CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-4-3

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains **38** questions. **All** questions are **compulsory**.

(ii) This question paper is divided into **five** Sections – **A, B, C, D** and **E**.

(iii) In **Section A**, Questions no. **1 to 18** are multiple choice questions (MCQs) and questions number **19** and **20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B**, Questions no. **21 to 25** are very short answer (VSA) type questions, carrying **2** marks each.

(v) In **Section C**, Questions no. **26 to 31** are short answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D**, Questions no. **32 to 35** are long answer (LA) type questions carrying **5** marks each.

(vii) In **Section E**, Questions no. **36 to 38** are case study based questions carrying **4** marks each.

(viii) There is no overall choice. However, an internal choice has been provided in 2 questions in Section B, 3 questions in Section C, 2 questions in Section D and 2 questions in Section E.

(ix) Use of calculators **is not** allowed.

Section A (20 marks)

This section comprises multiple choice questions (MCQs) of 1 mark each.

1. If A is a 3×4 matrix and B is a matrix such that $A'B$ and AB' are both defined, then the order of the matrix B is: [1 mark]

(A) 3×4

(B) 3×3

(C) 4×4

(D) 4×3

Correct Answer: (A) 3×4

Solution:

Concept:

- The transpose interchanges rows and columns: a 3×4 matrix has transpose of order 4×3.
- A matrix product exists exactly when the number of columns of its first factor equals the number of rows of its second factor.

Step 1: Use the first product Let B have order $m \times n$. Since A' has order 4×3, the product $A'B$ is defined only when $m = 3$. It would then have order $4 \times n$.

Step 2: Use the second product The transpose B' has order $n \times m$. For AB', the four columns of A require $n = 4$. Thus both conditions must hold simultaneously. Option (B) fails the second product, option (C) fails the first, and option (D) fails both.

Step 3: Conclusion B has order 3×4. Hence, the correct option is (A).

Quick Tip to Solve: For two transpose-product conditions, write the unknown order as m by n before matching inner dimensions.

2. If the area of a triangle with vertices $(2, -6)$, $(5, 4)$ and $(k, 4)$ is 35 sq units, then k is [1 mark]

(A) 12

(B) -2

(C) $-12,\ -2$

(D) $12,\ -2$

Correct Answer: (D) $12,\ -2$

Solution:

Concept:

- The area of a triangle is $\frac{1}{2} \times$ base $\times$ perpendicular height.
- Lengths are non-negative, so a horizontal separation is the absolute difference of the x-coordinates.

Step 1: Identify the base and height The vertices $(5, 4)$ and $(k, 4)$ have the same y-coordinate, so their joining segment lies on $y = 4$. Its length is $|k - 5|$. The perpendicular distance of $(2, -6)$ from this

Quick Tip to Solve: When two vertices share a y-coordinate, use their horizontal separation as the base and retain both signs.

horizontal line is $|-6-4|=10$.

Step 2: Solve the absolute-value equation The given area yields

$$\frac{1}{2}|k-5|\times 10=35 \quad \Rightarrow \quad |k-5|=7.$$

Hence $k-5=7$ or $k-5=-7$. Both branches are necessary: options (A) and (B) give only one value each. The value -12 in option (C) gives area 85, so that option fails.

Step 3: Conclusion $k=12$ or $k=-2$. Hence, the correct option is (D).

Quick Tip to Solve: When two vertices share a y-coordinate, use their horizontal separation as the base and retain both signs.

3. If $f(x)=2|x|+3|\sin x|+6$, then the right hand derivative of $f(x)$ at $x=0$ is: [1 mark]

(A) 6

(B) 5

(C) 3

(D) 2

Correct Answer: (B) 5

Solution:

Concept:

- The right hand derivative is $f'_+(0)=\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}$.
- For sufficiently small positive h, both h and $\sin h$ are positive, and $\lim_{h\to 0}\sin h/h=1$.

Step 1: Remove absolute values on the right Here $f(0)=6$. When $h>0$ is close to zero, $|h|=h$ and $|\sin h|=\sin h$. Therefore $f(h)-f(0)=2h+3\sin h$. The constant term cancels; it cannot contribute a derivative of 6.

Step 2: Evaluate the limit It follows that

$$f'_+(0)=\lim_{h\to 0^+}\left(2+3\frac{\sin h}{h}\right)=2+3.$$

The value 3 omits the contribution from $2|x|$, while 2 omits the contribution from $3|\sin x|$. Taking the right side is essential because the absolute-value signs depend on the side of zero.

Step 3: Conclusion The right hand derivative is 5. Hence, the correct option is (B).

Quick Tip to Solve: For a right derivative at zero, remove absolute values using positive increments before taking the limit.

4. If $x\begin{pmatrix}1\\2\end{pmatrix}+y\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}4\\9\end{pmatrix}$**, then:** [1 mark]

(A) $x=1,\ y=2$

(B) $x=2,\ y=1$

(C) $x=1,\ y=-1$

(D) $x=3,\ y=2$

Correct Answer: (B) $x=2,\ y=1$

Solution:

Concept:

- Equal column matrices have equal corresponding entries.
- The resulting two linear equations can be solved by elimination; the pair must satisfy both entries.

Step 1: Compare corresponding entries From the first entry of the printed matrix equation, $x+2y=4$. From the second entry, $2x+5y=9$. Both equations are required because matrix equality applies entry by entry; satisfying just one row does not establish a solution.

Step 2: Eliminate and check the options Subtract twice the first equation from the second:

$$(2x+5y)-2(x+2y)=9-8,$$

giving $y=1$. Substitution into $x+2y=4$ gives $x=2$. Option (A) produces first entry 5, option (C) produces -1, and option (D) produces 7, so all three fail the required first entry 4.

Step 3: Conclusion $x=2,\ y=1$. Hence, the correct option is (B).

Quick Tip to Solve: Convert a column-matrix equality into one equation per row, then eliminate the variable with matched coefficients.

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5. If a matrix $A=\begin{pmatrix}1 & 2 & 3\end{pmatrix}$**, then the matrix** AA' **(where** A' **is the transpose of** A**) is:** [1 mark]

(A) 14

(B) $\begin{pmatrix}1&0&0\\0&2&0\\0&0&3\end{pmatrix}$

(C) $\begin{pmatrix}1&2&3\\2&3&1\\3&1&2\end{pmatrix}$

(D) $\begin{pmatrix}14\end{pmatrix}$

Correct Answer: (D) $\begin{pmatrix}14\end{pmatrix}$

Solution:

Concept:

- The transpose of a row matrix is a column matrix with the same entries.

Quick Tip to Solve: Check the product order before calculation; a row times its transpose gives a one-entry matrix.

- A 1×3 matrix multiplied by a 3×1 matrix produces a 1×1 matrix.

Step 1: Determine the order first For $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$, the transpose is $A' = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Hence AA' has one row and one column. The two 3×3 matrices in options (B) and (C) therefore cannot represent this product.

Step 2: Compute the sole entry Row-by-column multiplication gives

$$AA' = \begin{pmatrix} 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 \end{pmatrix} = \begin{pmatrix} 14 \end{pmatrix}.$$

The printed option (A) gives the scalar 14, whereas the question asks for the matrix. Option (D) explicitly gives the required one-entry matrix. Retaining its brackets distinguishes the matrix from its single entry.

Step 3: Conclusion $AA' = \begin{pmatrix} 14 \end{pmatrix}$. Hence, the correct option is (D).

Quick Tip to Solve: Check the product order before calculation; a row times its transpose gives a one-entry matrix.

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6. The product $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is equal to: [1 mark]

(A) $\begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$

(B) $\begin{pmatrix} (a+b)^2 & 0 \\ (a+b)^2 & 0 \end{pmatrix}$

(C) $\begin{pmatrix} a^2+b^2 & 0 \\ a^2+b^2 & 0 \end{pmatrix}$

(D) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

Correct Answer: (A) $\begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$

Solution:

Concept:

- Each entry of a matrix product is the dot product of a row of the first matrix and a column of the second.
- The scalar products satisfy $ab = ba$, which cancels the off-diagonal terms here.

Step 1: Compute the first row The upper-left entry is $a \cdot a + b \cdot b = a^2 + b^2$. The upper-right entry is $a(-b) + ba = -ab + ab = 0$. Thus the first row of the product is $\begin{pmatrix} a^2+b^2 & 0 \end{pmatrix}$.

Step 2: Compute the second row and compare The lower-left entry is $(-b)a + ab = 0$. The lower-right

Quick Tip to Solve: Compute one diagonal and one off-diagonal entry first; opposite signs cancel the mixed products.

entry is $(-b)(-b) + a^2 = a^2 + b^2$. Option (B) incorrectly includes a cross term through $(a+b)^2$ and has the wrong lower row. Option (C) places the nonzero term in the lower-left position. Option (D) has diagonal entries a, b, not the computed squared sum.

Step 3: Conclusion The product is $\begin{pmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{pmatrix}$. Hence, the correct option is (A).

Quick Tip to Solve: Compute one diagonal and one off-diagonal entry first; opposite signs cancel the mixed products.

7. Distance of the point (p, q, r) from y-axis is: [1 mark]

(A) q

(B) $|q|$

(C) $|q| + |r|$

(D) $\sqrt{p^2 + r^2}$

Correct Answer: (D) $\sqrt{p^2 + r^2}$

Solution:

Concept:

- A general point on the y-axis is $(0, t, 0)$, where t is any real number.
- The distance from a point to an axis is its smallest Euclidean distance to a point of that axis.

Step 1: Write the squared distance Let $P = (p, q, r)$ and $Q = (0, t, 0)$ be on the y-axis. The distance formula gives

$$PQ^2 = p^2 + (q - t)^2 + r^2.$$

The x-coordinate and z-coordinate measure displacement perpendicular to the y-axis; changing t can remove the displacement along that axis.

Step 2: Find the minimum and test alternatives Since $(q-t)^2 \geq 0$, the smallest squared distance occurs at $t = q$. The perpendicular foot is therefore $(0, q, 0)$. Options (A) and (B) depend on the coordinate along the y-axis instead of perpendicular displacement. Option (C) adds absolute coordinates and is not the Euclidean perpendicular length; it also uses the wrong coordinate q.

Step 3: Conclusion The distance is $\sqrt{p^2 + r^2}$. Hence, the correct option is (D).

Quick Tip to Solve: For distance to a coordinate axis, discard the coordinate along that axis and square the other two.

8. The solution set of the inequation $3x + 5y < 7$ is: [1 mark]

(A) whole xy-plane except the points lying on the line $3x + 5y = 7$.

(B) whole xy-plane along with the points lying on the line $3x + 5y = 7$.

(C) open half plane containing the origin except the points of line $3x + 5y = 7$.

(D) open half plane not containing the origin.

Correct Answer: (C) open half plane containing the origin except the points of line $3x + 5y = 7$.

Solution:

Concept:

- A linear equation $ax + by = c$ divides the plane into two half planes.
- A strict inequality excludes the boundary line; a test point identifies the required side.

Step 1: Identify the boundary The boundary of the solution set is $3x + 5y = 7$. Since the sign is strictly less than, no point on this line satisfies the given inequation. Thus the solution is an open half plane, not the whole plane.

Step 2: Test the origin and compare options At $(0, 0)$, $3(0) + 5(0) = 0 < 7$, so the required half plane contains the origin. Options (A) and (B) incorrectly include both sides of the line, while (D) selects the side that does not contain the origin.

Step 3: Conclusion The solution is the open half plane containing the origin, excluding the boundary line. Hence, the correct option is (C).

Quick Tip to Solve: For a strict linear inequality, exclude the boundary and test the origin whenever it is not on that boundary.

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9. If $\int_0^a 3x^2\,dx = 8$, then the value of $'a'$ is: [1 mark]

(A) 2

(B) 4

(C) 8

(D) 10

Correct Answer: (A) 2

Solution:

Concept:

- The power rule gives $\int 3x^2\,dx = x^3 + C$.
- A definite integral is evaluated as the antiderivative at the upper limit minus its value at the lower limit.

Step 1: Evaluate the definite integral Using the antiderivative of $3x^2$,

$$\int_0^a 3x^2\,dx = [x^3]_0^a = a^3 - 0 = a^3.$$

The lower-limit term vanishes, so no constant of integration enters this equation.

