CBSE Class 12 Mathematics Question Paper 2023 with Solution PDF-Set 65-5-1

Question Paper with Solutions

Conducted by the Central Board of Secondary Education (CBSE)

Time Allowed: 3 hours **Maximum Marks: 80**

General Instructions

(i) This question paper contains 38 questions. All questions are compulsory.

(ii) Question paper is divided into **FIVE** Sections – Section **A, B, C, D** and **E**.

(iii) In **Section A** – Question Number **1** to **18** are Multiple Choice Questions (MCQ) type and Question Number **19 & 20** are Assertion-Reason based questions of **1** mark each.

(iv) In **Section B** – Question Number **21** to **25** are Very Short Answer (VSA) type questions of **2** marks each.

(v) In **Section C** – Question Number **26** to **31** are Short Answer (SA) type questions, carrying **3** marks each.

(vi) In **Section D** – Question Number **32** to **35** are Long Answer (LA) type questions carrying **5** marks each.

(vii) In **Section E** – Question Number **36** to **38** are case study based questions carrying **4** marks each where **2 VSA** type questions are of **1** mark each and **1 SA** type question is of **2** marks. Internal choice is provided in **2** marks question in each case-study.

(viii) There is no overall choice. However, an internal choice has been provided in **2** questions in Section – **B**, **3** questions in Section – **C**, **2** questions in Section – **D** and **2** questions in Section – **E**.

(ix) Use of calculators is **NOT** allowed.

Section A (20 marks)

(Multiple Choice Questions)

Each question carries 1 mark.

1. Let $A = \{3, 5\}$. Then the number of reflexive relations on A is: [1 mark]

(A) 2

(B) 4

(C) 0

(D) 8

Correct Answer: (B) 4

Solution:

Concept:

- A relation on a set is a subset of its Cartesian product. Reflexivity requires every diagonal pair (a, a) to belong to the relation.
- For a set of n elements, the $n^2 - n$ off-diagonal pairs are independently included or excluded.

Step 1: List the compulsory pairs Here $A = \{3, 5\}$, so $A \times A = \{(3, 3), (3, 5), (5, 3), (5, 5)\}$. Every reflexive relation must contain both $(3, 3)$ and $(5, 5)$. These two pairs are fixed, so they contribute no independent choices.

Step 2: Count the optional pairs The remaining pairs $(3, 5)$ and $(5, 3)$ may each be present or absent. Thus the number is $2 \times 2 = 4$. The options 2 and 8 count the wrong number of free pairs; 0 is impossible since the diagonal relation itself is reflexive.

Step 3: Conclusion There are 4 reflexive relations. Hence, the correct option is (B).

Quick Tip to Solve: Fix all diagonal pairs first; count binary choices only for the off-diagonal pairs.

---

2. $\sin\left[\frac{\pi}{3} + \sin^{-1}\left(\frac{1}{2}\right)\right]$ is equal to: [1 mark]

(A) 1

(B) $\frac{1}{2}$

(C) $\frac{1}{3}$

(D) $\frac{1}{4}$

Correct Answer: (A) 1

Solution:

Concept:

- The principal range of $\sin^{-1} t$ is $[-\pi/2, \pi/2]$.
- Use the exact value $\sin(\pi/6) = 1/2$ and evaluate the resulting angle.

Step 1: Evaluate the inverse sine Since $\pi/6$ lies in the principal range and its sine is $1/2$, we have

Quick Tip to Solve: Replace the inverse sine by its principal angle before adding the two angles.

$\sin^{-1}(1/2) = \pi/6$. The inverse sine returns an angle, not the input number $1/2$.

Step 2: Evaluate the outer sine The angle inside the outer sine is $\pi/3 + \pi/6 = \pi/2$. Therefore $\sin[\pi/3 + \sin^{-1}(1/2)] = \sin(\pi/2) = 1$. The remaining options $1/2, 1/3, 1/4$ do not equal the sine of this angle.

Step 3: Conclusion The required value is 1. Hence, the correct option is (A).

Quick Tip to Solve: Replace the inverse sine by its principal angle before adding the two angles.

3. For a square matrix A, if $A^2 - A + I = O$, then A^{-1} equals: [1 mark]

(A) A

(B) $A + I$

(C) $I - A$

(D) $A - I$

Correct Answer: (C) $I - A$

Solution:

Concept:

- An inverse of a square matrix A is a matrix C satisfying $AC = CA = I$.
- Rearrange a matrix polynomial to exhibit a product equal to the identity.

Step 1: Rearrange the given identity From $A^2 - A + I = O$, move the quadratic and linear terms to obtain $A - A^2 = I$. Factoring gives $A(I - A) = I$. Since $I - A$ is a polynomial in A, it commutes with A.

Step 2: Identify and check the inverse Also $(I - A)A = A - A^2 = I$, so $I - A$ is the inverse. The proposed $A - I$ gives product $-I$, not I. The matrices A and $A + I$ do not yield the required identity from the given equation and are not the general inverse.

Step 3: Conclusion $A^{-1} = I - A$. Hence, the correct option is (C).

Quick Tip to Solve: Rearrange the polynomial until one factor is A and the product is exactly I.

4. If $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix}$ and $A = B^2$, then x equals: [1 mark]

(A) ± 1

(B) -1

(C) 1

(D) 2

Correct Answer: (C) 1

Solution:

Concept:

- Matrix multiplication uses the dot product of each row with each column.
- Equal matrices have equal entries in every corresponding position.

Step 1: Square the given matrix For $B = \begin{pmatrix} x & 0 \\ 1 & 1 \end{pmatrix}$, multiplication gives

$$B^2 = \begin{pmatrix} x^2 & 0 \\ x+1 & 1 \end{pmatrix}.$$

The lower-left entry contains two terms: $1 \cdot x + 1 \cdot 1 = x + 1$.

Step 2: Compare with A Since $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = B^2$, we require both $x^2 = 1$ and $x + 1 = 2$. The second equation gives $x = 1$, which satisfies the first. Although -1 satisfies the diagonal equation, it fails the lower-left entry. The value 2 fails both equations.

Step 3: Conclusion $x = 1$. Hence, the correct option is (C).

Quick Tip to Solve: Check an off-diagonal entry too; the diagonal equation alone leaves a false sign choice.

5. If $\begin{vmatrix} \alpha & 3 & 4 \\ 1 & 2 & 1 \\ 1 & 4 & 1 \end{vmatrix} = 0$, then the value of α is: [1 mark]

(A) 1

(B) 2

(C) 3

(D) 4

Correct Answer: (D) 4

Solution:

Concept:

- Replacing a row by that row minus another row does not change the determinant.
- Expand along a row with two zero entries to reduce the calculation.

Quick Tip to Solve: Subtract the nearly identical rows first and remember the negative cofactor sign at (3,2).

Step 1: Simplify the third row Apply $R_3 \to R_3 - R_2$. The determinant becomes

$$\begin{vmatrix} \alpha & 3 & 4 \\ 1 & 2 & 1 \\ 0 & 2 & 0 \end{vmatrix}.$$

This row operation preserves its value and creates only one nonzero entry in the third row.

Step 2: Expand and solve The cofactor sign at position $(3,2)$ is negative. Thus the determinant equals $-2\begin{vmatrix} \alpha & 4 \\ 1 & 1 \end{vmatrix} = -2(\alpha - 4)$. Equating it to zero gives $\alpha = 4$. Substitution of 1, 2 or 3 leaves a nonzero determinant, so none of those options satisfies the condition.

Step 3: Conclusion $\alpha = 4$. Hence, the correct option is (D).

Quick Tip to Solve: Subtract the nearly identical rows first and remember the negative cofactor sign at (3,2).

6. The derivative of x^{2x} w.r.t. x is: [1 mark]

(A) x^{2x-1}

(B) $2x^{2x}\log x$

(C) $2x^{2x}(1+\log x)$

(D) $2x^{2x}(1-\log x)$

Correct Answer: (C) $2x^{2x}(1+\log x)$

Solution:

Concept:

- Logarithmic differentiation handles powers whose base and exponent both depend on x.
- For real logarithmic differentiation take $x > 0$, with $\log x$ denoting the natural logarithm.

Step 1: Take logarithms Let $y = x^{2x}$. Then $\log y = 2x\log x$. Differentiating both sides and applying the product rule gives

$$\frac{y'}{y} = 2\log x + 2x\frac{1}{x} = 2\log x + 2.$$

Both terms are needed because the exponent also varies.

Step 2: Restore y and compare Multiplying by $y = x^{2x}$ yields $y' = 2x^{2x}(1+\log x)$. Option (B) misses the derivative of $\log x$; option (D) has an incorrect minus sign. Option (A), x^{2x-1}, cannot follow from a constant-exponent power rule because $2x$ is variable.

Step 3: Conclusion $\frac{d}{dx}x^{2x} = 2x^{2x}(1+\log x)$. Hence, the correct option is (C).

Quick Tip to Solve: For a variable raised to a variable power, take logs and use the product rule.

7. The function $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x, is continuous at: [1 mark]

(A) $x = 1$

(B) $x = 1.5$

(C) $x = -2$

(D) $x = 4$

Correct Answer: (B) $x = 1.5$

Solution:

Concept:

- The greatest integer function $[x]$ equals the integer n on $[n, n+1)$.
- Continuity at a requires both one-sided limits to equal $f(a)$.

Step 1: Check the noninteger point In a sufficiently small neighbourhood of 1.5, every point lies between 1 and 2. Hence $[x] = 1$ throughout that neighbourhood. Both one-sided limits at 1.5 equal 1, which is also $f(1.5)$.

Step 2: Check the integer alternatives At any integer n, the left-hand limit is $n - 1$, while the right-hand limit and $f(n)$ equal n. Thus a jump occurs at each integer. In particular, 1, -2 and 4 are all discontinuity points; the negative integer follows the same rule as positive integers.

Step 3: Conclusion The function is continuous at $x = 1.5$. Hence, the correct option is (B).

Quick Tip to Solve: A greatest integer function is locally constant between integers and jumps at each integer.

---

8. If $x = A\cos 4t + B\sin 4t$, then $\dfrac{d^2x}{dt^2}$ is equal to: [1 mark]

(A) x

(B) $-x$

(C) $16x$

(D) $-16x$

Correct Answer: (D) $-16x$

Solution:

Concept:

- The chain rule gives $\frac{d}{dt}\cos kt = -k\sin kt$ and $\frac{d}{dt}\sin kt = k\cos kt$.
- Applying the derivative twice multiplies either sine or cosine of kt by $-k^2$.

Step 1: Differentiate once Treat A and B as constants. The inner argument $4t$ contributes a factor of 4, so

$$\frac{dx}{dt} = -4A\sin 4t + 4B\cos 4t.$$

Step 2: Differentiate again and compare A second differentiation contributes another factor of 4 and

Quick Tip to Solve: For any linear combination of sine and cosine with frequency k, the second derivative is -k squared times the original expression.

reverses the sign of each original trigonometric term:

$$\frac{d^2x}{dt^2} = -16A\cos 4t - 16B\sin 4t = -16(A\cos 4t + B\sin 4t).$$

Thus options (A) and (B) omit the frequency squared, while (C) has the wrong sign.

Step 3: Conclusion $\frac{d^2x}{dt^2} = -16x$. Hence, the correct option is (D).

Quick Tip to Solve: For any linear combination of sine and cosine with frequency k, the second derivative is -k squared times the original expression.

9. The interval in which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing, is: [1 mark]

(A) $(-1, \infty)$

(B) $(-2, -1)$

(C) $(-\infty, -2)$

(D) $[-1, 1]$

Correct Answer: (B) $(-2, -1)$

Solution:

Concept:

- A differentiable function is decreasing on an interval where $f\prime(x) < 0$.
- The sign of a factored quadratic changes at each distinct simple root.

Step 1: Find the derivative Differentiating each polynomial term gives

$$f\prime(x) = 6x^2 + 18x + 12 = 6(x+1)(x+2).$$

The derivative vanishes at $x = -2$ and $x = -1$, which separate the real line into three sign intervals.

