CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-4-2

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.

(iii) In **Section A**, Questions no. **1** to **18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21** to **25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26** to **31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32** to **35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36** to **38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculators is not allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$**, then the value of** $(2x + y - z)$ **is:** [1 mark]

(A) 1

(B) 2

(C) 3

(D) 5

Correct Answer: (D) 5

Solution:

Concept:

- Matrix equality means that corresponding entries are equal. Multiply each row of the coefficient matrix by the column of unknowns.
- For an upper triangular system, back substitution starts with the last row and proceeds upwards.

Step 1: Read the three scalar equations The matrix equation gives

$$x + y + z = 6, \qquad y + z = 3, \qquad z = 2.$$

Thus the last row fixes z immediately; no inverse matrix is needed.

Step 2: Back substitute and evaluate With $z = 2$, the second equation gives $y = 3 - 2 = 1$. The first gives $x = 6 - 1 - 2 = 3$. Therefore

$$2x + y - z = 2(3) + 1 - 2 = 5.$$

Options (A), (B) and (C) give 1, 2 and 3, respectively, and do not equal the required expression. In particular, option (C) is the value of x, not the requested combination.

Step 3: Conclusion The value of $2x + y - z$ is $\boxed{5}$. Hence, the correct option is (D).

Quick Tip to Solve: For this upper triangular matrix, find z, then y, then x; evaluate the requested expression last.

2. If a matrix $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, then the matrix AA' (where A' is the transpose of A) is: [1 mark]

(A) 14

(B) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

(C) $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$

(D) $\begin{pmatrix} 14 \end{pmatrix}$

Correct Answer: (D) $\begin{pmatrix} 14 \end{pmatrix}$

Solution:

Concept:

- The transpose interchanges rows and columns, so a 1×3 row matrix has a 3×1 transpose.
- The product of a 1×3 matrix and a 3×1 matrix has order 1×1; its entry is the row-column dot product.

Step 1: Form the transpose For the given matrix,

$$A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}, \qquad A\prime = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

The product $AA\prime$ is therefore a one-entry matrix, not a matrix of order three.

Step 2: Multiply the row and column The unique entry is the sum of the three products:

$$AA\prime = \begin{pmatrix} 1(1) + 2(2) + 3(3) \end{pmatrix} = \begin{pmatrix} 14 \end{pmatrix}.$$

Options (B) and (C) have the wrong order. Option (A) is the scalar 14; the question asks for a matrix, which must retain its brackets as in option (D).

Step 3: Conclusion The required matrix is $\boxed{\begin{pmatrix} 14 \end{pmatrix}}$. Hence, the correct option is (D).

Quick Tip to Solve: Check the product order first: a row times its transpose gives a 1×1 matrix of squared entries summed.

3. If $x\begin{pmatrix}1\\2\end{pmatrix}+y\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}4\\9\end{pmatrix}$, then: [1 mark]

(A) $x=1,\ y=2$

(B) $x=2,\ y=1$

(C) $x=1,\ y=-1$

(D) $x=3,\ y=2$

Correct Answer: (B) $x=2,\ y=1$

Solution:

Concept:

- Equality of column matrices gives equality of their corresponding components.
- Two linear equations in x and y can be solved by elimination and then checked by substitution.

Step 1: Compare the entries The entries give $x+2y=4$ and $2x+5y=9$. Thus the column equation is equivalent to

$$x+2y=4, \qquad 2x+5y=9.$$

Both equations must hold for a proposed answer.

Step 2: Eliminate and check Subtract twice the first equation from the second:

$$(2x+5y)-2(x+2y)=9-8, \qquad y=1.$$

Hence $x=4-2=2$. Substitution yields $2+2(1)=4$ and $2(2)+5(1)=9$. Options (A), (C) and (D) give first components 5, -1 and 7, respectively, so none satisfies the stated column equality.

Step 3: Conclusion $\boxed{x=2,\ y=1}$. Hence, the correct option is (B).

Quick Tip to Solve: Subtract twice the first component equation from the second to eliminate x immediately.

4. If A is a square matrix and $A^2=A$, then $(I+A)^2-3A$ is equal to: [1 mark]

(A) I

(B) A

(C) $2A$

(D) $3I$

Correct Answer: (A) I

Solution:

Concept:

- The identity matrix satisfies $I^2=I$ and $IA=AI=A$.
- The idempotent relation $A^2=A$ replaces the square by the matrix itself.

Step 1: Expand the square Expand by matrix multiplication, keeping every term:

$$(I+A)^2=I^2+IA+AI+A^2=I+2A+A^2.$$

Quick Tip to Solve: Expand first, then replace A^2 by A; the coefficients $2+1-3$ cancel exactly.

This expansion is valid because the identity matrix commutes with every square matrix of its order.

Step 2: Use the given identity Subtracting $3A$ and using $A^2 = A$ gives

$$(I + A)^2 - 3A = I + 2A + A^2 - 3A = I + A^2 - A = I.$$

Option (A) holds for every permitted matrix. Options (B) and (C) wrongly retain a term in A; (D) incorrectly multiplies the identity by three. For example, the permitted choice $A = 0$ also rules out all three alternatives.

Step 3: Conclusion The expression equals $\boxed{I}$. Hence, the correct option is (A).

Quick Tip to Solve: Expand first, then replace A^2 by A; the coefficients $2 + 1 - 3$ cancel exactly.

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5. The value of the determinant $\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$ **is** [1 mark]

(A) 47

(B) -79

(C) 49

(D) -51

Correct Answer: (A) 47

Solution:

Concept:

- For expansion along the first row, the cofactor signs are $+, -, +$.
- A 2×2 determinant satisfies $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

Step 1: Expand along the first row Let the required determinant be D. Then

$$D = 2\begin{vmatrix} 1 & 1 \\ 8 & 1 \end{vmatrix} - 7\begin{vmatrix} 1 & 1 \\ 10 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 10 & 8 \end{vmatrix}.$$

The middle contribution has a minus sign because its cofactor position is $(1, 2)$.

Step 2: Evaluate the minors The three minors are $1 - 8 = -7$, $1 - 10 = -9$, and $8 - 10 = -2$. Hence

$$D = 2(-7) - 7(-9) + (-2) = -14 + 63 - 2 = 47.$$

The other printed values, -79, 49, and -51, are not this signed cofactor sum; dropping the last minor would incorrectly produce 49.

Step 3: Conclusion The determinant equals $\boxed{47}$. Hence, the correct option is (A).

Quick Tip to Solve: Write the alternating cofactor signs before arithmetic; here the negative middle minor produces a positive contribution.

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6. The function $f(x) = |x|$ is [1 mark]

(A) continuous and differentiable everywhere.

(B) continuous and differentiable nowhere.

(C) continuous everywhere, but differentiable everywhere except at $x = 0$.

(D) continuous everywhere, but differentiable nowhere.

Correct Answer: (C) continuous everywhere, but differentiable everywhere except at $x = 0$.

Solution:

Concept:

- The absolute value function is $f(x) = x$ for $x \geq 0$ and $f(x) = -x$ for $x < 0$.
- Differentiability at a point requires equal left-hand and right-hand derivatives there.

Step 1: Check continuity Away from zero, f is linear and therefore continuous.

At zero,

$$\lim_{x \to 0^-} |x| = 0 = \lim_{x \to 0^+} |x| = f(0).$$

Thus the function is continuous at zero as well, so it is continuous everywhere.

Step 2: Check differentiability For $x < 0$, $f\prime(x) = -1$, and for $x > 0$, $f\prime(x) = 1$. At zero, the left derivative is -1 and the right derivative is 1; they differ. Thus only zero is a point of non-differentiability. Option (A) wrongly includes zero; (B) denies continuity; (D) wrongly denies differentiability at every nonzero point.

Step 3: Conclusion The function is continuous everywhere and differentiable everywhere except at $x = 0$. Hence, the correct option is (C).

Quick Tip to Solve: For $|x|$, check the corner at zero using one-sided slopes; continuity alone does not guarantee differentiability.

7. If $y = \log(\sin e^x)$, then $\dfrac{dy}{dx}$ is: [1 mark]

(A) $\cot e^x$

(B) $\operatorname{cosec} e^x$

(C) $e^x \cot e^x$

(D) $e^x \operatorname{cosec} e^x$

Correct Answer: (C) $e^x \cot e^x$

Solution:

Concept:

- The chain rule is $\dfrac{d}{dx} \log u = \dfrac{u\prime}{u}$ where the real logarithm requires $u > 0$.
- Use $\dfrac{d}{dx} \sin v = \cos v \, v\prime$ and $\dfrac{d}{dx} e^x = e^x$.

Step 1: Differentiate the logarithm For $y = \log(\sin e^x)$, the real domain consists of points where

Quick Tip to Solve: Work from the outside inward: logarithm, sine, then exponential; multiply all three derivative factors.

$\sin(e^x) > 0$. At every such point,

$$\frac{dy}{dx} = \frac{1}{\sin e^x}\frac{d}{dx}(\sin e^x).$$

Step 2: Apply the inner chain rule Since the argument of sine is e^x, its derivative contributes another factor e^x. Hence

$$\frac{dy}{dx} = \frac{e^x \cos e^x}{\sin e^x} = e^x \cot e^x.$$

Option (A) misses the derivative of e^x. Options (B) and (D) use cosecant and omit the cosine factor obtained by differentiating sine.

Step 3: Conclusion $\boxed{\frac{dy}{dx} = e^x \cot e^x}$. Hence, the correct option is (C).

Quick Tip to Solve: Work from the outside inward: logarithm, sine, then exponential; multiply all three derivative factors.

8. $\int e^{5\log x}\,dx$ **is equal to:** [1 mark]

(A) $\frac{x^5}{5} + C$

(B) $\frac{x^6}{6} + C$

(C) $5x^4 + C$

(D) $6x^5 + C$

Correct Answer: (B) $\frac{x^6}{6} + C$

Solution:

Concept:

- Natural logarithms satisfy $e^{\log x} = x$ for $x > 0$.
- The power rule is $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, for $n \neq -1$.

Step 1: Simplify the integrand On the domain $x > 0$, the logarithm law gives $5\log x = \log(x^5)$. Therefore $e^{5\log x} = e^{\log(x^5)} = x^5$. The exponential and logarithm cancel before integration; the factor 5 is an exponent, not a coefficient to integrate separately.

Step 2: Apply the power rule and check the options We obtain

$$\int e^{5\log x}\,dx = \int x^5\,dx = \frac{x^6}{6} + C.$$

Differentiating option (B) gives x^5, as required. The derivatives of (A), (C) and (D) are respectively x^4, $20x^3$ and $30x^4$; none equals the integrand throughout its domain.

Step 3: Conclusion The integral is $\boxed{\frac{x^6}{6} + C}$. Hence, the correct option is (B).

Quick Tip to Solve: Simplify an exponential containing a logarithm before using the power rule.

9. $\int_0^4 (e^{2x} + x)\,dx$ **is equal to:** [1 mark]

(A) $\dfrac{15 + e^8}{2}$

(B) $\dfrac{16 - e^8}{2}$

(C) $\dfrac{e^8 - 15}{2}$

(D) $\dfrac{-e^8 - 15}{2}$

Correct Answer: (A) $\dfrac{15 + e^8}{2}$

Solution:

Concept:

- Integrate a sum term by term and evaluate its antiderivative at the upper limit minus the lower limit.
- For a constant $a \neq 0$, $\int e^{ax}\,dx = \frac{e^{ax}}{a} + C$.

Step 1: Find an antiderivative The exponential term needs the reciprocal of its inner derivative, while the second term follows the power rule. Thus an antiderivative of $e^{2x} + x$ is $F(x) = \frac{e^{2x}}{2} + \frac{x^2}{2}$. Indeed, $F\prime(x) = e^{2x} + x$.