Quick Tip to Solve: Integrating the derivative $3x^2$ restores x^3; take the real cube root after applying the limits.

Step 2: Solve and test the choices The condition becomes $a^3 = 8$, which has the unique real solution $a = 2$. The other offered values $4, 8, 10$ give integrals $64, 512, 1000$, respectively, none of which equals 8.

Step 3: Conclusion $a = 2$. Hence, the correct option is (A).

Quick Tip to Solve: Integrating the derivative $3x^2$ restores x^3; take the real cube root after applying the limits.

10. The sine of the angle between the vectors $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + 2\hat{k}$ is: [1 mark]

(A) $\sqrt{\frac{5}{21}}$

(B) $\frac{5}{\sqrt{21}}$

(C) $\sqrt{\frac{3}{21}}$

(D) $\frac{4}{\sqrt{21}}$

Correct Answer: (A) $\sqrt{\frac{5}{21}}$

Solution:

Concept:

- For nonzero vectors, $\sin\theta = |\vec{a} \times \vec{b}|/(|\vec{a}||\vec{b}|)$, where $0 \le \theta \le \pi$.
- The magnitude of a vector (u, v, w) is $\sqrt{u^2 + v^2 + w^2}$.

Step 1: Find the cross product and magnitudes For $\vec{a} = (3, 1, 2)$ and $\vec{b} = (1, 1, 2)$,

$$\vec{a} \times \vec{b} = (0, -4, 2), \qquad |\vec{a} \times \vec{b}| = \sqrt{20}.$$

Also, $|\vec{a}| = \sqrt{14}$ and $|\vec{b}| = \sqrt{6}$.

Step 2: Evaluate the sine and check the options Therefore

$$\sin\theta = \frac{\sqrt{20}}{\sqrt{14}\sqrt{6}} = \sqrt{\frac{5}{21}}.$$

Options (B) and (D), namely $5/\sqrt{21}$ and $4/\sqrt{21}$, do not equal this value; (B) even exceeds one. Option (C) has squared value $3/21$, whereas the required squared sine is $5/21$.

Step 3: Conclusion $\sin\theta = \sqrt{5/21}$. Hence, the correct option is (A).

Quick Tip to Solve: Compute the squared sine as $|\vec{a} \times \vec{b}|^2/(|\vec{a}|^2|\vec{b}|^2)$ to simplify radicals quickly.

11. The order and degree (if defined) of the differential equation $\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^3 = x\sin\left(\frac{dy}{dx}\right)$ respectively are: [1 mark]

(A) $2,\ 2$

(B) $1,\ 3$

(C) $2,\ 3$

(D) 2, degree not defined

Correct Answer: (D) 2, degree not defined

Solution:

Concept:

- The order is the order of the highest derivative appearing in a differential equation.
- Degree is defined only when the differential equation is polynomial in its derivatives.

Step 1: Determine the order The equation contains dy/dx and d^2y/dx^2. The highest derivative is the second derivative, so the order is 2. The outer square on this derivative does not increase its order.

Step 2: Check whether the degree exists The expression $x\sin(dy/dx)$ contains a trigonometric function of a derivative. Consequently, the equation is not polynomial in its derivatives, and its degree is not defined. One cannot assign degree 2 from the square of the second derivative, or degree 3 from the cube of the first derivative. Option (B) also has an incorrect order.

Step 3: Conclusion The order is 2 and the degree is not defined. Hence, the correct option is (D).

Quick Tip to Solve: Before reading a derivative exponent as the degree, check for derivatives inside trigonometric or other non-polynomial functions.

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12. $\int e^{5\log x}\,dx$ is equal to: [1 mark]

(A) $\frac{x^5}{5} + C$

(B) $\frac{x^6}{6} + C$

(C) $5x^4 + C$

(D) $6x^5 + C$

Correct Answer: (B) $\frac{x^6}{6} + C$

Solution:

Concept:

- For $x > 0$, $e^{5\log x} = e^{\log(x^5)} = x^5$, with the natural logarithm.
- The power rule is $\int x^n\,dx = x^{n+1}/(n+1) + C$ for $n \neq -1$.

Step 1: Simplify the integrand The logarithm requires $x > 0$. Using $5\log x = \log(x^5)$ and the inverse relationship between the exponential and natural logarithm gives $e^{5\log x} = x^5$.

Quick Tip to Solve: First cancel the exponential with the natural logarithm, then increase the power by one and divide by that new power.

Step 2: Integrate and verify the choices Thus

$$\int e^{5\log x}\,dx = \int x^5\,dx = \frac{x^6}{6} + C.$$

Its derivative is x^5, as required. By contrast, derivatives of options (A), (C) and (D) are x^4, $20x^3$ and $30x^4$, respectively, so they do not reproduce the integrand.

Step 3: Conclusion The integral is $x^6/6 + C$, for $x > 0$. Hence, the correct option is (B).

Quick Tip to Solve: First cancel the exponential with the natural logarithm, then increase the power by one and divide by that new power.

13. A unit vector along the vector $4\hat{i} - 3\hat{k}$ is: [1 mark]

(A) $\frac{1}{7}(4\hat{i} - 3\hat{k})$

(B) $\frac{1}{5}(4\hat{i} - 3\hat{k})$

(C) $\frac{1}{\sqrt{7}}(4\hat{i} - 3\hat{k})$

(D) $\frac{1}{\sqrt{5}}(4\hat{i} - 3\hat{k})$

Correct Answer: (B) $\frac{1}{5}(4\hat{i} - 3\hat{k})$

Solution:

Concept:

- A unit vector along a nonzero vector $\vec{v}$ is $\vec{v}/|\vec{v}|$.
- Dividing by the positive magnitude preserves direction and makes the resulting magnitude one.

Step 1: Compute the magnitude For $\vec{v} = 4\hat{i} - 3\hat{k}$, the components are $(4, 0, -3)$. Hence

$$|\vec{v}| = \sqrt{4^2 + 0^2 + (-3)^2} = \sqrt{25} = 5.$$

The magnitude is not the sum of the absolute component values.

Step 2: Normalize and test the alternatives Dividing by 5 gives $\hat{v} = (4\hat{i} - 3\hat{k})/5$, whose magnitude is $5/5 = 1$. The alternatives (A), (C) and (D) have magnitudes $5/7$, $5/\sqrt{7}$ and $5/\sqrt{5}$, respectively, so none is a unit vector.

Step 3: Conclusion A unit vector along the given vector is $\frac{1}{5}(4\hat{i} - 3\hat{k})$. Hence, the correct option is (B).

Quick Tip to Solve: Use the 3-4-5 magnitude here, keeping the negative sign of the $\hat{k}$ component when normalizing.

14. Which of the following points satisfies both the inequations $2x + y \leq 10$ and $x + 2y \geq 8$? [1 mark]

(A) $(-2, 4)$

(B) $(3, 2)$

(C) $(-5, 6)$

(D) $(4, 2)$

Correct Answer: (D) $(4, 2)$

Solution:

Concept:

- A point satisfies two simultaneous inequalities only if both conditions hold at that point.
- For each offered (x, y), evaluate $2x + y$ and $x + 2y$ separately.

Step 1: Check the first three points For (A), $(-2, 4)$, the values are 0 and 6; the second inequality fails. For (B), $(3, 2)$, they are 8 and 7, so again the second fails. For (C), $(-5, 6)$, they are -4 and 7, so the second also fails. Each of these points satisfies only the first condition.

Step 2: Check the remaining point For (D), $(4, 2)$,

$$2x + y = 2(4) + 2 = 10 \leq 10, \qquad x + 2y = 4 + 2(2) = 8 \geq 8.$$

Both equalities are allowed because the inequalities are non-strict. Thus this point belongs to both required closed half planes.

Step 3: Conclusion The point $(4, 2)$ satisfies both inequations. Hence, the correct option is (D).

Quick Tip to Solve: Test the second inequality first here: it eliminates the first three options immediately; then verify the first for the remaining point.

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15. If $y = \sin^2(x^3)$, then $\frac{dy}{dx}$ is equal to: [1 mark]

(A) $2 \sin x^3 \cos x^3$

(B) $3x^3 \sin x^3 \cos x^3$

(C) $6x^2 \sin x^3 \cos x^3$

(D) $2x^2 \sin^2(x^3)$

Correct Answer: (C) $6x^2 \sin x^3 \cos x^3$

Solution:

Concept:

- The chain rule differentiates each successive layer: if $y = f(u)$ and $u = g(x)$, then $dy/dx = f\prime(u)g\prime(x)$.
- For a squared sine, $d(\sin^2 u)/du = 2 \sin u \cos u$.

Step 1: Identify the inner variable Put $u = x^3$. Then $y = (\sin u)^2$ and $du/dx = 3x^2$. Both the square and the argument of the sine depend on x, so differentiating only the outer square would omit a factor.

Quick Tip to Solve: Differentiate the three layers in order: square, sine, then the cubic argument.

Step 2: Apply the chain rule and compare

$$\frac{dy}{dx} = 2\sin u \cos u \frac{du}{dx} = 6x^2 \sin(x^3)\cos(x^3).$$

Option (A) omits $3x^2$; option (B) uses the wrong power and coefficient; option (D) incorrectly retains a squared sine instead of the sine-cosine product.

Step 3: Conclusion $\frac{dy}{dx} = 6x^2 \sin(x^3)\cos(x^3)$. Hence, the correct option is (C).

Quick Tip to Solve: Differentiate the three layers in order: square, sine, then the cubic argument.

16. The point $(x, y, 0)$ on the xy-plane divides the line segment joining the points $(1, 2, 3)$ and $(3, 2, 1)$ in the ratio: [1 mark]

(A) $1 : 2$ internally

(B) $2 : 1$ internally

(C) $3 : 1$ internally

(D) $3 : 1$ externally

Correct Answer: (D) $3 : 1$ externally

Solution:

Concept:

- A point on the xy-plane has third coordinate $z = 0$.
- For external division of AB in the positive ratio $m : n$, the point is $(mB - nA)/(m - n)$.

Step 1: Rule out internal division Let $A = (1, 2, 3)$ and $B = (3, 2, 1)$. Both endpoint heights are positive. An internal division in ratio $m : n$, with $m, n > 0$, has $z = (m + 3n)/(m + n) > 0$. Thus none of the internally dividing options can lie on the xy-plane.

Step 2: Determine the external ratio External division gives

$$0 = \frac{m(1) - n(3)}{m - n}, \qquad m = 3n.$$

Hence $AP : PB = m : n = 3 : 1$. As a check, the point is $P = (4, 2, 0)$, obtained from $(3B - A)/2$, beyond B.

Step 3: Conclusion The point divides the segment in the ratio $3 : 1$ externally. Hence, the correct option is (D).

Quick Tip to Solve: Use the zero third coordinate first; positive endpoint heights immediately exclude internal division.

17. The events E and F are independent. If $P(E) = 0.3$ and $P(E \cup F) = 0.5$, then $P(E/F) - P(F/E)$ equals: [1 mark]

(A) $\frac{1}{7}$

(B) $\frac{2}{7}$

(C) $\frac{3}{35}$

(D) $\frac{1}{70}$

Correct Answer: (D) $\frac{1}{70}$

Solution:

Concept:

- For independent events, $P(E \cap F) = P(E)P(F)$, and conditioning on either event leaves the probability of the other unchanged.
- The addition rule is $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Step 1: Find the probability of F Write $p = P(F)$. The addition rule and independence give

$$0.5 = 0.3 + p - 0.3p = 0.3 + 0.7p.$$

Therefore $p = 0.2/0.7 = 2/7$. Both conditioning probabilities are defined since $P(E)$ and $P(F)$ are positive.

Step 2: Evaluate the required difference Independence gives $P(E/F) = 3/10$ and $P(F/E) = 2/7$. Thus

$$P(E/F) - P(F/E) = \frac{3}{10} - \frac{2}{7} = \frac{21 - 20}{70} = \frac{1}{70}.$$

Option (B) is merely $P(F)$, while (A) and (C) do not equal this difference.

Step 3: Conclusion $P(E/F) - P(F/E) = \frac{1}{70}$. Hence, the correct option is (D).