Step 2: Determine the negative interval For $x < -2$, both factors are negative and their product is positive. For $-2 < x < -1$, $x + 2 > 0$ but $x + 1 < 0$, so the derivative is negative. For $x > -1$, both are positive. Therefore (A) and (C) are increasing intervals, while (D) also has positive derivative in its interior.

Step 3: Conclusion The decreasing interval is $(-2, -1)$. Hence, the correct option is (B).

Quick Tip to Solve: Factor the derivative first; a positive quadratic with two distinct roots is negative between its roots.

10. $\displaystyle\int \frac{\sec x}{\sec x - \tan x}\,dx$ **equals:** [1 mark]

(A) $\sec x - \tan x + c$

(B) $\sec x + \tan x + c$

(C) $\tan x - \sec x + c$

(D) $-(\sec x + \tan x) + c$

Correct Answer: (B) $\sec x + \tan x + c$

Solution:

Concept:

- The identity $\sec^2 x - \tan^2 x = 1$ allows rationalisation of the denominator.
- Use $\int \sec^2 x\,dx = \tan x + C$ and $\int \sec x \tan x\,dx = \sec x + C$.

Step 1: Simplify the printed integrand Multiply numerator and denominator by $\sec x + \tan x$. On the domain where the given trigonometric functions exist,

$$\frac{\sec x}{\sec x - \tan x} = \frac{\sec x(\sec x + \tan x)}{\sec^2 x - \tan^2 x} = \sec^2 x + \sec x \tan x.$$

Step 2: Integrate and test the alternatives Integrating the two terms gives

$$\int (\sec^2 x + \sec x \tan x)\,dx = \tan x + \sec x + C.$$

The derivatives of (A), (C), and (D) are respectively $\sec x \tan x - \sec^2 x$, $\sec^2 x - \sec x \tan x$, and $-\sec x \tan x - \sec^2 x$; none matches the simplified integrand.

Step 3: Conclusion The integral is $\sec x + \tan x + C$. Hence, the correct option is (B).

Quick Tip to Solve: Multiply by the conjugate sec x plus tan x; the denominator becomes one by the Pythagorean identity.

---

11. The value of $\displaystyle\int_{-1}^{1} \frac{|x-2|}{x-2}\,dx,\ x \neq 2$**, is:** [1 mark]

(A) 1

(B) -1

(C) 2

(D) -2

Correct Answer: (D) -2

Solution:

Concept:

- For a negative number u, its absolute value is $|u| = -u$.
- The integral of a constant k from a to b is $k(b-a)$.

Quick Tip to Solve: Locate the absolute-value sign-change point before splitting an integral; here it lies outside the integration interval.

Step 1: Check the sign throughout the interval On the entire interval $[-1, 1]$, we have $x - 2 \leq -1 < 0$. Therefore $|x - 2| = -(x - 2)$, and

$$\frac{|x-2|}{x-2} = -1.$$

The excluded point $x = 2$ is outside the limits, so it introduces no break in this integral.

Step 2: Evaluate the constant integral It follows that

$$\int_{-1}^{1} \frac{|x-2|}{x-2}\, dx = \int_{-1}^{1} (-1)\, dx = [-x]_{-1}^{1} = -1 - 1 = -2.$$

Option (B) is only the integrand value, whereas (A) and (C) have the wrong sign. The interval length is 2, which must also be included.

Step 3: Conclusion The value is -2. Hence, the correct option is (D).

Quick Tip to Solve: Locate the absolute-value sign-change point before splitting an integral; here it lies outside the integration interval.

---

12. The sum of the order and the degree of the differential equation $\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^3\right]$ is: [1 mark]

(A) 2

(B) 3

(C) 5

(D) 0

Correct Answer: (B) 3

Solution:

Concept:

- The order is the highest derivative present in a differential equation.
- The degree is the power of the highest-order derivative when the equation is polynomial in its derivatives.

Step 1: Expand the derivative Apply the chain rule to the cube of dy/dx:

$$\frac{d}{dx}\left[\left(\frac{dy}{dx}\right)^3\right] = 3\left(\frac{dy}{dx}\right)^2 \frac{d^2y}{dx^2}.$$

The highest derivative is the second derivative, so the order is 2.

Step 2: Identify the degree The expanded expression is polynomial in derivatives. Its highest-order derivative d^2y/dx^2 occurs to the first power, giving degree 1. The square on the first derivative does not determine the degree because it is a lower-order derivative. Thus the required sum is $2 + 1 = 3$: option (A) omits the degree, (C) confuses the cube with the degree, and (D) ignores both definitions.

Step 3: Conclusion The sum of order and degree is 3. Hence, the correct option is (B).

Quick Tip to Solve: Expand any derivative of a composite expression before identifying the highest derivative and its exponent.

13. Two vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are collinear if: [1 mark]

(A) $a_1b_1 + a_2b_2 + a_3b_3 = 0$

(B) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

(C) $a_1 = b_1,\ a_2 = b_2,\ a_3 = b_3$

(D) $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$

Correct Answer: (B) $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$

Solution:

Concept:

- Two nonzero vectors are collinear exactly when one is a scalar multiple of the other: $\vec{a} = \lambda\vec{b}$.
- Comparing components gives $a_i = \lambda b_i$; when denominators are nonzero, all component ratios equal λ.

Step 1: Compare corresponding components The scalar-multiple condition means

$$a_1 = \lambda b_1, \qquad a_2 = \lambda b_2, \qquad a_3 = \lambda b_3.$$

Hence the printed ratio condition $a_1/b_1 = a_2/b_2 = a_3/b_3$ expresses collinearity when those ratios are defined. If a component of $\vec{b}$ is zero, use the scalar-multiple statement instead of dividing by it.

Step 2: Distinguish the other conditions Option (A) states that the dot product is zero, which describes perpendicular nonzero vectors. Option (C) describes equal vectors, a special case of collinearity but not its general condition; for example $\vec{a} = 2\vec{b}$ is also collinear. Option (D) merely equates component sums and does not force proportional components.

Step 3: Conclusion The corresponding component ratios are equal. Hence, the correct option is (B).

Quick Tip to Solve: Use a scalar multiple to test collinearity; use a zero dot product to test perpendicularity.

14. The magnitude of the vector $6\hat{i} - 2\hat{j} + 3\hat{k}$ is: [1 mark]

(A) 1

(B) 5

(C) 7

(D) 12

Correct Answer: (C) 7

Solution:

Concept:

- For $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$, the magnitude is $|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$.
- The unit vectors $\hat{i}, \hat{j}, \hat{k}$ are mutually perpendicular, so their squared contributions add.

Step 1: Substitute the components Here the three components are $6, -2, 3$. Squaring each component

Quick Tip to Solve: Square all three components, add them, then take the positive square root; never add the components directly.

removes its sign:

$$|6\hat{i} - 2\hat{j} + 3\hat{k}| = \sqrt{6^2 + (-2)^2 + 3^2} = \sqrt{36 + 4 + 9}.$$

Step 2: Take the nonnegative square root The sum of the squared components is 49, so the magnitude is $\sqrt{49} = 7$. Magnitude is always nonnegative. The alternatives 1, 5 and 12 have squares 1, 25 and 144, none equal to the required squared magnitude 49; adding signed components instead of their squares would not give the Euclidean length.

Step 3: Conclusion The magnitude is 7. Hence, the correct option is (C).

Quick Tip to Solve: Square all three components, add them, then take the positive square root; never add the components directly.

15. If a line makes angles of $90°$, $135°$ and $45°$ with the x, y and z axes respectively, then its direction cosines are: [1 mark]

(A) $0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$

(B) $-\frac{1}{\sqrt{2}},\ 0,\ \frac{1}{\sqrt{2}}$

(C) $\frac{1}{\sqrt{2}},\ 0,\ -\frac{1}{\sqrt{2}}$

(D) $0,\ \frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$

Correct Answer: (A) $0,\ -\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}$

Solution:

Concept:

- Direction cosines are $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$, where the angles are measured from the positive coordinate axes.
- They satisfy $l^2 + m^2 + n^2 = 1$; the signs depend on the given angles.

Step 1: Evaluate in axis order For the given angles,

$$l = \cos 90° = 0, \qquad m = \cos 135° = -\frac{1}{\sqrt{2}}, \qquad n = \cos 45° = \frac{1}{\sqrt{2}}.$$

The obtuse angle with the positive y-axis makes its cosine negative.

Step 2: Check normalization and options The squared cosines sum to $0 + \frac{1}{2} + \frac{1}{2} = 1$. Option (A) has exactly the required order and signs. Options (B) and (C) place zero at the y-coordinate instead of the x-coordinate. Option (D) has a positive y-cosine, contrary to $135°$.

Step 3: Conclusion The direction cosines are $\boxed{(0, -1/\sqrt{2}, 1/\sqrt{2})}$. Hence, the correct option is (A).

Quick Tip to Solve: Write the cosines in x, y, z order; an obtuse axis angle gives a negative cosine.

16. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$ is: [1 mark]

(A) 0°

(B) 30°

(C) 45°

(D) 90°

Correct Answer: (D) 90°

Solution:

Concept:

- For direction vectors $\mathbf{a}$ and $\mathbf{b}$, the acute angle satisfies $\cos\theta = |\mathbf{a}\cdot\mathbf{b}|/(|\mathbf{a}||\mathbf{b}|)$.
- In equal-multiple equations, direction ratios are reciprocals of the coefficients, not the coefficients themselves.

Step 1: Read direction ratios correctly Put $2x = 3y = -z = t$. Then $(x, y, z) = t(1/2, 1/3, -1)$, so a direction vector is $\mathbf{a} = (3, 2, -6)$. Similarly, putting $6x = -y = -4z = s$ gives $(x, y, z) = s(1/6, -1, -1/4)$; a direction vector is $\mathbf{b} = (2, -12, -3)$.

Step 2: Compute the scalar product

$$\mathbf{a}\cdot\mathbf{b} = 3(2) + 2(-12) + (-6)(-3) = 6 - 24 + 18 = 0.$$

Both vectors are nonzero, so their scalar product vanishes only when they are perpendicular. Thus $\cos\theta = 0$. The cosines for options (A), (B) and (C) are respectively $1, \sqrt{3}/2, 1/\sqrt{2}$, all nonzero.

Step 3: Conclusion The angle is $\boxed{90^\circ}$. Hence, the correct option is (D).

Quick Tip to Solve: Convert equal multiples into reciprocal direction ratios before taking the scalar product.

---

17. If for any two events A and B, $P(A) = \frac{4}{5}$ and $P(A \cap B) = \frac{7}{10}$, then $P(B/A)$ is equal to: [1 mark]

(A) $\frac{1}{10}$

(B) $\frac{1}{8}$

(C) $\frac{7}{8}$

(D) $\frac{17}{20}$

Correct Answer: (C) $\frac{7}{8}$

Solution:

Concept:

- Conditional probability is $P(B \mid A) = P(A \cap B)/P(A)$, provided $P(A) > 0$.
- The given event after the slash is the condition, so the denominator here is $P(A)$.

Step 1: Identify the conditioning event The notation $P(B/A)$ means the probability of B given that

Quick Tip to Solve: In P(B/A), use P(A) as the denominator and divide the intersection probability by it.

A has occurred. We are given $P(A) = 4/5 > 0$, so the conditional-probability formula applies directly; knowledge of $P(B)$ is unnecessary.

Step 2: Substitute and compare the options

$$P(B \mid A) = \frac{7/10}{4/5} = \frac{7}{10} \times \frac{5}{4} = \frac{7}{8}.$$

This lies between zero and one, as required. Option (A), $1/10$, is the difference of the given probabilities, not their ratio. Option (B), $1/8$, is the complementary conditional probability. Option (D), $17/20$, is also different from the required ratio.

Step 3: Conclusion $\boxed{P(B/A) = 7/8}$. Hence, the correct option is (C).