Step 2: Substitute both limits and compare By the fundamental theorem of calculus,

$$\begin{aligned}\int_0^4 (e^{2x} + x)\,dx &= \left[\frac{e^{2x}}{2} + \frac{x^2}{2}\right]_0^4 \\ &= \frac{e^8}{2} + 8 - \frac{1}{2} \\ &= \frac{e^8 + 15}{2}.\end{aligned}$$

Option (A) agrees. Option (C) reverses the constant contribution; (B) and (D) have a negative coefficient of e^8, inconsistent with integrating the positive exponential term.

Step 3: Conclusion The value is $\boxed{\dfrac{15 + e^8}{2}}$. Hence, the correct option is (A).

Quick Tip to Solve: Remember that the exponential contributes 1/2 at the lower limit, even when x is zero.

10. A unit vector along the vector $4\hat{i} - 3\hat{k}$ is: [1 mark]

(A) $\frac{1}{7}(4\hat{i} - 3\hat{k})$

(B) $\frac{1}{5}(4\hat{i} - 3\hat{k})$

(C) $\frac{1}{\sqrt{7}}(4\hat{i} - 3\hat{k})$

(D) $\frac{1}{\sqrt{5}}(4\hat{i} - 3\hat{k})$

Correct Answer: (B) $\frac{1}{5}(4\hat{i} - 3\hat{k})$

Solution:

Concept:

- A unit vector has magnitude 1.
- For a nonzero vector $\vec{v}$, the unit vector in its direction is $\vec{v}/|\vec{v}|$.

Step 1: Find the magnitude Let $\vec{v} = 4\hat{i} - 3\hat{k}$. Its components are $(4, 0, -3)$, so

$$|\vec{v}| = \sqrt{4^2 + 0^2 + (-3)^2} = \sqrt{25} = 5.$$

The negative sign of the third component does not make its contribution to the squared magnitude negative.

Step 2: Normalize and test the alternatives Dividing by the positive magnitude preserves direction and gives $\frac{1}{5}(4\hat{i} - 3\hat{k})$. Its magnitude is $5/5 = 1$. The magnitudes of the vectors in (A), (C) and (D) are respectively $5/7$, $5/\sqrt{7}$ and $5/\sqrt{5} = \sqrt{5}$, so they are not unit vectors.

Step 3: Conclusion The required vector is $\boxed{\frac{1}{5}(4\hat{i} - 3\hat{k})}$. Hence, the correct option is (B).

Quick Tip to Solve: Divide by the square root of the sum of squared components, not their sum.

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11. If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then $\vec{a} \cdot \vec{b} \geq 0$ only when: [1 mark]

(A) $0 < \theta < \frac{\pi}{2}$

(B) $0 \leq \theta \leq \frac{\pi}{2}$

(C) $0 < \theta < \pi$

(D) $0 \leq \theta \leq \pi$

Correct Answer: (B) $0 \leq \theta \leq \frac{\pi}{2}$

Solution:

Concept:

- For nonzero vectors, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$.
- The angle between two vectors lies in $[0, \pi]$; the magnitudes are positive.

Step 1: Determine the sign condition Because $|\vec{a}||\vec{b}| > 0$, the condition $\vec{a} \cdot \vec{b} \geq 0$ is equivalent to

Quick Tip to Solve: A nonnegative dot product allows a right angle, so retain the equality at pi/2.

$\cos\theta \geq 0$. Within the standard interval $0 \leq \theta \leq \pi$, cosine is nonnegative exactly for $0 \leq \theta \leq \pi/2$.

Step 2: Check endpoints and options At $\theta = 0$, the dot product is positive; at $\theta = \pi/2$, it is zero and still satisfies the inequality. Hence both endpoints must be included, which excludes (A) as the complete range. Options (C) and (D) include obtuse angles, for which cosine and the dot product are negative. Option (B) gives precisely the allowed range.

Step 3: Conclusion The range is $\boxed{0 \leq \theta \leq \frac{\pi}{2}}$. Hence, the correct option is (B).

Quick Tip to Solve: A nonnegative dot product allows a right angle, so retain the equality at pi/2.

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12. The integrating factor for solving the differential equation $x\frac{dy}{dx} - y = 2x^2$ is: [1 mark]

(A) e^{-y}

(B) e^{-x}

(C) x

(D) $\frac{1}{x}$

Correct Answer: (D) $\frac{1}{x}$

Solution:

Concept:

- A linear differential equation has standard form $\frac{dy}{dx} + P(x)y = Q(x)$.
- Its integrating factor is $e^{\int P(x)\,dx}$, up to a nonzero constant multiplier.

Step 1: Put the equation in standard form On an interval with $x \neq 0$, divide the entire equation $x\frac{dy}{dx} - y = 2x^2$ by x. This gives

$$\frac{dy}{dx} - \frac{1}{x}y = 2x,$$

so $P(x) = -1/x$. Reading P directly from the original equation would miss this division.

Step 2: Calculate and verify the integrating factor Now $\int P(x)\,dx = -\log|x|$, so the standard exponential gives $1/|x|$. On either interval of fixed sign this differs from $1/x$ by only a constant factor, hence $1/x$ is a valid integrating factor. Multiplication gives $\frac{1}{x}y' - \frac{y}{x^2} = 2$, whose left side is $\frac{d}{dx}(y/x)$. Options (A) and (B) are unrelated exponential factors, while (C) has the opposite logarithmic sign.

Step 3: Conclusion A suitable integrating factor is $\boxed{\frac{1}{x}}$, for $x \neq 0$. Hence, the correct option is (D).

Quick Tip to Solve: First make the coefficient of dy/dx equal to one, then integrate the coefficient of y.

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13. The number of solutions of the differential equation $\frac{dy}{dx} = \frac{y+1}{x-1}$, when $y(1) = 2$, is: [1 mark]

(A) zero

(B) one

(C) two

(D) infinite

Correct Answer: (A) zero

Solution:

Concept:

- A solution of an initial value problem must satisfy the differential equation on an interval containing its initial point.
- For $x \neq 1$, separating variables leads to $y + 1 = C(x - 1)$ for the nonconstant family.

Step 1: Examine the initial point The differential equation is $y' = (y + 1)/(x - 1)$. At the prescribed point $(x, y) = (1, 2)$, its right side would be $3/0$, which is undefined. Thus a differentiable solution cannot satisfy this equation on an interval containing the initial point.

Step 2: Check the integrated family Away from $x = 1$, integration yields $\log|y + 1| = \log|x - 1| + K$, or $y + 1 = C(x - 1)$. Any continuous extension of this family to $x = 1$ has $y(1) = -1$, never 2. The constant solution $y = -1$ also fails the given initial value. Consequently neither one, two nor infinitely many solutions can meet the stated condition.

Step 3: Conclusion The number of solutions is $\boxed{0}$. Hence, the correct option is (A).

Quick Tip to Solve: Check whether the differential equation is defined at the initial point before finding its constant.

14. Distance of the point (p, q, r) from y-axis is: [1 mark]

(A) q

(B) $|q|$

(C) $|q| + |r|$

(D) $\sqrt{p^2 + r^2}$

Correct Answer: (D) $\sqrt{p^2 + r^2}$

Solution:

Concept:

- Every point on the y-axis has coordinates $(0, t, 0)$, with t real.
- The distance from a point to an axis is its least distance to any point on that axis.

Step 1: Express the distance to an arbitrary point Let $P = (p, q, r)$ and take $Q = (0, t, 0)$ on the y-axis. The distance formula gives

$$PQ^2 = p^2 + (q - t)^2 + r^2.$$

Only the middle term changes as Q moves along the axis; the first and third terms measure displacement perpendicular to it.

Quick Tip to Solve: For distance to a coordinate axis, omit that axis coordinate and square the other two.

Step 2: Minimize and compare options Since $(q-t)^2 \geq 0$, the least squared distance occurs at $t = q$, so the nearest point is $(0, q, 0)$. The minimum distance is $\sqrt{p^2+r^2}$. Options (A) and (B) use the coordinate along the axis rather than perpendicular displacement; (C) is a sum of absolute coordinates and does not give this Euclidean distance.

Step 3: Conclusion The distance is $\boxed{\sqrt{p^2+r^2}}$. Hence, the correct option is (D).

Quick Tip to Solve: For distance to a coordinate axis, omit that axis coordinate and square the other two.

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15. If the direction cosines of a line are $\left(\frac{1}{a}, \frac{1}{a}, \frac{1}{a}\right)$, then: [1 mark]

(A) $0 < a < 1$

(B) $a > 2$

(C) $a > 0$

(D) $a = \pm\sqrt{3}$

Correct Answer: (D) $a = \pm\sqrt{3}$

Solution:

Concept:

- Direction cosines l, m, n are the cosines of the angles a line makes with the positive coordinate axes; they satisfy $l^2 + m^2 + n^2 = 1$.
- Here $l = m = n = 1/a$, where $a \neq 0$. Both signs of a direction cosine are possible.

Step 1: Apply the normalization identity Substitute the three given direction cosines into the identity:

$$\frac{1}{a^2} + \frac{1}{a^2} + \frac{1}{a^2} = 1.$$

Thus $3/a^2 = 1$, so $a^2 = 3$.

Step 2: Check the alternatives Taking square roots gives $a = \sqrt{3}$ or $a = -\sqrt{3}$. Option (A) cannot hold since the positive value exceeds 1. Option (B) cannot hold since $\sqrt{3} < 2$. Option (C) omits the permissible negative value; changing all three signs simply reverses the direction along the line. Option (D) includes both values.

Step 3: Conclusion $\boxed{a = \pm\sqrt{3}}$. Hence, the correct option is (D).

Quick Tip to Solve: For equal direction cosines, add their three squares before taking either square root.

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16. For two events A and B, if $P(A) = 0.4$, $P(B) = 0.8$ and $P(B/A) = 0.6$, then $P(A \cup B)$ is: [1 mark]

(A) 0.24

(B) 0.3

(C) 0.48

(D) 0.96

Correct Answer: (D) 0.96

Solution:

Concept:

- For $P(A) > 0$, conditional probability gives $P(B \mid A) = P(A \cap B)/P(A)$.
- The addition rule is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$; subtract the overlap once.

Step 1: Find the intersection probability The printed notation $P(B/A)$ means $P(B \mid A)$. Therefore

$$P(A \cap B) = P(A)P(B \mid A) = 0.4 \times 0.6 = 0.24.$$

This is the probability that both events occur, not that at least one occurs.

Step 2: Use the addition rule and compare options

$$P(A \cup B) = 0.4 + 0.8 - 0.24 = 0.96.$$

Option (A), 0.24, is the intersection. Options (B) and (C), 0.3 and 0.48, are less than $P(B) = 0.8$, so neither can be the union, which includes all of B. The result agrees with option (D) and lies between 0.8 and 1.

Step 3: Conclusion $\boxed{P(A \cup B) = 0.96}$. Hence, the correct option is (D).

Quick Tip to Solve: Convert the conditional probability into the overlap before applying the union formula.

17. Which of the following points satisfies both the inequations $2x + y \leq 10$ and $x + 2y \geq 8$? [1 mark]

(A) $(-2, 4)$

(B) $(3, 2)$

(C) $(-5, 6)$

(D) $(4, 2)$

Correct Answer: (D) $(4, 2)$

Solution:

Concept:

- A point satisfies two simultaneous inequations only when substituting its coordinates makes both statements true.
- The symbols $\leq$ and $\geq$ include equality, so a point on either boundary can be a solution.