Quick Tip to Solve: First find P(F) using the union; independence then removes both conditional-probability calculations.

18. The integrating factor for solving the differential equation $x\dfrac{dy}{dx} - y = 2x^2$ is: [1 mark]

(A) e^{-y}

(B) e^{-x}

(C) x

(D) $\dfrac{1}{x}$

Correct Answer: (D) $\dfrac{1}{x}$

Solution:

Concept:

- A linear equation in standard form $dy/dx + P(x)y = Q(x)$ has integrating factor $e^{\int P(x)\,dx}$.
- The coefficient of dy/dx must be one before identifying $P(x)$.

Step 1: Put the equation in standard form On an interval where $x \neq 0$, divide $x\,dy/dx - y = 2x^2$ by x:

$$\frac{dy}{dx} - \frac{1}{x}y = 2x.$$

Thus $P(x) = -1/x$, rather than the unnormalised coefficient -1.

Step 2: Calculate and check the factor

$$\text{I.F.} = e^{-\int dx/x} = e^{-\ln|x|} = \frac{1}{|x|}.$$

A nonzero constant multiple is also an integrating factor; on either fixed-sign interval we may choose $1/x$. Indeed, multiplying the standard equation by $1/x$ makes its left side $d(y/x)/dx$. Neither exponential option does this, and x has the wrong sign in its logarithmic derivative.

Step 3: Conclusion A suitable integrating factor is $\dfrac{1}{x}$, for $x \neq 0$. Hence, the correct option is (D).

Quick Tip to Solve: Always divide by the coefficient of dy/dx before integrating the coefficient of y.

19. Assertion (A): The lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ are perpendicular, when $\vec{b_1} \cdot \vec{b_2} = 0$.

Reason (R): The angle θ between the lines $\vec{r} = \vec{a_1} + \lambda\vec{b_1}$ and $\vec{r} = \vec{a_2} + \mu\vec{b_2}$ is given by $\cos\theta = \dfrac{\vec{b_1} \cdot \vec{b_2}}{|\vec{b_1}|\,|\vec{b_2}|}$.

[1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

Solution:

Concept:

- The nonzero vectors $\vec{b}_1$ and $\vec{b}_2$ specify the directions of the two lines.
- The dot-product identity is $\vec{b}_1 \cdot \vec{b}_2 = |\vec{b}_1|\,|\vec{b}_2|\cos\theta$.

Step 1: Judge the Assertion If $\vec{b}_1 \cdot \vec{b}_2 = 0$, the two nonzero direction vectors are orthogonal. The lines therefore have perpendicular directions, which is the angle criterion used for perpendicular lines here. Hence Assertion (A) is true.

Step 2: Judge the Reason Dividing the dot-product identity by the positive product of the vector magnitudes gives

$$\cos\theta = \frac{\vec{b}_1 \cdot \vec{b}_2}{|\vec{b}_1|\,|\vec{b}_2|}.$$

This is exactly the stated formula for the angle between the direction vectors, so Reason (R) is true.

Step 3: Check the explanation Substituting the zero dot product into the formula yields $\cos\theta = 0$, hence $\theta = \pi/2$. The Reason directly establishes the Assertion. Option (B) wrongly denies this explanatory link, and options (C) and (D) incorrectly declare one of the true statements false.

Step 4: Conclusion Both statements are true, and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

Quick Tip to Solve: Substitute the asserted dot product into the reason before deciding whether it explains the assertion.

20. Assertion (A): All trigonometric functions have their inverses over their respective domains.
Reason (R): The inverse of $\tan^{-1} x$ exists for some $x \in \mathbb{R}$. [1 mark]

(A) Both Assertion (A) and Reason (R) are true and Reason (R) is the correct explanation of the Assertion (A).

(B) Both Assertion (A) and Reason (R) are true, but Reason (R) is not the correct explanation of the Assertion (A).

(C) Assertion (A) is true and Reason (R) is false.

(D) Assertion (A) is false and Reason (R) is true.

Correct Answer: (D) Assertion (A) is false and Reason (R) is true.

Solution:

Concept:

- A function has an inverse function on its image only when it is one-one on the chosen domain.
- Trigonometric functions require suitable domain restrictions; $\tan^{-1}$ itself maps $\mathbb{R}$ one-one onto $(-\pi/2, \pi/2)$.

Step 1: Judge the Assertion On their full natural domains, trigonometric functions repeat values and are not one-one. For example, $\sin 0 = \sin \pi = 0$, although $0 \neq \pi$. Therefore they do not all possess inverse functions over their respective unrestricted domains. Assertion (A) is false.

Step 2: Judge the Reason The function $f(x) = \tan^{-1} x$ is strictly increasing on $\mathbb{R}$, with range $(-\pi/2, \pi/2)$. Its inverse is $f^{-1}(t) = \tan t$ on that range. In particular $\tan(\tan^{-1} x) = x$ for every real x, so the stated existence for some real x is true.

Step 3: Relate the statements The invertibility of the already restricted inverse-tangent function cannot establish invertibility of all trigonometric functions on their full domains. The truth pattern is false Assertion and true Reason; options (A), (B) and (C) all incorrectly call the Assertion true.

Step 4: Conclusion The assertion is false and the reason is true. Hence, the correct option is (D).

Quick Tip to Solve: Distinguish the full domain of a trigonometric function from the restricted domain used to define its inverse.

Section B (10 marks)

This section comprises very short answer (VSA) type questions of 2 marks each.

21. If $xy = e^{x-y}$, then show that $\dfrac{dy}{dx} = \dfrac{y(x-1)}{x(y+1)}$. [2 marks]

Solution:

Concept:

- Implicit differentiation treats y as a differentiable function of x. The product rule gives $d(xy)/dx = y + x\, dy/dx$.
- The chain rule gives $d(e^{x-y})/dx = e^{x-y}(1 - dy/dx)$.

Quick Tip to Solve: After differentiating, replace the exponential by xy before collecting all terms containing dy/dx.

Step 1: Differentiate both sides Put $y\prime = dy/dx$. Differentiating the given relation $xy = e^{x-y}$ with respect to x, using the product rule on the left and the chain rule on the right, gives

$$y + xy\prime = e^{x-y}(1 - y\prime).$$

The factor $1 - y\prime$ appears because both x and y vary with x.

Step 2: Substitute the original relation and collect terms Replace e^{x-y} by xy, as given in the question:

$$\begin{aligned} y + xy\prime &= xy - xyy\prime, \\ xy\prime + xyy\prime &= xy - y, \\ x(1+y)y\prime &= y(x-1). \end{aligned}$$

Thus division by $x(y+1)$ produces the required derivative wherever y is differentiable as a function of x. Since $xy = e^{x-y} > 0$, $x \neq 0$; the derivative formula applies where $y \neq -1$.

Step 3: Conclusion Hence $\dfrac{dy}{dx} = \dfrac{y(x-1)}{x(y+1)}$, as required, wherever $x(y+1) \neq 0$.

Quick Tip to Solve: After differentiating, replace the exponential by xy before collecting all terms containing dy/dx.

22(a). Find the domain of $y = \sin^{-1}(x^2 - 4)$. [2 marks]

Solution:

Concept:

- The real function $\sin^{-1} u$ is defined exactly when $-1 \leq u \leq 1$. Its argument, rather than its output, determines the domain.
- For a nonnegative number a, the condition $x^2 \geq a$ means $x \leq -\sqrt{a}$ or $x \geq \sqrt{a}$.

Step 1: Apply the restriction on the argument Here the argument of the inverse sine is $x^2 - 4$. Hence a real value of y exists precisely when

$$-1 \leq x^2 - 4 \leq 1.$$

Adding 4 to all three members gives

$$3 \leq x^2 \leq 5.$$

Both endpoints are allowed, because the inverse sine is defined at -1 and 1.

Step 2: Convert the square inequalities to intervals The upper bound gives $-\sqrt{5} \leq x \leq \sqrt{5}$, whereas the lower bound gives $x \leq -\sqrt{3}$ or $x \geq \sqrt{3}$. Taking their intersection leaves two closed intervals. Values between $-\sqrt{3}$ and $\sqrt{3}$ make the argument smaller than -1; values outside $[-\sqrt{5}, \sqrt{5}]$ make it greater than 1.

Step 3: Conclusion The domain is $\boxed{[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]}$.

Quick Tip to Solve: First bound the entire inverse-sine argument between -1 and 1, then split the square inequality into negative and positive intervals.

OR

22(b). Evaluate: $\cos^{-1}\left[\cos\left(-\frac{7\pi}{3}\right)\right]$. [2 marks]

Solution:

Concept:

- The principal value of $\cos^{-1} u$ lies in $[0, \pi]$; thus $\cos^{-1}(\cos\theta)$ need not equal θ.
- Cosine is even and has period 2π: $\cos(-\theta) = \cos\theta$ and $\cos(\theta + 2\pi) = \cos\theta$.

Step 1: Reduce the angle inside the cosine Adding one full revolution to the given angle does not change its cosine:

$$\cos\left(-\frac{7\pi}{3}\right) = \cos\left(-\frac{7\pi}{3} + 2\pi\right) = \cos\left(-\frac{\pi}{3}\right).$$

Since cosine is even, this is $\cos(\pi/3) = 1/2$. The negative sign in the original angle therefore does not produce a negative cosine.

Step 2: Choose the principal inverse-cosine value The requested expression is now $\cos^{-1}(1/2)$. We seek the angle in $[0, \pi]$ whose cosine is $1/2$. The angle $\pi/3$ belongs to that interval and has the required cosine. Angles such as $-\pi/3$ and $7\pi/3$ have the same cosine but lie outside the principal range, so they cannot be the value of the inverse cosine.

Step 3: Conclusion $\boxed{\cos^{-1}[\cos(-7\pi/3)] = \pi/3}$.

Quick Tip to Solve: Reduce the cosine angle modulo 2π, then select the equivalent angle in the principal range $[0, \pi]$.

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23. If the projection of the vector $\hat{i} + \hat{j} + \hat{k}$ on the vector $p\hat{i} + \hat{j} - 2\hat{k}$ is $\frac{1}{3}$, then find the value(s) of p. [2 marks]

Solution:

Concept:

- The scalar projection of $\vec{a}$ on a nonzero vector $\vec{b}$ is $\vec{a} \cdot \vec{b}/|\vec{b}|$. It retains the sign of the dot product.
- For $\vec{b} = p\hat{i} + \hat{j} - 2\hat{k}$, its magnitude is $\sqrt{p^2 + 5}$, which is positive for every real p.

Step 1: Form the projection equation Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = p\hat{i} + \hat{j} - 2\hat{k}$. Then $\vec{a} \cdot \vec{b} = p + 1 - 2 = p - 1$. The given projection yields

$$\frac{p-1}{\sqrt{p^2+5}} = \frac{1}{3}, \qquad 3(p-1) = \sqrt{p^2+5}.$$

The right side is positive, so a valid solution must satisfy $p > 1$.

Quick Tip to Solve: Before squaring a scalar-projection equation, record the sign of the numerator and check every root in the original equation.

Step 2: Solve and check the candidates Squaring gives

$$\begin{aligned} 9(p-1)^2 &= p^2+5, \\ 8p^2-18p+4 &= 0, \\ 4p^2-9p+2 &= 0, \\ (4p-1)(p-2) &= 0. \end{aligned}$$

Thus the candidates are $p = 1/4$ and $p = 2$. For $p = 2$, the original projection is $1/\sqrt{9} = 1/3$. For $p = 1/4$, it is $(-3/4)/(9/4) = -1/3$, which fails the given positive projection. This extraneous root was introduced by squaring.

Step 3: Conclusion The required value is $\boxed{p = 2}$.

Quick Tip to Solve: Before squaring a scalar-projection equation, record the sign of the numerator and check every root in the original equation.

24. Find the point on the curve $y^2 = 8x$ for which the abscissa and ordinate change at the same rate. [2 marks]

Solution:

Concept:

- If the coordinates depend on time t, implicit differentiation of $y^2 = 8x$ gives $2y\,dy/dt = 8\,dx/dt$.
- Equal rates of change at a moving point mean $dy/dt = dx/dt$, equivalently the curve has slope $dy/dx = 1$.