Quick Tip to Solve: In P(B/A), use P(A) as the denominator and divide the intersection probability by it.

18. Five fair coins are tossed simultaneously. The probability of the event that at least one head comes up is: [1 mark]

(A) $\frac{27}{32}$

(B) $\frac{5}{32}$

(C) $\frac{31}{32}$

(D) $\frac{1}{32}$

Correct Answer: (C) $\frac{31}{32}$

Solution:

Concept:

- For any event E, $P(E) = 1 - P(E^c)$, where E^c is its complement.
- Five independent fair coins give $2^5 = 32$ equally likely ordered outcomes.

Step 1: Use the complementary event Let E be the event of at least one head. Its complement is no heads at all, which means that all five coins show tails. There is exactly one such outcome, $TTTTT$, among the 32 equally likely outcomes.

Step 2: Calculate and reject other options

$$P(E^c) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}, \qquad P(E) = 1 - \frac{1}{32} = \frac{31}{32}.$$

Option (D) counts the excluded all-tail outcome. Option (B), $5/32$, counts exactly one head rather than at least one. Option (A), $27/32$, also does not include all 31 outcomes containing a head.

Step 3: Conclusion The required probability is $\boxed{31/32}$. Hence, the correct option is (C).

Quick Tip to Solve: For at least one success, subtract the probability of no successes from one.

19. Assertion (A): Two coins are tossed simultaneously. The probability of getting two heads, if it is known that at least one head comes up, is $\frac{1}{3}$.

Reason (R): Let E and F be two events with a random experiment, then $P(F/E) = \frac{P(E \cap F)}{P(E)}$.

[1 mark]

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).

(B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).

(C) (A) is true and (R) is false.

(D) (A) is false, but (R) is true.

Correct Answer: (A) Both (A) and (R) are true and (R) is the correct explanation of (A).

Solution:

Concept:

- Conditional probability obeys $P(F \mid E) = P(E \cap F)/P(E)$ for $P(E) > 0$.
- Two fair coins have equally likely outcomes HH, HT, TH, TT.

Step 1: Judge the assertion Let $E = \{HH, HT, TH\}$ denote at least one head and $F = \{HH\}$ denote two heads. Once E is known, the possible outcomes are HH, HT, TH, all equally likely. Exactly one gives two heads. Therefore $P(F \mid E) = 1/3$, so Assertion (A) is true.

Step 2: Judge the reason The reason states the standard conditional-probability formula. In this experiment $P(E) = 3/4 > 0$ and $P(E \cap F) = P(F) = 1/4$. Thus its hypotheses are satisfied, and Reason (R) is true.

Step 3: Check the explanatory link Applying the reason directly gives

$$P(F \mid E) = \frac{P(E \cap F)}{P(E)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Therefore the formula in the reason proves the numerical claim in the assertion. Option (B) wrongly denies that link, while options (C) and (D) incorrectly call one of the statements false.

Step 4: Conclusion Both statements are true and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

Quick Tip to Solve: Restrict the sample space to HH, HT and TH when at least one head is given.

20. Assertion (A): $\displaystyle\int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\,dx = 3.$

Reason (R): $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$ [1 mark]

(A) Both (A) and (R) are true and (R) is the correct explanation of (A).

(B) Both (A) and (R) are true, but (R) is not the correct explanation of (A).

(C) (A) is true and (R) is false.

(D) (A) is false, but (R) is true.

Correct Answer: (A) Both (A) and (R) are true and (R) is the correct explanation of (A).

Solution:

Concept:

- The reflection property is $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.
- When $f(x) + f(a+b-x) = 1$, adding the two equal integrals gives $2I = b - a$.

Step 1: Judge the assertion Let

$$I = \int_2^8 \frac{\sqrt{10-x}}{\sqrt{x}+\sqrt{10-x}}\,dx.$$

The integrand is continuous on $[2, 8]$, with a positive denominator. Reflecting x to $10 - x$ gives

$$I = \int_2^8 \frac{\sqrt{x}}{\sqrt{10-x}+\sqrt{x}}\,dx.$$

Adding the original and reflected forms yields $2I = \int_2^8 1\,dx = 6$, so $I = 3$. Assertion (A) is true.

Step 2: Judge the reason For an integrable function, substitute $u = a + b - x$. Then $dx = -du$, and the limits $x = a, b$ become $u = b, a$. Reversing these limits cancels the minus sign and proves the stated reflection identity. Thus Reason (R) is true.

Step 3: Check the explanatory link Here $a + b = 2 + 8 = 10$, so the reason interchanges precisely the two square roots in the integrand. Their complementary fractions add to one, proving the assertion. Option (B) wrongly denies this explanation; options (C) and (D) wrongly label a true statement false.

Step 4: Conclusion Both statements are true and Reason (R) correctly explains Assertion (A). Hence, the correct option is (A).

Quick Tip to Solve: When the bounds sum to 10, replace x by 10-x and add the two integral forms.

Section B (10 marks)

This section comprises of Very Short Answer (VSA) type questions of 2 marks each.

21. Write the domain and range (principal value branch) of the following function: $f(x) = \tan^{-1} x$.

[2 marks]

Solution:

Concept:

- The principal inverse $\tan^{-1}$ is defined by restricting $\tan y$ to $-\pi/2 < y < \pi/2$.
- On this interval, $\tan y$ is continuous, strictly increasing and takes every real value exactly once.

Step 1: Determine the domain Write $y = \tan^{-1} x$, so $x = \tan y$ with y on the principal branch. As y increases through $(-\pi/2, \pi/2)$, $\tan y$ takes all real values, approaching $-\infty$ at the left boundary and $+\infty$ at the right boundary. Hence an inverse value exists for every real input x, and the domain is $\mathbb{R} = (-\infty, \infty)$.

Step 2: Determine the principal-value range The output of the inverse must lie in the interval chosen to make tangent one-to-one. Thus $-\pi/2 < \tan^{-1} x < \pi/2$ for every real x. The endpoints are excluded: tangent is undefined at $\pm\pi/2$, and no finite real input gives either endpoint as an inverse value. For example, $\tan^{-1} 0 = 0$ lies within this interval. Therefore the principal-value range is the open interval $(-\pi/2, \pi/2)$.

Step 3: Conclusion $\boxed{\text{Domain}(f) = \mathbb{R}, \qquad \text{Range}(f) = (-\pi/2, \pi/2)}$.

Quick Tip to Solve: For an inverse, exchange the domain and range of the restricted original function.

22(a). If $f(x) = \begin{cases} x^2, & x \geq 1 \\ x, & x < 1 \end{cases}$, then show that f is not differentiable at $x = 1$. [2 marks]

Solution:

Concept:

- Differentiability at a point requires finite, equal left-hand and right-hand derivatives. Here $f(1) = 1$.
- The derivative at $x = 1$ is tested using $\lim_{h\to 0} \dfrac{f(1+h) - f(1)}{h}$; use the branch appropriate to the sign of h.

Step 1: Check the left-hand derivative For $h < 0$, we have $1 + h < 1$, so the linear branch gives $f(1 + h) = 1 + h$. Thus

$$f\prime_{-}(1) = \lim_{h\to 0^-} \frac{(1+h) - 1}{h} = \lim_{h\to 0^-} 1 = 1.$$

This is the slope when the point is approached from the left.

Step 2: Check the right-hand derivative For $h > 0$, the quadratic branch applies and $f(1+h) = (1+h)^2$.

Quick Tip to Solve: At the junction, use the left branch for negative increments and the right branch for positive increments; matching values do not imply matching slopes.

Hence

$$f\prime_+(1) = \lim_{h\to 0^+} \frac{(1+h)^2 - 1}{h} = \lim_{h\to 0^+} (2+h) = 2.$$

Both branches approach $1 = f(1)$, so the function is continuous at the junction; however, continuity alone does not guarantee differentiability. The two one-sided slopes are unequal.

Step 3: Conclusion Since $f\prime_-(1) = 1 \neq 2 = f\prime_+(1)$, f is not differentiable at $x = 1$.

Quick Tip to Solve: At the junction, use the left branch for negative increments and the right branch for positive increments; matching values do not imply matching slopes.

OR

22(b). Find the value(s) of λ, if the function $f(x) = \begin{cases} \frac{\sin^2 \lambda x}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ is continuous at $x = 0$. [2 marks]

Solution:

Concept:

- Continuity at zero requires $\lim_{x\to 0} f(x) = f(0) = 1$.
- The standard trigonometric limit is $\lim_{t\to 0} \sin t/t = 1$, with angles measured in radians. The parameter λ is a fixed real number.

Step 1: Evaluate the limiting value For $\lambda \neq 0$ and $x \neq 0$, factor the expression as

$$\frac{\sin^2(\lambda x)}{x^2} = \lambda^2 \left(\frac{\sin(\lambda x)}{\lambda x}\right)^2.$$

As $x \to 0$, $\lambda x \to 0$ from the appropriate side, and the ratio tends to 1. Therefore the two-sided limit is λ^2. If $\lambda = 0$, the numerator vanishes for every nonzero x, so the limit is zero, again equal to λ^2.

Step 2: Apply continuity and check both values The limiting value must equal the separately defined value at zero:

$$\lambda^2 = 1 \quad \Longrightarrow \quad (\lambda - 1)(\lambda + 1) = 0.$$

For either candidate, squaring the sine makes the sign of λ immaterial, and the limit equals 1. The value $\lambda = 0$ does not satisfy continuity because its limit is zero while $f(0) = 1$.

Step 3: Conclusion The required values are $\boxed{\lambda = 1 \text{ or } \lambda = -1}$.

Quick Tip to Solve: Factor out the square of the parameter before using the sine limit; keep both roots when the continuity condition is $\lambda^2 = 1$.

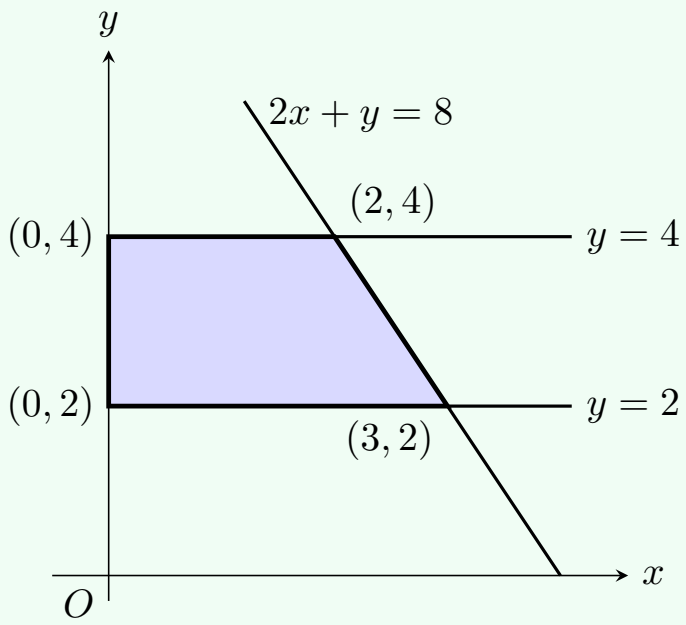
23. Sketch the region bounded by the lines $2x + y = 8$, $y = 2$, $y = 4$ and the y-axis. Hence, obtain its area using integration. [2 marks]

Solution:

Concept:

- The area between a right boundary $x_R(y)$ and a left boundary $x_L(y)$ is $\int [x_R(y) - x_L(y)]\, dy$.
- Here the horizontal boundaries are $y = 2$ and $y = 4$. The left boundary is the y-axis, $x = 0$, and the right boundary is $x = (8 - y)/2$.

Step 1: Locate and sketch the bounded region On the y-axis the horizontal lines give $(0, 2)$ and $(0, 4)$. Substituting $y = 2$ and $y = 4$ into $2x + y = 8$ gives $(3, 2)$ and $(2, 4)$, respectively. Thus the region is the quadrilateral with vertices $(0, 2), (3, 2), (2, 4), (0, 4)$, shaded in the sketch. For every y between 2 and 4, a horizontal strip extends from the axis to the sloping line.