Step 1: Test options (A), (B) and (C) For $(-2, 4)$, $2x + y = 0 \leq 10$, but $x + 2y = 6 < 8$. For $(3, 2)$, $2x + y = 8 \leq 10$, but $x + 2y = 7 < 8$. For $(-5, 6)$, $2x + y = -4 \leq 10$, but $x + 2y = 7 < 8$. Thus each

Quick Tip to Solve: Test the second inequation first here: it rejects the first three options immediately.

of options (A), (B) and (C) fails the second condition.

Step 2: Test option (D) For $(4, 2)$,

$$2x + y = 2(4) + 2 = 10 \leq 10, \qquad x + 2y = 4 + 2(2) = 8 \geq 8.$$

Both conditions hold at equality. Equality is allowed in both printed inequations, so this boundary point belongs to the simultaneous solution set.

Step 3: Conclusion $\boxed{(4, 2)}$ satisfies both inequations. Hence, the correct option is (D).

Quick Tip to Solve: Test the second inequation first here: it rejects the first three options immediately.

18. The solution set of the inequation $3x + 5y < 7$ is: [1 mark]

(A) whole xy-plane except the points lying on the line $3x + 5y = 7$.

(B) whole xy-plane along with the points lying on the line $3x + 5y = 7$.

(C) open half plane containing the origin except the points of line $3x + 5y = 7$.

(D) open half plane not containing the origin.

Correct Answer: (C) open half plane containing the origin except the points of line $3x + 5y = 7$.

Solution:

Concept:

- A linear equation $3x + 5y = 7$ divides the plane into two half planes.
- A strict inequality excludes its boundary line; a test point away from the line identifies the required side.

Step 1: Test which side contains the solutions At the origin $(0, 0)$, the left side is $3(0) + 5(0) = 0 < 7$. Thus the origin satisfies the inequality and the solution lies on the side of the line containing the origin. Points on the other side have $3x + 5y > 7$.

Step 2: Exclude the boundary and check the choices Every point on $3x + 5y = 7$ gives equality, which is excluded by the strict sign $<$. Therefore the required region is an open half plane containing the origin. Options (A) and (B) incorrectly include points from both sides of the line, while option (D) selects the opposite side. Option (C) describes the required region.

Step 3: Conclusion The solution is the open half plane containing the origin, excluding $3x + 5y = 7$. Hence, the correct option is (C).

Quick Tip to Solve: Use the origin as a test point whenever it does not lie on the boundary line.

19. Assertion (A): All trigonometric functions have their inverses over their respective domains.
Reason (R): The inverse of $\tan^{-1} x$ exists for some $x \in \mathbb{R}$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (D) Assertion (A) is false and Reason (R) is true.

Solution:

Concept:

- A function has an inverse function on its domain and image only when it is one-one.
- Periodic trigonometric functions require suitable domain restrictions to become one-one; $\tan^{-1} x$ denotes the principal inverse tangent.

Step 1: Judge the Assertion On their full respective domains, trigonometric functions are not one-one. For example, $\sin 0 = \sin \pi = 0$, although $0 \neq \pi$. Hence the assertion that all have inverses on those domains is false. Inverse trigonometric functions are defined after suitable restrictions.

Step 2: Judge the Reason The function $g(x) = \tan^{-1} x$ maps $\mathbb{R}$ one-one onto $(-\pi/2, \pi/2)$. Its inverse is $\tan t$ on that interval, since $\tan(\tan^{-1} x) = x$ for every real x. Therefore the stated existence for some real x is true.

Step 3: Decide the explanatory relationship The reason concerns the already one-one inverse tangent function. It does not establish invertibility of the original trigonometric functions on their full domains. With Assertion false and Reason true, only option (D) matches; options (A), (B) and (C) all require a true Assertion.

Step 4: Conclusion The Assertion is false and the Reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Distinguish tangent on its full domain from principal inverse tangent on the real line.

20. Assertion (A): The lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are perpendicular, when $\vec{b_1} \cdot \vec{b_2} = 0$.

Reason (R): The angle θ between the lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by $\cos\theta = \dfrac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|}$.

[1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- The nonzero vectors $\vec{b_1}, \vec{b_2}$ give the directions of the respective lines.
- For direction vectors, $\vec{b_1} \cdot \vec{b_2} = |\vec{b_1}||\vec{b_2}|\cos\theta$, where θ is their angle.

Step 1: Judge the Assertion If $\vec{b_1} \cdot \vec{b_2} = 0$, the two nonzero direction vectors are orthogonal. Thus the angle between the directions of the lines is $90°$; the lines are perpendicular in the sense of the angle between lines. The Assertion is true.

Step 2: Judge the Reason Dividing the dot-product identity by the nonzero product of magnitudes gives

$$\cos\theta = \frac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|}.$$

This is precisely the printed relation, so the Reason is true.

Step 3: Check that the Reason explains the Assertion Substituting the hypothesis $\vec{b_1} \cdot \vec{b_2} = 0$ into the Reason gives $\cos\theta = 0$, hence $\theta = 90°$. This directly establishes the Assertion. Option (B) misses this explanatory link, and options (C) and (D) incorrectly declare one statement false.

Step 4: Conclusion Both statements are true and the Reason correctly explains the Assertion. Hence, the correct option is (A).

Quick Tip to Solve: Identify the direction vectors first; the position vectors do not determine the angle.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21. Find the interval in which the function $f(x) = 2x^3 - 3x$ is strictly increasing. [2 marks]

Solution:

Concept:

- A differentiable function is strictly increasing on an interval if its derivative is positive throughout the interior of that interval.
- For $f(x) = 2x^3 - 3x$, the power rule gives $f\prime(x) = 6x^2 - 3$. Critical points separate intervals on which this derivative has a constant sign.

Step 1: Differentiate and solve the sign condition

$$f\prime(x) = 6x^2 - 3 = 3(2x^2 - 1).$$

For increasing behaviour, solve

$$\begin{aligned} 6x^2 - 3 > 0 &\iff x^2 > \frac{1}{2}, \\ &\iff x < -\frac{1}{\sqrt{2}} \text{ or } x > \frac{1}{\sqrt{2}}. \end{aligned}$$

Squaring does not permit only the positive branch: both outer intervals must be retained.

Step 2: Interpret the intervals The derivative vanishes at $x = \pm 1/\sqrt{2}$. It is positive to the left of the negative critical point and to the right of the positive critical point, and negative between them. Thus the two open intervals obtained above are the conventional intervals of increase. Each critical endpoint may also be included in its adjacent increasing interval, since a zero derivative at that single endpoint does not destroy strict increase. The two separate increasing intervals must not be interpreted as one interval spanning the decreasing middle portion.

Step 3: Conclusion The function is strictly increasing on $\boxed{(-\infty, -1/\sqrt{2})}$ and $\boxed{(1/\sqrt{2}, \infty)}$. The respective finite endpoints may also be included.

Quick Tip to Solve: For a cubic with an upward-opening quadratic derivative, check both outer sign intervals.

22(a). Find the vector equation of the line passing through the point $(2, 1, 3)$ and perpendicular to both the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$. [2 marks]

Solution:

Concept:

- A direction vector of a line in symmetric form is given by its denominators.
- A vector perpendicular to two non-parallel vectors is their cross product. A line through position vector $\vec{a}$ with direction $\vec{d}$ has equation $\vec{r} = \vec{a} + \lambda\vec{d}$.

Quick Tip to Solve: Read direction vectors from the denominators, then take their cross product; numerator offsets are not needed.

Step 1: Identify the two direction vectors The given lines have direction vectors $\vec{b}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$ and $\vec{b}_2 = -3\hat{i} + 2\hat{j} + 5\hat{k}$. The constants in their numerators locate points on those lines; they do not affect their directions. Thus the direction of the required line must be orthogonal to both $\vec{b}_1$ and $\vec{b}_2$.

Step 2: Compute a perpendicular direction Using the cross product,

$$\vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & 2 & 5 \end{vmatrix} = 4\hat{i} - 14\hat{j} + 8\hat{k}.$$

We may divide by 2 and use $\vec{d} = 2\hat{i} - 7\hat{j} + 4\hat{k}$. Indeed, $2 - 14 + 12 = 0$ and $-6 - 14 + 20 = 0$, so its dot product with each given direction is zero. The required point has position vector $2\hat{i} + \hat{j} + 3\hat{k}$.

Step 3: Conclusion $\boxed{\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k}), \quad \lambda \in \mathbb{R}}$.

Quick Tip to Solve: Read direction vectors from the denominators, then take their cross product; numerator offsets are not needed.

OR

22(b). The equations of a line are $5x - 3 = 15y + 7 = 3 - 10z$. Write the direction cosines of the line and find the coordinates of a point through which it passes. [2 marks]

Solution:

Concept:

- Direction cosines are components of a unit direction vector: $l = a/\sqrt{a^2 + b^2 + c^2}$, and similarly for m, n.
- Equating all three expressions to a parameter gives a point and a direction vector. Reversing direction changes all three cosine signs together.

Step 1: Express the coordinates using one parameter Let $5x - 3 = 15y + 7 = 3 - 10z = t$. Solving separately gives

$$x = \frac{3}{5} + \frac{t}{5}, \qquad y = -\frac{7}{15} + \frac{t}{15}, \qquad z = \frac{3}{10} - \frac{t}{10}.$$

Therefore a direction vector is proportional to $(1/5, 1/15, -1/10)$. Multiplication by 30 gives the convenient direction ratios $(6, 2, -3)$.

Step 2: Normalize and select a point The length of $(6, 2, -3)$ is $\sqrt{36 + 4 + 9} = 7$. Dividing each direction ratio by 7 gives a unit vector. Taking $t = 0$ in the parametrisation gives the point $(3/5, -7/15, 3/10)$. Substituting this point makes each of the three original expressions zero, confirming it lies on the line.

Step 3: Conclusion The direction cosines are $\boxed{(6/7,\ 2/7,\ -3/7)}$, and a point is $\boxed{(3/5,\ -7/15,\ 3/10)}$. The opposite orientation has direction cosines $(-6/7, -2/7, 3/7)$.

Quick Tip to Solve: Use a common parameter first; the coefficients of that parameter, not the coefficients of x, y and z, give direction ratios.

23(a). Find the domain of $y = \sin^{-1}(x^2 - 4)$. [2 marks]

Solution:

Concept:

- The real function $\sin^{-1} u$ is defined exactly for $-1 \le u \le 1$.
- When $a \le x^2 \le b$ with $0 < a < b$, both positive and negative values of x must be included.

Step 1: Restrict the argument of inverse sine For $y = \sin^{-1}(x^2 - 4)$ to be real, its complete argument must lie in $[-1, 1]$. Hence

$$-1 \le x^2 - 4 \le 1.$$

Adding 4 throughout, without reversing either inequality, gives $3 \le x^2 \le 5$. This condition is necessary and sufficient because inverse sine is defined at both endpoints of its domain.

Step 2: Translate the squared bounds to intervals The bound $x^2 \ge 3$ means $x \le -\sqrt{3}$ or $x \ge \sqrt{3}$. The bound $x^2 \le 5$ means $-\sqrt{5} \le x \le \sqrt{5}$. Intersecting these conditions gives the two intervals $[-\sqrt{5}, -\sqrt{3}]$ and $[\sqrt{3}, \sqrt{5}]$. Their endpoints are included: the inverse-sine argument is either -1 or 1 there.

Step 3: Conclusion The domain is $\boxed{[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]}$.

Quick Tip to Solve: First bound the entire inverse-sine argument between -1 and 1; retain both signs when solving bounds on x squared.