Step 1: Express the equal-rate condition Differentiate the equation of the curve with respect to time:

$$2y\frac{dy}{dt} = 8\frac{dx}{dt}.$$

At the required point, the abscissa x and ordinate y change at the same rate. Writing their common nonzero rate as v, the equation becomes $2yv = 8v$. Cancelling v gives $2y = 8$, so the ordinate is $y = 4$.

Step 2: Find the abscissa and verify the slope Substitute $y = 4$ into the original curve:

$$4^2 = 8x \quad \Longrightarrow \quad x = 2.$$

As a check, implicit differentiation with respect to x gives $dy/dx = 4/y$. At $(2, 4)$, this slope is 1, so an increase of the abscissa produces the same instantaneous increase of the ordinate. The negative ordinate -4 would instead give slope -1, corresponding to opposite rates.

Step 3: Conclusion The required point is $\boxed{(2, 4)}$.

Quick Tip to Solve: Equal rates of abscissa and ordinate give slope 1; impose this on the implicit derivative before finding the coordinates.

25(a). Find the vector equation of the line passing through the point $(2, 1, 3)$ and perpendicular to both the lines $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x}{-3} = \frac{y}{2} = \frac{z}{5}$. [2 marks]

Solution:

Concept:

- The denominators in a symmetric line equation give a direction vector. Here the two direction vectors are $\vec{a} = (1, 2, 3)$ and $\vec{b} = (-3, 2, 5)$.
- A vector perpendicular to two nonparallel vectors is their cross product. A line through position vector $\vec{r}_0$ with direction $\vec{d}$ has equation $\vec{r} = \vec{r}_0 + \lambda\vec{d}$, $\lambda \in \mathbb{R}$.

Step 1: Compute a perpendicular direction Use the cross product of the two given directions:

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -3 & 2 & 5 \end{vmatrix} \\ &= (10-6)\hat{i} - (5+9)\hat{j} + (2+6)\hat{k} \\ &= 4\hat{i} - 14\hat{j} + 8\hat{k}.\end{aligned}$$

Dividing by 2 gives the simpler nonzero direction $\vec{d} = 2\hat{i} - 7\hat{j} + 4\hat{k}$. Scaling a direction vector does not change the line.

Step 2: Use the given point and check perpendicularity The position vector of $(2, 1, 3)$ is $2\hat{i} + \hat{j} + 3\hat{k}$. Attach the direction $\vec{d}$ to this point. To verify the two perpendicular directions, calculate

$$\begin{aligned}\vec{d} \cdot \vec{a} &= 2 - 14 + 12 = 0, \\ \vec{d} \cdot \vec{b} &= -6 - 14 + 20 = 0.\end{aligned}$$

Thus this direction is orthogonal to each given line direction, and setting the line parameter to zero gives exactly the prescribed point.

Step 3: Conclusion The vector equation is $\boxed{\vec{r} = (2\hat{i} + \hat{j} + 3\hat{k}) + \lambda(2\hat{i} - 7\hat{j} + 4\hat{k}),\ \lambda \in \mathbb{R}}$.

Quick Tip to Solve: Take the cross product of the denominator vectors, simplify its common factor, and check both dot products are zero.

OR

25(b). The equations of a line are $5x - 3 = 15y + 7 = 3 - 10z$. Write the direction cosines of the line and find the coordinates of a point through which it passes. [2 marks]

Solution:

Concept:

- Direction ratios are the coefficients of the parameter in the parametric equations of a line. They describe its direction, rather than the coordinates of a point.

Quick Tip to Solve: Set the common expression equal to a parameter first; take the reciprocals of the coefficients, retaining the sign of the z-term.

- If direction ratios are a, b, c, the direction cosines are $(a, b, c)/\sqrt{a^2+b^2+c^2}$. Their squares add to 1.

Step 1: Introduce a common parameter Put $5x - 3 = 15y + 7 = 3 - 10z = t$. Solving the three equations separately gives

$$x = \frac{3}{5} + \frac{t}{5}, \qquad y = -\frac{7}{15} + \frac{t}{15}, \qquad z = \frac{3}{10} - \frac{t}{10}.$$

Thus a direction vector has components $1/5, 1/15, -1/10$. Multiplying all three by 30, the direction ratios are $6, 2, -3$. Notice the negative sign in the third ratio: increasing the parameter decreases z.

Step 2: Normalize and identify a point The length of the direction vector $(6, 2, -3)$ is

$$\sqrt{6^2 + 2^2 + (-3)^2} = \sqrt{49} = 7.$$

Hence its unit direction vector is $(6/7, 2/7, -3/7)$. Setting $t = 0$ in the parametric equations gives the point $(3/5, -7/15, 3/10)$. Substitution in the original equation makes all three expressions zero, verifying that this point lies on the line. Reversing all three direction cosines represents the same line with the opposite orientation.

Step 3: Conclusion The direction cosines are $\boxed{6/7,\ 2/7,\ -3/7}$, and one point on the line is $\boxed{(3/5, -7/15, 3/10)}$.

Quick Tip to Solve: Set the common expression equal to a parameter first; take the reciprocals of the coefficients, retaining the sign of the z-term.

Section C (18 marks)

This section comprises short answer (SA) type questions of 3 marks each.

26. Find: $\displaystyle\int \frac{2}{(1-x)(1+x^2)}\,dx$. [3 marks]

Solution:

Concept:

- For a linear factor and an irreducible quadratic factor, use partial fractions of the form $A/(1-x) + (Bx+C)/(1+x^2)$.
- The basic antiderivatives are $\int dx/(1-x) = -\log|1-x|$, $\int 2x\,dx/(1+x^2) = \log(1+x^2)$, and $\int dx/(1+x^2) = \tan^{-1} x$.

Step 1: Determine the partial fractions Write

$$\frac{2}{(1-x)(1+x^2)} = \frac{A}{1-x} + \frac{Bx+C}{1+x^2}.$$

Quick Tip to Solve: Use a linear numerator over the quadratic factor; after decomposition, separate its x-term from its constant term.

Multiplying by $(1-x)(1+x^2)$ yields

$$2 = A(1+x^2) + (Bx+C)(1-x) = (A+C) + (B-C)x + (A-B)x^2.$$

Comparing constant, linear and quadratic coefficients gives $A+C=2$, $B-C=0$, and $A-B=0$. Therefore $A=B=C=1$, so the integrand becomes $1/(1-x) + (x+1)/(1+x^2)$.

Step 2: Integrate the separated terms The integral is therefore

$$\begin{aligned} I &= \int \frac{dx}{1-x} + \int \frac{x\,dx}{1+x^2} + \int \frac{dx}{1+x^2} \\ &= -\log|1-x| + \frac{1}{2}\log(1+x^2) + \tan^{-1} x + C. \end{aligned}$$

The factor $1/2$ in the second term is needed because the derivative of $1+x^2$ is $2x$. The negative sign in the first term follows from the derivative of $1-x$.

Step 3: Check the antiderivative Differentiating the expression gives $1/(1-x)+x/(1+x^2)+1/(1+x^2)$. Combining these fractions gives numerator $(1+x^2)+(x+1)(1-x)=2$, exactly the original integrand. The antiderivative applies on any interval that does not cross $x=1$, where the integrand is undefined.

Step 4: Conclusion

$$\boxed{\int \frac{2\,dx}{(1-x)(1+x^2)} = -\log|1-x| + \frac{1}{2}\log(1+x^2) + \tan^{-1} x + C.}$$

Quick Tip to Solve: Use a linear numerator over the quadratic factor; after decomposition, separate its x-term from its constant term.

27(a). Evaluate: $\displaystyle\int_{1/3}^{1} \frac{(x-x^3)^{1/3}}{x^4}\,dx.$ [3 marks]

Solution:

Concept:

- On the interval $1/3 \le x \le 1$, both x and $1-x^2$ are nonnegative, so the real cube root can be factored safely.
- Choose a substitution whose derivative accounts for the remaining power of x: $u = x^{-2} - 1$ gives $du = -2x^{-3}\,dx$.

Step 1: Rewrite to reveal the substitution Let $I = \int_{1/3}^{1} (x-x^3)^{1/3} x^{-4}\,dx$. Since $x - x^3 = x^3(x^{-2}-1)$ and $x > 0$,

$$(x-x^3)^{1/3} = x(x^{-2}-1)^{1/3}.$$

Therefore

$$I = \int_{1/3}^{1} (x^{-2}-1)^{1/3} x^{-3}\,dx.$$

Quick Tip to Solve: Factor the cube-root argument as x cubed times a bracket; the substitution x to the power minus two then absorbs the denominator.

This rewriting is exact and turns the remaining factor $x^{-3}\,dx$ into a constant multiple of the differential of $x^{-2}-1$.

Step 2: Change the variable and limits Put $u = x^{-2}-1$, so $du = -2x^{-3}\,dx$. At $x = 1/3$, $u = 9-1 = 8$; at $x = 1$, $u = 0$. Consequently,

$$I = -\frac{1}{2}\int_{8}^{0} u^{1/3}\,du = \frac{1}{2}\int_{0}^{8} u^{1/3}\,du.$$

Reversing the limits removes the minus sign. This also agrees with the nonnegative original integrand, so the resulting integral must be nonnegative.

Step 3: Evaluate the power integral Using the power rule,

$$I = \frac{1}{2}\left[\frac{u^{4/3}}{4/3}\right]_{0}^{8} = \frac{3}{8}\left(8^{4/3}-0\right).$$

Here $8^{1/3} = 2$, and thus $8^{4/3} = 2^4 = 16$. Equivalently, an antiderivative in the original variable is $-\frac{3}{8}(x^{-2}-1)^{4/3}$; differentiating it gives $x^{-3}(x^{-2}-1)^{1/3}$, which verifies the substitution.

Step 4: Conclusion The value of the definite integral is $\boxed{6}$.

Quick Tip to Solve: Factor the cube-root argument as x cubed times a bracket; the substitution x to the power minus two then absorbs the denominator.

OR

27(b). Evaluate: $\displaystyle\int_{1}^{3}\left(|x-1|+|x-2|\right)dx$. [3 marks]

Solution:

Concept:

- For real t, $|t| = t$ if $t \geq 0$, and $|t| = -t$ if $t < 0$. Split the interval at a zero where a modulus changes sign.
- Definite integration is additive: $\int_1^3 f(x)\,dx = \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx$.

Step 1: Remove each modulus on the correct interval Throughout $[1,3]$, $x-1 \geq 0$, so $|x-1| = x-1$. The expression $x-2$ is negative for $1 \leq x < 2$ and nonnegative for $2 \leq x \leq 3$. Thus

$$|x-1|+|x-2| = \begin{cases} (x-1)+(2-x) = 1, & 1 \leq x \leq 2, \\ (x-1)+(x-2) = 2x-3, & 2 \leq x \leq 3. \end{cases}$$

Both formulas give 1 at $x = 2$, so the split is consistent at the common endpoint.

Quick Tip to Solve: Locate the zeros inside the integration interval before removing modulus signs; only x equals two requires an interior split here.

Step 2: Integrate over the two subintervals Hence

$$\begin{aligned} I &= \int_1^2 1\,dx + \int_2^3 (2x-3)\,dx \\ &= [x]_1^2 + [x^2 - 3x]_2^3 \\ &= (2-1) + \big((9-9) - (4-6)\big) \\ &= 1 + 2. \end{aligned}$$

In the first part the two distances add to the constant distance between 1 and 2. In the second part both expressions inside the moduli are nonnegative, and their sum increases linearly.

Step 3: Verify by integrating the moduli separately As a check, $\int_1^3 |x-1|\,dx = [(x-1)^2/2]_1^3 = 2$. Also,

$$\int_1^3 |x-2|\,dx = \int_1^2 (2-x)\,dx + \int_2^3 (x-2)\,dx = \frac{1}{2} + \frac{1}{2} = 1.$$

The separate contributions agree with the result of the interval split. Omitting that split would incorrectly treat $x-2$ as positive on $[1,2]$.

Step 4: Conclusion The required definite integral is $\boxed{3}$.

Quick Tip to Solve: Locate the zeros inside the integration interval before removing modulus signs; only x equals two requires an interior split here.