The shaded quadrilateral bounded by the four given lines.

Step 2: Integrate the horizontal strip widths The width of a strip is $(8 - y)/2 - 0$, so its infinitesimal area is $(4 - y/2)\, dy$. Therefore

$$\begin{aligned} A &= \int_2^4 \left(4 - \frac{y}{2}\right) dy \\ &= \left[4y - \frac{y^2}{4}\right]_2^4 \\ &= (16 - 4) - (8 - 1) = 12 - 7. \end{aligned}$$

This integrates positive strip widths throughout the entire shaded region.

Step 3: Conclusion The area of the bounded region is $\boxed{5 \text{ square units}}$.

Quick Tip to Solve: Use horizontal strips: the fixed limits are 2 and 4, and the strip width is immediately $(8 - y)/2$.

24(a). **If the vectors $\vec{a}$ and $\vec{b}$ are such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{2}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector, then find the angle between $\vec{a}$ and $\vec{b}$.** [2 marks]

Solution:

Concept:

- For nonzero vectors with included angle θ, $|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta$, where $0 \le \theta \le \pi$.
- A unit vector has magnitude 1; therefore the given cross product has magnitude 1, irrespective of its direction.

Step 1: Substitute the given magnitudes Using $|\vec{a}| = 3$ and $|\vec{b}| = 2/3$, we obtain

$$1 = 3 \cdot \frac{2}{3} \sin\theta = 2\sin\theta.$$

Dividing by 2 gives $\sin\theta = 1/2$. The vectors are nonzero, so their included angle is well defined. The unit-vector condition concerns the cross product, and does not say that either original vector has unit magnitude.

Step 2: Find every included angle In the interval $[0, \pi]$, sine is positive in both the first and the second quadrants. Its value is $1/2$ at

$$\theta = \frac{\pi}{6} \quad \text{and} \quad \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}.$$

The cross-product magnitude alone cannot distinguish these supplementary angles: both give $2\sin\theta = 1$. No dot-product sign or acute-angle restriction is stated, so both possibilities must be retained.

Step 3: Conclusion The angle between the vectors can be $\boxed{30^\circ \text{ or } 150^\circ}$.

Quick Tip to Solve: A cross-product magnitude determines the sine of the angle; retain the supplementary angle unless the question specifies an acute angle.

OR

24(b). **Find the area of a parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.** [2 marks]

Solution:

Concept:

- The area of a parallelogram with adjacent side vectors $\vec{a}$ and $\vec{b}$ is $|\vec{a} \times \vec{b}|$, because this magnitude equals base times perpendicular height.
- For $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$, its magnitude is $\sqrt{c_1^2 + c_2^2 + c_3^2}$.

Step 1: Calculate the cross product Using the given vectors as the second and third rows of the

Quick Tip to Solve: Expand the cross-product determinant with the minus sign on the middle cofactor, then take its magnitude.

determinant,

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$
$$= \hat{i}[(-1)(1) - 3(-7)] - \hat{j}[(1)(1) - 3(2)] + \hat{k}[(1)(-7) - (-1)(2)]$$
$$= 20\hat{i} + 5\hat{j} - 5\hat{k}.$$

The middle cofactor carries a minus sign; thus its component is $+5$, not -5.

Step 2: Take the magnitude The parallelogram is non-degenerate since the cross product is nonzero. Therefore its area is

$$|\vec{a} \times \vec{b}| = \sqrt{20^2 + 5^2 + (-5)^2} = \sqrt{450} = \sqrt{225 \cdot 2}.$$

This quantity is an area, so it is expressed in square units.

Step 3: Conclusion The area of the parallelogram is $\boxed{15\sqrt{2} \text{ square units}}$.

Quick Tip to Solve: Expand the cross-product determinant with the minus sign on the middle cofactor, then take its magnitude.

25. Find the vector and the cartesian equations of a line that passes through the point $A(1, 2, -1)$ and is parallel to the line $5x - 25 = 14 - 7y = 35z$. [2 marks]

Solution:

Concept:

- A line through position vector $\vec{a}$, parallel to a nonzero vector $\vec{d}$, has vector equation $\vec{r} = \vec{a} + \lambda\vec{d}$, where $\lambda \in \mathbb{R}$.
- The symmetric Cartesian equation uses the direction ratios as denominators: $(x - x_0)/d_1 = (y - y_0)/d_2 = (z - z_0)/d_3$.

Step 1: Extract the direction ratios Let the common value in the given equation be t. Then

$$5x - 25 = t, \qquad 14 - 7y = t, \qquad 35z = t.$$

Solving each relation for its coordinate gives

$$x = 5 + \frac{t}{5}, \qquad y = 2 - \frac{t}{7}, \qquad z = \frac{t}{35}.$$

Thus the direction ratios are proportional to $1/5, -1/7, 1/35$. Multiplying all three by 35 gives $7, -5, 1$. The negative sign in the second ratio comes from the coefficient -7 of y.

Step 2: Use the prescribed point For a parallel line the direction ratios stay the same, while its position vector is that of $A(1, 2, -1)$. Hence its parametric equations are

$$x = 1 + 7\lambda, \qquad y = 2 - 5\lambda, \qquad z = -1 + \lambda.$$

Quick Tip to Solve: Introduce a common parameter first: coefficients in the given equal expressions are not themselves direction ratios.

At $\lambda = 0$ these give the required point. Eliminating the same parameter gives the symmetric form, and the coefficient of λ gives its vector form.

Step 3: Conclusion The required equations are

$$\boxed{\vec{r} = (\hat{i} + 2\hat{j} - \hat{k}) + \lambda(7\hat{i} - 5\hat{j} + \hat{k}), \quad \lambda \in \mathbb{R}}$$

and

$$\boxed{\frac{x-1}{7} = \frac{y-2}{-5} = z + 1}.$$

Quick Tip to Solve: Introduce a common parameter first: coefficients in the given equal expressions are not themselves direction ratios.

Section C (18 marks)

This section comprises of Short Answer (SA) type questions of 3 marks each.

26. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix}$, then show that $A^3 - 23A - 40I = O$. [3 marks]

Solution:

Concept:

- Matrix multiplication uses row-by-column products: $(AB)_{ij} = \sum_k a_{ik} b_{kj}$. Matrix powers mean repeated matrix multiplication.
- The identity matrix I has diagonal entries 1 and off-diagonal entries 0; O is the zero matrix.

Step 1: Compute the square of the matrix Multiplying the given matrix by itself,

$$A^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{pmatrix}.$$

For example, its first row is obtained from

$$1 + 6 + 12 = 19, \qquad 2 - 4 + 6 = 4, \qquad 3 + 2 + 3 = 8.$$

Its second row is $(3{-}6{+}4,\ 6{+}4{+}2,\ 9{-}2{+}1) = (1, 12, 8)$; its third row is $(4{+}6{+}4,\ 8{-}4{+}2,\ 12{+}2{+}1) = (14, 6, 15)$.

Step 2: Compute the cube Now multiply A^2 by A, again using row-by-column products:

$$A^3 = \begin{pmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{pmatrix}.$$

Quick Tip to Solve: Calculate the square first, then compare the cube with the full matrix of 23A+40I; add 40 only on the diagonal.

For example, the first row gives $19+12+32=63$, $38-8+16=46$, and $57+4+8=69$. The middle diagonal entry is $2-24+16=-6$.

Step 3: Evaluate the polynomial The scalar multiple and identity term give

$$23A+40I=\begin{pmatrix}23 & 46 & 69\\ 69 & -46 & 23\\ 92 & 46 & 23\end{pmatrix}+\begin{pmatrix}40 & 0 & 0\\ 0 & 40 & 0\\ 0 & 0 & 40\end{pmatrix}=\begin{pmatrix}63 & 46 & 69\\ 69 & -6 & 23\\ 92 & 46 & 63\end{pmatrix}.$$

Every entry equals the corresponding entry of A^3, so their entrywise difference is zero.

Step 4: Conclusion Hence $\boxed{A^3-23A-40I=O}$, as required.

Quick Tip to Solve: Calculate the square first, then compare the cube with the full matrix of 23A+40I; add 40 only on the diagonal.

27(a). Differentiate $\sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$ **w.r.t.** $\sin^{-1}\left(2x\sqrt{1-x^2}\right)$**.** [3 marks]

Solution:

Concept:

- For functions $u(x)$ and $v(x)$, $du/dv=(du/dx)/(dv/dx)$ on a differentiable branch with $dv/dx\neq 0$.
- Principal inverse trigonometric values must be respected: $\sqrt{x^2}=|x|$, and $\sqrt{(1-2x^2)^2}=|1-2x^2|$.

Step 1: Differentiate the first function Put

$$u=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right),\qquad v=\sin^{-1}\left(2x\sqrt{1-x^2}\right).$$

The real domain of u is $-1<x<1$. Its argument is positive and at least 1. With $f=(1-x^2)^{-1/2}$, we have $f'=x(1-x^2)^{-3/2}$ and

$$\sqrt{f^2-1}=\frac{|x|}{\sqrt{1-x^2}}.$$

Consequently, for $x\neq 0$,

$$\frac{du}{dx}=\frac{f'}{f\sqrt{f^2-1}}=\frac{x}{|x|\sqrt{1-x^2}}.$$

Equivalently $u=\sin^{-1}|x|$, which also shows its cusp at zero.

Step 2: Differentiate the second function For $g=2x\sqrt{1-x^2}$, the product rule gives

$$g'=\frac{2(1-2x^2)}{\sqrt{1-x^2}},\qquad 1-g^2=1-4x^2(1-x^2)=(1-2x^2)^2.$$

Thus, on intervals excluding $x=\pm 1/\sqrt{2}$,

$$\frac{dv}{dx}=\frac{2(1-2x^2)}{\sqrt{1-x^2}\,|1-2x^2|}.$$

Quick Tip to Solve: When simplifying the inverse sine of a double-angle expression, check its principal range before cancelling absolute values.

Keeping the absolute value prevents an incorrect use of $\sin^{-1}(\sin 2\theta) = 2\theta$ outside its principal interval.

Step 3: Divide the derivatives and identify the branches The ratio simplifies to

$$\frac{du}{dv} = \frac{x\,|1-2x^2|}{2|x|(1-2x^2)}.$$

It equals $1/2$ on $(-1, -1/\sqrt{2})$ and $(0, 1/\sqrt{2})$, and $-1/2$ on $(-1/\sqrt{2}, 0)$ and $(1/\sqrt{2}, 1)$. At zero, u has a cusp. At $\pm 1/\sqrt{2}$, v turns at an inverse-sine endpoint, giving different branch slopes. Therefore an ordinary two-sided derivative with respect to v is not defined there.

Step 4: Conclusion On the commonly used branch $0 < x < 1/\sqrt{2}$, $\boxed{du/dv = 1/2}$. In general, on the differentiable intervals of the printed functions,

$$\boxed{\frac{du}{dv} = \frac{x\,|1-2x^2|}{2|x|(1-2x^2)}}, \quad -1 < x < 1, \quad x \notin \{0, \pm 1/\sqrt{2}\}.$$

Quick Tip to Solve: When simplifying the inverse sine of a double-angle expression, check its principal range before cancelling absolute values.

OR

27(b). If $y = \tan x + \sec x$, then prove that $\dfrac{d^2y}{dx^2} = \dfrac{\cos x}{(1-\sin x)^2}$. [3 marks]

Solution:

Concept:

- The second derivative is obtained by differentiating the first derivative once more: $y'' = \dfrac{d}{dx}(y')$.
- Use $\tan x = \dfrac{\sin x}{\cos x}$, $\sec x = \dfrac{1}{\cos x}$, and $\cos^2 x = (1-\sin x)(1+\sin x)$. The given function is defined wherever $\cos x \neq 0$.

Step 1: Rewrite the given function Combining the two terms over a common denominator gives

$$y = \frac{1+\sin x}{\cos x}.$$

This form lets us differentiate using the quotient rule and then simplify with the Pythagorean identity. All the following steps apply on any interval where the original function is defined; in particular, $1 - \sin x \neq 0$ there.

