -1/x\,dx} = e^{-\ln x} = \frac{1}{x}.$$

Multiply the equation by $1/x$:

$$\frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \frac{1}{x}.$$

By the quotient rule, the left side equals $d(y/x)/dx$. Integrating both sides therefore gives

$$\frac{y}{x} = \ln x + C.$$

Step 3: Apply the initial value and verify Substitute $x = 1$ and $y = 0$: $0 = \ln 1 + C$, hence $C = 0$. Thus the selected curve is $y = x \ln x$. Differentiation gives $dy/dx = \ln x + 1$, while the right side of the equation becomes

$$\frac{x + x\ln x}{x} = 1 + \ln x.$$

Both sides agree, and $y(1) = 1 \ln 1 = 0$. The solution therefore satisfies the differential equation and the stated condition on its appropriate domain.

Step 4: Conclusion The particular solution is

$$\boxed{y = x \ln x, \qquad x > 0.}$$

Quick Tip to Solve: After dividing the linear equation by x, recognize the derivative of y/x; the condition at x=1 then makes the constant vanish.

OR

29(b). Find the general solution of the differential equation $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$. [3 marks]

Solution:

Concept:

- Separate the variables by collecting the factors involving y with dy, and those involving x with dx.
- Put $t = \tan y$, so that $dt = \sec^2 y\, dy$. The resulting equation is homogeneous and linear in t.

Step 1: Substitute and separate The printed differential equation is

$$e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0.$$

Substituting $t = \tan y$ gives

$$(1 - e^x)\, dt = -e^x t\, dx.$$

On an interval where $x \neq 0$ and $t \neq 0$, divide by $(1 - e^x)t$:

$$\frac{dt}{t} = \frac{-e^x}{1 - e^x}\, dx.$$

The numerator on the right is the derivative of its denominator, so both sides have logarithmic antiderivatives.

Step 2: Integrate and check the solution Integration gives

$$\ln |t| = \ln |1 - e^x| + C_1.$$

Exponentiating and incorporating signs into an arbitrary real constant gives $t = C(1 - e^x)$. The division by t excluded $t = 0$; however, the constant solutions $y = n\pi$ satisfy the original equation and are included by choosing $C = 0$. To verify the family, differentiation yields

$$\sec^2 y\, dy = -Ce^x\, dx.$$

Its substitution into the original left-hand side gives

$$e^x C(1 - e^x)\, dx - Ce^x(1 - e^x)\, dx = 0.$$

Thus the family satisfies the equation wherever its trigonometric functions are defined, including its continuous extension through $x = 0$. Different branches of y differ by an integer multiple of π.

Step 3: Conclusion The general solution is $\boxed{\tan y = C(1 - e^x)}$, where C is an arbitrary real constant and $\cos y \neq 0$.

Quick Tip to Solve: Set $t = \tan y$ to replace $\sec^2 y\, dy$ by dt, then recognise $-e^x$ as the derivative of $1 - e^x$.

30. Solve the following linear programming problem graphically:

Minimise: $z = -3x + 4y$

subject to the constraints $x + 2y \leq 8,\ 3x + 2y \leq 12,\ x, y \geq 0$**.** [3 marks]

Solution:

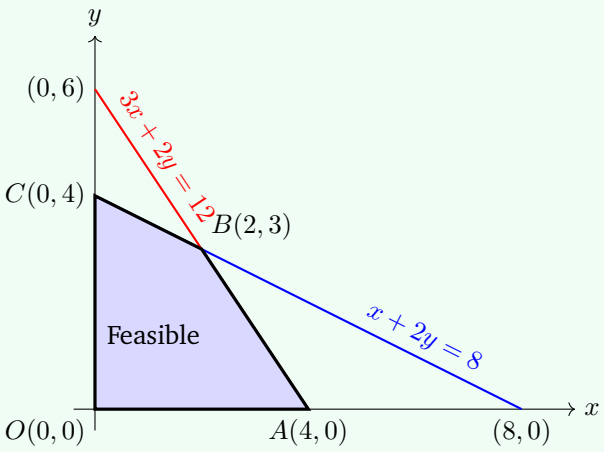
Concept:

- A linear inequality represents a half-plane. The feasible region is the common part of all constraint half-planes, including the non-negative quadrant.
- A linear objective attains its minimum on a non-empty bounded feasible polygon at a corner point. Evaluate $z = -3x + 4y$ at every vertex.

Step 1: Plot the boundary lines and identify the region For $x + 2y = 8$, the intercepts are $(8, 0)$ and $(0, 4)$. For $3x + 2y = 12$, they are $(4, 0)$ and $(0, 6)$. Join each pair to draw the corresponding line. Shade the side of each line containing the origin, restricted to the first quadrant. The intersection of the two oblique lines satisfies

$$x + 2y = 8,$$
$$3x + 2y = 12.$$

Subtracting gives $2x = 4$, so $x = 2$ and $y = 3$. Thus the common shaded region is the bounded quadrilateral with vertices $O(0, 0)$, $A(4, 0)$, $B(2, 3)$, and $C(0, 4)$.



The shaded quadrilateral is the feasible region.