28. Solve the following linear programming problem graphically:

Maximise $z = 5x + 3y$

subject to the constraints $3x + 5y \le 15,\ 5x + 2y \le 10,\ x, y \ge 0$**.** [3 marks]

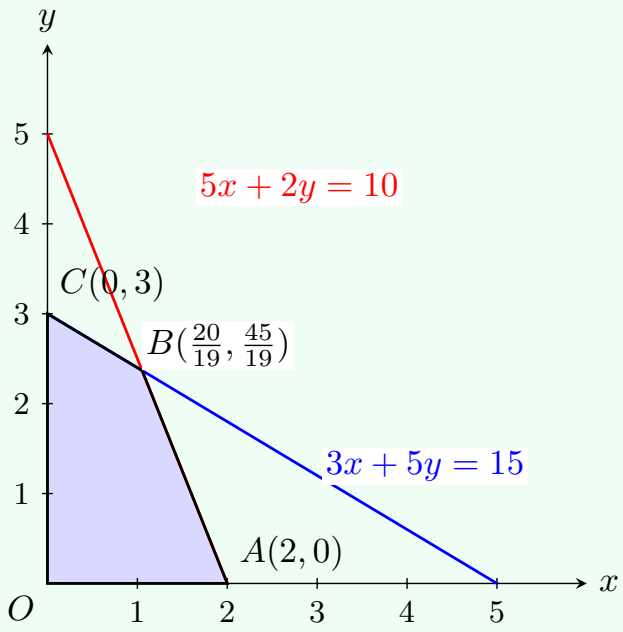
Solution:

Concept:

- The feasible region is the common region satisfying all four constraints. Nonnegativity restricts it to the first quadrant.
- A linear objective function attains its maximum over a nonempty bounded polygonal region at a corner point. Here $z = 5x + 3y$.

Step 1: Draw the boundary lines and identify the feasible region The line $3x+5y = 15$ passes through $(5,0)$ and $(0,3)$; the line $5x + 2y = 10$ passes through $(2,0)$ and $(0,5)$. Substituting the origin into each inequality gives $0 \le 15$ and $0 \le 10$, respectively. Hence select the half-plane containing the origin for each line. Their intersection in the first quadrant is the shaded bounded quadrilateral $OABC$, with $O = (0,0)$, $A = (2,0)$, $C = (0,3)$, and B the intersection of the two boundary lines.

Quick Tip to Solve: Test (0,0) to choose the half-planes, then evaluate every feasible vertex; the maximum here occurs at the intersection of the two lines.



Shaded quadrilateral OABC: the feasible region.

Step 2: Find the intersection vertex Solving the two equations simultaneously gives

$$\begin{aligned} 3x + 5y = 15, \quad & 5x + 2y = 10, \\ 6x + 10y = 30, \quad & 25x + 10y = 50, \\ 19x = 20, \quad & x = \frac{20}{19}. \end{aligned}$$

Substitute into the second equation: $2y = 10 - 100/19 = 90/19$. Thus $y = 45/19$, and $B = (20/19, 45/19)$. This point satisfies both constraints with equality and lies in the first quadrant.

Step 3: Compare the objective values at all corners Apply the corner-point method:

Corner	$z = 5x + 3y$
$O = (0, 0)$	0
$A = (2, 0)$	10
$B = (20/19, 45/19)$	$100/19 + 135/19 = 235/19$
$C = (0, 3)$	9

The region is bounded, so the largest entry is a finite maximum. Since $235/19 > 10 > 9 > 0$, the intersection vertex gives the largest objective value.

Step 4: Conclusion The maximum is $\boxed{z_{\max} = 235/19}$ at $\boxed{(x, y) = (20/19, 45/19)}$.

Quick Tip to Solve: Test (0,0) to choose the half-planes, then evaluate every feasible vertex; the maximum here occurs at the intersection of the two lines.

29. From a lot of 30 bulbs which include 6 defective bulbs, a sample of 2 bulbs is drawn at random one by one with replacement. Find the probability distribution of the number of defective bulbs and hence find the mean number of defective bulbs. [3 marks]

Solution:

Concept:

- A random variable assigns a number to each outcome. Let X denote the number of defective bulbs in the two draws; its possible values are $0, 1, 2$.
- Replacement keeps the composition of the lot unchanged, so the two draws are independent. The mean is $E(X) = \sum_x xP(X = x)$.

Step 1: Find the probabilities on each draw There are 6 defective and $30 - 6 = 24$ good bulbs. Write D for a defective bulb and G for a good bulb. On each draw,

$$P(D) = \frac{6}{30} = \frac{1}{5}, \qquad P(G) = \frac{24}{30} = \frac{4}{5}.$$

Because the first bulb is replaced before drawing the second, these probabilities apply to both draws, regardless of the first outcome. Thus the probability of any ordered pair is the product of its two draw probabilities.

Step 2: Construct the probability distribution For $X = 0$, both bulbs must be good: $P(X = 0) = P(GG) = (4/5)^2 = 16/25$. For $X = 1$, either DG or GD occurs. These outcomes are disjoint, so $P(X = 1) = (1/5)(4/5) + (4/5)(1/5) = 8/25$. For $X = 2$, both bulbs are defective: $P(X = 2) = P(DD) = (1/5)^2 = 1/25$. Therefore,

x	0	1	2
$P(X = x)$	$16/25$	$8/25$	$1/25$

All probabilities are nonnegative and their sum is $(16 + 8 + 1)/25 = 1$, as required.

Step 3: Calculate the mean number of defective bulbs Multiply each possible value by its probability and add:

$$E(X) = 0\left(\frac{16}{25}\right) + 1\left(\frac{8}{25}\right) + 2\left(\frac{1}{25}\right) = \frac{10}{25} = \frac{2}{5}.$$

The mean describes the average defective count over repeated samples of two bulbs; it need not itself be one of the integer values observed in an individual sample.

Step 4: Conclusion The distribution is $\boxed{P(X = 0) = 16/25,\ P(X = 1) = 8/25,\ P(X = 2) = 1/25}$. The mean number of defective bulbs is $\boxed{2/5 = 0.4}$.

Quick Tip to Solve: With replacement, keep the defective probability at 1/5 for both draws; count both DG and GD when calculating exactly one defective bulb.

30(a). Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$, $y(1) = 0$. [3 marks]

Solution:

Concept:

- A homogeneous first-order equation has its right side as a function of y/x. Use $y = vx$, where v is a function of x, so $dy/dx = v + x\,dv/dx$.
- A particular solution is obtained by determining the integration constant from the given initial condition. Since the equation is undefined at $x = 0$, use the interval $x > 0$ containing the initial point $x = 1$.

Step 1: Apply the homogeneous substitution Rewrite the given equation as

$$\frac{dy}{dx} = 1 + \frac{y}{x}.$$

Set $y = vx$, so $y/x = v$. Differentiating the product carefully gives $dy/dx = v + x\,dv/dx$. Substitute this into the equation:

$$v + x\frac{dv}{dx} = 1 + v \quad \Longrightarrow \quad x\frac{dv}{dx} = 1.$$

The terms v cancel; no division by v is needed, so no possible zero value of v is lost.

Step 2: Integrate and use the initial condition Separate the variables and integrate:

$$dv = \frac{dx}{x}, \qquad v = \ln|x| + C.$$

Since $v = y/x$, the general relation is $y = x(\ln|x| + C)$. The initial condition $y(1) = 0$ gives $0 = 1(\ln 1 + C) = C$. The interval containing 1 is $x > 0$, on which $\ln|x| = \ln x$, yielding the required particular relation.

Step 3: Check the equation and the initial value For $y = x\ln x$, differentiation gives $dy/dx = \ln x + 1$. The right side of the original equation becomes $(x + y)/x = 1 + y/x = 1 + \ln x$, exactly the same expression. Also $y(1) = 1\ln 1 = 0$. Hence both the differential equation and the stated initial condition are satisfied on $x > 0$.

Step 4: Conclusion The particular solution is $\boxed{y = x\ln x, \quad x > 0}$.

Quick Tip to Solve: Set y=vx and cancel v before integrating; then apply y(1)=0, which immediately makes the integration constant zero.

OR

30(b). Find the general solution of the differential equation $e^x \tan y\,dx + (1 - e^x)\sec^2 y\,dy = 0$. [3 marks]

Solution:

Concept:

- Use the substitution $u = \tan y$, so that $du = \sec^2 y\,dy$. It converts the given equation into a differential equation involving only x and u.

Quick Tip to Solve: Recognize $\sec^2 y\,dy$ as $d(\tan y)$, and retain the zero solution when separating variables.

- For a separable differential equation, collect the dependent variable with its differential and integrate both sides. An arbitrary constant represents the family of solutions.

Step 1: Reduce and separate the equation The printed equation is

$$e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0.$$

Putting $u = \tan y$ gives

$$(1 - e^x) \, du + e^x u \, dx = 0.$$

For $u \neq 0$ and $x \neq 0$, separation yields

$$\frac{du}{u} = -\frac{e^x}{1 - e^x} \, dx.$$

The minus sign is essential: the derivative of $1 - e^x$ is $-e^x$.

Step 2: Integrate and restore the dependent variable Set $v = 1 - e^x$, with $dv = -e^x \, dx$. Thus

$$\int \frac{du}{u} = \int \frac{dv}{v}, \qquad \log |u| = \log |1 - e^x| + K.$$

On an interval where the quantities have fixed signs, exponentiation and absorption of the sign into an arbitrary real constant give

$$u = C(1 - e^x).$$

The case $u = 0$, temporarily excluded during separation, is also a solution and is included by taking $C = 0$. Substituting $u = \tan y$ restores the original variable.

Step 3: Check the family in the original equation Differentiation of $\tan y = C(1 - e^x)$ gives $\sec^2 y \, dy = -Ce^x \, dx$. Therefore the original left side is

$$e^x C(1 - e^x) \, dx - (1 - e^x) C e^x \, dx = 0.$$

This verifies the family on every branch where $\tan y$ is defined; the final relation also remains meaningful at $x = 0$.

Step 4: Conclusion The general solution is $\boxed{\tan y = C(1 - e^x)}$, where C is an arbitrary real constant.

Quick Tip to Solve: Recognize $\sec^2 y \, dy$ as $d(\tan y)$, and retain the zero solution when separating variables.

31(a). Evaluate: $\displaystyle\int_{\pi/4}^{\pi/2} e^{2x} \left(\frac{1 - \sin 2x}{1 - \cos 2x} \right) dx$. [3 marks]

Solution:

Concept:

- The double-angle identities $1 - \cos 2x = 2 \sin^2 x$ and $\sin 2x = 2 \sin x \cos x$ simplify the trigonometric quotient.

Quick Tip to Solve: After double-angle simplification, compare the integrand with the product-rule derivative of $e^{2x} \cot x$.

- The product rule gives $\frac{d}{dx}(e^{2x}\cot x) = e^{2x}(2\cot x - \csc^2 x)$. An integrand matching this derivative has an immediate antiderivative.

Step 1: Simplify the trigonometric factor Let the required integral be I. Throughout $[\pi/4, \pi/2]$, $\sin x \neq 0$, so the denominator is nonzero and the following algebra is valid:

$$\begin{aligned}\frac{1-\sin 2x}{1-\cos 2x} &= \frac{1-2\sin x\cos x}{2\sin^2 x}\\ &= \frac{1}{2}\csc^2 x - \cot x.\end{aligned}$$

Consequently,

$$I = \int_{\pi/4}^{\pi/2} e^{2x}\left(\frac{1}{2}\csc^2 x - \cot x\right)dx.$$

Keep both terms together: their combination is suited to the derivative of an exponential times a cotangent.

Step 2: Recognize the exact derivative Using the product rule and $\frac{d}{dx}\cot x = -\csc^2 x$, we obtain

$$\begin{aligned}\frac{d}{dx}\left(-\frac{1}{2}e^{2x}\cot x\right) &= -e^{2x}\cot x + \frac{1}{2}e^{2x}\csc^2 x\\ &= e^{2x}\left(\frac{1}{2}\csc^2 x - \cot x\right).\end{aligned}$$

This reproduces the entire transformed integrand, including its signs and factor of one half. Hence no integration by parts or extra substitution is needed.