Step 2: Find and simplify the first derivative Apply the quotient rule carefully, noting that the

Quick Tip to Solve: Simplify the first derivative to $1/(1-\sin x)$ before applying the chain rule a second time.

derivative of the denominator is $-\sin x$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos x \cos x - (1+\sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin x + \sin^2 x}{\cos^2 x} \\ &= \frac{1+\sin x}{\cos^2 x} = \frac{1}{1-\sin x}.\end{aligned}$$

The cancellation is valid because $\cos x \neq 0$ implies $1+\sin x \neq 0$. Simplifying before differentiating again avoids a lengthy product-rule calculation.

Step 3: Differentiate once more Write the simplified first derivative as $(1-\sin x)^{-1}$. By the chain rule,

$$\begin{aligned}\frac{d^2y}{dx^2} &= (-1)(1-\sin x)^{-2}\frac{d}{dx}(1-\sin x) \\ &= (-1)(1-\sin x)^{-2}(-\cos x).\end{aligned}$$

The two negative factors cancel. This expression has exactly the denominator required in the question.

Step 4: Conclusion

$$\boxed{\frac{d^2y}{dx^2} = \frac{\cos x}{(1-\sin x)^2}}$$

Hence the stated identity is proved wherever $\tan x + \sec x$ is defined.

Quick Tip to Solve: Simplify the first derivative to $1/(1-\sin x)$ before applying the chain rule a second time.

28(a). Evaluate: $\displaystyle\int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx$. [3 marks]

Solution:

Concept:

- For a continuous function f on $[a,b]$, the reflection property is $\displaystyle\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$.
- Here $\sin(2\pi - x) = -\sin x$, while $e^{\sin x} > 0$, so the denominator never vanishes on the interval.

Step 1: Reflect the interval Let

$$I = \int_0^{2\pi} \frac{1}{1+e^{\sin x}}\,dx.$$

Use the substitution $t = 2\pi - x$. Then $dx = -dt$; the bounds change from $0, 2\pi$ to $2\pi, 0$. Reversing these bounds absorbs the minus sign, so the same integral becomes

$$I = \int_0^{2\pi} \frac{1}{1+e^{-\sin t}}\,dt.$$

Renaming the integration variable t as x does not change the value of this definite integral.

Step 2: Put the reflected integrand in a useful form Multiply numerator and denominator by $e^{\sin x}$,

Quick Tip to Solve: Pair x with $2\pi - x$: the sine changes sign and the two exponential fractions add to one.

which is nonzero:

$$I = \int_0^{2\pi} \frac{e^{\sin x}}{1 + e^{\sin x}}\, dx.$$

The original and reflected integrands now have the same denominator. Their numerators add to that denominator, so their sum is identically one. This is why reflection is preferable to attempting an indefinite antiderivative.

Step 3: Add the two equal representations By linearity of definite integrals,

$$\begin{aligned} 2I &= \int_0^{2\pi} \left(\frac{1}{1 + e^{\sin x}} + \frac{e^{\sin x}}{1 + e^{\sin x}} \right) dx \\ &= \int_0^{2\pi} 1\, dx = [x]_0^{2\pi} = 2\pi. \end{aligned}$$

There is no extra constant of integration in a definite integral. Dividing this equality by two evaluates the requested quantity.

Step 4: Conclusion

$$\boxed{\int_0^{2\pi} \frac{1}{1 + e^{\sin x}}\, dx = \pi}.$$

Quick Tip to Solve: Pair x with $2\pi - x$: the sine changes sign and the two exponential fractions add to one.

OR

28(b). Find: $\displaystyle\int \frac{x^4}{(x-1)(x^2+1)}\, dx.$ [3 marks]

Solution:

Concept:

- For an improper rational function, first divide the numerator by the denominator. Decompose the remaining proper fraction into partial fractions.
- Use $\int \frac{dx}{x-1} = \ln|x-1|$, $\int \frac{x\,dx}{x^2+1} = \frac{1}{2}\ln(x^2+1)$, and $\int \frac{dx}{x^2+1} = \tan^{-1} x$.

Step 1: Divide the polynomials The denominator is $(x-1)(x^2+1) = x^3 - x^2 + x - 1$. Since the numerator has higher degree, polynomial division is necessary. The identity

$$x^4 = (x+1)(x-1)(x^2+1) + 1$$

gives

$$\frac{x^4}{(x-1)(x^2+1)} = x + 1 + \frac{1}{(x-1)(x^2+1)}.$$

The remainder is the constant 1, so the final fraction is proper.

Quick Tip to Solve: Check the division quickly using $(x+1)(x-1)(x^2+1) = x^4 - 1$; the remainder is exactly one.

Step 2: Determine the partial fractions Set

$$\frac{1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+D}{x^2+1}.$$

Multiplying by the denominator gives

$$1 = A(x^2+1) + (Bx+D)(x-1).$$

Equating coefficients of x^2, x and the constant term yields

$$A + B = 0, \qquad D - B = 0, \qquad A - D = 1.$$

Thus $A = \frac{1}{2}$, $B = D = -\frac{1}{2}$, and the original integrand is

$$x + 1 + \frac{1}{2(x-1)} - \frac{x}{2(x^2+1)} - \frac{1}{2(x^2+1)}.$$

Step 3: Integrate each term Integrate the polynomial and the three proper fractions separately:

$$\int x\,dx = \frac{x^2}{2}, \qquad \int 1\,dx = x,$$
$$\int \frac{dx}{2(x-1)} = \frac{1}{2}\ln|x-1|,$$
$$-\frac{1}{2}\int \frac{x\,dx}{x^2+1} = -\frac{1}{4}\ln(x^2+1),$$
$$-\frac{1}{2}\int \frac{dx}{x^2+1} = -\frac{1}{2}\tan^{-1}x.$$

For the logarithmic term involving x^2+1, the substitution $u = x^2+1$ gives $du = 2x\,dx$. Since $x^2+1 > 0$, no absolute-value sign is needed for that logarithm. The constant of integration is arbitrary on each interval avoiding $x = 1$.

Step 4: Conclusion

$$\boxed{\frac{x^2}{2} + x + \frac{1}{2}\ln|x-1| - \frac{1}{4}\ln(x^2+1) - \frac{1}{2}\tan^{-1}x + C}$$

Quick Tip to Solve: Check the division quickly using $(x+1)(x-1)(x^2+1) = x^4 - 1$; the remainder is exactly one.

29. Find the area of the following region using integration: $\{(x, y) : y^2 \leq 2x$ and $y \geq x - 4\}$. [3 marks]

Solution:

Concept:

- The inequalities determine the horizontal boundaries: $y^2 \leq 2x$ gives $x \geq y^2/2$, while $y \geq x - 4$ gives $x \leq y + 4$.
- For horizontal strips, the area is $A = \int_{y_1}^{y_2} (x_{\text{right}} - x_{\text{left}})\, dy$, where y_1, y_2 are the ordinates of the intersections.

Step 1: Determine the intersection ordinates The bounding curves are the parabola $x = y^2/2$ and the straight line $x = y + 4$. At an intersection their values of x agree, so

$$\frac{y^2}{2} = y + 4,$$
$$y^2 - 2y - 8 = 0,$$
$$(y - 4)(y + 2) = 0.$$

Thus the ordinates are $y = -2$ and $y = 4$, corresponding to the intersection points $(2, -2)$ and $(8, 4)$. These supply the limits of integration.

Step 2: Set up horizontal strips At each fixed y, both inequalities hold precisely when

$$\frac{y^2}{2} \leq x \leq y + 4.$$

The interval is nonempty when $y^2 - 2y - 8 \leq 0$, namely $-2 \leq y \leq 4$. Hence the line is the right boundary and the parabola is the left boundary throughout the required region. The strip width is $y + 4 - y^2/2$; integrating with respect to y avoids splitting the area into separate pieces.

Step 3: Evaluate the area integral Therefore

$$\begin{aligned} A &= \int_{-2}^{4} \left(y + 4 - \frac{y^2}{2} \right) dy \\ &= \left[\frac{y^2}{2} + 4y - \frac{y^3}{6} \right]_{-2}^{4} \\ &= \left(8 + 16 - \frac{64}{6} \right) - \left(2 - 8 + \frac{8}{6} \right) \\ &= \frac{40}{3} - \left(-\frac{14}{3} \right) = \frac{54}{3}. \end{aligned}$$

The integrand is nonnegative over the full interval, so this integral directly represents the area, with no signed-area cancellation.

Step 4: Conclusion The area of the given region is $\boxed{18 \text{ square units}}$.

Quick Tip to Solve: Rewrite both inequalities as bounds on x; horizontal strips then give one integral from $y = -2$ to $y = 4$.

30(a). **Find the coordinates of the foot of the perpendicular drawn from the point $P(0,2,3)$ to the line $\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}$.** [3 marks]

Solution:

Concept:

- A point on a line in symmetric form can be found by setting each ratio equal to a parameter t. The denominators give a direction vector of the line.
- If H is the foot from P and $\vec{d}$ is the line direction, then $\overrightarrow{PH}\cdot\vec{d}=0$, since the joining vector is perpendicular to the line.

Step 1: Write a general point on the given line Set

$$\frac{x+3}{5}=\frac{y-1}{2}=\frac{z+4}{3}=t.$$

Then any point on this line has coordinates

$$H=(-3+5t,\ 1+2t,\ -4+3t).$$

The line passes through $A=(-3,1,-4)$ and has direction vector $\vec{d}=(5,2,3)$. Taking the foot in this parameterised form automatically ensures that it lies on the given line.

Step 2: Use the perpendicularity condition For $P=(0,2,3)$, the vector from P to H is

$$\overrightarrow{PH}=(-3+5t,\ -1+2t,\ -7+3t).$$

It must be perpendicular to $(5,2,3)$. Consequently,

$$\begin{aligned}5(-3+5t)+2(-1+2t)+3(-7+3t)&=0,\\ -15+25t-2+4t-21+9t&=0,\\ 38t-38&=0.\end{aligned}$$

Thus $t=1$. The coefficient $38=5^2+2^2+3^2$ is positive, so the condition determines a unique foot.

Step 3: Find and check the coordinates Substituting $t=1$ gives

$$H=(-3+5,\ 1+2,\ -4+3)=(2,3,-1).$$

To verify that this is on the line, its three ratios are $5/5=2/2=3/3=1$. Also, $\overrightarrow{PH}=(2,1,-4)$, and

$$(2,1,-4)\cdot(5,2,3)=10+2-12=0.$$

Thus the point satisfies both requirements: membership of the given line and perpendicularity of the segment joining it to P.

Step 4: Conclusion The foot of the perpendicular is $\boxed{H(2,3,-1)}$.

Quick Tip to Solve: Parameterise the line first and dot the vector from the external point to the general point with $(5,2,3)$.

OR

30(b). Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ satisfy the condition $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Evaluate the quantity $\mu=\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}$, if $|\vec{a}|=3$, $|\vec{b}|=4$ and $|\vec{c}|=2$. [3 marks]

Solution:

Concept:

- For vectors, the scalar square satisfies $\vec{v}\cdot\vec{v}=|\vec{v}|^2$, where $|\vec{v}|$ is its magnitude.
- The dot product distributes over addition and is symmetric, so expanding $|\vec{a}+\vec{b}+\vec{c}|^2$ produces each pairwise dot product twice.

Step 1: Square the given vector relation The condition is $\vec{a}+\vec{b}+\vec{c}=\vec{0}$. Taking the dot product of each side with itself gives

$$(\vec{a}+\vec{b}+\vec{c})\cdot(\vec{a}+\vec{b}+\vec{c})=0.$$

This is a scalar equation. It connects the known magnitudes to the required sum of dot products without requiring the directions or angles of the individual vectors.