Step 2: Evaluate the objective at the corner points The vertex values are

Vertex (x, y)	$z = -3x + 4y$
$(0, 0)$	0
$(4, 0)$	-12
$(2, 3)$	6
$(0, 4)$	16

As a check, the constraints give $3x \leq 12 - 2y$, and hence $-3x + 4y \geq -12 + 6y \geq -12$. Equality requires $y = 0$ and $x = 4$, confirming that only the corner A attains the minimum.

Step 3: Conclusion The minimum value is $\boxed{z_{\min} = -12}$, attained at $\boxed{(x, y) = (4, 0)}$.

Quick Tip to Solve: Subtract the two boundary equations to get $B(2, 3)$, then compare the objective at all four feasible vertices.

31. From a lot of 30 bulbs which include 6 defective bulbs, a sample of 2 bulbs is drawn at random one by one with replacement. Find the probability distribution of the number of defective bulbs and hence find the mean number of defective bulbs. [3 marks]

Solution:

Concept:

- With replacement, the composition of the lot is restored after each draw. The two draws are independent and the probability of a defective bulb remains $p = 6/30 = 1/5$.
- If X counts defective bulbs in two independent draws, then $P(X = r) = \binom{2}{r} p^r (1-p)^{2-r}$, for $r = 0, 1, 2$. Its mean is $E(X) = \sum_r r P(X = r)$.

Step 1: Identify the outcomes and their probabilities Let D denote drawing a defective bulb and G denote drawing a good bulb. There are 6 defective and $30 - 6 = 24$ good bulbs, so

$$P(D) = \frac{6}{30} = \frac{1}{5}, \qquad P(G) = \frac{24}{30} = \frac{4}{5}.$$

Let X be the number of defective bulbs among the two draws. Its possible values are $0, 1, 2$. Because the first bulb is replaced before the second draw, each ordered pair has probability equal to the product of its two draw probabilities.

Step 2: Form the probability distribution For $X = 0$, both bulbs must be good: $P(GG) = (4/5)^2 = 16/25$. For $X = 1$, the disjoint outcomes DG and GD are both possible, giving

$$P(X = 1) = \frac{1}{5}\frac{4}{5} + \frac{4}{5}\frac{1}{5} = \frac{8}{25}.$$

For $X = 2$, both bulbs must be defective: $P(DD) = (1/5)^2 = 1/25$. Therefore the distribution is

x	0	1	2
$P(X = x)$	$16/25$	$8/25$	$1/25$

The probabilities are non-negative and their sum is $(16 + 8 + 1)/25 = 1$, as required.

Step 3: Calculate the mean Multiply each possible count by its probability and add:

$$E(X) = 0\frac{16}{25} + 1\frac{8}{25} + 2\frac{1}{25} = \frac{10}{25} = \frac{2}{5}.$$

This also agrees with the binomial mean $np = 2(1/5)$. A mean is an average over repeated samples, so it need not equal one of the integer counts in a single sample.

Step 4: Conclusion The distribution is $P(X = 0) = 16/25$, $P(X = 1) = 8/25$, $P(X = 2) = 1/25$; the mean number of defective bulbs is $\boxed{2/5 = 0.4}$.

Quick Tip to Solve: With replacement, keep the defective probability at $1/5$ on both draws; include both DG and GD for exactly one defective bulb.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32. Find the inverse of the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{pmatrix}$. Using the inverse, A^{-1}, solve the system of linear equations

$x - y + 2z = 1;\ 2y - 3z = 1;\ 3x - 2y + 4z = 3$. [5 marks]

Solution:

Concept:

- A square matrix has an inverse when its determinant is non-zero. For such a matrix, $A^{-1} = \dfrac{\text{adj}\, A}{|A|}$.
- The cofactor C_{ij} is $(-1)^{i+j}$ times the minor obtained by deleting row i and column j; the adjoint is the transpose of the cofactor matrix.
- Writing a linear system as $AX = B$, with X the column of unknowns, gives the unique solution $X = A^{-1}B$.

Step 1: Evaluate the determinant Expanding along the first row, with alternating signs, gives

$$\begin{aligned} |A| &= 1(8-6) - (-1)(0+9) + 2(0-6) \\ &= 2 + 9 - 12 = -1. \end{aligned}$$

Since the determinant is non-zero, the inverse exists and the given system has a unique solution.

Step 2: Calculate all the cofactors Retaining the cofactor signs separately from the signs of the entries, we obtain

$$\begin{aligned} &C_{11} = 8 - 6 = 2, && C_{12} = -(0+9) = -9, && C_{13} = 0 - 6 = -6, \\ &C_{21} = -(-4+4) = 0, && C_{22} = 4 - 6 = -2, && C_{23} = -(-2+3) = -1, \\ &C_{31} = 3 - 4 = -1, && C_{32} = -(-3-0) = 3, && C_{33} = 2 - 0 = 2. \end{aligned}$$

Thus the cofactor matrix is $C = \begin{pmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{pmatrix}$. It must be transposed before forming the inverse.

Step 3: Form the adjoint and inverse The transpose of the cofactor matrix gives

$$\text{adj}\, A = C^T = \begin{pmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{pmatrix}.$$

Dividing every entry by $|A| = -1$, we get

$$A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}.$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; here division by -1 changes every sign. Check the resulting unknowns in all three equations.