Step 3: Apply the definite limits The fundamental theorem of calculus now gives

$$\begin{aligned}I &= \left[-\frac{1}{2}e^{2x}\cot x\right]_{\pi/4}^{\pi/2}\\ &= -\frac{1}{2}e^{\pi}\cot\frac{\pi}{2} + \frac{1}{2}e^{\pi/2}\cot\frac{\pi}{4}.\end{aligned}$$

Use $\cot(\pi/2) = 0$ and $\cot(\pi/4) = 1$. The upper-limit term vanishes, while subtracting the negative lower-limit term leaves a positive value, consistent with the nonnegative original integrand.

Step 4: Conclusion Therefore, $\boxed{I = \frac{1}{2}e^{\pi/2}}$.

Quick Tip to Solve: After double-angle simplification, compare the integrand with the product-rule derivative of $e^{2x}\cot x$.

OR

31(b). Evaluate: $\displaystyle\int_{-2}^{2} \frac{x^2}{1+5^x}\,dx$. [3 marks]

Solution:

Concept:

- On a symmetric interval, substitution $x = -t$ gives $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} f(-x)\,dx$.
- The identities $5^{-x} = 1/5^x$ and $\frac{1}{1+5^x} + \frac{1}{1+5^{-x}} = 1$ allow the reflected integrands to be added.

Step 1: Reflect the integral about zero Write

$$I = \int_{-2}^{2} \frac{x^2}{1+5^x}\,dx.$$

The denominator is positive for every real x, so the integrand is continuous throughout the interval. Substitute $x = -t$. Then $dx = -dt$, and the limits change from $-2, 2$ to $2, -2$, respectively:

$$I = \int_{2}^{-2} \frac{t^2}{1+5^{-t}}(-dt) = \int_{-2}^{2} \frac{t^2}{1+5^{-t}}\,dt.$$

Rename the dummy variable t as x. This equality uses both the reversal of limits and the negative differential; missing either would give a wrong sign.

Step 2: Add the two equal representations Since $5^{-x} = 1/5^x$, the reflected representation becomes

$$I = \int_{-2}^{2} \frac{x^2 5^x}{1+5^x}\,dx.$$

Adding this to the original expression yields

$$\begin{aligned} 2I &= \int_{-2}^{2} \left(\frac{x^2}{1+5^x} + \frac{x^2 5^x}{1+5^x} \right) dx \\ &= \int_{-2}^{2} x^2\,dx. \end{aligned}$$

The factor x^2 is unchanged by reflection because it is even. The original integrand itself is not even, so symmetry should be used through this addition rather than by doubling its integral over $[0, 2]$.

Step 3: Evaluate the polynomial integral Now the exponential terms have cancelled:

$$2I = \left[\frac{x^3}{3} \right]_{-2}^{2} = \frac{8}{3} - \left(-\frac{8}{3} \right) = \frac{16}{3}.$$

Dividing this equality by two gives the value of the original integral. The result is positive, as required because the numerator is nonnegative and the denominator is positive throughout the interval.

Step 4: Conclusion The required integral is $\boxed{\frac{8}{3}}$.

Quick Tip to Solve: For an even numerator over symmetric limits, pair the denominators $1 + a^x$ and $1 + a^{-x}$; their reciprocal sum is one.

Section D (20 marks)

This section comprises long answer (LA) type questions of 5 marks each.

32(a). Find the image of the point $(2, -1, 5)$ in the line $\frac{x-11}{10} = \frac{y+2}{-4} = \frac{z+8}{-11}$. [5 marks]

Solution:

Concept:

- The image of a point in a line is its reflection about that line. The foot H of the perpendicular from the point P is the midpoint of the segment joining P and its image $P\prime$.
- If $\mathbf{d}$ is the direction vector of the line, then $\overrightarrow{PH} \cdot \mathbf{d} = 0$. The midpoint relation gives $P\prime = 2H - P$, coordinate by coordinate.

Step 1: Parameterise the line and its perpendicular foot Let $P = (2, -1, 5)$. Set the common ratio in the line equation equal to t. Then

$$H = (11 + 10t, -2 - 4t, -8 - 11t).$$

The direction vector of the line is $\mathbf{d} = (10, -4, -11)$. This form automatically ensures that the foot lies on the given line. The negative denominators in the symmetric equation produce the negative signs in its second and third coordinates. Subtracting the coordinates of P gives

$$\overrightarrow{PH} = (9 + 10t, -1 - 4t, -13 - 11t).$$

Step 2: Use perpendicularity to determine the parameter For the perpendicular foot, the scalar product of $\overrightarrow{PH}$ with the direction vector must be zero. Thus

$$\begin{aligned} 10(9 + 10t) - 4(-1 - 4t) - 11(-13 - 11t) &= 0, \\ 90 + 100t + 4 + 16t + 143 + 121t &= 0, \\ 237 + 237t &= 0. \end{aligned}$$

Hence $t = -1$, and substitution gives $H = (1, 2, 3)$. All three ratios in the original line equation evaluated at H equal -1, which checks its membership of the line.

Step 3: Apply the midpoint condition and check the construction Write the image as $P\prime = (a, b, c)$. Since H is the midpoint of $PP\prime$,

$$\frac{2+a}{2} = 1, \qquad \frac{-1+b}{2} = 2, \qquad \frac{5+c}{2} = 3.$$

Solving gives $a = 0$, $b = 5$, and $c = 1$. To verify the reflection, $\overrightarrow{PH} = (-1, 3, -2)$ has scalar product $-10 - 12 + 22 = 0$ with $\mathbf{d}$. Also $\overrightarrow{HP\prime} = (-1, 3, -2)$, so $P, H, P\prime$ are collinear and $PH = HP\prime$. Thus the segment is perpendicular to the given line and bisected by it, as reflection requires.

Step 4: Conclusion The image of the given point in the line is $\boxed{(0, 5, 1)}$.

Quick Tip to Solve: Find the perpendicular foot using one scalar-product equation, then subtract the original point from twice the foot.

OR

32(b). Vertices B and C of $\triangle ABC$ lie on the line $\frac{x+2}{2} = \frac{y-1}{1} = \frac{z}{4}$. Find the area of $\triangle ABC$ given that point A has coordinates $(1, -1, 2)$ and the line segment BC has length of 5 units. [5 marks]

Solution:

Concept:

- The area of a triangle is $\frac{1}{2}bh$, where b is the base length and h is the perpendicular distance of the opposite vertex from the line containing that base.
- For a point A, a point D on a line, and its direction vector $\mathbf{d}$, the perpendicular distance is $h = \frac{|\overrightarrow{DA} \times \mathbf{d}|}{|\mathbf{d}|}$. The cross product measures the area of the corresponding parallelogram.

Step 1: Identify the base line and the vector to the vertex The vertices B and C lie on the given line, so the required altitude is the perpendicular distance of $A = (1, -1, 2)$ from this line. The common ratio gives the parametric form

$$(x, y, z) = (-2 + 2t, 1 + t, 4t).$$

Taking $t = 0$, choose $D = (-2, 1, 0)$ on the line. Its direction vector is $\mathbf{d} = (2, 1, 4)$. Therefore

$$\overrightarrow{DA} = (1 + 2, -1 - 1, 2 - 0) = (3, -2, 2).$$

The exact locations of B and C are unnecessary: their fixed separation provides the base length, and their common line fixes the altitude.

Step 2: Calculate the cross product and the perpendicular distance Using the determinant expansion,

$$\begin{aligned} \overrightarrow{DA} \times \mathbf{d} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 2 \\ 2 & 1 & 4 \end{vmatrix} \\ &= (-8 - 2)\mathbf{i} - (12 - 4)\mathbf{j} + (3 + 4)\mathbf{k} \\ &= -10\mathbf{i} - 8\mathbf{j} + 7\mathbf{k}. \end{aligned}$$

Consequently, $|\overrightarrow{DA} \times \mathbf{d}| = \sqrt{100 + 64 + 49} = \sqrt{213}$, while $|\mathbf{d}| = \sqrt{4 + 1 + 16} = \sqrt{21}$. The altitude is

$$h = \frac{\sqrt{213}}{\sqrt{21}} = \sqrt{\frac{71}{7}} \text{ units.}$$

This distance is measured from A to the entire line containing BC; the perpendicular foot need not lie inside the segment.

Step 3: Use the base length and verify the altitude The given base length is $BC = 5$ units. Therefore the area is

$$\frac{1}{2} \times 5 \times \sqrt{\frac{71}{7}} = \frac{5\sqrt{497}}{14} \text{ square units.}$$

As an independent check, $|\overrightarrow{DA}|^2 = 17$ and $\overrightarrow{DA} \cdot \mathbf{d} = 6 - 2 + 8 = 12$. Subtracting the squared projection

Quick Tip to Solve: When two vertices lie on a known line and their separation is given, compute the distance of the third vertex from that line.

along the line gives

$$h^2 = 17 - \frac{12^2}{21} = \frac{213}{21} = \frac{71}{7},$$

which agrees with the cross-product calculation. Hence the altitude and the resulting area use the printed coordinates and direction ratios consistently.

Step 4: Conclusion The area of $\triangle ABC$ is $\boxed{\frac{5\sqrt{497}}{14} \text{ square units}}$.

Quick Tip to Solve: When two vertices lie on a known line and their separation is given, compute the distance of the third vertex from that line.

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33. Find the inverse of the matrix $A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{pmatrix}$**. Using the inverse,** A^{-1}**, solve the system of linear equations** $x - y + 2z = 1$; $2y - 3z = 1$; $3x - 2y + 4z = 3$**.** [5 marks]

Solution:

Concept:

- For a square matrix A, the inverse exists when its determinant $|A|$ is nonzero, and $A^{-1} = \frac{\operatorname{adj} A}{|A|}$.
- The cofactor C_{ij} is $(-1)^{i+j}$ times the minor obtained by deleting row i and column j. The adjugate is the transpose of the cofactor matrix.
- A system with coefficient matrix A, unknown column X, and constant column B is $AX = B$. Multiplication on the left by A^{-1} gives $X = A^{-1}B$.

Step 1: Calculate the determinant Expanding along the first row and retaining the alternating signs,

$$\begin{aligned} |A| &= 1(8-6) - (-1)(0+9) + 2(0-6) \\ &= 2 + 9 - 12 = -1. \end{aligned}$$

Since the determinant is nonzero, the inverse exists and the given system has a unique solution.

Step 2: Calculate the cofactors and adjugate The nine cofactors, computed with their signs, are

$$\begin{aligned} &C_{11} = 8 - 6 = 2, && C_{12} = -(0+9) = -9, && C_{13} = 0 - 6 = -6, \\ &C_{21} = -(-4+4) = 0, && C_{22} = 4 - 6 = -2, && C_{23} = -(-2+3) = -1, \\ &C_{31} = 3 - 4 = -1, && C_{32} = -(-3-0) = 3, && C_{33} = 2 - 0 = 2. \end{aligned}$$

Therefore, transposing the cofactor matrix gives

$$\operatorname{adj} A = \begin{pmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{pmatrix}.$$

Quick Tip to Solve: Here the determinant is -1, so simply negate the transposed cofactor matrix; then multiply its rows by the constant column (1,1,3).

Step 3: Obtain the inverse Dividing every entry of the adjugate by the determinant -1, we obtain

$$A^{-1} = \frac{1}{-1}\begin{pmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}.$$

The transpose in the adjugate step is essential: dividing the cofactor matrix itself by the determinant would not give the required inverse.

Step 4: Solve using the inverse and verify In the order x, y, z, the coefficient matrix of the three equations is exactly A. Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2+0+3 \\ 9+2-9 \\ 6+1-6 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}.$$

Substitution checks all three equations: $1-2+2=1$, $2(2)-3(1)=1$, and $3(1)-2(2)+4(1)=3$. Hence the inverse calculation and the system solution agree.

Step 5: Conclusion $A^{-1} = \begin{pmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{pmatrix}$, and the solution is $\boxed{x=1,\ y=2,\ z=1}$.

Quick Tip to Solve: Here the determinant is -1, so simply negate the transposed cofactor matrix; then multiply its rows by the constant column (1,1,3).