OR

23(b). Evaluate: $\cos^{-1}\left[\cos\left(-\frac{7\pi}{3}\right)\right]$. [2 marks]

Solution:

Concept:

- The principal range of $\cos^{-1}$ is $[0, \pi]$; its value must lie in that interval.
- Cosine is even and periodic: $\cos(-\theta) = \cos\theta$ and $\cos(\theta + 2\pi) = \cos\theta$.

Step 1: Evaluate the inner cosine The supplied angle $-7\pi/3$ lies outside the principal range of inverse cosine. Use evenness and periodicity to evaluate its cosine:

$$\cos\left(-\frac{7\pi}{3}\right) = \cos\left(\frac{7\pi}{3}\right) = \cos\left(2\pi + \frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}.$$

Thus the expression becomes $\cos^{-1}(1/2)$.

Step 2: Choose the angle in the principal range Within $[0, \pi]$, the unique angle having cosine $1/2$ is $\pi/3$, since $\cos(\pi/3) = 1/2$. Although many angles have that cosine, inverse cosine returns only the principal one. In particular, one cannot cancel $\cos^{-1}$ and $\cos$ to obtain $-7\pi/3$, because cancellation in that order requires an input angle in $[0, \pi]$.

Step 3: Conclusion $\boxed{\cos^{-1}[\cos(-7\pi/3)] = \pi/3}$.

Quick Tip to Solve: Reduce the inner cosine first, then select its unique angle between 0 and pi; do not cancel the two functions directly.

24. If $(x^2+y^2)^2 = xy$, then find $\frac{dy}{dx}$. [2 marks]

Solution:

Concept:

- For an implicit relation, differentiate with respect to x, treating y as a differentiable function of x.
- The chain rule gives $d(y^2)/dx = 2y\,dy/dx$, and the product rule gives $d(xy)/dx = y + x\,dy/dx$.

Step 1: Apply the chain rule and product rule Write $y\prime = dy/dx$. Differentiating $(x^2+y^2)^2 = xy$ with respect to x gives

$$2(x^2+y^2)(2x+2yy\prime) = y + xy\prime.$$

The factor $2(x^2+y^2)$ comes from differentiating the outer square; $2x+2yy\prime$ is the derivative of its inner expression. On the right, both factors in xy vary with x.

Step 2: Collect the derivative terms Expanding and collecting the terms containing $y\prime$,

$$4x(x^2+y^2) + 4y(x^2+y^2)y\prime = y + xy\prime,$$
$$\{4y(x^2+y^2) - x\}y\prime = y - 4x(x^2+y^2).$$

Divide by $4y(x^2+y^2)-x$ at points where it is nonzero. This preserves the implicit relation and isolates the required derivative.

Step 3: Conclusion $\boxed{\frac{dy}{dx} = \frac{y-4x(x^2+y^2)}{4y(x^2+y^2)-x}}$, wherever the denominator is nonzero.

Quick Tip to Solve: Keep dy/dx beside every differentiated y term, then move all terms containing that derivative to the same side.

25. If $\vec{a} = 4\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$, then find a unit vector along the vector $\vec{a} \times \vec{b}$. [2 marks]

Solution:

Concept:

- For non-parallel vectors, the cross product is perpendicular to both vectors. Its components follow from the determinant with first row $\hat{i}, \hat{j}, \hat{k}$.
- A unit vector along a nonzero vector $\vec{c}$ is $\vec{c}/|\vec{c}|$, where $|\vec{c}|$ is its magnitude. Dividing by the positive magnitude preserves its direction.

Step 1: Calculate the cross product Using the given components and expanding along the first row,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -1 & 1 \\ 2 & -2 & 1 \end{vmatrix}$$
$$= \hat{i}(-1+2) - \hat{j}(4-2) + \hat{k}(-8+2)$$
$$= \hat{i} - 2\hat{j} - 6\hat{k}.$$

Quick Tip to Solve: Expand the cross-product determinant with signs +, -, +, then divide by its magnitude.

The minus sign before the middle cofactor is essential.

Step 2: Normalize the resulting vector The magnitude is

$$|\vec{a} \times \vec{b}| = \sqrt{1^2 + (-2)^2 + (-6)^2} = \sqrt{41}.$$

Thus dividing each component by $\sqrt{41}$ produces a vector of magnitude one. Its dot products with $\vec{a}$ and $\vec{b}$ are proportional to $4 + 2 - 6 = 0$ and $2 + 4 - 6 = 0$, respectively, confirming perpendicularity.

Step 3: Conclusion A unit vector along $\vec{a} \times \vec{b}$ is $\boxed{\dfrac{\hat{i} - 2\hat{j} - 6\hat{k}}{\sqrt{41}}}$.

Quick Tip to Solve: Expand the cross-product determinant with signs +, -, +, then divide by its magnitude.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26. Find: $\displaystyle\int \frac{x^2}{x^2 + 6x + 12}\, dx$. [3 marks]

Solution:

Concept:

- For a rational function whose numerator and denominator have equal degree, first divide to separate a polynomial term.
- Use $\int D\prime(x)/D(x)\, dx = \ln|D(x)| + C$ and $\int du/(u^2 + a^2) = a^{-1}\tan^{-1}(u/a) + C$, with $a > 0$.

Step 1: Separate the derivative of the denominator Let $D = x^2 + 6x + 12$, so $D\prime = 2x + 6$. The numerator can be written as

$$x^2 = D - (6x + 12) = D - 3(2x + 6) + 6.$$

Therefore

$$\frac{x^2}{D} = 1 - 3\frac{2x + 6}{D} + \frac{6}{D}.$$

This decomposition isolates the logarithmic contribution without substituting into the entire numerator.

Step 2: Integrate the remaining quadratic term Complete the square:

$$D = (x + 3)^2 + 3.$$

With $u = x + 3$, the remaining integral becomes

$$\int \frac{6}{(x+3)^2 + 3}\, dx = \frac{6}{\sqrt{3}}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) = 2\sqrt{3}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right).$$

Since $D \geq 3$ for real x, the logarithm may be written as $\ln D$ without absolute-value signs.

Quick Tip to Solve: After division, express the linear remainder using the derivative 2x+6 before completing the square.

Step 3: Combine and verify the antiderivative Integrating the first two terms gives $x - 3\ln D$. The derivative of the inverse-tangent contribution is $6/D$; hence the derivative of the combined expression is

$$1 - \frac{3(2x+6)}{D} + \frac{6}{D} = \frac{D - 6x - 12}{D} = \frac{x^2}{D},$$

exactly the original integrand. Add an arbitrary real constant C.

Step 4: Conclusion

$$\boxed{\int \frac{x^2}{x^2+6x+12}\,dx = x - 3\ln(x^2+6x+12) + 2\sqrt{3}\tan^{-1}\left(\frac{x+3}{\sqrt{3}}\right) + C.}$$

Quick Tip to Solve: After division, express the linear remainder using the derivative 2x+6 before completing the square.

27. Two fair dice are thrown simultaneously. If X denotes the number of sixes, find the mean of X. [3 marks]

Solution:

Concept:

- For a discrete random variable X, its mean is $E(X) = \sum_x xP(X = x)$; possible values must be weighted by their probabilities.
- Two fair independent dice have $6 \times 6 = 36$ equally likely ordered outcomes. A six on either die has probability $1/6$.

Step 1: Find the possible values and their probabilities The number of sixes X can be $0, 1, 2$. For $X = 0$, both dice must show one of the five non-six faces, giving $5 \times 5 = 25$ outcomes. For $X = 1$, the six may occur on either die, while the other die has five choices, giving $2 \times 5 = 10$ outcomes. For $X = 2$, only $(6, 6)$ qualifies. Thus

$$P(X = 0) = \frac{25}{36}, \qquad P(X = 1) = \frac{10}{36}, \qquad P(X = 2) = \frac{1}{36}.$$

Their sum is $36/36 = 1$, so the distribution accounts for all outcomes.

Step 2: Calculate the expectation Weight each possible number of sixes by its probability:

$$\begin{aligned} E(X) &= 0 \cdot \frac{25}{36} + 1 \cdot \frac{10}{36} + 2 \cdot \frac{1}{36} \\ &= \frac{12}{36} = \frac{1}{3}. \end{aligned}$$

Although the observed count is always an integer, the mean over many repetitions need not be an integer. The two-sixes outcome contributes twice its probability because it contains two sixes.

Step 3: Check using indicators Let I_1 and I_2 equal one when the first and second die, respectively, shows a six, and zero otherwise. Then $X = I_1 + I_2$. Each indicator has expectation $1/6$, so linearity gives $E(X) = 1/6 + 1/6 = 1/3$, agreeing with the distribution calculation.

Quick Tip to Solve: For exactly one six, count both orders: (six, non-six) and (non-six, six).

Step 4: Conclusion The mean number of sixes is $\boxed{E(X) = \frac{1}{3}}$.

Quick Tip to Solve: For exactly one six, count both orders: (six, non-six) and (non-six, six).

28(a). Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$, $y(1) = 0$. [3 marks]

Solution:

Concept:

- A homogeneous equation $dy/dx = F(y/x)$ can be reduced using $y = vx$, where v is a function of x; then $dy/dx = v + x\,dv/dx$.
- Use the initial condition to determine the integration constant. Because the equation divides by x, its solution interval cannot cross $x = 0$.

Step 1: Reduce the equation by substitution The given equation is

$$\frac{dy}{dx} = \frac{x+y}{x} = 1 + \frac{y}{x}.$$

Put $y = vx$. Differentiating by the product rule gives $dy/dx = v + x\,dv/dx$. Substitution therefore yields

$$v + x\frac{dv}{dx} = 1 + v, \qquad x\frac{dv}{dx} = 1.$$

The terms involving v cancel; omitting the product-rule term would give an incorrect differential equation.

Step 2: Integrate and apply the initial condition For $x \neq 0$, separate the variables:

$$dv = \frac{dx}{x}, \qquad v = \ln|x| + C.$$

Hence $y = x(\ln|x| + C)$. The initial point is $x = 1 > 0$, so the relevant connected interval is $x > 0$. Using $y(1) = 0$ gives

$$0 = 1(\ln 1 + C) = C.$$

Thus on this interval, the expression reduces to $y = x \ln x$.

Step 3: Verify the particular solution Differentiate $y = x \ln x$ for $x > 0$:

$$\frac{dy}{dx} = \ln x + 1.$$

Substituting the same expression on the right gives

$$(x+y)/x = 1 + (x \ln x)/x = 1 + \ln x.$$

Both sides agree, and $y(1) = 1 \ln 1 = 0$. The domain restriction is required by the original equation and the initial point, rather than being an arbitrary restriction.

Quick Tip to Solve: In y=vx, retain both terms v+x dv/dx; the v terms then cancel immediately.

Step 4: Conclusion The required particular solution is $\boxed{y = x \ln x, \quad x > 0}$.

Quick Tip to Solve: In y=vx, retain both terms v+x dv/dx; the v terms then cancel immediately.

OR

28(b). Find the general solution of the differential equation $e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0$. [3 marks]

Solution:

Concept:

- Use the substitution $t = \tan y$, so that $dt = \sec^2 y \, dy$. This converts the equation into a separable equation in x and t.
- For a separable equation, collect the dependent variable with its differential and then integrate both sides. The identity $\int du/u = \log|u| + C$ applies on intervals where $u \neq 0$.

Step 1: Transform and separate the variables The given equation becomes

$$e^x t \, dx + (1 - e^x) \, dt = 0.$$

Hence, for $t \neq 0$ and $x \neq 0$,

$$\frac{dt}{t} = \frac{e^x}{e^x - 1} \, dx.$$

The denominator is accompanied by its derivative e^x, so the right-hand integral is logarithmic. Division by t temporarily excludes $t = 0$; we will retain that solution in the final family.