Step 4: Apply the inverse to the system In the order x, y, z, the coefficient matrix is exactly A, so

$$AX = B, \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

Premultiplying by the inverse gives

$$X = A^{-1}B = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$
$$= \begin{pmatrix} -2+3 \\ 9+2-9 \\ 6+1-6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Substitution checks all three equations: $1 - 2 + 2 = 1$, $2(2) - 3(1) = 1$, and $3(1) - 2(2) + 4(1) = 3$.

Step 5: Conclusion $A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}$, and the solution is $\boxed{x = 1,\ y = 2,\ z = 1}$.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; here division by -1 changes every sign. Check the resulting unknowns in all three equations.

33. Using integration, find the area of the region bounded by the parabola $y^2 = 4ax$ and its latus rectum. [5 marks]

Solution:

Concept:

- For the standard parabola $y^2 = 4ax$, where $a > 0$, the focus is $(a, 0)$. Its latus rectum is the chord through the focus perpendicular to the axis, so its line is $x = a$.
- Integration with respect to y adds horizontal strips: the area between a right boundary $x_R(y)$ and a left boundary $x_L(y)$ is $\int [x_R(y) - x_L(y)]\, dy$.
- The parabola and its latus rectum are symmetric about the x-axis; therefore the area above the axis equals the area below it.

Step 1: Locate the bounding chord and its endpoints The axis of $y^2 = 4ax$ is the x-axis. A chord through its focus and perpendicular to that axis is vertical, hence the latus rectum is $x = a$. Substituting this in the parabola gives

$$y^2 = 4a(a) = 4a^2, \qquad y = \pm 2a.$$

Thus the endpoints are $(a, 2a)$ and $(a, -2a)$. The bounded region extends from $y = -2a$ to $y = 2a$, with its left edge on the parabola and its right edge on the chord.

Step 2: Express the width of a horizontal strip Rearranging the parabola in a form suitable for

Quick Tip to Solve: Use horizontal strips between x = y squared divided by 4a and x = a; the endpoints have y = plus or minus 2a, so symmetry halves the integration work.

horizontal strips yields $x = \dfrac{y^2}{4a}$. Therefore a strip at height y has width

$$x_R - x_L = a - \frac{y^2}{4a}.$$

For $-2a \le y \le 2a$, this width is non-negative. It is largest at $y = 0$, where it equals a, and decreases to zero at both chord endpoints. These observations check that the chosen difference describes the enclosed region rather than the exterior.

Step 3: Set up and evaluate the area integral The strip area is $\left(a - \dfrac{y^2}{4a}\right) dy$. Adding all such strips and using symmetry gives

$$\begin{aligned} S &= \int_{-2a}^{2a} \left(a - \frac{y^2}{4a}\right) dy \\ &= 2\int_{0}^{2a} \left(a - \frac{y^2}{4a}\right) dy \\ &= 2\left[ay - \frac{y^3}{12a}\right]_0^{2a} \\ &= 2\left(2a^2 - \frac{8a^3}{12a}\right) \\ &= 2\left(2a^2 - \frac{2a^2}{3}\right). \end{aligned}$$

The lower-limit contribution is zero. The factor 2 accounts for the equal region below the axis; omitting it would give only the upper half of the required area.

Step 4: Check the area by vertical strips Equivalently, $0 \le x \le a$, and the upper and lower curves are $y = 2\sqrt{ax}$ and $y = -2\sqrt{ax}$. Hence the vertical-strip height is $4\sqrt{ax}$, giving

$$\int_0^a 4\sqrt{ax}\, dx = 4\sqrt{a}\left[\frac{2x^{3/2}}{3}\right]_0^a = \frac{8a^2}{3}.$$

This independently agrees with the horizontal-strip evaluation. The result has the dimension of area, and is smaller than the enclosing rectangle's area $a \times 4a = 4a^2$, as expected.

Step 5: Conclusion The area bounded by the parabola and its latus rectum is $\boxed{\dfrac{8a^2}{3} \text{ square units}}$.

Quick Tip to Solve: Use horizontal strips between x = y squared divided by 4a and x = a; the endpoints have y = plus or minus 2a, so symmetry halves the integration work.

34(a). If $\mathbb{N}$ denotes the set of all natural numbers and R is the relation on $\mathbb{N}\times\mathbb{N}$ defined by $(a,b)\,R\,(c,d)$, if $ad(b+c)=bc(a+d)$. Show that R is an equivalence relation. [5 marks]

Solution:

Concept:

- An equivalence relation is a relation that is reflexive, symmetric and transitive. These properties must hold for every element of the underlying set.
- Here the elements are ordered pairs of positive natural numbers. All denominators used below are therefore nonzero.
- Rewriting the defining condition as equality of $h(a,b)=\dfrac{1}{b}-\dfrac{1}{a}$ allows all three properties to be checked directly.

Step 1: Obtain an equivalent condition Take any $(a,b),(c,d)\in\mathbb{N}\times\mathbb{N}$. Divide the defining equality by the nonzero product $abcd$:

$$\begin{aligned} ad(b+c) &= bc(a+d)\\ \frac{1}{c}+\frac{1}{b} &= \frac{1}{d}+\frac{1}{a}\\ \frac{1}{b}-\frac{1}{a} &= \frac{1}{d}-\frac{1}{c}. \end{aligned}$$

Each algebraic step is reversible. Therefore $(a,b)R(c,d)$ if and only if $h(a,b)=h(c,d)$. This equivalence will be used in all three checks.