Step 2: Expand and identify the required quantity By distributivity of the dot product,

$$\begin{aligned}0&=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2\\&\quad+2\big(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}\big).\end{aligned}$$

The factor 2 occurs because, for example, $\vec{a}\cdot\vec{b}$ and $\vec{b}\cdot\vec{a}$ are equal and both appear in the expansion. With the quantity in parentheses denoted by μ, the relation becomes

$$2\mu=-\big(|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2\big).$$

Step 3: Substitute the magnitudes Using the given magnitudes $3, 4, 2$,

$$\begin{aligned}2\mu&=-(3^2+4^2+2^2)\\&=-(9+16+4)=-29,\\\mu&=-\frac{29}{2}.\end{aligned}$$

The negative sign is consistent with the zero vector sum: the pairwise cross terms must cancel the positive sum of the three squared magnitudes. In particular, dot products cannot be replaced by products of magnitudes, since the angles between these vectors are not zero.

Step 4: Conclusion The required value is $\boxed{\mu=-\frac{29}{2}}$.

Quick Tip to Solve: Square the zero vector sum; the sum of all pairwise dot products is minus half the sum of squared magnitudes.

31. Find the distance between the lines:

$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$

$\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(4\hat{i} + 6\hat{j} + 12\hat{k})$ [3 marks]

Solution:

Concept:

- For parallel lines through position vectors $\vec{a}_1, \vec{a}_2$ with common direction $\vec{b}$, the perpendicular distance is $d = \dfrac{|(\vec{a}_2 - \vec{a}_1) \times \vec{b}|}{|\vec{b}|}$.
- Multiplying a direction vector by a nonzero scalar leaves its direction unchanged. The cross product gives the area of a parallelogram whose height is the required distance.

Step 1: Check the directions of the lines Choose points with position vectors $\vec{a}_1 = (1, 2, -4)$ and $\vec{a}_2 = (3, 3, -5)$. The first direction vector is $\vec{b} = (2, 3, 6)$. The second direction vector is $(4, 6, 12) = 2\vec{b}$, so the two lines are parallel. Thus their distance is the perpendicular separation, found using one common direction vector.

Step 2: Calculate the cross product and its magnitude The vector joining the selected points is $\vec{a}_2 - \vec{a}_1 = (2, 1, -1)$. Therefore,

$$(\vec{a}_2 - \vec{a}_1) \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 2 & 3 & 6 \end{vmatrix} = 9\hat{i} - 14\hat{j} + 4\hat{k}.$$

Its magnitude is $\sqrt{9^2 + (-14)^2 + 4^2} = \sqrt{293}$. Also, $|\vec{b}| = \sqrt{2^2 + 3^2 + 6^2} = 7$. Dividing the area represented by the cross product by the base length $|\vec{b}|$ yields the required perpendicular height. For verification, the projection formula gives the squared distance as $6 - \dfrac{(4 + 3 - 6)^2}{49} = \dfrac{293}{49}$, agreeing with this calculation.

Step 3: Conclusion The distance between the lines is $\boxed{\dfrac{\sqrt{293}}{7} \text{ units}}$.

Quick Tip to Solve: Check for proportional direction vectors before selecting a distance formula, and use either common direction vector consistently.

Section D (20 marks)

This section comprises of Long Answer (LA) type questions of 5 marks each.

32(a). The median of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ cm/s. Find the rate at which its side is increasing. [5 marks]

Solution:

Concept:

- A median joins a vertex to the midpoint of the opposite side. In an equilateral triangle, this median is also an altitude and bisects the triangle into two congruent right triangles.
- Related rates are obtained by differentiating a geometric relation with respect to time t. If a is the side length and m the median length, their rates are da/dt and dm/dt.

Step 1: Express the median in terms of the side Let the side of the equilateral triangle at time t be a cm, and its median be m cm. Since all three sides are equal, the median is perpendicular to the opposite side and divides that side into two segments of length $a/2$. Each resulting right triangle has hypotenuse a, one leg $a/2$, and the other leg m. By Pythagoras' theorem,

$$m^2 + \left(\frac{a}{2}\right)^2 = a^2, \qquad m^2 = \frac{3a^2}{4}.$$

Both lengths are positive, so the required geometric relation is $m = \dfrac{\sqrt{3}}{2}a$. This proportionality holds at each instant because the triangle remains equilateral.

Step 2: Differentiate the relation with respect to time Both a and m change with time; $\sqrt{3}/2$ is constant. Differentiating therefore gives

$$\frac{dm}{dt} = \frac{\sqrt{3}}{2}\frac{da}{dt}, \qquad \frac{da}{dt} = \frac{2}{\sqrt{3}}\frac{dm}{dt}.$$

The statement that the median is increasing specifies the positive rate $\dfrac{dm}{dt} = 2\sqrt{3}$ cm/s. Substituting into the relation between the rates,

$$\frac{da}{dt} = \frac{2}{\sqrt{3}}\left(2\sqrt{3}\right) = 4 \text{ cm/s}.$$

The positive sign confirms that the side increases. No particular initial side length is needed: the median is always the same constant fraction of the side, so their rates have the same constant ratio. Conversely, multiplying the calculated side rate by $\sqrt{3}/2$ returns $2\sqrt{3}$ cm/s, verifying the given rate.

Step 3: Conclusion The side of the equilateral triangle is increasing at $\boxed{4 \text{ cm/s}}$.

Quick Tip to Solve: Use the equilateral-triangle relation $m = (\sqrt{3}/2)a$ before differentiating; a constant length ratio also gives the rate ratio.

OR

32(b). Sum of two numbers is 5. If the sum of the cubes of these numbers is least, then find the sum of the squares of these numbers. [5 marks]

Solution:

Concept:

- Express both numbers in terms of one variable using their fixed sum. If one number is x, the other is $5 - x$, where $x \in \mathbb{R}$.
- At a stationary point of a differentiable function $S(x)$, $S'(x) = 0$. If $S''(x) > 0$, the stationary point is a minimum.
- The identity $x^2 + y^2 = (x + y)^2 - 2xy$ gives the required sum of squares once the minimizing numbers are known.

Step 1: Form the function to be minimized Let the two numbers be x and y. Since $x + y = 5$, we have $y = 5 - x$. Their sum of cubes is therefore

$$\begin{aligned} S(x) &= x^3 + (5 - x)^3 \\ &= x^3 + 125 - 75x + 15x^2 - x^3 \\ &= 15x^2 - 75x + 125. \end{aligned}$$

Thus the fixed sum cancels the cubic terms, leaving a quadratic.

Step 2: Locate the stationary point Differentiating gives $S'(x) = 30x - 75$. The necessary condition for an interior minimum is $S'(x) = 0$, so $30x - 75 = 0$, whence $x = \frac{75}{30} = \frac{5}{2}$. Since the given sum is fixed throughout, this determines the pair uniquely. The second number is then $y = 5 - \frac{5}{2} = \frac{5}{2}$.

Step 3: Verify that the least value is attained Since $S''(x) = 30 > 0$, the stationary point is a minimum. Completing the square verifies that it is the absolute minimum:

$$S(x) = 15\left(x - \frac{5}{2}\right)^2 + \frac{125}{4}.$$

The square is nonnegative for all real x, with equality only at $x = \frac{5}{2}$. This proves that the minimum is global.

Step 4: Calculate the requested sum of squares Using the minimizing values in the expression requested in the question,

$$x^2 + y^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2 = \frac{25}{4} + \frac{25}{4} = \frac{25}{2}.$$

Step 5: Conclusion The sum of cubes is least when both numbers are $\frac{5}{2}$. The required sum of their squares is $\boxed{\frac{25}{2}}$.

Quick Tip to Solve: Substitute the second number as $5 - x$; the cubic terms cancel, leaving an upward-opening quadratic whose vertex gives the minimizing pair.

33. Evaluate: $\int_0^{\pi/2} \sin 2x \, \tan^{-1}(\sin x)\, dx$**.** [5 marks]

Solution:

Concept:

- Use $\sin 2x = 2\sin x \cos x$. The substitution $t = \sin x$ then gives $dt = \cos x\, dx$ and changes the limits from $0, \frac{\pi}{2}$ to $0, 1$.
- Integration by parts states $\int u\, dv = uv - \int v\, du$. Here $\tan^{-1} t$ denotes the principal inverse tangent and $\frac{d}{dt}\tan^{-1} t = \frac{1}{1+t^2}$.
- Simplify the rational expression using $\frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}$, so the remaining integrals are elementary.

Step 1: Transform the definite integral Denote the given integral by I. Apply the double-angle identity to obtain

$$I = 2\int_0^{\pi/2} \sin x \cos x \, \tan^{-1}(\sin x)\, dx.$$

Put $t = \sin x$, giving $dt = \cos x\, dx$. At the lower limit $x = 0$, $t = 0$, and at the upper limit $x = \frac{\pi}{2}$, $t = 1$. Consequently,

$$I = 2\int_0^1 t \tan^{-1} t\, dt.$$

The substitution is increasing on this interval, so the order of the transformed limits stays the same. All functions involved are continuous on the interval of integration.

Step 2: Integrate by parts Choose $u = \tan^{-1} t$ and $dv = 2t\, dt$. Then $du = \frac{dt}{1+t^2}$ and $v = t^2$. Integration by parts gives

$$I = \left[t^2 \tan^{-1} t\right]_0^1 - \int_0^1 \frac{t^2}{1+t^2}\, dt.$$

The boundary term at $t = 1$ is $\tan^{-1} 1 = \frac{\pi}{4}$, whereas the term at $t = 0$ is zero. Thus $I = \frac{\pi}{4} - \int_0^1 \frac{t^2}{1+t^2}\, dt$.

Step 3: Evaluate the remaining rational integral Write $t^2 = (1 + t^2) - 1$. Dividing by $1 + t^2$, which is positive on $[0, 1]$, yields

$$\frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}.$$

Therefore,

$$\begin{aligned}\int_0^1 \frac{t^2}{1+t^2}\, dt &= \left[t - \tan^{-1} t\right]_0^1 \\ &= 1 - \frac{\pi}{4}.\end{aligned}$$

Substituting this value gives $I = \frac{\pi}{4} - (1 - \frac{\pi}{4})$. The two inverse-tangent endpoint contributions add, because the remaining integral is subtracted.

Step 4: Conclusion Hence the value of the given integral is $\boxed{\frac{\pi}{2} - 1}$.

Quick Tip to Solve: Replace $\sin 2x$ by $2\sin x \cos x$ and set $t = \sin x$ before integrating by parts; take the inverse tangent as the differentiated factor.

34. Solve the following Linear Programming Problem graphically:

Maximize $P = 70x + 40y$ subject to the constraints $3x + 2y \leq 9$, $3x + y \leq 9$, $x \geq 0,\ y \geq 0$. [5 marks]

Solution:

Concept:

- Each linear inequality represents a half-plane. The feasible region is their common intersection in the first quadrant, since $x \geq 0$ and $y \geq 0$.
- For a bounded, nonempty feasible polygon, the maximum of a linear objective function occurs at a corner point. Here the objective is $P = 70x + 40y$.

Step 1: Plot both boundary lines Replace the inequalities by equalities to obtain the boundary lines. For $3x + 2y = 9$, putting $y = 0$ gives $x = 3$, and putting $x = 0$ gives $y = 9/2$. Thus this line joins $A(3, 0)$ and $B(0, 9/2)$. For $3x + y = 9$, the intercepts are $A(3, 0)$ and $C(0, 9)$. Plot these two lines on the same coordinate axes, including their boundaries because the inequalities contain equality.