Step 2: Integrate and write the constant in algebraic form Integrating both sides gives

$$\log|t| = \log|e^x - 1| + C_1.$$

Therefore $t = C(e^x - 1)$, where the arbitrary real constant C absorbs the sign and the nonzero exponential of the integration constant. Substituting back gives $\tan y = C(e^x - 1)$. The value $C = 0$ also solves the original equation and restores the solution lost while dividing by t.

Step 3: Verify the family in the original equation Differentiating the proposed family gives $\sec^2 y \, dy = Ce^x \, dx$. Substitution in the original left-hand side gives

$$Ce^x(e^x - 1) \, dx + (1 - e^x)Ce^x \, dx = 0.$$

Thus the family satisfies the equation directly, including at $x = 0$, on any branch where $\tan y$ is defined. No initial condition is given, so the constant remains arbitrary.

Step 4: Conclusion The general solution is $\boxed{\tan y = C(e^x - 1)}$, where C is an arbitrary real constant.

Quick Tip to Solve: Substitute $t = \tan y$ immediately; its differential is exactly the $\sec^2 y \, dy$ term.

29(a). Evaluate: $\displaystyle\int_{\pi/4}^{\pi/2} e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$. [3 marks]

Solution:

Concept:

- The double-angle identities $\sin 2x = 2\sin x\cos x$ and $1-\cos 2x = 2\sin^2 x$ convert the quotient into a combination of $\csc^2 x$ and $\cot x$.
- The product rule gives $\frac{d}{dx}(e^{2x}\cot x) = e^{2x}(2\cot x - \csc^2 x)$. Recognising this derivative avoids a lengthy integration by parts.

Step 1: Simplify the trigonometric quotient Write the required integral as I. Throughout the interval $[\pi/4, \pi/2]$, $\sin x \neq 0$, so all the following divisions are valid. Using the double-angle identities,

$$\frac{1-\sin 2x}{1-\cos 2x} = \frac{1-2\sin x\cos x}{2\sin^2 x} = \frac{1}{2}\csc^2 x - \cot x.$$

Consequently,

$$I = \int_{\pi/4}^{\pi/2} e^{2x}\left(\frac{1}{2}\csc^2 x - \cot x\right)dx.$$

Step 2: Recognise the complete derivative Since $(\cot x)' = -\csc^2 x$, the product rule gives

$$\frac{d}{dx}\left(-\frac{1}{2}e^{2x}\cot x\right) = e^{2x}\left(\frac{1}{2}\csc^2 x - \cot x\right).$$

Thus a single product supplies the antiderivative of the entire integrand. The negative sign and the factor $1/2$ are essential: differentiating only $e^{2x}\cot x$ would give the negative of twice the required integrand.

Step 3: Evaluate at both limits By the fundamental theorem of calculus,

$$\begin{aligned} I &= \left[-\frac{1}{2}e^{2x}\cot x\right]_{\pi/4}^{\pi/2} \\ &= -\frac{1}{2}e^{\pi}\cot\frac{\pi}{2} + \frac{1}{2}e^{\pi/2}\cot\frac{\pi}{4} \\ &= 0 + \frac{1}{2}e^{\pi/2}. \end{aligned}$$

Here $\cot(\pi/2) = 0$ and $\cot(\pi/4) = 1$; the exponential at the lower limit is $e^{\pi/2}$, not $e^{\pi/4}$.

Step 4: Conclusion The value of the definite integral is $\boxed{\frac{1}{2}e^{\pi/2}}$.

Quick Tip to Solve: After using double-angle identities, match $\csc^2 x - 2\cot x$ with the derivative of $-e^{2x}\cot x$.

OR

29(b). Evaluate: $\displaystyle\int_{-2}^{2} \frac{x^2}{1+5^x}\,dx$. [3 marks]

Solution:

Concept:

- For a symmetric interval, the substitution $x = -t$ gives $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} f(-x)\,dx$. Adding these two forms can simplify the denominator.
- The reciprocal exponent identity $5^{-x} = 1/5^x$ implies $\frac{1}{1+5^x} + \frac{1}{1+5^{-x}} = 1$. Also, x^2 is an even function.

Step 1: Reflect the integral across the origin Let

$$I = \int_{-2}^{2} \frac{x^2}{1+5^x}\,dx.$$

With $x = -t$, we have $dx = -dt$, and the limits $-2, 2$ become $2, -2$. Reversing the limits absorbs the minus sign, giving

$$I = \int_{-2}^{2} \frac{t^2}{1+5^{-t}}\,dt = \int_{-2}^{2} \frac{x^2}{1+5^{-x}}\,dx.$$

The integration variable is a dummy variable, so the last renaming changes nothing.

Step 2: Add the original and reflected forms Adding the two expressions for I,

$$2I = \int_{-2}^{2} x^2 \left(\frac{1}{1+5^x} + \frac{1}{1+5^{-x}}\right) dx.$$

Since $5^x > 0$, both denominators are nonzero. Multiplying the numerator and denominator of the second fraction by 5^x yields

$$\frac{1}{1+5^{-x}} = \frac{5^x}{1+5^x}.$$

The bracket is therefore $(1+5^x)/(1+5^x) = 1$, and hence $2I = \int_{-2}^{2} x^2\,dx$.

Step 3: Integrate the polynomial Now apply the elementary antiderivative $\int x^2 dx = x^3/3$:

$$2I = \left[\frac{x^3}{3}\right]_{-2}^{2} = \frac{8}{3} - \left(-\frac{8}{3}\right) = \frac{16}{3}.$$

Dividing by two gives the required value. The original integrand itself is not even; it is the sum with its reflected form that becomes the even polynomial x^2.

Step 4: Conclusion The value of the definite integral is $\boxed{\frac{8}{3}}$.

Quick Tip to Solve: On symmetric limits, replace x by $-x$ and add; the two reciprocal exponential denominators then cancel.

30(a). Find: $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}\,dx$. [3 marks]

Solution:

Concept:

- Use substitution when the numerator is the derivative of an exponential expression: if $t = e^x$, then $dt = e^x\,dx$.
- Completing the square gives the standard integral $\int (a^2-u^2)^{-1/2}\,du = \sin^{-1}(u/a) + C$, where $a > 0$ and $|u| < a$.

Step 1: Remove the exponential by substitution Let $t = e^x$. Since $dt = e^x\,dx$, the required integral becomes

$$I = \int \frac{dt}{\sqrt{5-4t-t^2}}.$$

This substitution absorbs the entire numerator; no extra exponential factor remains.

Step 2: Complete the square and apply the standard form The expression under the radical can be written as

$$5-4t-t^2 = 9-(t+2)^2.$$

Now put $u = t+2$, so that $du = dt$. Thus

$$I = \int \frac{du}{\sqrt{9-u^2}} = \sin^{-1}\left(\frac{u}{3}\right) + C.$$

Substituting back successively gives the antiderivative in terms of x. For a real integrand the radicand must be positive. Because $e^x > 0$, the condition $9-(e^x+2)^2 > 0$ reduces to $e^x < 1$, or $x < 0$.

Step 3: Check by differentiation On the real domain $x < 0$, differentiation gives

$$\frac{d}{dx}\sin^{-1}\left(\frac{e^x+2}{3}\right) = \frac{e^x/3}{\sqrt{1-(e^x+2)^2/9}} = \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}.$$

This reproduces the printed integrand and checks both the shift and the scale in the inverse sine.

Step 4: Conclusion

$$\boxed{\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}\,dx = \sin^{-1}\left(\frac{e^x+2}{3}\right) + C}$$

C is an arbitrary constant; the real domain is $x < 0$.

Quick Tip to Solve: First replace e^x by a single variable; completing the square then exposes the inverse-sine form immediately.

OR

30(b). Evaluate: $\int_0^{\pi/2} \sqrt{\sin x}\, \cos^5 x\, dx$. [3 marks]

Solution:

Concept:

- For $t = \sin x$, the differential is $dt = \cos x\, dx$; keep one factor of $\cos x$ for this differential.
- Use $\cos^2 x = 1 - \sin^2 x$ to rewrite the remaining even power. On $[0, \pi/2]$, $\sin x \geq 0$, so the square root is real.

Step 1: Transform the integral and its limits Write $\cos^5 x = \cos^4 x \cos x$, and set $t = \sin x$. The limits $x = 0$ and $x = \pi/2$ become $t = 0$ and $t = 1$, respectively. Also,

$$\cos^4 x = (1 - \sin^2 x)^2 = (1 - t^2)^2.$$

Therefore the required definite integral is

$$I = \int_0^1 t^{1/2}(1 - t^2)^2\, dt.$$

Changing the limits along with the variable avoids any need to substitute back.

Step 2: Expand and integrate each power Expanding the square gives

$$\begin{aligned} I &= \int_0^1 \left(t^{1/2} - 2t^{5/2} + t^{9/2}\right) dt \\ &= \left[\frac{2}{3}t^{3/2} - \frac{4}{7}t^{7/2} + \frac{2}{11}t^{11/2}\right]_0^1. \end{aligned}$$

All three powers are greater than -1, so these integrals are finite at the lower limit. Each antiderivative term vanishes at $t = 0$.

Step 3: Evaluate the endpoints and simplify At the upper endpoint the bracket is

$$I = \frac{2}{3} - \frac{4}{7} + \frac{2}{11} = \frac{154 - 132 + 42}{231}.$$

The result is positive, consistent with the original nonnegative integrand. The negative middle term comes from expanding $(1 - t^2)^2$, not from a reversal of limits.

Step 4: Conclusion

$$\boxed{\int_0^{\pi/2} \sqrt{\sin x}\, \cos^5 x\, dx = \frac{64}{231}}$$

Quick Tip to Solve: With an odd power of cosine, reserve one cosine factor for $d(\sin x)$ and replace the remaining even power using the identity.

31. Solve the following linear programming problem graphically:

Maximise $z = -3x - 5y$

subject to the constraints $-2x + y \le 4,\ x + y \ge 3,\ x - 2y \le 2,\ x \ge 0,\ y \ge 0.$ [3 marks]

Solution:

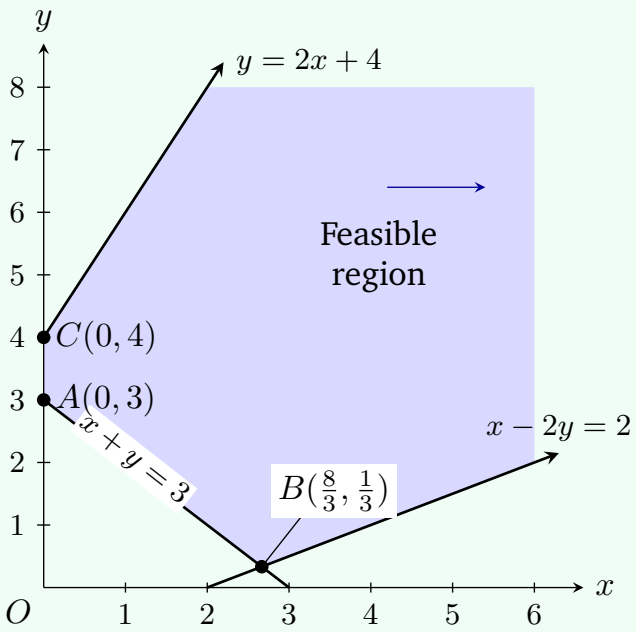
Concept:

- The feasible region is the intersection of all the constraint half-planes in the first quadrant. Its corner points are candidates for the optimum of a linear objective.
- An unbounded feasible region does not by itself imply an unbounded maximum. For $z = -3x - 5y$, verify that no feasible point improves on the best corner value.