34. Using integration, find the area of the region bounded by the parabola $y^2 = 4ax$ and its latus rectum. [5 marks]

Solution:

Concept:

- For the standard parabola $y^2 = 4ax$, with $a > 0$, the vertex is the origin, the focus is $(a, 0)$, and the latus rectum is the chord through the focus perpendicular to the axis, namely $x = a$.
- With horizontal strips, the area between a right boundary $x_R(y)$ and a left boundary $x_L(y)$ is $\int_{y_1}^{y_2} [x_R(y) - x_L(y)]\, dy$.
- The parabola is symmetric about the x-axis. Thus the part of the bounded region above the axis has the same area as the part below it.

Step 1: Find the latus rectum and its endpoints The latus rectum lies on $x = a$. To find its intersections with the parabola, substitute this value of x into the equation:

$$y^2 = 4a(a) = 4a^2, \qquad y = \pm 2a.$$

The endpoints are therefore $(a, 2a)$ and $(a, -2a)$. These ordinates provide the limits for integration with respect to y. The region extends from the parabolic arc to this vertical chord.

Quick Tip to Solve: Use horizontal strips: the vertical latus rectum is $x = a$, so integrate $a - y^2/(4a)$ from $-2a$ to $2a$.

Step 2: Express the width of a horizontal strip Write the parabola as $x = \dfrac{y^2}{4a}$. For $-2a \le y \le 2a$, this is the left boundary and $x = a$ is the right boundary. Consequently, a strip of thickness dy contributes

$$d\mathcal{A} = \left(a - \frac{y^2}{4a}\right) dy.$$

Its width is nonnegative throughout the interval, becomes zero at the chord endpoints, and equals a at $y = 0$. These checks confirm the correct order of subtraction. Since this width is an even function of y, the required integral may be evaluated over the upper half and doubled.

Step 3: Integrate over the bounded region Using the symmetry about the axis,

$$\begin{aligned}
\mathcal{A} &= \int_{-2a}^{2a} \left(a - \frac{y^2}{4a}\right) dy \\
&= 2\int_{0}^{2a} \left(a - \frac{y^2}{4a}\right) dy \\
&= 2\left[ay - \frac{y^3}{12a}\right]_0^{2a} \\
&= 2\left(2a^2 - \frac{8a^3}{12a}\right) \\
&= 2\left(2a^2 - \frac{2a^2}{3}\right) = \frac{8a^2}{3}.
\end{aligned}$$

Both halves of the region have been included. As a size check, the region lies inside the rectangle of width a and height $4a$, whose area is $4a^2$; the calculated area is smaller, as expected. The result also has dimensions of area, since a is a length.

Step 4: Conclusion The area bounded by the parabola and its latus rectum is $\boxed{\dfrac{8a^2}{3} \text{ square units}}$.

Quick Tip to Solve: Use horizontal strips: the vertical latus rectum is $x = a$, so integrate $a - y^2/(4a)$ from $-2a$ to $2a$.

35(a). **If N denotes the set of all natural numbers and R is the relation on $N \times N$ defined by $(a, b)\, R\, (c, d)$, if $ad(b + c) = bc(a + d)$. Show that R is an equivalence relation.** [5 marks]

Solution:

Concept:

- A relation on a set is an equivalence relation if it is reflexive, symmetric and transitive. Here its elements are ordered pairs in $N \times N$.
- Natural numbers are positive, so $abcd \neq 0$. Dividing the defining equation by $abcd$ converts it into equality of the same expression for the two ordered pairs.

Step 1: Rewrite the relation in a useful form Let $(a, b), (c, d) \in N \times N$. Since all four entries are

Quick Tip to Solve: Divide by the product of the four positive entries first: equality of reciprocal differences makes all three properties immediate.

positive, division by their product is valid. The given condition is equivalent to

$$\begin{aligned} ad(b+c) &= bc(a+d) \\ \frac{b+c}{bc} &= \frac{a+d}{ad} \\ \frac{1}{b}+\frac{1}{c} &= \frac{1}{a}+\frac{1}{d} \\ \frac{1}{a}-\frac{1}{b} &= \frac{1}{c}-\frac{1}{d}. \end{aligned}$$

Thus the relation holds exactly when the values of $h(a,b) = \frac{1}{a} - \frac{1}{b}$ and $h(c,d) = \frac{1}{c} - \frac{1}{d}$ agree. Each rearrangement is reversible, so this criterion is both necessary and sufficient.

Step 2: Verify reflexivity Take any $(a,b) \in N \times N$. Clearly

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a} - \frac{1}{b}.$$

Hence $(a,b)R(a,b)$ by the equivalent criterion. Directly, the original condition becomes $ab(b+a) = ba(a+b)$, which also holds. Since the pair was arbitrary, every element is related to itself and R is reflexive.

Step 3: Verify symmetry Suppose $(a,b)R(c,d)$. Then

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}.$$

Equality can be reversed, giving $\frac{1}{c} - \frac{1}{d} = \frac{1}{a} - \frac{1}{b}$. Applying the same criterion with the two pairs interchanged gives $(c,d)R(a,b)$. Thus every related pair can be reversed, and R is symmetric.

Step 4: Verify transitivity Let $(a,b),(c,d),(e,f) \in N \times N$, and suppose $(a,b)R(c,d)$ and $(c,d)R(e,f)$. These assumptions give

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{c} - \frac{1}{d}, \qquad \frac{1}{c} - \frac{1}{d} = \frac{1}{e} - \frac{1}{f}.$$

Combining the equalities yields $\frac{1}{a} - \frac{1}{b} = \frac{1}{e} - \frac{1}{f}$. Rearranging gives $\frac{1}{b} + \frac{1}{e} = \frac{1}{a} + \frac{1}{f}$. Multiplying by $abef$ gives $af(b+e) = be(a+f)$, precisely the defining condition for $(a,b)R(e,f)$. Therefore R is transitive.

Step 5: Conclusion The relation R is reflexive, symmetric and transitive on $N \times N$. Hence $\boxed{R \text{ is an equivalence relation.}}$

Quick Tip to Solve: Divide by the product of the four positive entries first: equality of reciprocal differences makes all three properties immediate.

OR

35(b). Let $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \to \mathbb{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. Show that f is a one-one function. Also, check whether f is an onto function or not. [5 marks]

Solution:

Concept:

- A function is one-one if $f(x_1) = f(x_2)$ forces $x_1 = x_2$ for all inputs in its domain.
- A function is onto its stated codomain if every element of that codomain has a preimage in the domain. A single unattained codomain value disproves surjectivity.

Step 1: Note the domain restriction Write $D = \mathbb{R} \setminus \{-\frac{4}{3}\}$. For any $x \in D$, the denominator $3x + 4$ is nonzero, so $f(x) = \frac{4x}{3x+4}$ is defined and real. The nonzero denominators allow safe cross-multiplication. The codomain is all of $\mathbb{R}$; onto requires every real output to have a permitted input.

Step 2: Prove that equal outputs require equal inputs Let $x_1, x_2 \in D$ and assume $f(x_1) = f(x_2)$. Then

$$\begin{aligned} \frac{4x_1}{3x_1+4} &= \frac{4x_2}{3x_2+4} \\ 4x_1(3x_2+4) &= 4x_2(3x_1+4) \\ 12x_1x_2 + 16x_1 &= 12x_1x_2 + 16x_2 \\ 16x_1 &= 16x_2 \\ x_1 &= x_2. \end{aligned}$$

Since the inputs were arbitrary, no two distinct inputs have the same output. Thus f is one-one on D.

Step 3: Test the real output that cannot be attained The number $\frac{4}{3}$ belongs to the codomain. If it had a preimage $x \in D$, then

$$\begin{aligned} \frac{4x}{3x+4} &= \frac{4}{3} \\ 12x &= 4(3x+4) \\ 12x &= 12x + 16, \end{aligned}$$

which would imply $0 = 16$, a contradiction. Hence $\frac{4}{3}$ has no preimage, and the function is not onto $\mathbb{R}$.

Step 4: Determine the range to complete the check For any real $y \neq \frac{4}{3}$, solve $y = \frac{4x}{3x+4}$:

$$(4 - 3y)x = 4y, \qquad x = \frac{4y}{4-3y}.$$

The denominator is nonzero. This candidate also lies in the domain because

$$x + \frac{4}{3} = \frac{4y}{4-3y} + \frac{4}{3} = \frac{16}{3(4-3y)} \neq 0.$$

Moreover, $3x + 4 = \frac{16}{4-3y}$, so substitution gives $f(x) = \frac{16y/(4-3y)}{16/(4-3y)} = y$. Thus every real value except $\frac{4}{3}$ is attained, and the range is exactly $\mathbb{R} \setminus \{\frac{4}{3}\}$.

Step 5: Conclusion $\boxed{f \text{ is one-one but is not onto } \mathbb{R}.}$ Its range is $\mathbb{R} \setminus \{\frac{4}{3}\}$, since $\frac{4}{3}$ has no preimage.

Quick Tip to Solve: For onto, solve the output equation for the input and check both the exceptional output and whether the candidate input is allowed.

Section E (12 marks)

This section comprises 3 case study based questions of 4 marks each.

36. A building contractor undertakes a job to construct 4 flats on a plot along with parking area. Due to strike the probability of many construction workers not being present for the job is 0.65. The probability that many are not present and still the work gets completed on time is 0.35. The probability that work will be completed on time when all workers are present is 0.80.

Let: E_1: represent the event when many workers were not present for the job;

E_2: represent the event when all workers were present; and

E: represent completing the construction work on time.

Based on the above information, answer the following questions:

[4 marks]

36(i). What is the probability that all the workers are present for the job? [1 mark]

Solution:

Concept:

- Complementary events exhaust the possibilities: $P(E_1) + P(E_2) = 1$. Here E_1 denotes many workers being absent and E_2 denotes all workers being present.
- The probability of an event lies between zero and one; subtracting the probability of its complement from one gives the required probability.

Step 1: Identify the complementary event The case study classifies worker attendance into the two events E_1 and E_2. Therefore, within this stated model, all workers being present is the complement of many workers being absent. The given attendance probability is $P(E_1) = 0.65$.

Step 2: Apply the complement rule Using the total probability of the two attendance possibilities,

$$P(E_2) = 1 - P(E_1) = 1 - 0.65 = 0.35.$$

The probabilities relating to completion of the work are not needed because the question asks only about attendance.

Step 3: Conclusion The probability that all workers are present is $\boxed{0.35}$.

Quick Tip to Solve: For attendance alone, use the complement of 0.65; do not involve the completion probability 0.80.

36(ii). What is the probability that construction will be completed on time? [1 mark]

Solution:

Concept:

- For the mutually exclusive and exhaustive attendance events, $P(E) = P(E \cap E_1) + P(E \cap E_2)$.
- Convert a conditional probability to a joint probability using $P(E \cap E_2) = P(E_2)P(E \mid E_2)$.

Quick Tip to Solve: The phrase "not present and still completed" supplies a joint probability, so add it directly to 0.35 times 0.80.

Step 1: Find the joint probability for full attendance From the attendance data, $P(E_2) = 1 - 0.65 = 0.35$. The probability of timely completion when all workers are present is $P(E \mid E_2) = 0.80$. Hence

$$P(E \cap E_2) = 0.35 \times 0.80 = 0.28.$$

Step 2: Add the two disjoint completion cases The statement gives $P(E \cap E_1) = 0.35$ directly: it is the probability that many are absent and the work still finishes on time. Thus

$$P(E) = 0.35 + 0.28 = 0.63.$$

The value 0.35 in this joint event must not be multiplied again by 0.65.

Step 3: Conclusion The probability of completing construction on time is $\boxed{0.63}$.

Quick Tip to Solve: The phrase "not present and still completed" supplies a joint probability, so add it directly to 0.35 times 0.80.

36(iii)(a). What is the probability that many workers are not present given that the construction work is completed on time? [2 marks]

Solution:

Concept:

- Conditional probability is $P(A \mid B) = P(A \cap B)/P(B)$, provided $P(B) > 0$.
- The attendance cases partition the completion event: $P(E) = P(E \cap E_1) + P(E \cap E_2)$.