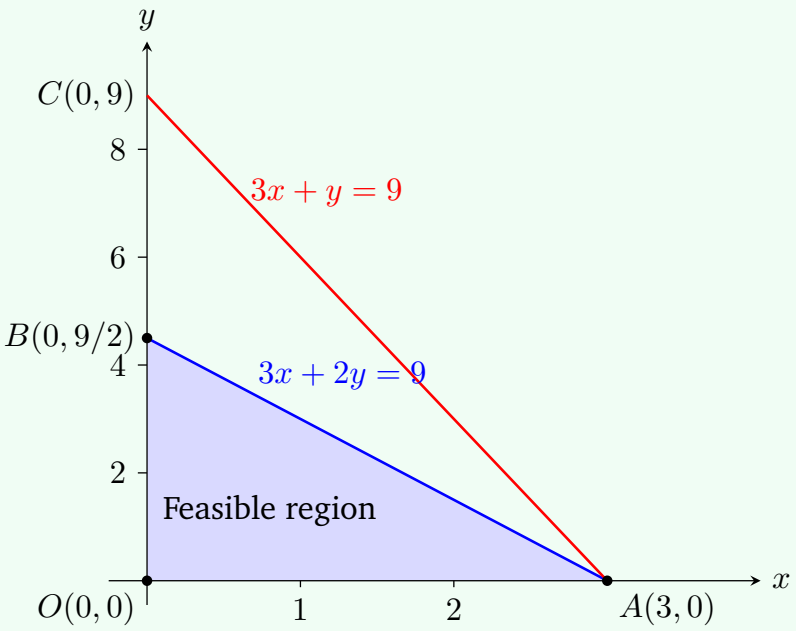
Step 2: Identify the common feasible region The origin satisfies both inequalities, since substituting $(0, 0)$ gives $0 \leq 9$ in each case. Hence take the side of each line containing the origin and restrict it to the first quadrant. To check how the boundaries intersect, subtract their equations:

$$(3x + 2y) - (3x + y) = 9 - 9,$$
$$y = 0, \qquad x = 3.$$

Also, for every point with $y \geq 0$,

$$3x + y \leq 3x + 2y \leq 9.$$

Thus the second constraint is automatically satisfied wherever the first holds in the first quadrant. The common feasible region is the closed shaded triangle OAB, with vertices $O(0, 0)$, $A(3, 0)$ and $B(0, 9/2)$. It is nonempty and bounded.



The shaded triangle OAB is the common feasible region.

Quick Tip to Solve: In the first quadrant, notice that $3x + y \leq 3x + 2y$: the second constraint adds no new edge, so test only the three vertices of the triangle under $3x + 2y = 9$.

Step 3: Evaluate the objective at each vertex Apply the corner-point method to the three vertices of the shaded region:

Corner point	$P = 70x + 40y$
$O(0,0)$	$70(0) + 40(0) = 0$
$A(3,0)$	$70(3) + 40(0) = 210$
$B(0,9/2)$	$70(0) + 40(9/2) = 180$

Comparing these values gives $210 > 180 > 0$. Along the sloping edge AB, substituting $y = (9 - 3x)/2$ also gives $P = 180 + 10x$, which increases as x runs from 0 to 3. This confirms that the maximizing vertex is unique.

Step 4: Conclusion The maximum value is $\boxed{P_{\max} = 210}$, attained at $\boxed{x = 3,\ y = 0}$.

Quick Tip to Solve: In the first quadrant, notice that $3x + y \le 3x + 2y$: the second constraint adds no new edge, so test only the three vertices of the triangle under $3x + 2y = 9$.

35(a). In answering a question on a multiple choice test, a student either knows the answer or guesses. Let $\frac{3}{5}$ be the probability that he knows the answer and $\frac{2}{5}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be correct with probability $\frac{1}{3}$, what is the probability that the student knows the answer, given that he answered it correctly? [5 marks]

Solution:

Concept:

- Bayes' theorem finds the probability of a cause after observing an outcome. For events K and C, $P(K \mid C) = P(K \cap C)/P(C)$, provided $P(C) > 0$.
- If knowing and guessing are mutually exclusive and exhaustive, the total probability of a correct response is $P(C) = P(K)P(C \mid K) + P(G)P(C \mid G)$, where G denotes guessing.

Step 1: Define the events and record the data Let K be the event that the student knows the answer, G the event that he guesses, and C the event that his response is correct. Every response belongs to exactly one of K or G. The given probabilities are

$$P(K) = \frac{3}{5}, \qquad P(G) = \frac{2}{5}, \qquad P(C \mid G) = \frac{1}{3}.$$

When the student knows the answer, he answers correctly, so $P(C \mid K) = 1$. Notice that the required probability is $P(K \mid C)$: correctness has already been observed, and we must determine how likely it is that the answer was known.

Step 2: Find the probability of a correct response A correct response can arise either from knowing the answer or from a successful guess. These two routes are disjoint. Hence, using the theorem of

Quick Tip to Solve: First add the probabilities of both routes to a correct answer; then divide the known-and-correct probability by this total.

total probability,

$$\begin{aligned} P(C) &= P(K)P(C \mid K) + P(G)P(C \mid G) \\ &= \frac{3}{5} \cdot 1 + \frac{2}{5} \cdot \frac{1}{3} \\ &= \frac{9}{15} + \frac{2}{15} = \frac{11}{15}. \end{aligned}$$

In particular, $P(K \cap C) = P(K)P(C \mid K) = 3/5$. The denominator must include successful guesses as well as known answers, since both produce a correct response.

Step 3: Apply Bayes' theorem Conditioning on a correct response restricts the sample space to event C. Therefore,

$$\begin{aligned} P(K \mid C) &= \frac{P(K \cap C)}{P(C)} \\ &= \frac{3/5}{11/15} = \frac{3}{5} \cdot \frac{15}{11} = \frac{9}{11}. \end{aligned}$$

As a check, the conditional probability of guessing is $(2/15)/(11/15) = 2/11$. These two conditional probabilities add to one, as they should. Also, observing a correct response increases the probability of knowing from $3/5$ to $9/11$.

Step 4: Conclusion The probability that the student knows the answer, given that he answered correctly, is $\boxed{\frac{9}{11}}$.

Quick Tip to Solve: First add the probabilities of both routes to a correct answer; then divide the known-and-correct probability by this total.

OR

35(b). A box contains 10 tickets, 2 of which carry a prize of ₹ 8 each, 5 of which carry a prize of ₹ 4 each, and the remaining 3 carry a prize of ₹ 2 each. If one ticket is drawn at random, find the mean value of the prize. [5 marks]

Solution:

Concept:

- A discrete random variable assigns a numerical value to each possible outcome. Its probability distribution lists each distinct value together with the probability of obtaining that value.
- The mean or expectation is the probability-weighted average $E(X) = \sum_i x_i P(X = x_i)$, where x_i are the possible values of the random variable X.

Step 1: Choose the random variable Let X denote the value, in rupees, of the prize on the ticket drawn. The possible prize values are 2, 4 and 8. Since the ticket is drawn at random from the box, each of the ten individual tickets is equally likely to be selected. Thus the probability of a particular prize value is the number of tickets carrying that prize divided by the total number of tickets. The three prize groups contain 3, 5 and 2 tickets respectively, and their total is $3 + 5 + 2 = 10$.

Quick Tip to Solve: Weight each prize by its ticket count, add the total prize value, and divide by the total number of tickets.

Step 2: Write and check the probability distribution The distribution of X is

x	2	4	8
$P(X=x)$	$\frac{3}{10}$	$\frac{5}{10}$	$\frac{2}{10}$

These probabilities are all non-negative and satisfy

$$\frac{3}{10}+\frac{5}{10}+\frac{2}{10}=1.$$

The prize values themselves are not equally likely: five tickets carry the prize of four rupees, whereas only two carry the prize of eight rupees. Consequently, an unweighted average of the three distinct prize values would not represent a random draw from this box.

Step 3: Calculate the expected prize Multiply each prize value by its probability and add the products:

$$\begin{aligned} E(X) &= 2\left(\frac{3}{10}\right)+4\left(\frac{5}{10}\right)+8\left(\frac{2}{10}\right) \\ &= \frac{6}{10}+\frac{20}{10}+\frac{16}{10} \\ &= \frac{42}{10}=4.2. \end{aligned}$$

Equivalently, the total prize value on all ten tickets is $3\times 2+5\times 4+2\times 8=42$ rupees; dividing this by ten gives the same mean. The mean lies between the smallest and largest prizes, which provides a useful consistency check. It represents the average prize value over many random draws under the same distribution; it need not itself be printed on any ticket.

Step 4: Conclusion The mean value of the prize is $\boxed{₹\,4.20}$.

Quick Tip to Solve: Weight each prize by its ticket count, add the total prize value, and divide by the total number of tickets.

Section E (12 marks)

This section comprises of 3 case study/passage-based questions of 4 marks each with two sub-parts. First two case study questions have three sub – parts (I), (II), (III) of marks 1, 1, 2 respectively. The third case study question has two sub – parts (I) and (II) of marks 2 each.

36. An organization conducted a bike race under two different categories - Boys and Girls. There were 28 participants in all. Among all of them, finally three from category 1 and two from category 2 were selected for the final race. Ravi forms two sets B and G with these participants for his college project.

Let $B = \{b_1, b_2, b_3\}$ and $G = \{g_1, g_2\}$, where B represents the set of Boys selected and G the set of Girls selected for the final race.

Based on the above information, answer the following questions. [4 marks]

36(I). How many relations are possible from B to G? [1 mark]

Solution:

Concept:

- A relation from B to G is any subset of the Cartesian product $B \times G$.
- A set with n elements has 2^n subsets, including the empty set and the entire set.

Step 1: Count the available ordered pairs The domain set has three boys and the target set has two girls. Thus $|B \times G| = |B||G| = 3 \times 2 = 6$. Each pair consists of one boy first and one girl second; reversing the order would describe a different Cartesian product.

Step 2: Count all subsets For each of these six ordered pairs, a relation either includes that pair or excludes it. These choices are independent, giving 2^6 subsets. No restriction of having exactly one image is imposed on a relation, so both the empty relation and relations containing several pairs for one boy are counted.

Step 3: Conclusion Hence the number of relations from B to G is $\boxed{64}$.

Quick Tip to Solve: Count ordered pairs first, then raise 2 to that count when counting relations.

36(II). Among all the possible relations from B to G, how many functions can be formed from B to G? [1 mark]

Solution:

Concept:

- A function assigns exactly one element of the codomain to each element of the domain.
- For finite sets with m domain elements and n codomain elements, the number of functions is n^m.

Step 1: Choose the image of each boy Each of b_1, b_2, b_3 must have precisely one image in $G = \{g_1, g_2\}$. There are two choices for the image of each boy. Different boys may have the same image; neither injectivity nor surjectivity is required here.

Step 2: Apply the multiplication principle The three choices of images are independent. Therefore the total number of functions is $2 \times 2 \times 2 = 2^3$. Unlike a general relation, a function cannot omit a boy or assign both girls to the same boy, so counting all subsets of $B \times G$ would overcount.

Quick Tip to Solve: For functions, use the codomain size as the base and the domain size as the exponent.

Step 3: Conclusion The number of functions from B to G is $\boxed{8}$.

Quick Tip to Solve: For functions, use the codomain size as the base and the domain size as the exponent.

36(III)(a). Let $R : B \to B$ be defined by $R = \{(x, y) : x$ and y are students of the same sex$\}$. Check if R is an equivalence relation. [2 marks]

Solution:

Concept:

- An equivalence relation must be reflexive, symmetric and transitive on its underlying set.
- Every member of $B = \{b_1, b_2, b_3\}$ is a boy, so any two members are of the same sex.

Step 1: Identify the relation and check reflexivity The definition applies to pairs (x, y) with $x, y \in B$. Since all three elements represent boys, every ordered pair satisfies the condition. Thus $R = B \times B$. In particular, for every $x \in B$, the pair (x, x) belongs to R, because a student is of the same sex as himself. Hence R is reflexive.

Step 2: Check symmetry Let $(x, y) \in R$. Then x and y are students of the same sex. The statement is unchanged when the students are interchanged; therefore $(y, x) \in R$. This holds for every pair in R, so the relation is symmetric.

Step 3: Check transitivity Let $(x, y) \in R$ and $(y, z) \in R$. Then x, y, z all belong to B and are boys. Consequently x and z are of the same sex, giving $(x, z) \in R$. Thus R is transitive as well.