Step 1: Draw the constraint lines and feasible region The three boundary lines are

$$y = 2x + 4, \qquad y = 3 - x, \qquad y = \frac{x - 2}{2}.$$

Their required half-planes are, respectively, below the first line and above the other two. Restrict the graph to $x \ge 0,\ y \ge 0$. The common region extends indefinitely to the right between the rays on $y = 2x + 4$ and $y = (x - 2)/2$. The shaded portion of the graph shows this unbounded feasible region.



Feasible region with corners A, B, and C; the shaded region continues to the right.

Step 2: Find its corner points On the y-axis the feasible interval is $3 \le y \le 4$, giving $A = (0, 3)$ and $C = (0, 4)$. The lower sloping boundaries intersect where

$$x + y = 3, \qquad x - 2y = 2.$$

Substituting $x = 3 - y$ gives $3 - 3y = 2$, hence $y = 1/3$ and $x = 8/3$. Thus $B = (8/3, 1/3)$. There is no feasible point on the x-axis: that would require both $x \ge 3$ and $x \le 2$.

Quick Tip to Solve: When the objective has negative coefficients, examine the lower boundary carefully; verify a bound before accepting an optimum on an unbounded region.

Step 3: Evaluate the objective and verify the global maximum The corner values are

Corner point	$z = -3x - 5y$
$A = (0, 3)$	-15
$B = (8/3, 1/3)$	$-29/3$
$C = (0, 4)$	-20

To check the unbounded region, combine the two lower constraints. They give

$$\begin{aligned} 3x + 5y &= \frac{11}{3}(x + y) - \frac{2}{3}(x - 2y) \\ &\geq \frac{11}{3}(3) - \frac{2}{3}(2) = \frac{29}{3}. \end{aligned}$$

Therefore every feasible point satisfies $z \leq -29/3$. Equality requires both boundaries to be equalities, which occurs at B. This confirms the corner comparison over the entire region.

Step 4: Conclusion The maximum value is $\boxed{z_{\max} = -29/3}$, attained uniquely at $\boxed{(x, y) = (8/3, 1/3)}$.

Quick Tip to Solve: When the objective has negative coefficients, examine the lower boundary carefully; verify a bound before accepting an optimum on an unbounded region.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32. Using integration, find the area of the region bounded by the circle $x^2 + y^2 = 16$, line $y = x$ and y-axis, but lying in the 1st quadrant. [5 marks]

Solution:

Concept:

- In the first quadrant, the upper boundary of the circle is $y = \sqrt{16 - x^2}$, and its radius is 4.
- A vertical strip between two curves has area $(y_{\text{upper}} - y_{\text{lower}})\,dx$; integrating these strips between the intersection limits gives the bounded area.
- Use $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}(x/a) + C$, with inverse sine measured in radians.

Step 1: Locate the boundary points The positive y-axis meets the circle at $(0, 4)$, while the line $y = x$ meets this axis at the origin. Substitution of $y = x$ in the circle gives

$$x^2 + x^2 = 16 \quad \Longrightarrow \quad x^2 = 8.$$

Taking the positive value because the region lies in the first quadrant, the line-circle intersection is $(2\sqrt{2}, 2\sqrt{2})$. The enclosed boundary runs up the positive y-axis, along the circular arc to this intersection and back along the line to the origin. Thus its horizontal limits are 0 and $2\sqrt{2}$.

Step 2: Set up the area integral On this interval the circular arc is the upper curve and $y = x$ is the lower curve. To verify their order, observe that $x^2 \leq 8$ implies $16 - x^2 \geq x^2$, and both square roots are

Quick Tip to Solve: Find the line-circle intersection first, subtract the line from the upper arc, and evaluate inverse sine in radians.

nonnegative. Hence $\sqrt{16-x^2} \geq x$. Each vertical strip therefore has height $\sqrt{16-x^2} - x$, so

$$A = \int_0^{2\sqrt{2}} \left(\sqrt{16-x^2} - x\right) dx.$$

The subtraction removes the triangular region below the line from the region below the arc over the same horizontal interval.

Step 3: Evaluate at the limits With $a = 4$ in the standard antiderivative,

$$A = \left[\frac{x}{2}\sqrt{16-x^2} + 8\sin^{-1}\left(\frac{x}{4}\right) - \frac{x^2}{2}\right]_0^{2\sqrt{2}}.$$

At the upper limit, the square root equals $2\sqrt{2}$, and $\sin^{-1}(\sqrt{2}/2) = \pi/4$. At the lower limit all three terms vanish. Consequently,

$$A = \frac{2\sqrt{2}}{2}(2\sqrt{2}) + 8\left(\frac{\pi}{4}\right) - \frac{(2\sqrt{2})^2}{2} = 4 + 2\pi - 4.$$

The two algebraic terms cancel, leaving only the inverse-sine contribution.

Step 4: Conclusion The required area is $\boxed{2\pi \text{ square units}}$.

Quick Tip to Solve: Find the line-circle intersection first, subtract the line from the upper arc, and evaluate inverse sine in radians.

33(a). Show that the following lines do not intersect each other:
$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5}$; $\frac{x+2}{4} = \frac{y-1}{3} = \frac{z+1}{-2}$. [5 marks]

Solution:

Concept:

- A symmetric line equation $(x-x_0)/a = (y-y_0)/b = (z-z_0)/c = t$ represents points (x_0+at, y_0+bt, z_0+ct), with direction vector (a,b,c).
- Two lines intersect only if one point belongs to both: their three coordinate equations must hold simultaneously for the same pair of parameters.
- Nonparallel lines without a common point are skew. Nonproportional direction vectors establish that the lines are not parallel, but do not by themselves exclude intersection.

Step 1: Write the first line parametrically Introduce a real parameter λ for the first line:

$$\frac{x-1}{3} = \frac{y+1}{2} = \frac{z-1}{5} = \lambda.$$

Its coordinates are therefore

$$x = 1+3\lambda, \qquad y = -1+2\lambda, \qquad z = 1+5\lambda.$$

Quick Tip to Solve: Solve two coordinate equations for the line parameters, then check the third; matching only two coordinates does not prove intersection in space.

Step 2: Write the second line and impose a common point Using an independent real parameter μ, the second line is

$$x = -2 + 4\mu, \qquad y = 1 + 3\mu, \qquad z = -1 - 2\mu.$$

If a common point exists, equating its corresponding coordinates gives all three necessary conditions:

$$\begin{aligned} 3\lambda - 4\mu &= -3, \\ 2\lambda - 3\mu &= 2, \\ 5\lambda + 2\mu &= -2. \end{aligned}$$

The third condition must be checked after solving the first two.

Step 3: Solve the first two conditions Multiply the first equation by 2 and the second by 3:

$$6\lambda - 8\mu = -6, \qquad 6\lambda - 9\mu = 6.$$

Subtracting the second from the first yields $\mu = -12$. Substitution in $2\lambda - 3\mu = 2$ gives $2\lambda + 36 = 2$, so $\lambda = -17$. These parameters are unique.

Step 4: Check the remaining coordinate For the first line the candidate has $z = 1 + 5(-17) = -84$, whereas the second line gives $z = -1 - 2(-12) = 23$. These unequal values contradict the third condition. Thus no pair of parameters satisfies all three equations simultaneously. Also, the direction vectors $(3, 2, 5)$ and $(4, 3, -2)$ are not proportional, since $3/4 \neq 2/3$.

Step 5: Conclusion The two lines have no common point and hence $\boxed{\text{do not intersect}}$. Since they are also nonparallel, they are skew lines.

Quick Tip to Solve: Solve two coordinate equations for the line parameters, then check the third; matching only two coordinates does not prove intersection in space.

OR

33(b). Find the angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$. [5 marks]

Solution:

Concept:

- The direction ratios of a line are proportional to the coefficients of its parameter in its parametric equations. In a symmetric equation, the coefficients multiplying the coordinates are not themselves the direction ratios.
- For direction vectors $\mathbf{u}$ and $\mathbf{v}$, the angle θ between the two undirected lines satisfies $\cos\theta = \dfrac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}||\mathbf{v}|}$. A zero scalar product of two nonzero direction vectors means that the lines are perpendicular.

Step 1: Find direction ratios of the first line Put $2x = 3y = -z = t$. Then

$$x = \frac{t}{2}, \qquad y = \frac{t}{3}, \qquad z = -t.$$

Quick Tip to Solve: First take reciprocals of the coordinate coefficients to obtain direction ratios, clear fractions, and test the scalar product: a zero result immediately gives a right angle.

Thus a direction vector is $\left(\frac{1}{2}, \frac{1}{3}, -1\right)$. Multiplying all three components by the same nonzero number, namely 6, gives the simpler direction vector $\mathbf{u} = (3, 2, -6)$. The negative sign in the third component comes from $-z = t$, so it must be retained.

Step 2: Find direction ratios of the second line Use an independent parameter s and put $6x = -y = -4z = s$. This gives

$$x = \frac{s}{6}, \qquad y = -s, \qquad z = -\frac{s}{4}.$$

The direction ratios are therefore $\frac{1}{6}, -1, -\frac{1}{4}$. Multiplying each component by 12, we obtain $\mathbf{v} = (2, -12, -3)$. Both the second and the third components are negative; using the coefficients $6, -1, -4$ directly would give a different, incorrect direction.

Step 3: Apply the scalar-product formula Compute the numerator and the two lengths separately:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= 3(2) + 2(-12) + (-6)(-3) \\ &= 6 - 24 + 18 = 0, \\ |\mathbf{u}| &= \sqrt{3^2 + 2^2 + (-6)^2} = 7, \\ |\mathbf{v}| &= \sqrt{2^2 + (-12)^2 + (-3)^2} = \sqrt{157}. \end{aligned}$$

Both lengths are positive, so division is valid. Hence

$$\cos\theta = \frac{|0|}{7\sqrt{157}} = 0.$$

The angle between undirected lines is chosen in $[0, \frac{\pi}{2}]$; the value in this interval whose cosine is zero is $\frac{\pi}{2}$.

Step 4: Conclusion The angle between the given lines is $\boxed{\theta = \frac{\pi}{2} = 90^\circ}$. Hence the lines are perpendicular.

Quick Tip to Solve: First take reciprocals of the coordinate coefficients to obtain direction ratios, clear fractions, and test the scalar product: a zero result immediately gives a right angle.

34(a). If N denotes the set of all natural numbers and R is the relation on $N \times N$ defined by $(a, b)\, R\, (c, d)$, if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation. [5 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation when it is reflexive, symmetric and transitive. Here the elements of the set are ordered pairs in $N \times N$, so each property must be proved for such pairs.
- All natural numbers in the question are positive. We may divide by their product and rewrite the defining equation as equality of a real-valued expression attached to each ordered pair.

Step 1: Rewrite the defining condition For any $(a, b), (c, d) \in N \times N$, divide the defining equation by

Quick Tip to Solve: Divide by the product of the four positive entries and group reciprocals by ordered pair. Equality of the same pair expression makes all three proofs direct.

the nonzero product $abcd$:

$$\frac{b+c}{bc} = \frac{a+d}{ad},$$
$$\frac{1}{c} + \frac{1}{b} = \frac{1}{d} + \frac{1}{a},$$
$$\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}.$$

Define $F(a,b) = \frac{1}{a} - \frac{1}{b}$. These reversible steps show that

$$(a,b)R(c,d) \iff F(a,b) = F(c,d).$$

We now check the three properties using this equivalent condition.

Step 2: Prove reflexivity Take any $(a,b) \in N \times N$. Clearly $F(a,b) = F(a,b)$, so $(a,b)R(a,b)$. In the original equation this follows by putting $c = a$ and $d = b$, since both sides then equal $ab(a+b)$. Thus every pair is related to itself, and R is reflexive.