Step 2: Prove reflexivity Let (a,b) be any element of $\mathbb{N}\times\mathbb{N}$. Since $h(a,b)=h(a,b)$, the equivalent condition gives $(a,b)R(a,b)$. Direct substitution into the printed condition also gives $ab(b+a)=ba(a+b)$, which is true. Thus every ordered pair is related to itself, so R is reflexive.

Step 3: Prove symmetry Suppose $(a,b)R(c,d)$. The rewritten condition gives $h(a,b)=h(c,d)$. Reversing this equality gives $h(c,d)=h(a,b)$, so $(c,d)R(a,b)$. Directly, the condition for the reversed pairs is $cb(d+a)=da(c+b)$, which is just the original equality with its sides exchanged. This implication holds for arbitrary pairs; hence R is symmetric.

Step 4: Prove transitivity Suppose $(a,b)R(c,d)$ and $(c,d)R(e,f)$, with all three pairs in the given set. Then

$$\frac{1}{b}-\frac{1}{a}=\frac{1}{d}-\frac{1}{c},\qquad \frac{1}{d}-\frac{1}{c}=\frac{1}{f}-\frac{1}{e}.$$

Combining the equalities gives $\dfrac{1}{b}-\dfrac{1}{a}=\dfrac{1}{f}-\dfrac{1}{e}$. Rearrangement yields $\dfrac{1}{b}+\dfrac{1}{e}=\dfrac{1}{a}+\dfrac{1}{f}$. Multiplying by $abef$, which is nonzero, gives $af(b+e)=be(a+f)$. This is precisely the condition for $(a,b)R(e,f)$. Therefore R is transitive.

Step 5: Conclusion Since R is reflexive, symmetric and transitive on $\mathbb{N}\times\mathbb{N}$, it is an equivalence relation.

Quick Tip to Solve: Divide by the product of the four entries first: equality of one expression makes the transitivity proof immediate.

OR

34(b). Let $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \to \mathbb{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. Show that f is a one-one function. Also, check whether f is an onto function or not. [5 marks]

Solution:

Concept:

- A function is one-one if equality of the images of any two elements of its domain implies equality of those elements.
- A function is onto if every element of its stated codomain has a preimage in the domain. Its range must therefore equal its codomain.
- For $f(x) = \frac{4x}{3x+4}$, the domain excludes $-\frac{4}{3}$, so its denominator is nonzero. This permits cross-multiplication when comparing images.

Step 1: Compare the images of two arbitrary inputs Let $x_1, x_2 \in \mathbb{R} \setminus \{-\frac{4}{3}\}$ and suppose $f(x_1) = f(x_2)$. Then

$$\frac{4x_1}{3x_1+4} = \frac{4x_2}{3x_2+4}.$$

Both denominators are nonzero by the domain restriction. Cross-multiplication and cancellation of the common nonzero factor 4 give

$$\begin{aligned} x_1(3x_2+4) &= x_2(3x_1+4) \\ 3x_1x_2 + 4x_1 &= 3x_1x_2 + 4x_2 \\ 4x_1 &= 4x_2 \\ x_1 &= x_2. \end{aligned}$$

Thus equal images force equal inputs for any two domain elements, establishing that f is one-one.

Step 2: Identify a codomain value with no preimage The codomain is all of $\mathbb{R}$, so it includes $\frac{4}{3}$. If some domain element x satisfied $f(x) = \frac{4}{3}$, then

$$\frac{4x}{3x+4} = \frac{4}{3} \quad \Longrightarrow \quad 12x = 4(3x+4) = 12x + 16.$$

This would require $0 = 16$, an impossibility. Therefore $\frac{4}{3}$ has no preimage, which is enough to show that the function is not onto the stated codomain.

Step 3: Determine the full range For completeness, let $y \in \mathbb{R}$ with $y \neq \frac{4}{3}$. Solving $y = f(x)$ gives

$$\begin{aligned} y(3x+4) &= 4x \\ (4-3y)x &= 4y \\ x &= \frac{4y}{4-3y}. \end{aligned}$$

Quick Tip to Solve: For a rational function, solve $y = f(x)$ and inspect the denominator in the expression for x to locate a missing range value.

This input exists because $4 - 3y \neq 0$. Moreover,

$$3x + 4 = \frac{12y + 4(4 - 3y)}{4 - 3y} = \frac{16}{4 - 3y} \neq 0,$$

so x lies in the domain. Substitution gives

$$f(x) = \frac{16y/(4 - 3y)}{16/(4 - 3y)} = y.$$

Thus every real value except $\frac{4}{3}$ is attained, and the range is exactly $\mathbb{R} \setminus \{\frac{4}{3}\}$.

Step 4: Conclusion The function f is one-one but is not onto $\mathbb{R}$, since its range is $\mathbb{R} \setminus \{\frac{4}{3}\}$.

Quick Tip to Solve: For a rational function, solve $y = f(x)$ and inspect the denominator in the expression for x to locate a missing range value.

35(a). Show that the following lines do not intersect each other:
$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}, \frac{x+2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$. [5 marks]

Solution:

Concept:

- A line in symmetric form is written parametrically by equating all three ratios to the same real parameter. This expresses every point of the line using one variable.
- Two lines intersect only if there is a pair of parameter values giving the same x, y and z coordinates simultaneously. Agreement in just two coordinates is insufficient in three dimensions.
- Direction vectors are parallel only when one is a scalar multiple of the other. Nonparallel lines that have no common point are called skew lines.