Step 1: Calculate the probability of timely completion The case study gives $P(E_1) = 0.65$, $P(E \cap E_1) = 0.35$ and $P(E \mid E_2) = 0.80$. Since E_1 and E_2 form the two attendance cases, $P(E_2) = 1 - 0.65 = 0.35$. Consequently,

$$\begin{aligned} P(E \cap E_2) &= P(E_2)P(E \mid E_2) \\ &= 0.35(0.80) = 0.28, \\ P(E) &= 0.35 + 0.28 = 0.63. \end{aligned}$$

Step 2: Condition on work being completed We now restrict attention to the event E, namely completed construction. Among these outcomes, the ones with many workers absent form $E_1 \cap E$. Therefore

$$P(E_1 \mid E) = \frac{P(E_1 \cap E)}{P(E)} = \frac{0.35}{0.63} = \frac{35}{63} = \frac{5}{9}.$$

Notice that the given 0.35 is already a joint probability, so it is the numerator itself. The denominator is positive, as required for conditional probability.

Step 3: Conclusion The required probability is $\boxed{\frac{5}{9}}$.

Quick Tip to Solve: When reversing the condition, divide the given joint probability 0.35 by the overall completion probability 0.63.

OR

36(iii)(b). What is the probability that all workers were present given that the construction job was completed on time? [2 marks]

Solution:

Concept:

- Conditional probability is $P(A \mid B) = P(A \cap B)/P(B)$, with $P(B) > 0$.
- Use $P(E \cap E_2) = P(E_2)P(E \mid E_2)$ before reversing the given condition.

Step 1: Find the joint and overall completion probabilities The probability of full attendance is $P(E_2) = 1 - P(E_1) = 1 - 0.65 = 0.35$. Given full attendance, completion has probability 0.80, so

$$P(E_2 \cap E) = 0.35 \times 0.80 = 0.28.$$

The other attendance case contributes the given joint probability $P(E_1 \cap E) = 0.35$. Since these cases are disjoint and exhaustive,

$$P(E) = 0.35 + 0.28 = 0.63.$$

Step 2: Apply the reversed conditional probability The question conditions on completion and asks about full attendance. Therefore

$$P(E_2 \mid E) = \frac{P(E_2 \cap E)}{P(E)} = \frac{0.28}{0.63} = \frac{28}{63} = \frac{4}{9}.$$

The value 0.80 is $P(E \mid E_2)$, so it cannot itself answer the reversed question. As a check, the two posterior attendance probabilities add to one: $5/9 + 4/9 = 1$.

Step 3: Conclusion The required probability is $\boxed{\frac{4}{9}}$.

Quick Tip to Solve: Compute full attendance and timely completion together as 0.35 times 0.80, then divide by 0.63.

37. Let $f(x)$ be a real valued function. Then its

- **Left Hand Derivative (L.H.D.):** $Lf'(a) = \lim\limits_{h\to 0} \dfrac{f(a-h)-f(a)}{-h}$
- **Right Hand Derivative (R.H.D.):** $Rf'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$

Also, a function $f(x)$ is said to be differentiable at $x = a$ if its L.H.D. and R.H.D. at $x = a$ exist and both are equal. For the function

$$f(x) = \begin{cases} |x-3|, & x \geq 1 \\ \dfrac{x^2}{4} - \dfrac{3x}{2} + \dfrac{13}{4}, & x < 1 \end{cases}$$

answer the following questions: [4 marks]

37(i). What is R.H.D. of $f(x)$ at $x = 1$? [1 mark]

Solution:

Concept:

- The right-hand derivative uses values with $x > 1$: $Rf'(1) = \lim_{h\to 0^+}[f(1+h) - f(1)]/h$.
- For $1 \leq x < 3$, $|x-3| = 3 - x$; thus the relevant branch is linear near 1.

Step 1: Choose the branch on the right At the junction, $f(1) = |1-3| = 2$. For $0 < h < 2$, the point $1 + h$ lies between 1 and 3, so $f(1+h) = |h-2| = 2 - h$. The absolute value must be simplified with this local sign before forming the quotient.

Step 2: Evaluate the difference quotient

$$Rf'(1) = \lim_{h\to 0^+} \frac{(2-h)-2}{h} = \lim_{h\to 0^+} (-1) = -1.$$

The quotient is constant for sufficiently small positive h, so the right-hand limit exists.

Step 3: Conclusion The right-hand derivative at $x = 1$ is $\boxed{-1}$.

Quick Tip to Solve: Near the junction on the right, simplify $|x-3|$ as $3 - x$ before differentiating.

37(ii). What is L.H.D. of $f(x)$ at $x = 1$? [1 mark]

Solution:

Concept:

- The left-hand derivative is $Lf'(1) = \lim_{h\to 0^+}[f(1-h) - f(1)]/(-h)$.
- Use the quadratic branch for $1 - h < 1$, but evaluate $f(1)$ from the branch that includes equality.

Step 1: Evaluate the function values The value at the junction is $f(1) = |1-3| = 2$. For $h > 0$, use the quadratic branch to obtain

$$f(1-h) = \frac{(1-h)^2}{4} - \frac{3(1-h)}{2} + \frac{13}{4} = 2 + h + \frac{h^2}{4}.$$

The constant terms sum to 2, agreeing with the function value at the junction.

Quick Tip to Solve: When using $f(1-h)$, retain the denominator $-h$ to avoid reversing the derivative sign.

Step 2: Take the left-hand limit Substitute into the defining quotient, remembering its denominator is $-h$:

$$Lf'(1) = \lim_{h\to 0^+} \frac{h + h^2/4}{-h} = \lim_{h\to 0^+} \left(-1 - \frac{h}{4}\right) = -1.$$

Thus this finite one-sided derivative exists.

Step 3: Conclusion The left-hand derivative at $x = 1$ is $\boxed{-1}$.

Quick Tip to Solve: When using $f(1-h)$, retain the denominator $-h$ to avoid reversing the derivative sign.

37(iii)(a). Check if the function $f(x)$ is differentiable at $x = 1$. [2 marks]

Solution:

Concept:

- Differentiability at a junction requires both finite one-sided derivatives to exist and be equal.
- Continuity should be checked using the function value at the junction; here $f(1) = 2$.

Step 1: Check continuity at the junction The branch including $x = 1$ gives $f(1) = |1 - 3| = 2$. The quadratic branch has left-hand limit

$$\lim_{x\to 1^-} f(x) = \frac{1}{4} - \frac{3}{2} + \frac{13}{4} = 2,$$

and the absolute-value branch has right-hand limit $\lim_{x\to 1^+} |x - 3| = 2$. Both limits agree with $f(1)$, so the function is continuous there.

Step 2: Compute and compare the derivatives For small positive h, $f(1-h) = 2 + h + h^2/4$ and $f(1+h) = 2 - h$. Hence

$$Lf'(1) = \lim_{h\to 0^+} \frac{h + h^2/4}{-h} = -1,$$

$$Rf'(1) = \lim_{h\to 0^+} \frac{-h}{h} = -1.$$

These two finite limits are equal. Continuity alone would not suffice, but equality of the one-sided derivatives establishes differentiability.

Step 3: Conclusion The function is differentiable at $x = 1$, and $\boxed{f'(1) = -1}$.

Quick Tip to Solve: At a piecewise junction, compare both one-sided derivatives; continuity alone does not settle differentiability.

OR

37(iii)(b). Find $f'(2)$ and $f'(-1)$. [2 marks]

Solution:

Concept:

- To differentiate a piecewise function away from a junction, use the branch valid in a neighbourhood of the point.
- For $1 < x < 3$, $|x-3| = 3-x$; for $x < 1$, differentiate the given quadratic term by term.

Step 1: Find the derivative at x = 2 Since 2 lies strictly between 1 and 3, a sufficiently small neighbourhood of 2 uses $f(x) = |x-3| = 3-x$. Therefore

$$f'(x) = \frac{d}{dx}(3-x) = -1 \quad (1 < x < 3), \qquad f'(2) = -1.$$

The absolute-value corner at 3 does not affect differentiability at 2; the sign of $x-3$ is negative throughout this neighbourhood.

Step 2: Find the derivative at x = -1 Since $-1 < 1$, the applicable branch is $f(x) = x^2/4 - 3x/2 + 13/4$. Its derivative is

$$f'(x) = \frac{x}{2} - \frac{3}{2}.$$

Substituting the requested point gives

$$f'(-1) = -\frac{1}{2} - \frac{3}{2} = -2.$$

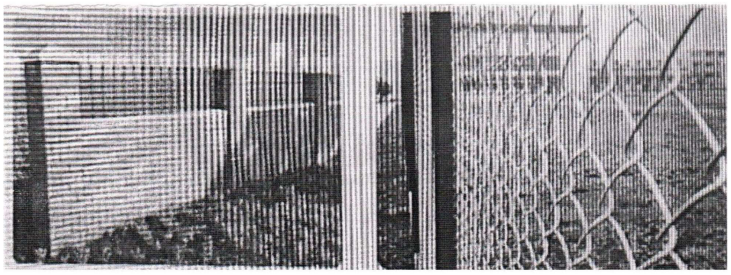
The constant $13/4$ contributes zero, and no junction comparison is necessary because -1 lies inside this branch.

Step 3: Conclusion The required derivatives are $\boxed{f'(2) = -1}$ and $\boxed{f'(-1) = -2}$.

Quick Tip to Solve: Check the interval first: at 2 use $3-x$, while at -1 use the quadratic branch.

38(i). Sooraj's father wants to construct a rectangular garden using a brick wall on one side of the garden and wire fencing for the other three sides as shown in the figure. He has 200 metres of fencing wire.

(i) Let 'x' metres denote the length of the side of the garden perpendicular to the brick wall and 'y' metres denote the length of the side parallel to the brick wall. Determine the relation representing the total length of fencing wire and also write $A(x)$, the area of the garden. [2 marks]



Solution:

Concept:

- The area of a rectangle is length times breadth: $A = xy$, where x and y are its adjacent side lengths in metres.
- Only three sides require wire: the two equal sides perpendicular to the wall and the opposite side parallel to it.

Step 1: Count the fenced sides The brick wall forms one complete side of the garden, so no fencing wire is needed along that side. The two perpendicular sides each have length x metres, while the fenced side parallel to the wall has length y metres. Since all 200 metres of wire are used, their combined length gives

$$x + x + y = 200.$$

Solving this relation for the parallel side gives $y = 200 - 2x$.

Step 2: Express the area in one variable Substitute the expression for y into the rectangular area formula:

$$A(x) = xy = x(200 - 2x) = 200x - 2x^2.$$

Both side lengths must be positive for a garden of nonzero area. Hence $x > 0$ and $200 - 2x > 0$, so the physical domain is $0 < x < 100$. The area is measured in square metres, whereas the fencing constraint is measured in metres.

Step 3: Conclusion The wire-length relation is $\boxed{2x + y = 200}$, and $\boxed{A(x) = 200x - 2x^2 \text{ m}^2}$, for $0 < x < 100$.

Quick Tip to Solve: Count only the three fenced sides before eliminating y; the wall removes one length from the usual perimeter.

38(ii). Determine the maximum value of $A(x)$. [2 marks]

Solution:

Concept:

- The fencing constraint $2x + y = 200$ gives the area function $A(x) = 200x - 2x^2$, with $0 < x < 100$.
- For a differentiable area function, solve $A\prime(x) = 0$ for a stationary point. A negative second derivative verifies a maximum.

Step 1: Find the stationary dimensions Differentiate the area with respect to the perpendicular side length x:

$$A\prime(x) = 200 - 4x.$$

At an interior maximum, $A\prime(x) = 0$, so $200 - 4x = 0$ gives $x = 50$ metres. This lies in the physical interval $0 < x < 100$. The corresponding parallel side length is $y = 200 - 2(50) = 100$ metres; thus the dimensions satisfy the original wire constraint.

Step 2: Verify and evaluate the maximum The second derivative is $A\prime\prime(x) = -4 < 0$ throughout the domain. Therefore the quadratic is strictly concave and its stationary point gives the greatest area. Substitution yields

$$A(50) = 200(50) - 2(50)^2 = 10000 - 5000 = 5000.$$

Equivalently, $A(x) = 5000 - 2(x - 50)^2$, so no other feasible value of x produces a larger area.

Step 3: Conclusion The maximum garden area is $\boxed{5000\ \text{m}^2}$, attained when $x = 50$ m and $y = 100$ m.

Quick Tip to Solve: For a rectangle fenced on three sides, the optimal parallel side is twice either perpendicular side.