Step 3: Prove symmetry Let $(a,b),(c,d) \in N \times N$, and suppose $(a,b)R(c,d)$. Then $F(a,b) = F(c,d)$. Reversing this equality gives $F(c,d) = F(a,b)$, so $(c,d)R(a,b)$. This holds for any two ordered pairs; therefore R is symmetric.

Step 4: Prove transitivity Take any three ordered pairs $(a,b),(c,d),(e,f) \in N \times N$. Suppose $(a,b)R(c,d)$ and $(c,d)R(e,f)$. These assumptions imply

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d} \quad \text{and} \quad \frac{1}{c} - \frac{1}{d} = \frac{1}{e} - \frac{1}{f}.$$

Consequently $\frac{1}{a} - \frac{1}{b} = \frac{1}{e} - \frac{1}{f}$. Rearranging and multiplying by the positive product $abef$, we get

$$\frac{1}{e} + \frac{1}{b} = \frac{1}{f} + \frac{1}{a} \implies af(b+e) = be(a+f).$$

This is the original defining condition for $(a,b)R(e,f)$. Thus the first pair is related to the third whenever both assumptions hold, and R is transitive.

Step 5: Conclusion The relation R is reflexive, symmetric and transitive. Hence it is an equivalence relation on $N \times N$.

Quick Tip to Solve: Divide by the product of the four positive entries and group reciprocals by ordered pair. Equality of the same pair expression makes all three proofs direct.

OR

34(b). Let $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \to \mathbb{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. Show that f is a one-one function. Also, check whether f is an onto function or not. [5 marks]

Solution:

Concept:

- A function is one-one if equal function values force equal inputs: $f(a) = f(b) \Rightarrow a = b$.
- A function is onto its codomain if every element of the codomain has a preimage in its domain. The range is the set of actual output values.
- For a rational function, cross-multiplication is valid only when both denominators are nonzero; the excluded input must also be checked when finding a preimage.

Step 1: Establish the domain restrictions Write $D = \mathbb{R} \setminus \{-4/3\}$. For every $x \in D$, $3x + 4 \neq 0$, so the given formula $f(x) = 4x/(3x + 4)$ defines a real output. Let $a, b \in D$; then both denominators used below are nonzero. We prove injectivity directly without requiring continuity or a derivative.

Step 2: Prove that equal outputs give equal inputs Suppose $f(a) = f(b)$. Cross-multiplying the equality gives

$$\begin{aligned} \frac{4a}{3a+4} &= \frac{4b}{3b+4}, \\ 4a(3b+4) &= 4b(3a+4), \\ 12ab + 16a &= 12ab + 16b, \\ 16a &= 16b, \qquad a = b. \end{aligned}$$

Thus no two distinct inputs in the given domain can have the same image.

Step 3: Test whether every real number is attained Let $y \in \mathbb{R}$ and seek an input for which $y = f(x)$. Rearranging gives

$$y(3x + 4) = 4x, \qquad (4 - 3y)x = 4y.$$

If $y = 4/3$, this equation would say $0 = 16/3$, which is impossible. Equivalently, substituting $4/3$ for $f(x)$ gives $12x = 12x + 16$. Hence this element of the specified codomain has no preimage, which already disproves surjectivity.

Step 4: Identify the entire range For any $y \neq 4/3$, take $x = 4y/(4 - 3y)$. This input is permitted because

$$x + \frac{4}{3} = \frac{16}{3(4-3y)} \neq 0.$$

Furthermore, $3x + 4 = 16/(4 - 3y)$, so direct substitution yields

$$f(x) = \frac{16y/(4-3y)}{16/(4-3y)} = y.$$

Thus every real value other than $4/3$ occurs, and the range is precisely $\mathbb{R} \setminus \{4/3\}$.

Step 5: Conclusion The given function is $\boxed{\text{one-one but not onto } \mathbb{R}}$. Its range is $\boxed{\mathbb{R} \setminus \{4/3\}}$, since the codomain element $4/3$ has no preimage.

Quick Tip to Solve: Solve $y = 4x/(3x + 4)$ for x; the value making $4 - 3y = 0$ identifies the missing output, while cross-multiplication proves injectivity.

35. Find the inverse of the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{pmatrix}$. Using the inverse, A^{-1}, solve the system of linear equations $x - y + 2z = 1$; $2y - 3z = 1$; $3x - 2y + 4z = 3$. [5 marks]

Solution:

Concept:

- A square matrix A is invertible when $|A| \neq 0$. Its inverse is $A^{-1} = \text{adj}(A)/|A|$.
- The cofactor $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the determinant remaining after deleting row i and column j. The adjugate is the transpose of the cofactor matrix.
- A linear system with coefficient matrix A, unknown column X, and constant column B is $AX = B$; for an invertible A, its unique solution is $X = A^{-1}B$.

Step 1: Evaluate the determinant For the given matrix, expansion along its first row gives

$$\begin{aligned} |A| &= 1\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} - (-1)\begin{vmatrix} 0 & -3 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix} \\ &= (8-6) + (0+9) + 2(0-6) \\ &= 2 + 9 - 12 = -1. \end{aligned}$$

Since the determinant is nonzero, the inverse exists and the associated system has exactly one solution.

Step 2: Compute the cofactors and adjugate Applying the alternating cofactor signs gives all nine cofactors:

$$\begin{aligned} &C_{11} = 8 - 6 = 2, && C_{12} = -(0+9) = -9, && C_{13} = 0 - 6 = -6, \\ &C_{21} = -(-4+4) = 0, && C_{22} = 4 - 6 = -2, && C_{23} = -(-2+3) = -1, \\ &C_{31} = 3 - 4 = -1, && C_{32} = -(-3-0) = 3, && C_{33} = 2 - 0 = 2. \end{aligned}$$

Therefore, the cofactor matrix and its transpose are

$$C = \begin{pmatrix} 2 & -9 & -6 \\ 0 & -2 & -1 \\ -1 & 3 & 2 \end{pmatrix}, \qquad \text{adj}(A) = \begin{pmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{pmatrix}.$$

Transposition is essential: the cofactor matrix itself is not the adjugate.

Step 3: Obtain and verify the inverse Dividing the adjugate by $|A| = -1$, we obtain

$$A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}.$$

Quick Tip to Solve: Since $|A| = -1$, negate the transposed cofactor matrix to obtain the inverse; then multiply it by $(1, 1, 3)^T$ in the printed equation order.

As a check, multiplication of the original matrix by this result gives

$$AA^{-1} = \begin{pmatrix} -2-9+12 & 0-2+2 & 1+3-4 \\ 18-18 & 4-3 & -6+6 \\ -6-18+24 & -4+4 & 3+6-8 \end{pmatrix} = I_3.$$

Hence both the signs and the transposition in the inverse are verified.

Step 4: Use the inverse to solve the system Keeping the unknowns in the order x, y, z, the given equations become

$$AX = B, \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}.$$

The second equation has coefficient zero for x, exactly as in the second row of A. Premultiplying by the inverse gives

$$X = A^{-1}B = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2+0+3 \\ 9+2-9 \\ 6+1-6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Substitution in the original equations confirms $1-2+2=1$, $2(2)-3(1)=1$, and $3(1)-2(2)+4(1)=3$.

Step 5: Conclusion $\boxed{A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}}$, and the unique solution is $\boxed{x=1,\ y=2,\ z=1}$.

Quick Tip to Solve: Since $|A| = -1$, negate the transposed cofactor matrix to obtain the inverse; then multiply it by $(1,1,3)^T$ in the printed equation order.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36(i). A building contractor undertakes a job to construct 4 flats on a plot along with parking area. Due to strike the probability of many construction workers not being present for the job is 0.65. The probability that many are not present and still the work gets completed on time is 0.35. The probability that work will be completed on time when all workers are present is 0.80.

Let: E_1: represent the event when many workers were not present for the job;

E_2: represent the event when all workers were present; and

E: represent completing the construction work on time.

Based on the above information, answer the following questions:

(i) What is the probability that all the workers are present for the job? [1 mark]

Solution:

Concept:

- For the two workforce cases in this problem, E_1 and E_2 form a partition: they are mutually exclusive and exhaustive, so $P(E_1) + P(E_2) = 1$.
- A joint probability $P(E \cap E_i)$ measures both events occurring; a conditional probability satisfies $P(E \cap E_i) = P(E_i)P(E \mid E_i)$.

Step 1: Identify the complementary workforce cases The event E_1 represents many workers being absent and has probability 0.65. In the case-study model, its complementary case E_2 represents all workers being present. The probabilities of these two cases add to unity.

Step 2: Apply the complement rule Therefore,

$$P(E_2) = 1 - P(E_1) = 1 - 0.65 = 0.35.$$

The number 0.80 concerns timely completion conditional on attendance; it is not the probability of attendance itself.

Step 3: Conclusion The probability that all workers are present is $\boxed{0.35}$.

Quick Tip to Solve: For attendance alone, use the complementary workforce probabilities before using any completion data.

36(ii). What is the probability that construction will be completed on time? [1 mark]

Solution:

Concept:

- For the two workforce cases in this problem, E_1 and E_2 form a partition: they are mutually exclusive and exhaustive, so $P(E_1) + P(E_2) = 1$.
- A joint probability $P(E \cap E_i)$ measures both events occurring; a conditional probability satisfies $P(E \cap E_i) = P(E_i)P(E \mid E_i)$.

Quick Tip to Solve: The phrase "not present and still completed" supplies a joint probability; add it directly to the other joint probability.

Step 1: Compute timely completion with full attendance First, $P(E_2) = 1 - 0.65 = 0.35$. The given 0.80 is $P(E \mid E_2)$, so

$$P(E \cap E_2) = P(E_2)P(E \mid E_2) = 0.35(0.80) = 0.28.$$

Step 2: Combine the disjoint completion cases Timely completion occurs either with many workers absent or with all workers present. These cases are disjoint. The first joint probability is already given as $P(E \cap E_1) = 0.35$; it must not be multiplied by 0.65 again. Thus

$$P(E) = P(E \cap E_1) + P(E \cap E_2) = 0.35 + 0.28 = 0.63.$$

Step 3: Conclusion The probability of completion on time is $\boxed{0.63}$.

Quick Tip to Solve: The phrase "not present and still completed" supplies a joint probability; add it directly to the other joint probability.

36(iii)(a). What is the probability that many workers are not present given that the construction work is completed on time? [2 marks]

Solution:

Concept:

- For the two workforce cases in this problem, E_1 and E_2 form a partition: they are mutually exclusive and exhaustive, so $P(E_1) + P(E_2) = 1$.
- A joint probability $P(E \cap E_i)$ measures both events occurring; a conditional probability satisfies $P(E \cap E_i) = P(E_i)P(E \mid E_i)$.
- For $P(E) > 0$, Bayes conditioning gives $P(E_1 \mid E) = \dfrac{P(E_1 \cap E)}{P(E)}$.

Step 1: Find the probability of timely completion The two workforce cases have probabilities $P(E_1) = 0.65$ and $P(E_2) = 1 - 0.65 = 0.35$. With all workers present, the completion probability is 0.80. Hence

$$P(E \cap E_2) = 0.35 \times 0.80 = 0.28.$$

Adding the given joint probability for absence gives

$$P(E) = 0.35 + 0.28 = 0.63.$$

Step 2: Condition the absence event on completion We now restrict the sample space to construction jobs completed on time. Of probability mass 0.63, the portion with many workers absent is 0.35. Therefore

$$P(E_1 \mid E) = \frac{P(E_1 \cap E)}{P(E)} = \frac{0.35}{0.63} = \frac{35}{63} = \frac{5}{9}.$$

This reverses the order of conditioning: it is not $P(E \mid E_1)$, which would instead use 0.65 as its denominator.