Step 1: Write both lines in parametric form For the first line, put each ratio equal to λ. For the second line, put each ratio equal to μ. Their general points are therefore

$$\begin{aligned} L_1: &\quad x = 1 + 3\lambda, \quad y = -1 + 2\lambda, \quad z = 1 + 5\lambda, \\ L_2: &\quad x = -2 + 4\mu, \quad y = 1 + 3\mu, \quad z = -1 - 2\mu. \end{aligned}$$

The sign in the last expression follows from the denominator -2 in the printed second line. The two parameters are independent, since each belongs to its own line.

Step 2: Determine the only possible parameters from two coordinates Suppose that a common point exists. Equating its x and y coordinates gives

$$\begin{aligned} 3\lambda - 4\mu &= -3, \\ 2\lambda - 3\mu &= 2. \end{aligned}$$

Multiply the first equation by 2 and the second by 3. Subtracting the second resulting equation from

Quick Tip to Solve: Solve two coordinate equations for the line parameters, then test the third; matching two coordinates alone does not prove intersection.

the first gives

$$(6\lambda - 8\mu) - (6\lambda - 9\mu) = -6 - 6, \qquad \mu = -12.$$

Substitution in $2\lambda - 3\mu = 2$ yields $2\lambda + 36 = 2$, so $\lambda = -17$. Thus any intersection would have to use precisely this pair of parameters.

Step 3: Test the remaining coordinate At these parameters, the first line gives $z = 1 + 5(-17) = -84$, whereas the second gives $z = -1 - 2(-12) = 23$. These are unequal. Equivalently, the required third equation is $5\lambda + 2\mu = -2$, but its left side at the candidate parameters is $-85 - 24 = -109$. Both candidate points have $x = -50$ and $y = -35$, but their different z coordinates make them distinct points. Consequently no pair of parameters satisfies all three coordinate equations. Moreover, $(3, 2, 5)$ and $(4, 3, -2)$ are not proportional, so the lines are nonparallel.

Step 4: Conclusion The lines have no common point and therefore do not intersect each other. Since their direction vectors are nonparallel, they are skew lines.

Quick Tip to Solve: Solve two coordinate equations for the line parameters, then test the third; matching two coordinates alone does not prove intersection.

OR

35(b). Find the angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$. [5 marks]

Solution:

Concept:

- Direction ratios of a line are proportional to the coefficients of a parameter in its parametric equations. In a chain such as $ax = by = cz$, the direction ratios are $1/a, 1/b, 1/c$, rather than a, b, c.
- If $\mathbf{u}$ and $\mathbf{v}$ are nonzero direction vectors, the smaller angle θ between the lines satisfies $\cos\theta = |\mathbf{u} \cdot \mathbf{v}|/(|\mathbf{u}||\mathbf{v}|)$.
- Multiplying a direction vector by any nonzero scalar leaves the line direction unchanged. A zero dot product of two nonzero vectors establishes perpendicular directions.

Step 1: Obtain direction ratios of the first line Set $2x = 3y = -z = t$. Then the coordinates of a general point are

$$x = \frac{t}{2}, \qquad y = \frac{t}{3}, \qquad z = -t.$$

Hence a direction vector is $(1/2, 1/3, -1)$. Multiplication by 6 clears the fractions and gives the convenient direction vector

$$\mathbf{u} = (3, 2, -6).$$

These ratios preserve the negative sign in the relation $-z = t$. The coefficients $(2, 3, -1)$ in the original chain are not the direction ratios, because they multiply the coordinates rather than the parameter.

Step 2: Obtain direction ratios of the second line Independently set $6x = -y = -4z = s$. This gives

$$x = \frac{s}{6}, \qquad y = -s, \qquad z = -\frac{s}{4}.$$

Quick Tip to Solve: In a chain of equal coordinate multiples, take reciprocals for direction ratios and clear fractions before evaluating the dot product.

A direction vector is consequently $(1/6, -1, -1/4)$. Multiplying this vector by 12, we choose

$$\mathbf{v} = (2, -12, -3).$$

Both negative components follow directly from the printed equalities. In particular, $-4z = s$ gives $z = -s/4$, which must be retained when finding the angle.

Step 3: Apply the dot-product formula Compute the dot product and magnitudes using the integer direction vectors:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 3(2) + 2(-12) + (-6)(-3) \\ &= 6 - 24 + 18 = 0, \\ |\mathbf{u}| &= \sqrt{9 + 4 + 36} = 7, \\ |\mathbf{v}| &= \sqrt{4 + 144 + 9} = \sqrt{157}. \end{aligned}$$

Therefore $\cos\theta = |0|/(7\sqrt{157}) = 0$. The denominator is nonzero, so the vanishing numerator establishes perpendicularity without any ambiguity about vector orientation. Both lines also pass through the origin, as seen by setting their parameters equal to zero.

Step 4: Conclusion The angle between the two lines is $\boxed{\theta = 90^\circ}$.