Step 4: Conclusion Since R is reflexive, symmetric and transitive, $\boxed{R \text{ is an equivalence relation on } B}$.

Quick Tip to Solve: Use the stated underlying set: on this all-boy set, the relation is the universal relation.

OR

36(III)(b). A function $f : B \to G$ be defined by $f = \{(b_1, g_1), (b_2, g_2), (b_3, g_1)\}$. Check if f is bijective. Justify your answer. [2 marks]

Solution:

Concept:

- A function is bijective if and only if it is both one-one (injective) and onto (surjective).
- Injectivity requires distinct domain elements to have distinct images; surjectivity requires every codomain element to have a preimage.

Step 1: Read the images and test injectivity The given ordered pairs specify $f(b_1) = g_1$, $f(b_2) = g_2$ and $f(b_3) = g_1$. Each of the three domain elements has exactly one image, so this is a function. However, $b_1 \neq b_3$ while $f(b_1) = f(b_3) = g_1$. The two distinct boys therefore have the same image, which violates the condition for a one-one function.

Step 2: Test surjectivity The image set is $f(B) = \{g_1, g_2\} = G$. Specifically, g_1 has preimages b_1 and b_3, and g_2 has preimage b_2. Thus every element of the codomain is reached and f is onto. Reaching

Quick Tip to Solve: Check repeated images first: one repeated image of distinct inputs disproves bijectivity.

all girls alone is insufficient for bijectivity, because injectivity must hold simultaneously.

Step 3: Conclusion f is onto but not one-one, hence not bijective.

Quick Tip to Solve: Check repeated images first: one repeated image of distinct inputs disproves bijectivity.

---

37. Gautam buys 5 pens, 3 bags and 1 instrument box and pays a sum of ₹ 160. From the same shop, Vikram buys 2 pens, 1 bag and 3 instrument boxes and pays a sum of ₹ 190. Also Ankur buys 1 pen, 2 bags and 4 instrument boxes and pays a sum of ₹ 250.

Based on the above information, answer the following questions. [4 marks]

37(I). Convert the given above situation into a matrix equation of the form $AX = B$. [1 mark]

Solution:

Concept:

- Let x, y, z denote the prices, in rupees, of one pen, one bag and one instrument box, respectively.
- In $AX = B$, the rows of A contain purchase quantities in the same order as the unknowns in X.

Step 1: Form the three purchase equations Using the purchases made by Gautam, Vikram and Ankur in that order gives

$$5x + 3y + z = 160, \qquad 2x + y + 3z = 190, \qquad x + 2y + 4z = 250.$$

Every coefficient is the number of units purchased; each right-hand side is that customerś total payment.

Step 2: Arrange the coefficients and totals Take the unknown column as $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Keeping the same customer order, the coefficient rows are $(5, 3, 1)$, $(2, 1, 3)$ and $(1, 2, 4)$, while the payment column is $(160, 190, 250)^T$. Matrix multiplication then reproduces each purchase equation.

Step 3: Conclusion The required matrix equation is

$$\boxed{\begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 160 \\ 190 \\ 250 \end{pmatrix}.}$$

Quick Tip to Solve: Fix the order pen, bag, instrument box before writing every coefficient row.

37(II). Find $|A|$. [1 mark]

Solution:

Concept:

- The coefficient matrix is $A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}$.
- Expansion along the first row uses the cofactor sign pattern $+, -, +$.

Step 1: Expand along the first row Using the quantities in the purchase equations,

$$|A| = 5\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}.$$

The middle term is subtracted because the second entry in the first row has a negative cofactor sign.

Step 2: Evaluate the minors Each two by two determinant is the main-diagonal product minus the other-diagonal product. Thus

$$|A| = 5(4-6) - 3(8-3) + (4-1) = -10 - 15 + 3.$$

The payment amounts do not appear in this determinant because A is the coefficient matrix.

Step 3: Conclusion $\boxed{|A| = -22}$.

Quick Tip to Solve: Write the alternating cofactor signs before evaluating the three first-row minors.

37(III)(a). Find A^{-1}. [2 marks]

Solution:

Concept:

- For a nonsingular square matrix, $A^{-1} = \dfrac{1}{|A|}\operatorname{adj} A$.
- The adjugate is the transpose of the cofactor matrix; each cofactor includes its alternating sign.

Step 1: Find the determinant Here $A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}$. Expanding along its first row,

$$|A| = 5(4-6) - 3(8-3) + (4-1) = -22.$$

Since this is nonzero, the inverse exists and the adjugate formula is applicable.

Step 2: Calculate the cofactors and transpose The first row of cofactors is $(-2, -5, 3)$. The second row is $(-(12-2), 20-1, -(10-3)) = (-10, 19, -7)$, and the third is $(9-1, -(15-2), 5-6) = (8, -13, -1)$.

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; cofactors alone are not the adjugate.

Consequently

$$C = \begin{pmatrix} -2 & -5 & 3 \\ -10 & 19 & -7 \\ 8 & -13 & -1 \end{pmatrix}, \qquad \operatorname{adj} A = C^T = \begin{pmatrix} -2 & -10 & 8 \\ -5 & 19 & -13 \\ 3 & -7 & -1 \end{pmatrix}.$$

Step 3: Apply the inverse formula Dividing the adjugate by -22 changes all its signs. As a check, multiplying A by the numerator matrix $\begin{pmatrix} 2 & 10 & -8 \\ 5 & -19 & 13 \\ -3 & 7 & 1 \end{pmatrix}$ gives $22I$; therefore dividing this product by 22 gives the identity matrix, as required for an inverse.

Step 4: Conclusion

$$\boxed{A^{-1} = \frac{1}{22}\begin{pmatrix} 2 & 10 & -8 \\ 5 & -19 & 13 \\ -3 & 7 & 1 \end{pmatrix}.}$$

Quick Tip to Solve: Transpose the cofactor matrix before dividing by the determinant; cofactors alone are not the adjugate.

OR

37(III)(b). Determine $P = A^2 - 5A$. [2 marks]

Solution:

Concept:

- The coefficient matrix of the purchases, in the order pens, bags and instrument boxes, is $A = \begin{pmatrix} 5 & 3 & 1 \\ 2 & 1 & 3 \\ 1 & 2 & 4 \end{pmatrix}$.
- The entry in row i, column j of A^2 is the scalar product of row i of A with column j of A. Matrix subtraction is performed entry by entry.

Step 1: Calculate the square of the coefficient matrix Using rows of the first factor and columns of the second factor gives

$$A^2 = \begin{pmatrix} 25+6+1 & 15+3+2 & 5+9+4 \\ 10+2+3 & 6+1+6 & 2+3+12 \\ 5+4+4 & 3+2+8 & 1+6+16 \end{pmatrix} = \begin{pmatrix} 32 & 20 & 18 \\ 15 & 13 & 17 \\ 13 & 13 & 23 \end{pmatrix}.$$

For example, the $(1,2)$ entry is $5(3) + 3(1) + 1(2) = 20$. The multiplication is ordinary matrix multiplication, not squaring individual entries.

Quick Tip to Solve: First form all nine row-column products for A^2; only then subtract five times the matching entry of A.

Step 2: Form the scalar multiple and subtract corresponding entries The scalar multiple is

$$5A = \begin{pmatrix} 25 & 15 & 5 \\ 10 & 5 & 15 \\ 5 & 10 & 20 \end{pmatrix}.$$

Consequently the rows of $A^2 - 5A$ are obtained by subtracting $(25, 15, 5)$, $(10, 5, 15)$ and $(5, 10, 20)$ from the corresponding rows of A^2. For instance, its second row has entries $15 - 10$, $13 - 5$ and $17 - 15$.

Step 3: Conclusion Thus $P = A^2 - 5A = \begin{pmatrix} 7 & 5 & 13 \\ 5 & 8 & 2 \\ 8 & 3 & 3 \end{pmatrix}$.

Quick Tip to Solve: First form all nine row-column products for A^2; only then subtract five times the matching entry of A.

38. An equation involving derivatives of the dependent variable with respect to the independent variables is called a differential equation. A differential equation of the form $\frac{dy}{dx} = F(x, y)$ is said to be homogeneous if $F(x, y)$ is a homogeneous function of degree zero, whereas a function $F(x, y)$ is a homogeneous function of degree n if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$. To solve a homogeneous differential equation of the type $\frac{dy}{dx} = F(x, y) = g\left(\frac{y}{x}\right)$, we make the substitution $y = vx$ and then separate the variables.

Based on the above, answer the following questions. [4 marks]

38(I). Show that $(x^2 - y^2)\,dx + 2xy\,dy = 0$ is a differential equation of the type $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$. [2 marks]

Solution:

Concept:

- A first-order equation is homogeneous in the required sense when its right-hand side depends only on the ratio $v = y/x$.
- A homogeneous function of degree zero satisfies $F(\lambda x, \lambda y) = F(x, y)$, wherever the expressions are defined and $\lambda \neq 0$.

Step 1: Isolate the derivative Starting with the printed equation

$$(x^2 - y^2)\,dx + 2xy\,dy = 0,$$

divide by dx along a differentiable solution curve. This gives

$$2xy\frac{dy}{dx} = y^2 - x^2, \qquad \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

The explicit derivative form is considered where $xy \neq 0$, so its denominator is nonzero.

Quick Tip to Solve: Divide the numerator and denominator of the derivative by x^2 to expose the ratio y/x.

Step 2: Express the right-hand side using one ratio Divide both numerator and denominator by x^2:

$$\frac{y^2 - x^2}{2xy} = \frac{(y/x)^2 - 1}{2(y/x)}.$$

Set $v = y/x$ and define $g(v) = (v^2 - 1)/(2v)$ for $v \neq 0$. The right-hand side contains no separate occurrence of x or y. Equivalently, scaling both variables by any nonzero λ multiplies its numerator and denominator by λ^2, leaving their ratio unchanged.

Step 3: Conclusion Hence $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$, with $g(v) = \frac{v^2 - 1}{2v}$, so the equation has the required homogeneous form.

Quick Tip to Solve: Divide the numerator and denominator of the derivative by x^2 to expose the ratio y/x.

38(II). Solve the above equation to find its general solution. [2 marks]

Solution:

Concept:

- For a homogeneous equation, substitute $y = vx$; the product rule gives $dy/dx = v + x\,dv/dx$.
- After separation, use $\int 2v/(1 + v^2)\,dv = \ln(1 + v^2)$ and $\int dx/x = \ln|x|$.

Step 1: Substitute and separate the variables The preceding equation yields

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}.$$

On an interval where $xy \neq 0$, put $y = vx$. Therefore

$$\begin{aligned} v + x\frac{dv}{dx} &= \frac{v^2 - 1}{2v}, \\ x\frac{dv}{dx} &= -\frac{v^2 + 1}{2v}, \\ \frac{2v}{1 + v^2}\,dv &= -\frac{dx}{x}. \end{aligned}$$

Since $1 + v^2 > 0$ for real v, dividing by this expression loses no real constant-ratio solution.

Step 2: Integrate and return to the original variables Integrating gives

$$\ln(1 + v^2) = -\ln|x| + C_1.$$

Thus $|x|(1 + v^2)$ is constant. On an interval with a fixed sign of x, absorb that sign into an arbitrary constant C. Substitution of $v = y/x$ gives

$$x\left(1 + \frac{y^2}{x^2}\right) = C.$$

As a check, differentiating $x^2 + y^2 = Cx$ gives $2x + 2y\,dy/dx = C$. Eliminating $C = (x^2 + y^2)/x$

Quick Tip to Solve: After integrating with $y = vx$, multiply the resulting equation by x to remove the ratio and obtain a simple implicit solution.

recovers the original differential equation wherever $x \neq 0$.

Step 3: Conclusion The general solution is $\boxed{x^2 + y^2 = Cx}$, where C is an arbitrary constant; its differentiable branches satisfy the given equation.

Quick Tip to Solve: After integrating with $y = vx$, multiply the resulting equation by x to remove the ratio and obtain a simple implicit solution.