Quick Tip to Solve: For "absent given completed", divide the absence-and-completion joint probability by the total completion probability.

Step 3: Conclusion The required probability is $\boxed{\frac{5}{9}}$.

Quick Tip to Solve: For "absent given completed", divide the absence-and-completion joint probability by the total completion probability.

OR

36(iii)(b). What is the probability that all workers were present given that the construction job was completed on time? [2 marks]

Solution:

Concept:

- For the two workforce cases in this problem, E_1 and E_2 form a partition: they are mutually exclusive and exhaustive, so $P(E_1) + P(E_2) = 1$.
- A joint probability $P(E \cap E_i)$ measures both events occurring; a conditional probability satisfies $P(E \cap E_i) = P(E_i)P(E \mid E_i)$.
- For $P(E) > 0$, $P(E_2 \mid E) = \dfrac{P(E_2 \cap E)}{P(E)}$; the event after "given" determines the denominator.

Step 1: Calculate the relevant joint probability The probability of all workers being present is

$$P(E_2) = 1 - P(E_1) = 1 - 0.65 = 0.35.$$

Since $P(E \mid E_2) = 0.80$, the probability of full attendance together with timely completion is

$$P(E_2 \cap E) = P(E_2)P(E \mid E_2) = 0.35 \times 0.80 = 0.28.$$

Step 2: Normalize by all timely completions Completion can occur in either workforce case, so

$$P(E) = P(E \cap E_1) + P(E \cap E_2) = 0.35 + 0.28 = 0.63.$$

Therefore

$$P(E_2 \mid E) = \frac{0.28}{0.63} = \frac{28}{63} = \frac{4}{9}.$$

The given 0.80 has the reverse conditioning and is not the requested answer. As a check, the two posterior workforce probabilities add to $5/9 + 4/9 = 1$.

Step 3: Conclusion The probability that all workers were present, given timely completion, is $\boxed{\frac{4}{9}}$.

Quick Tip to Solve: Multiply full attendance by completion given attendance, then divide by the probability of all timely completions.

37(i). **Let $f(x)$ be a real valued function. Then its**

Left Hand Derivative (L.H.D.): $Lf'(a) = \lim\limits_{h\to 0} \dfrac{f(a-h)-f(a)}{-h}$

Right Hand Derivative (R.H.D.): $Rf'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

Also, a function $f(x)$ is said to be differentiable at $x = a$ if its L.H.D. and R.H.D. at $x = a$ exist and both are equal.

For the function $f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$

answer the following questions:

(i) What is R.H.D. of $f(x)$ at $x = 1$? [1 mark]

Solution:

Concept:

- The right-hand derivative at a is $f'_+(a) = \lim_{h\to 0^+} \dfrac{f(a+h)-f(a)}{h}$, where positive h approaches the point from the right.
- For $x < 3$, $|x-3| = 3-x$. Select the branch valid for nearby points before simplifying the difference quotient.

Step 1: Evaluate the function on the right of the join At $x = 1$, the branch $x \geq 1$ applies, giving $f(1) = |1-3| = 2$. For sufficiently small positive h, $1 \leq 1+h < 3$, so

$$f(1+h) = |1+h-3| = |h-2| = 2-h.$$

Step 2: Take the right-hand difference quotient Using the actual value $f(1) = 2$,

$$f'_+(1) = \lim_{h\to 0^+} \frac{(2-h)-2}{h} = \lim_{h\to 0^+} (-1) = -1.$$

The polynomial branch is used only for $x < 1$; it does not supply nearby right-hand values.

Step 3: Conclusion The R.H.D. of f at $x = 1$ is $\boxed{-1}$.

Quick Tip to Solve: Near 1 on the right, the absolute-value branch is still to the left of 3, so replace it by 3 minus x.

37(ii). **What is L.H.D. of $f(x)$ at $x = 1$?** [1 mark]

Solution:

Concept:

- The left-hand derivative is $Lf\prime(a) = \lim_{h\to 0^+} \dfrac{f(a-h)-f(a)}{-h}$.
- For $x < 1$, use the polynomial branch; the value $f(1)$ comes from the branch defined for $x \geq 1$.

Quick Tip to Solve: Use the polynomial at $1-h$, but use the stated upper branch to obtain $f(1)$.

Step 1: Identify the correct values At the joining point, $f(1) = |1-3| = 2$. For $h > 0$, $1-h < 1$, so

$$f(1-h) = \frac{(1-h)^2}{4} - \frac{3(1-h)}{2} + \frac{13}{4} = 2 + h + \frac{h^2}{4}.$$

Step 2: Evaluate the left difference quotient Subtract the actual value at the joining point before dividing by the negative increment:

$$Lf\prime(1) = \lim_{h\to 0^+} \frac{h + h^2/4}{-h} = \lim_{h\to 0^+} \left(-1 - \frac{h}{4}\right).$$

The negative denominator is essential because the point is approached from the left.

Step 3: Conclusion The left-hand derivative of f at $x = 1$ is $\boxed{-1}$.

Quick Tip to Solve: Use the polynomial at $1-h$, but use the stated upper branch to obtain $f(1)$.

37(iii)(a). Check if the function $f(x)$ is differentiable at $x = 1$. [2 marks]

Solution:

Concept:

- A function is differentiable at an interior point when its two one-sided derivatives exist and are equal.
- The polynomial branch applies on the left of 1. Near 1 on the right, $|x-3| = 3-x$, since $x < 3$.

Step 1: Compute the left-hand derivative The point value is $f(1) = 2$. For $h > 0$, the lower branch gives

$$f(1-h) = \frac{(1-h)^2}{4} - \frac{3(1-h)}{2} + \frac{13}{4} = 2 + h + \frac{h^2}{4}.$$

Hence

$$Lf\prime(1) = \lim_{h\to 0^+} \frac{f(1-h) - f(1)}{-h} = \lim_{h\to 0^+} \left(-1 - \frac{h}{4}\right) = -1.$$

Step 2: Compute the right-hand derivative and compare For sufficiently small $h > 0$, $1 + h < 3$, and the upper branch gives $f(1+h) = |1+h-3| = 2-h$. Thus

$$Rf\prime(1) = \lim_{h\to 0^+} \frac{(2-h) - 2}{h} = -1.$$

Both limits are finite and equal. Also, each branch approaches $2 = f(1)$, consistent with continuity at the joining point.

Step 3: Conclusion $Lf\prime(1) = Rf\prime(1) = -1$. Therefore f is differentiable at $x = 1$, with $\boxed{f\prime(1) = -1}$.

Quick Tip to Solve: Resolve the modulus locally: near $x = 1$, $|x-3|$ equals $3-x$, not $x-3$.

OR

37(iii)(b). Find $f'(2)$ and $f'(-1)$. [2 marks]

Solution:

Concept:

- At a point strictly inside a branch interval, differentiate the formula for that branch.
- The derivative of $|x-3|$ is -1 for $x<3$ and 1 for $x>3$. The modulus has its corner at 3, not at 2.

Step 1: Differentiate the branch containing 2 The point $x=2$ lies in the upper branch $x \geq 1$. Throughout a sufficiently small open interval around 2, $x<3$, so

$$f(x)=|x-3|=3-x, \qquad f\prime(x)=-1.$$

Thus the derivative can be evaluated using this linear expression; there is no joining point or modulus corner at 2.

Step 2: Differentiate the branch containing -1 The point $x=-1$ lies strictly in the interval $x<1$, where

$$f(x)=\frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4}.$$

By the power rule and the derivative of a constant,

$$f\prime(x)=\frac{x}{2}-\frac{3}{2}, \qquad f\prime(-1)=-\frac{1}{2}-\frac{3}{2}=-2.$$

Each differentiation uses a formula valid in a full neighbourhood of the specified point.

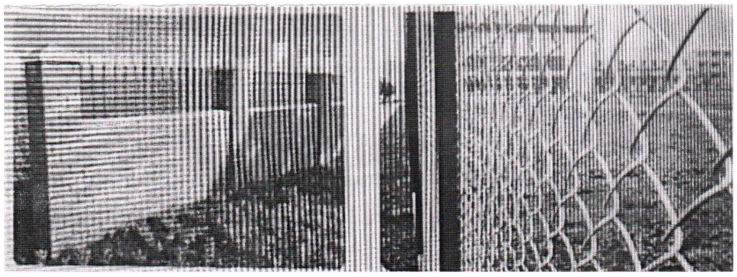
Step 3: Conclusion $\boxed{f\prime(2)=-1}$ and $\boxed{f\prime(-1)=-2}$.

Quick Tip to Solve: Choose the branch by the input value first, then remove the modulus using its local sign.

38(i). **Sooraj's father wants to construct a rectangular garden using a brick wall on one side of the garden and wire fencing for the other three sides as shown in the figure. He has 200 metres of fencing wire.**

Based on the above information, answer the following questions:

(i) Let x metres denote the length of the side of the garden perpendicular to the brick wall and y metres denote the length of the side parallel to the brick wall. Determine the relation representing the total length of fencing wire and also write $A(x)$, the area of the garden. [2 marks]



Solution:

Concept:

- The area of a rectangle with adjacent sides x and y is $A = xy$.
- The brick wall replaces one side of length y; the wire therefore covers two sides of length x and one of length y.

Step 1: Form the fencing equation The two sides perpendicular to the wall each have length x metres. The opposite side parallel to the wall has length y metres. As the entire available wire is used along these three sides,

$$x + x + y = 200, \qquad 2x + y = 200.$$

The wall side requires no wire and consequently is not included in this total.

Step 2: Express the area in one variable Solving the length constraint gives $y = 200 - 2x$. Substitute this into the rectangular area formula:

$$A(x) = x(200 - 2x) = 200x - 2x^2.$$

For a garden with positive side lengths, $x > 0$ and $200 - 2x > 0$, so the physically relevant domain is $0 < x < 100$. With x measured in metres, this expression gives the area in square metres.

Step 3: Conclusion The fencing relation is $\boxed{2x + y = 200}$, and $\boxed{A(x) = 200x - 2x^2 \text{ m}^2}$, for $0 < x < 100$.

Quick Tip to Solve: Count the two perpendicular sides separately; only one parallel side needs fencing wire.

38(ii). **Determine the maximum value of $A(x)$.** [2 marks]

Solution:

Concept:

- From the fencing constraint $2x + y = 200$, the area is $A(x) = 200x - 2x^2$, with $0 < x < 100$.

Quick Tip to Solve: For a three-sided fence, the optimal parallel side equals twice either perpendicular side.

- A stationary point satisfies $A\prime(x) = 0$. A negative second derivative establishes a strict local maximum; a downward quadratic also makes it the global maximum.

Step 1: Locate the stationary point Differentiate the area function with respect to the perpendicular side length:

$$A\prime(x) = 200 - 4x.$$

Set this derivative equal to zero:

$$200 - 4x = 0 \quad \Longrightarrow \quad x = 50.$$

This value lies in the admissible interval. The fencing constraint then gives $y = 200 - 2(50) = 100$ metres.

Step 2: Verify and evaluate the maximum The second derivative is $A\prime\prime(x) = -4 < 0$, so the stationary point is a maximum. In fact,

$$A(x) = 5000 - 2(x - 50)^2 \leq 5000,$$

which verifies that no admissible rectangle has a larger area. At $x = 50$,

$$A(50) = 200(50) - 2(50)^2 = 10000 - 5000 = 5000 \text{ m}^2.$$

Equality occurs with sides 50 metres and 100 metres.

Step 3: Conclusion The maximum garden area is $\boxed{5000 \text{ m}^2}$, attained when $x = 50$ m and $y = 100$ m.

Quick Tip to Solve: For a three-sided fence, the optimal parallel side equals twice either perpendicular side.