Quick Tip to Solve: In a chain of equal coordinate multiples, take reciprocals for direction ratios and clear fractions before evaluating the dot product.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. Case Study - 1

Let $f(x)$ be a real valued function. Then its

Left Hand Derivative (L.H.D.): $Lf'(a) = \lim\limits_{h \to 0} \dfrac{f(a-h) - f(a)}{-h}$

Right Hand Derivative (R.H.D.): $Rf'(a) = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

Also, a function $f(x)$ is said to be differentiable at $x = a$ if its L.H.D. and R.H.D. at $x = a$ exist and both are equal.

For the function $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$ **answer the following questions:** [4 marks]

36(i). What is R.H.D. of $f(x)$ at $x = 1$? [1 mark]

Solution:

Concept:

- The right-hand derivative uses values just greater than the point: $Rf\prime(a) = \lim_{h \to 0^+}[f(a+h) - f(a)]/h$.
- Near $x = 1$ on the right, $x < 3$, so $|x - 3| = 3 - x$.

Quick Tip to Solve: For a right derivative at 1, use small positive increments and simplify the absolute value before dividing.

Step 1: Choose the branch and endpoint value At the endpoint, the branch $x \geq 1$ gives $f(1) = |1-3| = 2$. For $0 < h < 2$, the same branch gives $f(1+h) = |h-2| = 2-h$; the absolute value must be simplified using the sign of its argument.

Step 2: Evaluate the difference quotient

$$Rf\prime(1) = \lim_{h \to 0^+} \frac{(2-h)-2}{h} = \lim_{h \to 0^+} (-1).$$

The quotient is constant for these positive increments, so its limit exists.

Step 3: Conclusion The R.H.D. of f at $x = 1$ is $\boxed{-1}$.

Quick Tip to Solve: For a right derivative at 1, use small positive increments and simplify the absolute value before dividing.

36(ii). What is L.H.D. of $f(x)$ at $x = 1$? [1 mark]

Solution:

Concept:

- The left-hand derivative is $Lf\prime(a) = \lim_{h \to 0^+} [f(a-h) - f(a)]/(-h)$.
- Use the polynomial branch for arguments below 1, but use the endpoint branch to compute $f(1)$.

Step 1: Substitute on the left side The endpoint value is $f(1) = |1-3| = 2$. For $h > 0$, the argument $1-h < 1$ belongs to the polynomial branch. Expanding gives

$$f(1-h) = \frac{(1-h)^2}{4} - \frac{3(1-h)}{2} + \frac{13}{4} = 2 + h + \frac{h^2}{4}.$$

Step 2: Take the left-hand limit

$$Lf\prime(1) = \lim_{h \to 0^+} \frac{h + h^2/4}{-h} = \lim_{h \to 0^+} \left(-1 - \frac{h}{4}\right).$$

The denominator is negative because the point is approached from the left.

Step 3: Conclusion The L.H.D. of f at $x = 1$ is $\boxed{-1}$.

Quick Tip to Solve: Keep the denominator as -h when using f(1-h); replacing it by h reverses the slope.

36(iii)(a). Check if the function $f(x)$ is differentiable at $x = 1$. [2 marks]

Solution:

Concept:

- A function is differentiable at a joining point when both finite one-sided derivatives exist and agree.
- Check continuity at the join and compute difference quotients using the actual endpoint value $f(1) = 2$.

Quick Tip to Solve: At a piecewise join, verify the endpoint value before comparing the two one-sided slopes.

Step 1: Check the values at the join On the left, the polynomial tends to $1/4 - 3/2 + 13/4 = 2$. On the right, $|x-3|$ tends to $|1-3| = 2$, which equals $f(1)$. Thus there is no jump at the joining point; both branches meet at the same function value.

Step 2: Compute the right-hand derivative For $0 < h < 2$, $f(1+h) = 2 - h$. Hence

$$Rf\prime(1) = \lim_{h \to 0^+} \frac{(2-h) - 2}{h} = -1.$$

The local expression is $3 - x$, rather than $x - 3$, since these nearby arguments are below 3.

Step 3: Compute the left-hand derivative Expanding the polynomial at $1 - h$ gives $f(1-h) = 2 + h + h^2/4$. Therefore

$$Lf\prime(1) = \lim_{h \to 0^+} \frac{h + h^2/4}{-h} = \lim_{h \to 0^+} \left(-1 - \frac{h}{4}\right) = -1.$$

Both limits are finite and have the same value.

Step 4: Conclusion $Lf\prime(1) = Rf\prime(1) = -1$; therefore f is differentiable at $x = 1$, with $\boxed{f\prime(1) = -1}$.

Quick Tip to Solve: At a piecewise join, verify the endpoint value before comparing the two one-sided slopes.

OR

36(iii)(b). Find $f'(2)$ and $f'(-1)$. [2 marks]

Solution:

Concept:

- At an interior point of a branch, differentiate that branch in a neighbourhood of the point.
- For $1 < x < 3$, $|x-3| = 3 - x$; for $x < 1$, differentiate the given polynomial term by term.

Step 1: Find the derivative at 2 The point 2 lies strictly between 1 and 3, so an entire small neighbourhood of 2 uses the absolute-value branch with negative argument. Thus $f(x) = 3 - x$ near 2, and

$$f\prime(x) = \frac{d}{dx}(3 - x) = -1 \quad (1 < x < 3).$$

The cusp of the absolute-value expression is at 3, not at the point being tested.

Step 2: Find the derivative at -1 The point -1 lies strictly below 1, so the polynomial branch applies on both sides near it. Using the power rule,

$$f\prime(x) = \frac{x}{2} - \frac{3}{2} \quad (x < 1).$$

Substitute $x = -1$:

$$f\prime(-1) = \frac{-1}{2} - \frac{3}{2} = -2.$$

No joining-point comparison is needed here because each specified point is interior to its applicable

Quick Tip to Solve: Locate each evaluation point relative to both the branch boundary 1 and the absolute-value cusp 3.

smooth branch.

Step 3: Conclusion $\boxed{f\prime(2) = -1, \qquad f\prime(-1) = -2}$.

Quick Tip to Solve: Locate each evaluation point relative to both the branch boundary 1 and the absolute-value cusp 3.

37. Case Study - 2

A building contractor undertakes a job to construct 4 flats on a plot along with parking area. Due to strike, the probability of many construction workers not being present for the job is 0.65. The probability that many are not present and still the work gets completed on time is 0.35. The probability that work will be completed on time when all workers are present is 0.80.

Let: E_1: represent the event when many workers were not present for the job;

E_2: represent the event when all workers were present; and

E: represent completing the construction work on time.

Based on the above information, answer the following questions: [4 marks]

37(i). What is the probability that all the workers are present for the job? [1 mark]

Solution:

Concept:

- In the case study, E_1 and E_2 are the two complementary worker-attendance cases.
- For complementary events, $P(E_1) + P(E_2) = 1$.

Step 1: Identify the probability supplied The stated probability that many workers are not present is $P(E_1) = 0.65$. The question asks for $P(E_2)$, the probability of the other attendance case, in which all workers are present.

Step 2: Apply the complement rule As the attendance cases form the partition used in the case study,

$$P(E_2) = 1 - P(E_1) = 1 - 0.65 = 0.35.$$

The given probability 0.80 concerns timely completion conditional on attendance, so it is not needed for this calculation.

Step 3: Conclusion The probability that all workers are present is $\boxed{0.35}$.

Quick Tip to Solve: For an attendance-only question, use the complement of 0.65 and leave completion probabilities aside.

37(ii). What is the probability that construction will be completed on time? [1 mark]

Solution:

Concept:

- The two attendance events E_1, E_2 partition the sample space used in the case study.
- For timely completion E, $P(E) = P(E \cap E_1) + P(E \cap E_2)$, and $P(E \cap E_2) = P(E_2)P(E \mid E_2)$.

Step 1: Compute the contribution when all attend The complement rule gives $P(E_2) = 1-0.65 = 0.35$. The conditional completion probability is $P(E \mid E_2) = 0.80$. Hence

$$P(E \cap E_2) = 0.35 \times 0.80 = 0.28.$$

Step 2: Add the disjoint completion cases The statement that workers are not present and still work finishes supplies the joint probability $P(E \cap E_1) = 0.35$, not a conditional probability. Therefore

$$P(E) = 0.35 + 0.28 = 0.63.$$

Multiplying the first 0.35 by 0.65 would incorrectly count the absence probability again.

Step 3: Conclusion The probability of completion on time is $\boxed{0.63}$.

Quick Tip to Solve: The words "not present and still completed" specify a joint probability; add it directly to the other joint case.

37(iii)(a). What is the probability that many workers are not present given that the construction work is completed on time? [2 marks]

Solution:

Concept:

- Conditional probability is $P(E_1 \mid E) = P(E_1 \cap E)/P(E)$, where $P(E) > 0$.
- The total probability of completion is the sum of its joint probabilities in the two attendance cases.

Step 1: Find the probability of timely completion The attendance cases in the case study give $P(E_2) = 1 - 0.65 = 0.35$. Since $P(E \mid E_2) = 0.80$, the probability of both full attendance and timely completion is $0.35 \times 0.80 = 0.28$. The other joint probability is already supplied as $P(E_1 \cap E) = 0.35$. Adding these disjoint cases gives

$$P(E) = 0.35 + 0.28 = 0.63.$$

Step 2: Condition on the stated completed work The information that construction finished on time restricts the sample space to E. Of those completed jobs, the ones with many workers absent are exactly $E_1 \cap E$. Therefore

$$P(E_1 \mid E) = \frac{P(E_1 \cap E)}{P(E)} = \frac{0.35}{0.63} = \frac{35}{63} = \frac{5}{9}.$$

Quick Tip to Solve: After "given completion", divide the joint absence-and-completion probability by total completion probability.

The prior absence probability 0.65 is not the numerator because it also includes jobs that were not completed on time.

Step 3: Conclusion The required probability is $\boxed{\frac{5}{9}}$.

Quick Tip to Solve: After "given completion", divide the joint absence-and-completion probability by total completion probability.

OR

37(iii)(b). What is the probability that all workers were present given that the construction job was completed on time? [2 marks]

Solution:

Concept:

- Conditional probability is $P(E_2 \mid E) = \dfrac{P(E_2 \cap E)}{P(E)}$, where E_2 means all workers were present and E means completion on time.
- For the two complementary attendance events, $P(E) = P(E_1 \cap E) + P(E_2 \cap E)$, and $P(E_2 \cap E) = P(E_2)P(E \mid E_2)$.

Step 1: Calculate the joint probability for all workers being present The probability of many workers being absent is $P(E_1) = 0.65$. Therefore $P(E_2) = 1 - 0.65 = 0.35$. Since the probability of timely completion when all workers are present is $P(E \mid E_2) = 0.80$,

$$P(E_2 \cap E) = 0.35 \times 0.80 = 0.28.$$

This is the probability that both all workers are present and the work finishes on time.

Step 2: Condition on timely completion The given value 0.35 is already the joint probability $P(E_1 \cap E)$; it must not be multiplied by 0.65 again. Thus

$$P(E) = 0.35 + 0.28 = 0.63.$$

Restricting the sample space to jobs completed on time gives

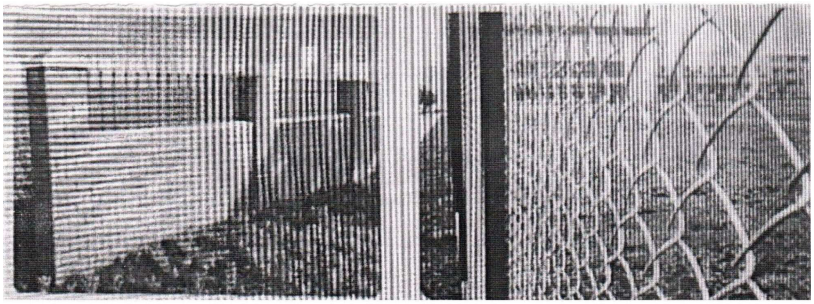
$$P(E_2 \mid E) = \frac{0.28}{0.63} = \frac{28}{63} = \frac{4}{9}.$$

Step 3: Conclusion The required probability that all workers were present given timely completion is $\boxed{\frac{4}{9}}$.

Quick Tip to Solve: Distinguish the joint probability 0.35 from the conditional probability 0.80 before applying Bayes' theorem.

38. Case Study - 3

Sooraj's father wants to construct a rectangular garden using a brick wall on one side of the garden and wire fencing for the other three sides as shown in the figure. He has 200 metres of fencing wire. Based on the above information, answer the following questions: [4 marks]



38(i). Let 'x' metres denote the length of the side of the garden perpendicular to the brick wall and 'y' metres denote the length of the side parallel to the brick wall. Determine the relation representing the total length of fencing wire and also write $A(x)$, the area of the garden. [2 marks]

Solution:

Concept:

- A rectangle of adjacent side lengths x and y has area $A = xy$. Here x is perpendicular and y is parallel to the wall.
- Only three sides require wire: two sides of length x and the side opposite the wall of length y. The wall supplies the fourth boundary.

Step 1: Write the fencing constraint The two sides perpendicular to the brick wall together use $x + x = 2x$ metres of wire. Of the two parallel sides, the side along the wall does not need fencing, so only the opposite side uses y metres. Using all 200 metres gives

$$2x + y = 200, \qquad y = 200 - 2x.$$

Counting both parallel sides would incorrectly treat the garden as requiring wire on all four sides.

Step 2: Express the area in terms of x Substitute the expression for y into the rectangular area formula:

$$A(x) = xy = x(200 - 2x) = 200x - 2x^2.$$

The area is measured in square metres. For a non-degenerate garden both dimensions must be positive: $x > 0$ and $200 - 2x > 0$, so the physical domain is $0 < x < 100$.

Step 3: Conclusion The fencing relation is $\boxed{2x + y = 200}$. The area, in square metres, is

$$\boxed{A(x) = 200x - 2x^2}, \qquad 0 < x < 100.$$

Quick Tip to Solve: Count the fenced sides before writing the constraint: the wall removes one y-side, leaving 2x+y.

38(ii). Determine the maximum value of $A(x)$. [2 marks]

Solution:

Concept:

- For a differentiable area function, an interior maximum can occur at a stationary point satisfying $A'(x) = 0$.
- The second derivative test gives a strict local maximum when $A''(x) < 0$. A quadratic with negative leading coefficient is concave throughout its domain.

Step 1: Find the stationary point of the area function With two perpendicular sides of length x and one fenced parallel side of length y, the wire constraint is $2x + y = 200$. Hence

$$A(x) = x(200 - 2x) = 200x - 2x^2, \qquad 0 < x < 100.$$

Differentiate and set the derivative equal to zero:

$$A'(x) = 200 - 4x = 0 \quad \Longrightarrow \quad x = 50.$$

This value lies within the admissible domain, and $y = 200 - 2(50) = 100$ metres.

Step 2: Verify the maximum and evaluate the area The second derivative is $A''(x) = -4 < 0$, so the stationary point is a maximum. In fact the function is concave throughout the interval: $A'(x) > 0$ for $x < 50$ and $A'(x) < 0$ for $x > 50$. Therefore this is its global maximum on the physical domain. Substitution gives

$$A(50) = 200(50) - 2(50)^2 = 10000 - 5000 = 5000 \text{ m}^2.$$

Step 3: Conclusion The maximum area is $\boxed{5000 \text{ m}^2}$, attained when the perpendicular sides are 50 metres long and the parallel side is 100 metres long.

Quick Tip to Solve: For a garden fenced on three sides, maximize x(200-2x); the optimal parallel side is twice the perpendicular side.